Fisher information of sequential measurements and the optimization in composite systems

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(Received 25 January 2023; revised 11 June 2023; accepted 23 June 2023; published 17 July 2023)

In parameter estimation, the saturation of the quantum Cramér-Rao bound (QCRB) requests one to perform projective measurements on the eigenvectors of the symmetric logarithmic derivative (SLD). However, it maybe infeasible to saturate the QCRB in composite systems on account of the technological difficulties. In practice, it is necessary to perform sequential measurements on different subsystems to obtain a precise estimation. In this paper, we derive the expression of Fisher information (FI) for a sequential measurements scheme in composite systems with N interacting subsystems. The result shows that the more subsystems used, the higher FI obtained due to the nonnegative property of FI. We illustrate the optimal measurements for different conditional density matrices in every subsystems are nonidentical, which depends on the previous outcomes. In addition, utilizing the convexity property of the quantum FI, we prove that the optimal projective measurements on the eigenvectors of the SLD of the subsystem reduced density matrix can be regarded as a trade off of the nonidentical optimal measurements of the conditional density matrices in every subsystems. In addition, importantly, we show the measurement sequence of the subsystem should be considered in the optimization to obtain a precise estimation for the sequential measurements scheme. Finally, we apply the theory to a nuclear spins control model where we concern the estimation of nuclear Larmor frequency. We should point out that the precision obtained by the sequential measurements scheme is no better than the theoretical result in the global system limited by the QCRB but more realistic, and better than the local measurements in one of the subsystems.

DOI: 10.1103/PhysRevA.108.012419

I. INTRODUCTION

Metrology, a basic statistical task, started from the pioneering work of Fisher for estimating the unknown parameters [1,2], which was advanced by quantum mechanics later. In quantum metrology, the estimation precision is bounded by the quantum Cramér-Rao bound (QCRB) [3–6], namely, the variance of the estimation is at least as high as the inverse of the quantum Fisher information (QFI).

A complete progress of quantum parameter estimation includes three steps: preparation, parametrization, and measurement. Achieving a high estimation precision is a significant project in production and practice. Therefore, it is necessary to optimize the above three steps. For preparation, one could employ quantum advantages, such as squeezing [7] and entanglement [8–10]. With an entangled quantum state, the precision can be improved from the classical shot-noise limit (scaling as $N^{-1/2}$) to the Heisenberg limit (scaling as N^{-1}), and the more resources used, the higher the precision obtained [11–13]. For parametrization, the unitary dynamic is the main approach to encoding the unknown parameters into a quantum states [14–18]. One can take the Hamiltonian extension or subtraction to obtain a higher precision [18]. In addition, the optimization of measurement is also vital to attain the QCRB.

It is known that if the quantum state is parameterized with a single unknown parameter, one such measurement is the projective measurements on the eigenvectors of the symmetric logarithmic derivative (SLD) [4,5]. However, the optimal measurement may be physically possible but technologically difficult in composite systems due to the multiformity and complexity of the quantum systems, such as two entangled particles with a long distance, thermodynamics [19–21], and optomechanical [22] systems. Therefore, the optimal global measurement is generally infeasible in composite systems [22]. Fortunately, the interaction leads to the entanglement above all the subsystems and the information of the unknown parameter is imprinted in the states of all the subsystems, so that the local measurements on the subsystems have attracted a great deal of interest on account of their accessibility in practice [23–25]. Recently, the authors of Ref. [26] showed a sequential measurements scheme with a many-body probe for estimating a local magnetic field upon its first qubit where the measurement was performed on the last one, which could reach the Heisenberg bound with a given measurement basis. In such a scheme, the QFI of the global system is unachievable. This gives rise to the following questions. What is the expression of the FI if one performs the sequential measurements scheme on different subsystems? How does one obtain a high precision in this scheme if possible? Additionally, should the sequence of the measurement on different subsystems be considered in the optimization?

In this paper, we will answer the above issues and optimize the sequential measurements scheme to obtain a precise estimation in composite systems. We first derive the expression of FI for sequential measurements with N interacting subsystems. It shows that the FI is a sum of different subsystems FI and the more subsystems used, the higher FI obtained due to the nonnegative property of FI. In addition, the optimization of the measurement on different subsystems is also considered.

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While one performs the local measurements in one subsystem, the density matrix collapses to the corresponding subspace, which is dependent on the previous measurement. Therefore, one needs to optimize the next measurement according to the outcomes. More specifically, the optimization should be done for different conditional density matrices in every subsystem and these optimal measurements are generally nonidentical. Then, utilizing the convexity property of the QFI, we prove that the optimal projective measurements on the eigenvectors of the SLD of the subsystem reduced density matrix can be regarded as a trade off of the nonidentical optimal measurements of the conditional density matrices in every subsystem. Finally, we point out that it is necessary to optimize the measurement sequence of the subsystems due to the fact that the FI is dependent on the previous outcomes. To illustrate our theory, we apply it to the nuclear spins control model where we are concerned with the estimation of the nuclear Larmor frequency.

This paper is organized as follows. We introduce the theoretical background, derive the expression of FI for the sequential measurements scheme, and give the trade off for the nonidentical optimal measurements of different conditional density matrices in Sec. II. In Sec. III, we apply the theory to the nuclear spins control model where we focus on the estimation of the nuclear Larmor frequency and compare the FI for different measurement schemes. Finally, we conclude and discuss the results in Sec. IV. The appendices are provided as a supplement for the derivation and discussion in the main text.

II. THEORETICAL FRAMEWORK

In this section, we first provide a summary of the necessary notations for single parameter quantum metrology theory, which will be used in the following. Then we give the general framework to calculate the FI of the sequential measurements scheme in composite systems with N interacting subsystems and give the optimal measurements for every subsystem. Next, we prove that the optimal projective measurements on the eigenvectors of the SLD of the reduced density matrix can be regarded as the trade off of the nonidentical optimal measurements of the conditional density matrices in every subsystem by means of the convexity property of the QFI. Finally, we illustrate that the measurement sequence of the subsystems should be considered in the optimization if we want to obtain as precise an estimation as possible.

A. Background

We consider an unknown parameter θ which is encoded into a parameterized density matrix ρ_θ and are concerned about the estimation of it. We perform the positive-operator-valued measurements (POVMs) $\{M_x\}$ satisfying $\sum_x M_x = \mathbb{I}$, where x is the outcome of M_x and $\mathbb{I}$ is the identity operator, and the corresponding probability distribution of the outcome x is $p_\theta(x) = \mathrm{Tr}(\rho_\theta M_x)$. For a statistical sample with n outcomes $X_n = \{x_1, \dots, x_n\}$, we can construct an unbiased estimator $\hat{\theta}(X_n)$, whose statistical average is $E[\hat{\theta}(X_n)] = \theta$, to estimate the value of θ. The variance of the unbiased estimator $\hat{\theta}$ is bounded by the Cramér-Rao bound as $\delta\hat{\theta}^2 \geqslant \frac{1}{nF_c}$, here $\delta\hat{\theta}^2 = E(\hat{\theta} - \theta)^2$ is the variance of the unbiased estimator and n is the number of repetitions for the procedure, which is assumed to be asymptotically large. In addition, F_c is the classical FI [1]

$$F_c = \sum_x \frac{1}{p_\theta(x)} \left[\frac{\partial p_\theta(x)}{\partial \theta}\right]^2. \tag{1}$$

The ultimate bound can be obtained upon maximizing the classical FI over the set of all possible POVMs. Then we get the quantum Cramér-Rao bound $(\delta\hat{\theta})^2 \geqslant \frac{1}{n\mathcal{F}_Q}$, where $\mathcal{F}_Q = \mathrm{Tr}(\rho_\theta L_\theta^2)$ is the QFI, L_θ is the SLD which obeys the operator equation $\frac{\partial \rho_\theta}{\partial \theta} = \frac{1}{2}(\rho_\theta L_\theta + L_\theta \rho_\theta)$ [4,5]. With the spectral decomposition of the density matrix $\rho_\theta = \sum_n \mu_n |\psi_n\rangle\langle\psi_n|$, we obtain the expression of the SLD, which can be expressed as $L_\theta = \sum_{n,m} \frac{2\langle\psi_n|\partial_\theta \rho_\theta|\psi_m\rangle}{\mu_n + \mu_m} |\psi_n\rangle\langle\psi_m|$ where $\mu_n + \mu_m \neq 0$. Accordingly, the optimal measurement M_x satisfies [4]

$$\begin{gathered} \mathrm{Im}(\mathrm{Tr}[\rho_\theta M_x L_\theta]) = 0, \\ \frac{\sqrt{M_x}\sqrt{\rho_\theta}}{\mathrm{Tr}[\rho_\theta M_x]} = \frac{\sqrt{M_x} L_\theta \sqrt{\rho_\theta}}{\mathrm{Tr}[\rho_\theta M_x L_\theta]}, \end{gathered} \tag{2}$$

where Im denotes the imaginary part. Equation (2) shows that the optimal measurement M_x are the projective measurements on the eigenvectors of the SLD. Next, we will give the expression of the FI for sequential measurements scheme in composite systems with N interacting subsystems.

B. FI of sequential measurements scheme in composite systems with N interacting subsystems

Consider a composite quantum system that consists of N interacting subsystems and the general time-independent Hamiltonian can be expressed as

$$H_N(\theta) = \sum_{\alpha=1}^{N} H_\alpha + H_{\mathrm{int}}, \tag{3}$$

where H_α is the Hamiltonian of the α_{th} subsystem and H_{int} is the interaction Hamiltonian of the N subsystems. The unknown Hamiltonian parameter θ is encoded into an initial density matrix ρ_0 by the unitary dynamic $\rho_\theta = U\rho_0 U^\dagger$, here, $U = e^{-iH_N(\theta)t}$ is the unitary operator, and we are concerned about the estimation of θ in the following.

While we perform the sequential measurements on the subsystems one by one to estimate the unknown θ as the schematic shown in Fig. 1, the FI is calculated as the following (the details are shown in Appendix A):

$$\begin{aligned} F(\rho_\theta) = {} & F(\rho_\theta^1 | M^1) + \sum_{n_i^1} p_\theta(n_i^1) F\big(\rho_\theta^{2|n_i^1} | M^{2|n_i^1}\big) \\ & + \cdots + \sum_{n_i^1,\dots,n_j^{N-2},n_k^{N-1}} p_\theta\big(n_i^1,\dots,n_j^{N-2},n_k^{N-1}\big) F\big(\rho_\theta^{N|n_i^1,\dots,n_j^{N-2},n_k^{N-1}} | M^{N|n_i^1,\dots,n_j^{N-2},n_k^{N-1}}\big), \end{aligned} \tag{4}$$

where

$$\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}} = \frac{\mathrm{Tr}_{\neq\alpha}\left[\left(M_i^1\otimes\cdots\otimes M_s^{\alpha-1|n_i^1,\ldots,n_l^{\alpha-2}}\otimes\mathbb{I}^\alpha\otimes\cdots\otimes\mathbb{I}^N\right)\rho_\theta\right]}{p_\theta\left(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}\right)}, \alpha\in[1,N], \tag{5}$$

is the conditional density matrix for the α_{th} subsystem with $\alpha-1$ outcomes $n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}$ [23], where $n_s^{\alpha-1}$ is the outcome in $(\alpha-1)_{th}$ subsystem with the measurement $M_s^{\alpha-1|n_i^1,\ldots,n_l^{\alpha-2}}$. The partial trace in Eq. (5) is over the subsystems except the α_{th}. While $\alpha=1$, $\rho_\theta^1=\mathrm{Tr}_{\neq1}(\rho_\theta)$ is the reduced density matrix of the first subsystem. $p_\theta(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1})=p_\theta^1(n_i^1)\cdots p_\theta^{\alpha-1}(n_s^{\alpha-1}|n_i^1,\ldots,n_l^{\alpha-2})$ is the joint probability for the $\alpha-1$ outcomes, and $p_\theta^{\alpha-1}(n_s^{\alpha-1}|n_i^1,\ldots,n_l^{\alpha-2})=\mathrm{Tr}(\rho_\theta^{\alpha-1|n_i^1,\ldots,n_l^{\alpha-2}}M_s^{\alpha-1|n_i^1,\ldots,n_l^{\alpha-2}})$ is the conditional probability for the outcome $n_s^{\alpha-1}$. $F(\rho_\theta^1|M^1)=\sum_{n_i^1}p_\theta^1(n_i^1)[\frac{\partial\ln p_\theta^1(n_i^1)}{\partial\theta}]^2$ is the FI of the reduced density matrix ρ_θ^1 for the first subsystem, $F(\rho_\theta^{2|n_i^1}|M^{2|n_i^1})=\sum_{n_j^2}p_\theta^2(n_j^2|n_i^1)[\frac{\partial\ln p_\theta^2(n_j^2|n_i^1)}{\partial\theta}]^2$ is the FI of the conditional density matrix $\rho_\theta^{2|n_i^1}$ for the second subsystem with outcome n_i^1 in the first one, and so on. Equation (4) shows that the FI of the sequential measurements is a sum of different subsystems FI and the more subsystems used, the higher the FI obtained on account of the nonnegative property of FI.

To obtain as a high an FI as possible, one should optimize the measurements performed on the subsystems with different conditional density matrices to obtain a precise estimation of the unknown parameter. According to Eq. (2), it is necessary to perform the projective measurements on the eigenvectors of SLD $L_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ to obtain the QFI of the conditional density matrices $\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ in every subsystem, where the corresponding SLD satisfies $\frac{\partial\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}}{\partial\theta}=\frac{1}{2}(L_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}+\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}L_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}})$. Therefore, the optimal measurements which are dependent on the previous outcomes are generally nonidentical. Next, we will prove the optimal projective measurements on the eigenvectors of the SLD of the subsystem reduced density matrix can be regarded as the trade off of the nonidentical optimal measurements of the corresponding conditional density matrices, which allows us to choose the identical measurement in every subsystems regardless of the previous outcomes, but the precision is no better than the nonidentical optimal measurement scheme.

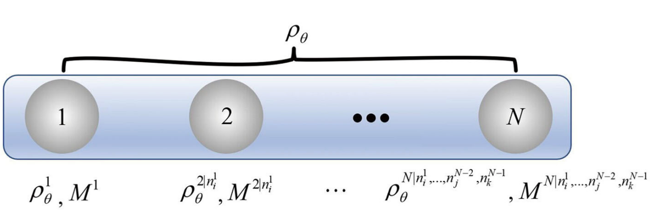

FIG. 1. A schematic of the sequential measurements scheme with N interacting subsystems, where ρ_θ is the parameterized density matrix and θ is the unknown parameter to be estimated. The measurements performed on different subsystems with conditional density matrices are dependent on the previous outcomes as listed in the schematic.

C. Trade off of the nonidentical optimal measurements of different conditional density matrices

In the previous subsection, we showed the FI of the sequential measurements scheme is a sum of the subsystems FI. To obtain a high FI in the subsystems, one should perform the projective measurements on the eigenvectors of the SLD $L_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ of different conditional density matrices $\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$, where the optimal measurements are dependent on the previous outcomes and generally nonidentical. Here, by means of the convexity property of the QFI, we show the optimal projective measurements on the eigenvectors of the SLD of the reduced density matrix ρ_θ^α (take the partial trace on ρ_θ over the subsystems except α_{th}) can be regarded as the trade off of the nonidentical optimal measurements of the conditional density matrices for every subsystem.

Notice the intrinsic relation between the conditional density matrices $\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ and the reduced density matrix ρ_θ^α for the α_{th} subsystem

$$\begin{aligned}\rho_\theta^\alpha &\equiv \mathrm{Tr}_{\neq\alpha}(\rho_\theta)\\ &= \sum_{n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}} p_\theta\left(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}\right)\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}},\end{aligned} \tag{6}$$

where the partial trace is over all the subsystems except α_{th}, $\sum_{n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}p_\theta(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1})=1$, Eq. (6) shows that the reduced density matrix ρ_θ^α of the α_{th} subsystem is the superposition of the conditional density matrix $\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ with the probability $p_\theta(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1})$. Utilizing the convexity property of the QFI [27–30]

$$\sum_j c_j\mathcal{F}_Q(\rho_\theta^j)\geqslant\mathcal{F}_Q\left(\sum_j c_j\rho_\theta^j\right), \tag{7}$$

where c_j is the weight of ρ_θ^j satisfying $\sum_j c_j=1$. Combining Eq. (6), we can obtain

$$\begin{aligned}&\sum_{n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}} p_\theta\left(n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}\right)\mathcal{F}_Q\left(\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}\right)\\ &\geqslant\mathcal{F}_Q(\rho_\theta^\alpha),\end{aligned} \tag{8}$$

where $\mathcal{F}_Q(\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}})=\mathrm{Tr}[\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}(L_\theta^{\alpha|n_i^1,\cdots,n_l^{\alpha-2},n_s^{\alpha-1}})^2]$ and $\mathcal{F}_Q(\rho_\theta^\alpha)=\mathrm{Tr}[\rho_\theta^\alpha(L_\theta^\alpha)^2]$ are the QFI of the conditional density matrices $\rho_\theta^{\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1}}$ and the reduced density matrix ρ_θ^α in the α_{th} subsystem, respectively, L_θ^α is the corresponding SLD satisfying

$\frac{\partial\rho_\theta^\alpha}{\partial\theta} = \frac{1}{2}(L_\theta^\alpha\rho_\theta^\alpha + \rho_\theta^\alpha L_\theta^\alpha)$. Equation (8) shows that the FI obtained by performing the local optimal measurements on the subsystems with reduced density matrix ρ_θ^α is no higher than the nonidentical optimal measurements with the different conditional density matrices. In addition, the optimal measurements for saturating $\mathcal{F}_Q(\rho_\theta^{\alpha|n_i^1,\dots,n_l^{\alpha-2},n_s^{\alpha-1}})$ and $\mathcal{F}_Q(\rho_\theta^\alpha)$ are the projective measurements on the eigenvectors of $L_\theta^{\alpha|n_i^1,\dots,n_l^{\alpha-2},n_s^{\alpha-1}}$ and L_θ^α, respectively. Therefore, the optimal projective measurements on the eigenvectors of the L_θ^α can be regarded as the trade off of the nonidentical optimal measurements of different conditional density matrices. With this identical measurement, the FI in Eq. (4) can be rewritten as

$$\tilde{F}(\rho_\theta) = F(\rho_\theta^1|M^1) + \sum_{n_i^1} p_\theta(n_i^1)F\left(\rho_\theta^{2|n_i^1}\middle|M^2\right) + \cdots + \sum_{n_i^1,\dots,n_j^{N-2},n_k^{N-1}} p_\theta(n_i^1,\dots,n_j^{N-2},n_k^{N-1})F\left(\rho_\theta^{N|n_i^1,\dots,n_j^{N-2},n_k^{N-1}}\middle|M^N\right), \tag{9}$$

where M^α denote the identical measurements for the conditional density matrices $\rho_\theta^{\alpha|n_i^1,\dots,n_l^{\alpha-2},n_s^{\alpha-1}}$ in α_{th} subsystem and the optimal identical measurements are independent on the previous outcomes only dependent on the subsystem reduced density matrix. And we should point out that the FI $\tilde{F}(\rho_\theta)$ in Eq. (9) is no higher than $F(\rho_\theta)$ given by Eq. (4) limited by the convexity property of QFI shown in inequality (8). In addition, we should point out that in order to obtain a high FI in Eq. (4), the measurement sequence of the subsystems should be considered in the optimization. In contrast, if one choose the identical measurement on the subsystem for different conditional density matrices no matter what the outcomes are in the previous, the sequence of the subsystem has no influence on the FI, and these will be shown in the next section with numerical results.

III. EXAMPLE

In the previous section, we showed the general framework for calculating the FI of the sequential measurements scheme in composite systems with N interacting subsystems and proved the trade off of the nonidentical optimal measurements of the corresponding conditional density matrices are the projective measurement on the SLD of the reduced density matrices in every subsystem. Here, we apply the theory to the nuclear spins control model where we are concerned with the estimation of nuclear Larmor frequency, compare the FI for the sequential measurements scheme and local measurements scheme, and examine the trade off relation of the nonidentical optimal measurements. In addition, considering that the FI of the sequential measurements scheme is dependent on the previous outcomes in Eq. (4), therefore, the measurement sequence of the subsystems should be considered if we want to obtain a precise estimation, which is shown with the numerical results in the following.

Control experiments of nuclei in solids are ubiquitous in quantum technology setups. The nitrogen-vacancy (NV) center in diamond has been found to be a powerful platform for various sensing applications [31–34]. The Hamiltonian of the NV center electronic spin and the nuclear spin with dipole-dipole interaction under an on-resonance drive is given by [35–38]

$$H(\omega_l) = \frac{\omega_l}{2}I_z + \frac{\omega_0}{2}\sigma_z + g\sigma_z I_x + \Omega_1\sigma_x\cos(\omega_0 t), \tag{10}$$

where ω_l is the Larmor frequency of the nuclear spin; ω_0 is the energy gap of the electronic spin; I_x, I_z, σ_x, and σ_z are the Pauli operators of the nuclei and the electronic spin, respectively. g is the electron-nucleus coupling strength and Ω_1 is the Rabi frequency of the drive. We move to the interaction picture to eliminate the time-dependence of the drive and make the rotating-wave-approximation assuming $\omega_0 \gg \Omega_1$ [38,39], we obtain

$$H_I(\omega_l) = \frac{\Omega_1}{2}\sigma_x + \frac{\omega_l}{2}I_z + g\sigma_z I_x. \tag{11}$$

Next, we employ Hamiltonian (11) to encode the concerned parameter ω_l into an initial density matrix $\rho_0 = |\Phi(0)\rangle\langle\Phi(0)|$, here $|\Phi(0)\rangle = \sin\phi|g^e, e^n\rangle + \cos\phi|e^e, g^n\rangle$, where $|g^\beta\rangle$ and $|e^\beta\rangle$, $\beta \in \{e, n\}$ are the ground and excited states of the electronic and nuclear spins, respectively. In addition, the parameterized density matrix is

$$\rho_{\omega_l} = U\rho_0 U^\dagger = |\Phi(\omega_l)\rangle\langle\Phi(\omega_l)|, \tag{12}$$

where $U = e^{-iH_I(\omega_l)t}$ is the unitary operator and

$$\begin{aligned}|\Phi(\omega_l)\rangle = \tfrac{1}{2}[&e^{-iE_1t}(\cos\theta_+ + \sin\theta_+)|E_1\rangle \\ &+ e^{-iE_2t}(\sin\varphi_+ - \cos\varphi_+)|E_2\rangle \\ &+ e^{-iE_3t}(\cos\theta_+ - \sin\theta_-)|E_3\rangle \\ &+ e^{-iE_4t}(\sin\varphi_- - \cos\varphi_-)|E_4\rangle],\end{aligned} \tag{13}$$

where $\tan\theta_+ = \frac{g}{\alpha+\sqrt{\alpha^2+g^2}}$, $\tan\varphi_+ = \frac{g}{\beta+\sqrt{\beta^2+g^2}}$, $\tan\theta_- = \frac{g}{\alpha-\sqrt{\alpha^2+g^2}}$, and $\tan\varphi_- = \frac{g}{\beta-\sqrt{\beta^2+g^2}}$, here, $\alpha = \frac{\omega_l-\Omega_1}{2}$ and $\beta = \frac{\omega_l+\Omega_1}{2}$, and we set $\phi = \frac{\pi}{4}$ in Eq. (13) for simplicity. E_i and $|E_i\rangle$ are the eigenvalues and eigenvectors of Hamiltonian (11), respectively. The details for calculating $|\Phi(\omega_l)\rangle$ are shown in the Appendix B.

Then we consider the local measurement M_i^1 on the first subsystem (the first subsystem could be either the electron or nucleus subsystem, and we will show the numerical results of different measurement sequences in the following), according to Eq. (5), the corresponding conditional density matrix of another subsystem can be expressed as

$$\rho_{\omega_l}^{2|n_i^1} = \frac{\mathrm{Tr}_1\left[\left(M_i^1 \otimes \mathbb{I}^2\right)\rho_{\omega_l}\right]}{p_{\omega_l}^1(n_i^1)}, \tag{14}$$

where $p_{\omega_l}^1(n_i^1) = \mathrm{Tr}[\rho_{\omega_l}(M_i^1 \otimes \mathbb{I}^2)]$ is the probability of the outcome n_i^1 in the first subsystem and $\mathbb{I}^2$ is the identity

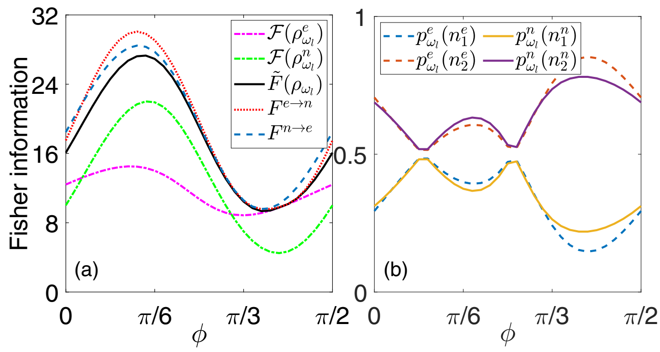

FIG. 2. (a) The FI of different measurement schemes with respect to ϕ of different initial states. The red dotted and blue dashed lines for $F^{e\to n}$ and $F^{n\to e}$ are the results of the sequential measurements scheme with different measurement sequence of the subsystems. The black solid line is the result of the trade off of the nonidentical optimal measurements on the eigenvectors of the conditional density matrix $\rho_{\omega_l}^{2|n_i^1}$ whatever the outcomes are in the first subsystem and $\tilde{F}(\rho_{\omega_l})$ is independent on the measurement sequence of the subsystem. The green dash-dotted and the magenta dash-dotted lines are the QFI with the local optimal measurements in one of the subsystem with the reduced density matrices $\rho_{\omega_l}^{\beta}$, $\beta\in\{e,n\}$, of different subsystems. (b) The probability $p_{\omega_l}^{\beta}(n_i^{\beta})$, of the measurement in the first subsystem for different measurement sequence where the dashed lines and solid lines correspond to $F^{e\to n}$ and $F^{n\to e}$, respectively, performing projective measurement M_i^{β}, on SLD of the corresponding reduced density matrices $\rho_{\omega_l}^{\beta}$ with outcomes n_i^{β}. The system parameters are chosen as $\Omega_1 = 3$, $\omega_l = 2$, $g = 2$, and $t = 10$.

operator of another subsystem. Next, we perform another local measurement $M_j^{2|n_i^1}$ and the joint probability distribution $p_{\omega_l}(n_i^1, n_j^2)$ of the sequential measurements with outcomes n_i^1 and n_j^2 is decomposed as

$$p_{\omega_l}(n_i^1, n_j^2) = p_{\omega_l}^1(n_i^1)p_{\omega_l}^2(n_j^2|n_i^1), \tag{15}$$

where $p_{\omega_l}^2(n_j^2|n_i^1) = \mathrm{Tr}(\rho_{\omega_l}^{2|n_i^1}M_j^{2|n_i^1})$ is the conditional probability of the outcome n_j^2. Submitting Eq. (15) into Eq. (4), we obtain the FI

$$F(\rho_{\omega_l}) = F(\rho_{\omega_l}^1|M^1) + \sum_{n_i^1} p_{\omega_l}^1(n_i^1)F(\rho_{\omega_l}^{2|n_i^1}|M^{2|n_i^1}), \tag{16}$$

where $\rho_{\omega_l}^1 = \mathrm{Tr}_2(\rho_{\omega_l})$ is the reduced density matrix of the first subsystem. Equation (16) includes two parts, one is the FI by performing the local measurement on the first subsystem with reduced density matrix $\rho_{\omega_l}^1$ and the other is the superposition of the FI with the conditional density matrices $\rho_{\omega_l}^{2|n_i^1}$ where the corresponding probability is $p_{\omega_l}^1(n_i^1)$. Obviously, the FI obtained by the sequential measurements scheme is higher than the local measurements on one subsystem due to the nonnegative of FI.

We show the numerical results of FI with the parameterized density matrix ρ_{ω_l} of Eq. (12) in Fig. 2(a) for different measurement schemes, where the measurements for the first subsystem are performed on the eigenvectors of the SLD of the reduced density matrix $\rho_{\omega_l}^1 = \mathrm{Tr}_2(\rho_{\omega_l})$. The subsequent measurements are performed on the eigenvectors of SLDs of the conditional density matrix $\rho_{\omega_l}^{2|n_i^1}$ and the reduced density matrix $\rho_{\omega_l}^2 = \mathrm{Tr}_1(\rho_{\omega_l})$ for nonidentical and identical measurements schemes, respectively. In the following, let us illustrate the results in Fig. 2(a) in detail, the red dotted and blue dashed lines are the results of the sequential measurement $F^{e\to n}$ and $F^{n\to e}$ with different measurement sequence of the two subsystems, respectively. The numerical results show that the FI in Eq. (4) is indeed dependent on the sequence of the subsystems. For comparison, we also give the result with the trade off of nonidentical optimal measurements, where the identical measurements are performed on the eigenvectors of SLD of the reduced density matrix in the second subsystem using the black solid line. The result is no higher than $F^{e\to n}$ and $F^{n\to e}$ limited by the convexity property of the QFI and independent on the measurement sequence of the subsystem. In addition, we show the QFI $\mathcal{F}^{\beta} = \mathrm{Tr}[\rho_{\omega_l}^{\beta}(L_{\omega_l}^{\beta})^2]$, $\beta\in\{e,n\}$, with the local measurements performed on the SLD of the reduced density matrices $\rho_{\omega_l}^{\beta}$ for different subsystems, which is always below the results of the sequential measurements scheme on account of the QFI of the reduced density matrix is just the maximum of the first term in Eq. (16) and the second term is always nonnegative due to the nonnegative property of FI. Finally, we give the probabilities of different projective measurements M_i^{β} on the eigenvectors of the SLD of corresponding reduced density matrixes in the first subsystem and the results are shown in Fig. 2(b), which satisfies $\sum_i p_{\omega_l}^{\beta}(n_i^{\beta}) = 1$ implied by $\sum_i M_i^{\beta} = \mathbb{I}^{\beta}$.

IV. CONCLUSION

In this paper, to obtain a precise estimation of the unknown parameter while the optimal global projective measurement is technologically difficult in composite systems, we derived the expression of FI for sequential measurements scheme in composite systems with N interacting subsystems. The result shows that the FI is a sum of different subsystems FI and the more subsystems used for the estimation, the higher FI obtained due to the nonnegative property of FI. We should point out that the precision shown here is no better than the theoretical result in the global system limited by the QCRB, but more realistic and better than the local measurements in one of the subsystems. Then we proved that the projective measurements on the eigenvectors of the SLD of subsystem reduced density matrix can be regarded as a trade off of the nonidentical optimal measurements of different conditional density matrices in every subsystem by means of the convexity property of the QFI. Importantly, we illustrated the measurement sequence of the subsystems in such that the sequential measurements scheme should be considered in the optimization to obtain a precise estimation. Finally, we applied our theory to the nuclear spins control model to estimate the nuclear Larmor frequency and the results showed the theory is applicative.

ACKNOWLEDGMENT

This work is supported by National Natural Science Foundation of China (NSFC) under Grant No. 12175033.

APPENDIX A: DERIVATION OF THE FI OF SEQUENTIAL MEASUREMENTS SCHEME IN EQ. (4)

In the Appendix, we give the derivation of the FI of sequential measurements scheme with three subsystems for the estimation of the unknown parameter θ and the FI of N subsystems could be concluded from this simple case.

For a parametrized density matrix ρ_θ, we consider sequential measurements with M_i^1, $M_j^{2|n_i^1}$, and $M_k^{3|n_i^1,n_j^2}$ for three arbitrary subsystems where n_i^1 and n_j^2 are the outcomes for the first and second subsystems, respectively. The subsequent measurements are dependent on the previous outcomes, for example, $M_j^{2|n_i^1}$ denotes the measurement in the second subsystem which is dependent on the outcome n_i^1 in the first subsystem. The joint probability distribution $p_\theta(n_i^1, n_j^2, n_k^3)$ is

$$p_\theta\left(n_i^1, n_j^2, n_k^3\right) = p_\theta^1\left(n_i^1\right) p_\theta^2\left(n_j^2\middle|n_i^1\right) p_\theta^3\left(n_k^3\middle|n_i^1, n_j^2\right), \tag{A1}$$

where $p_\theta^1(n_i^1) = \mathrm{Tr}(\rho_\theta^1 M_i^1)$, $p_\theta^2(n_j^2|n_i^1) = \mathrm{Tr}(\rho_\theta^{2|n_i^1} M_j^{2|n_i^1})$, and $p_\theta^3(n_k^3|n_i^1, n_j^2) = \mathrm{Tr}(\rho_\theta^{3|n_i^1,n_j^2} M_k^{3|n_i^1,n_j^2})$ are the probability with the above three measurements M_i^1, $M_j^{2|n_i^1}$, and $M_k^{3|n_i^1,n_j^2}$, respectively. Then submitting Eq. (A1) into the expression of FI in Eq. (1), we obtain

$$\begin{aligned}
F(\rho_\theta) &= \sum_{n_i^1,n_j^2,n_k^3} \frac{1}{p_\theta\left(n_i^1, n_j^2, n_k^3\right)} \left[\frac{\partial p_\theta\left(n_i^1, n_j^2, n_k^3\right)}{\partial\theta}\right]^2 \\
&= \sum_{n_i^1,n_j^2,n_k^3} \frac{\left[\frac{\partial p_\theta^1(n_i^1)}{\partial\theta} p_\theta^2\left(n_j^2|n_i^1\right) p_\theta^3\left(n_k^3|n_i^1, n_j^2\right) + p_\theta^1\left(n_i^1\right) \frac{\partial p_\theta^2(n_j^2|n_i^1)}{\partial\theta} p_\theta^3\left(n_k^3|n_i^1, n_j^2\right) + p_\theta^1\left(n_i^1\right) p_\theta^2\left(n_j^2|n_i^1\right) \frac{\partial p_\theta^3(n_k^3|n_i^1,n_j^2)}{\partial\theta}\right]^2}{p_\theta\left(n_i^1, n_j^2, n_k^3\right)} \\
&= \sum_{n_i^1,n_j^2,n_k^3} \frac{p_\theta^2\left(n_j^2|n_i^1\right) p_\theta^3\left(n_k^3|n_i^1, n_j^2\right)}{p_\theta^1\left(n_i^1\right)} \left[\frac{\partial p_\theta^1\left(n_i^1\right)}{\partial\theta}\right]^2 + \frac{p_\theta^1\left(n_i^1\right) p_\theta^3\left(n_k^3|n_i^1, n_j^2\right)}{p_\theta^2\left(n_j^2|n_i^1\right)} \left[\frac{\partial p_\theta^2\left(n_j^2|n_i^1\right)}{\partial\theta}\right]^2 \\
&\quad + \frac{p_\theta^1\left(n_i^1\right) p_\theta^2\left(n_j^2|n_i^1\right)}{p_\theta^3\left(n_k^3|n_i^1, n_j^2\right)} \left[\frac{\partial p_\theta^3\left(n_k^3|n_i^1, n_j^2\right)}{\partial\theta}\right]^2 \\
&= \sum_{n_i^1} \frac{\left[\frac{\partial p_\theta^1(n_i^1)}{\partial\theta}\right]^2}{p_\theta^1\left(n_i^1\right)} + \sum_{n_i^1} p_\theta^1\left(n_i^1\right) \sum_{n_j^2} \frac{\left[\frac{\partial p_\theta^2(n_j^2|n_i^1)}{\partial\theta}\right]^2}{p_\theta^2\left(n_j^2|n_i^1\right)} + \sum_{n_i^1,n_j^2} p_\theta^1\left(n_i^1\right) p_\theta^2\left(n_j^2|n_i^1\right) \sum_{n_k^3} \frac{\left[\frac{\partial p_\theta^3(n_k^3|n_i^1,n_j^2)}{\partial\theta}\right]^2}{p_\theta^3\left(n_k^3|n_i^1, n_j^2\right)} \\
&= F(\rho_\theta^1|M^1) + \sum_{n_i^1} p_\theta\left(n_i^1\right) F\left(\rho_\theta^{2|n_i^1}\middle|M^{2|n_i^1}\right) + \sum_{n_i^1,n_j^2} p_\theta\left(n_i^1, n_j^2\right) F\left(\rho_\theta^{3|n_i^1,n_j^2}\middle|M^{3|n_i^1,n_j^2}\right),
\end{aligned} \tag{A2}$$

where $p_\theta(n_i^1) = p_\theta^1(n_i^1)$, $p_\theta(n_i^1, n_j^2) = p_\theta^1(n_i^1) p_\theta^2(n_j^2|n_i^1)$, $F(\rho_\theta^1|M^1) = \sum_{n_i^1} \frac{1}{p_\theta^1(n_i^1)} [\frac{\partial p_\theta^1(n_i^1)}{\partial\theta}]^2$, $F(\rho_\theta^{2|n_i^1}|M^{2|n_i^1}) = \sum_{n_j^2} \frac{1}{p_\theta^2(n_j^2|n_i^1)} [\frac{\partial p_\theta^2(n_j^2|n_i^1)}{\partial\theta}]^2$, and $F(\rho_\theta^{3|n_i^1,n_j^2}|M^{3|n_i^1,n_j^2}) = \sum_{n_k^3} \frac{1}{p_\theta^3(n_k^3|n_i^1,n_j^2)} [\frac{\partial p_\theta^3(n_k^3|n_i^1,n_j^2)}{\partial\theta}]^2$. In addition, $\sum_{n_j^2} \frac{\partial p_\theta^2(n_j^2|n_i^1)}{\partial\theta} = 0$ and $\sum_{n_k^3} \frac{\partial p_\theta^3(n_k^3|n_i^1,n_j^2)}{\partial\theta} = 0$ were used in the derivation, which is implied by $\sum_{n_j^2} p_\theta^2(n_j^2|n_i^1) = 1$ and $\sum_{n_k^3} p_\theta^3(n_k^3|n_i^1, n_j^2) = 1$. Then the FI of the sequential measurements for N subsystems could be concluded as

$$F(\rho_\theta) = F(\rho_\theta^1|M^1) + \sum_{n_i^1} p_\theta\left(n_i^1\right) F\left(\rho_\theta^{2|n_i^1}\middle|M^{2|n_i^1}\right) + \cdots + \sum_{n_i^1,\ldots,n_j^{N-2},n_k^{N-1}} p_\theta\left(n_i^1, \ldots, n_j^{N-2}, n_k^{N-1}\right) F\left(\rho_\theta^{N|n_i^1,\ldots,n_j^{N-2},n_k^{N-1}}\middle|M^{N|n_i^1,\ldots,n_j^{N-2},n_k^{N-1}}\right), \tag{A3}$$

where $p_\theta(n_i^1, \ldots, n_j^{N-2}, n_k^{N-1}) = p_\theta^1(n_i^1) p_\theta^2(n_j^2|n_i^1) \cdots p_\theta^{N-1}(n_k^{N-1}|n_i^1, \ldots, n_j^{N-2})$. Also $\sum_{n_q^\alpha} \frac{\partial p_\theta^\alpha(n_q^\alpha|n_i^1,\ldots,n_l^{\alpha-2},n_s^{\alpha-1})}{\partial\theta} = 0, \alpha \in [2, N]$ were used implied by $\sum_{n_q^\alpha} p_\theta^\alpha(n_q^\alpha|n_i^1, \ldots, n_l^{\alpha-2}, n_s^{\alpha-1}) = 1$.

APPENDIX B: SOLVE THE EIGENVALUES AND EIGENVECTORS FOR HAMILTONIAN (11)

Here, we solve the eigenvalues and eigenvectors of the Hamiltonian in Eq. (11) used for encoding the nuclear Larmor frequency ω_I. We obtain the matrix form of the Hamiltonian (11) in the basis $|e^e, g^n\rangle$, $|e^e, e^n\rangle$, $|g^e, g^n\rangle$, and $|g^e, e^n\rangle$,

$$H_I = \begin{pmatrix} -A & g & B & 0 \\ g & A & 0 & B \\ B & 0 & -A & -g \\ 0 & B & -g & A \end{pmatrix}, \tag{B1}$$

where $A = \frac{\omega_l}{2}$, $B = \frac{\Omega_1}{2}$. The normalized eigenvectors are

$$|E_1\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}\cos\theta_+\\ -\sin\theta_+\\ \cos\theta_+\\ \sin\theta_+\end{pmatrix}, \quad |E_2\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}-\cos\varphi_+\\ \sin\varphi_+\\ \cos\varphi_+\\ \sin\varphi_+\end{pmatrix},$$
$$|E_3\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}\cos\theta_-\\ -\sin\theta_-\\ \cos\theta_-\\ \sin\theta_-\end{pmatrix}, \quad |E_4\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix}-\cos\varphi_-\\ \sin\varphi_-\\ \cos\varphi_-\\ \sin\varphi_-\end{pmatrix}, \tag{B2}$$

where

$$\tan\theta_+ = \frac{g}{\alpha + \sqrt{\alpha^2 + g^2}}, \quad \tan\varphi_+ = \frac{g}{\beta + \sqrt{\beta^2 + g^2}}, \quad \tan\theta_- = \frac{g}{\alpha - \sqrt{\alpha^2 + g^2}} \quad \tan\varphi_- = \frac{g}{\beta - \sqrt{\beta^2 + g^2}}, \tag{B3}$$

and $\alpha = A - B$ and $\beta = A + B$. In addition, θ_+, φ_+, θ_-, and φ_- satisfy

$$\cos\theta_+\cos\theta_- + \sin\theta_+\sin\theta_- = 0, \quad \cos\varphi_+\cos\varphi_- + \sin\varphi_+\sin\varphi_- = 0. \tag{B4}$$

The corresponding eigenvalues are

$$E_1 = -\sqrt{\alpha^2 + g^2}, \quad E_2 = -\sqrt{\beta^2 + g^2}, \quad E_3 = \sqrt{\alpha^2 + g^2}, \quad E_4 = \sqrt{\beta^2 + g^2}. \tag{B5}$$

Now, we have given all the eigenvalues and eigenvectors of the Hamiltonian (11). Therefore, we could obtain the parameterized state by projecting the initial state onto the eigenvectors in Eq. (B2). After some derivations and substitutions, we obtain the encoded states $|\Phi(\omega_l)\rangle$ in Eq. (13).

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