Locally stable sets with minimum cardinality

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The nonlocal set has received wide attention over recent years. Shortly before, Li and Wang arXiv:2202.09034 proposed the concept of a locally stable set: The only possible orthogonality preserving measurement on each subsystem is trivial. Locally stable sets present stronger nonlocality than those sets that are just locally indistinguishable. In this work, we focus on the constructions of locally stable sets in multipartite quantum systems. First, two lemmas are put forward to prove that an orthogonality-preserving local measurement must be trivial. Then we present the constructions of locally stable sets with minimum cardinality in bipartite quantum systems $\mathbb{C}^d \otimes \mathbb{C}^d$ $(d \geqslant 3)$ and $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ $(3 \leqslant d_1 \leqslant d_2)$. Moreover, for the multipartite quantum systems $(\mathbb{C}^d)^{\otimes n}$ $(d \geqslant 2)$ and $\otimes_{i=1}^n \mathbb{C}^{d_i}$ $(3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$, we also obtain $d+1$ and d_n+1 locally stable orthogonal states, respectively. Fortunately, our constructions reach the lower bound of the cardinality on the locally stable sets, which provides a positive and complete answer to an open problem raised in arXiv:2202.09034.

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I. INTRODUCTION

A set of orthogonal quantum states is locally indistinguishable if it is not possible to optimally distinguish the states by any sequence of local operations and classical communications (LOCC). In 1999, Bennett *et al.* [1] first presented a set of locally indistinguishable orthogonal product bases in $\mathbb{C}^3 \otimes \mathbb{C}^3$, which shows the phenomenon of nonlocality without entanglement. With the increasing research of nonlocality, there are many relevant references on locally indistinguishable orthogonal entangled states [2–12]. Especially, the locally indistinguishable orthogonal product states have attracted more attention [13–28]. Its closely related research branch, entanglement-assisted discrimination protocol, has also achieved fruitful results [29–31]. The local indistinguishability has wide applications in quantum cryptographic protocols such as secret sharing and data hiding [32–37]. That is the reason why so many scholars are engaged in the research of local discrimination of quantum states.

In 2019, Rout *et al.* [38] proposed the concept of genuine nonlocality based on local indistinguishability. Then many interesting results spring up like mushrooms [38–40]. Recently, Halder *et al.* [41] put forward the concept of strong nonlocality based on locally irreducible quantum states. A set of multipartite orthogonal product states is strongly nonlocal if it is locally irreducible in every bipartition. Many people began to engage in the research and a few strongly nonlocal sets were obtained [41–46].

An important method was provided to verify the local indistinguishability of orthogonal product states in Ref. [4], which showed the fact that no matter which party goes first, he (or she) can only perform a trivial measurement. Since 2014, a great deal of research (see Refs. [15–27]) on the locally indistinguishable sets of quantum states is based on the aforementioned observation. Li *et al.* [46] concentrated on the orthogonal sets of multipartite quantum states with the property: The only possible orthogonality preserving measurement on each subsystem is trivial. The set with such property is called a locally stable set. Note that locally stable sets are always locally indistinguishable. Hence they could be used to show some particular form of distinguishability-based nonlocality. Li *et al.* [46] also obtained a lower bound of the cardinality on the locally stable set, i.e., if $\mathcal{S}$ is a locally stable set of orthogonal pure states in $\mathcal{H} := \otimes_{i=1}^N \mathcal{H}_{A_i}$, whose local dimension is $\dim_{\mathbb{C}}(\mathcal{H}_{A_i}) = d_i$, then $|\mathcal{S}| \geqslant \max_i\{d_i+1\}$. They conjectured that this lower bound may be tight. That is, there may exist some locally stable set $\mathcal{S} \subseteq \mathcal{H}$ whose cardinality is exactly the aforementioned lower bound $\max_i\{d_i+1\}$. In this work, we will provide a positive answer to this conjecture. Adding any orthogonal states to a locally stable set (nonlocal set) forms a new set which is again locally stable (nonlocal). Hence it is interesting to find the optimal locally stable set in the sense that, removing any state from this set, it is impossible to achieve local stability again. Therefore, those locally stable sets in $\mathcal{H}$ with cardinality being $\max_i\{d_i+1\}$ are always optimal.

In the manuscript, we aim to construct locally stable sets whose cardinality reach the lower bound indicated in Ref. [46] for general multipartite quantum systems. Fortunately, we prove that there exist $d+1$ orthogonal states in $\mathbb{C}^d \otimes \mathbb{C}^d$ $(d \geqslant 3)$ and d_2+1 orthogonal states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ $(3 \leqslant d_1 \leqslant d_2)$ are locally stable. For the multipartite cases, we present two constructions of locally stable sets in multipartite quantum systems $(\mathbb{C}^d)^{\otimes n}$ $(d \geqslant 2)$ and $\otimes_{i=1}^n \mathbb{C}^{d_i}$ $(3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$, which contain $d+1$ and d_n+1 orthogonal states, respectively. All of the locally stable sets can reach the minimum cardinality on the locally stable set proposed in

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Ref. [46]. In addition, we found another structure of the smallest locally stable set in $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$ $(3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$, which is composed of genuine entangled states apart from one full product state.

II. PRELIMINARIES

Throughout this paper, we only consider pure states and we do not normalize states for simplicity. Here we take the computational basis $\{|i\rangle\}_{i=0}^{d_k-1}$ for each d_k-dimensional subsystem. For simplicity, we denote the state $\frac{1}{\sqrt{n}}(|i_1\rangle \pm |i_2\rangle \pm \cdots \pm |i_n\rangle)$ as $|i_1 \pm i_2 \pm \cdots \pm i_n\rangle$, $\mathbb{Z}_{d_k} = \{0, 1, \ldots, d_k - 1\}$, $(\mathbb{C}^d)^{\otimes n} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$, and $\otimes_{i=1}^{n}\mathbb{C}^{d_i} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$. In particular, it should be pointed out that the stopper state has the expression

$$|S\rangle = \otimes_{k=1}^{n}\left(\sum_{i_k\in\mathbb{Z}_{d_k}} |i_k\rangle_{A_k}\right). \tag{1}$$

For each integer $d \geqslant 2$, we denote $w_d = e^{\frac{2\pi\sqrt{-1}}{d}}$, i.e., a primitive dth root of unit.

All the participants perform positive operator-valued measures (POVM) on their local sites. Each kth subsystem's POVM element $M_k^\dagger M_k$ can be represented by a $d_k \times d_k$ matrix $E_k = (m_{a,b}^k)_{a,b\in\mathbb{Z}_{d_k}}$ in the computational basis. A POVM is called a trivial measurement if all its elements are proportional to the identity operator. To ensure the local distinguishability the postmeasurement states should remain orthogonal. We observe that in each locally distinguishable protocol, each local measurement must preserve the orthogonality of the states. Using this observation, there is a widely used method for deducing the local indistinguishability of an orthogonal set: To preserve the orthogonality of the states, each party could only perform trivial measurement. This method motivates the definition of locally stable.

Definition 1 (Locally indistinguishable) [1]. A set of orthogonal pure states in multipartite quantum systems is said to be locally indistinguishable, if it is not possible to distinguish the states by using LOCC.

Definition 2 (Locally irreducible) [41]. A set of orthogonal quantum states on $\mathcal{H} = \otimes_{i=1}^{n}\mathcal{H}_i$ with $n \geqslant 2$ and $\dim\mathcal{H}_i \geqslant 2$, $i = 1, 2, \ldots, n$ is locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements.

Definition 3 (Locally stable) [46]. An orthogonal set of pure states in multipartite quantum systems is said to be locally stable if the only possible orthogonality preserving measurement on the subsystems is trivial.

In Ref. [46], it is shown that locally stable sets are always locally irreducible and locally irreducible sets are always locally indistinguishable; the converse is not true. Therefore, locally stable sets present the strongest form of quantum nonlocality among the three classes: Locally indistinguishable sets, locally irreducible sets, and locally stable sets.

Given an orthogonal set $\mathcal{S} = \{|\phi_i\rangle\}_{i=1}^{N}$ of pure states in $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$, if the kth party starts with the first orthogonality preserving measurement whose measurement element is denoted as $E_k = (m_{a,b}^k)_{a,b\in\mathbb{Z}_{d_k}}$, then we have

$$\langle\phi_i|I_1 \otimes I_2 \otimes \cdots \otimes E_k \otimes \cdots \otimes I_n|\phi_j\rangle = 0 \tag{2}$$

for all different pairs $|\phi_i\rangle, |\phi_j\rangle \in \mathcal{S}$. Now we put forward two simple lemmas which are useful for deducing an orthogonality preserving measurement $E_k = (m_{a,b}^k)_{a,b\in\mathbb{Z}_{d_k}}$ to be a trivial one.

Lemma 1 (Zero entries). Fix $k \in \{1, 2, \ldots, n\}$. Suppose that

$$|\phi_i\rangle = \sum_{t=0}^{p_i-1} \omega_{p_i}^t \left|i_1^t\right\rangle_{A_1}\left|i_2^t\right\rangle_{A_2}\cdots\left|i_n^t\right\rangle_{A_n},$$

$$|\phi_j\rangle = \sum_{s=0}^{p_j-1} \omega_{p_j}^s \left|j_1^s\right\rangle_{A_1}\left|j_2^s\right\rangle_{A_2}\cdots\left|j_n^s\right\rangle_{A_n},$$

where $|i_1^t\rangle_{A_1}|i_2^t\rangle_{A_2}\cdots|i_n^t\rangle_{A_n}$ and $|j_1^s\rangle_{A_1}|j_2^s\rangle_{A_2}\cdots|j_n^s\rangle_{A_n}$ are mutually orthogonal and there is only one pair $(t_0, s_0) \in \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}$ such that

$$\prod_{\ell\neq k}\left\langle i_\ell^{t_0}\middle|j_\ell^{s_0}\right\rangle_{A_\ell} \neq 0.$$

Then the equation $\langle\phi_i|I_1 \otimes I_2 \otimes \cdots \otimes E_k \otimes \cdots \otimes I_n|\phi_j\rangle = 0$ implies that $m_{i_k^{t_0}, j_k^{s_0}}^k = 0$.

Lemma 2 (Diagonal entries). Fix $k \in \{1, 2, \ldots, n\}$. Let $|S\rangle$ be the stopper state defined in Eq. (1) and $|\phi_i\rangle = \sum_{t=0}^{p-1} \omega_p^t |i_1^t\rangle_{A_1}|i_2^t\rangle_{A_2}\cdots|i_n^t\rangle_{A_n}$, where there exist only two different values among $i_k^0, i_k^1, \ldots, i_k^{p-1}$, say $i_k^{t_0}$ and $i_k^{t_1}$. If all the off-diagonal entries of the matrix $E_k = (m_{a,b}^k)_{a,b\in\mathbb{Z}_{d_k}}$ are zeros, then the equation $\langle S|I_1 \otimes I_2 \otimes \cdots \otimes E_k \otimes \cdots \otimes I_n|\phi_i\rangle = 0$ implies that $m_{i_k^{t_0}, i_k^{t_0}}^k = m_{i_k^{t_1}, i_k^{t_1}}^k$.

The proofs of the above two Lemmas are given in Appendix A.

III. CONSTRUCTIONS IN BIPARTITE QUANTUM SYSTEMS

In this section, we propose the construction of locally stable sets with minimum cardinality in $\mathbb{C}^d \otimes \mathbb{C}^d$ $(d \geqslant 3)$ and $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ $(3 \leqslant d_1 \leqslant d_2)$.

A. Locally stable set in $\mathbb{C}^d \otimes \mathbb{C}^d$

Theorem 1. The following set $\mathcal{S}$ of $d + 1$ orthogonal states is locally stable in $\mathbb{C}^d \otimes \mathbb{C}^d$ [see Fig. 1(a) for an intuition of the example where $d = 5$]:

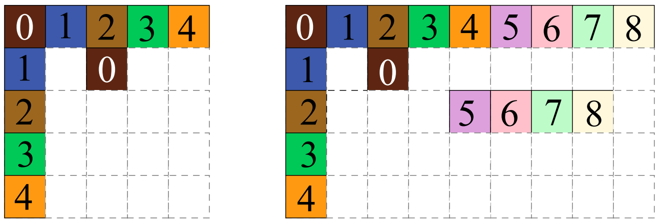
$$\begin{aligned}
|\phi_0\rangle &= |00\rangle_{AB} - |12\rangle_{AB},\\
|\phi_i\rangle &= |i0\rangle_{AB} - |0i\rangle_{AB},\\
|S\rangle &= |0 + \cdots + (d-1)\rangle_A |0 + \cdots + (d-1)\rangle_B,
\end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, d - 1$, $d \geqslant 3$.

Proof. First, we assume that Alice starts with the first measurement. Let $E_1 = (m_{a,b}^1)_{a,b\in\mathbb{Z}_d}$ represent an element of any orthogonality-preserving measurement performed by Alice. For each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|E_1 \otimes I_2|\phi\rangle = 0. \tag{4}$$

For $1 \leqslant i \neq j \leqslant d - 1$, considering Eq. (4) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$, we obtain $m_{i,j}^1 = m_{j,i}^1 = 0$ directly from Lemma 1.



(a) (b)

FIG. 1. Intuition of the structure of states we constructed in Eq. (3) and Eq. (6). Figure (a) corresponds to the systems $\mathbb{C}^5 \otimes \mathbb{C}^5$, while (b) corresponds to systems $\mathbb{C}^5 \otimes \mathbb{C}^9$. The squares indicated by the same color represent a unique state and the numbers represent the subscripts of the states. For example, the two orange squares (4,0) and (0,4) with label "4" correspond to the state $|\phi_4\rangle = |40\rangle_{AB} - |04\rangle_{AB}$; the two pink squares (0,6) and (2,5) with label "6" in the right side correspond to the state $|\phi_6\rangle = |06\rangle_{AB} - |25\rangle_{AB}$.

Now we consider Eq. (4) for the states $|\phi_0\rangle$ and $|\phi_i\rangle$ for $i = 1, 2, \ldots, d-1$. If $i \in \{1, 3, \ldots, d-1\}$, we can get $m^1_{0,i} = m^1_{i,0} = 0$ from Lemma 1. If $i = 2$, the corresponding Eq. (4) is just $(\langle 00| - \langle 12|)E_1 \otimes I_2(|20\rangle - |02\rangle) = 0$, i.e., $\langle 0|E_1|2\rangle\langle 0|I_2|0\rangle - \langle 0|E_1|0\rangle\langle 0|I_2|2\rangle - \langle 1|E_1|2\rangle\langle 2|I_2|0\rangle + \langle 1|E_1|0\rangle\langle 2|I_2|2\rangle = 0$, which gives rise to $m^1_{0,2} + m^1_{1,0} = 0$. Since $m^1_{0,1} = m^1_{1,0} = 0$, we can get $m^1_{0,2} = 0$. Thus $m^1_{0,i} = m^1_{i,0} = 0$ for $1 \leqslant i \leqslant d-1$. Therefore, the off-diagonal entries of E_1 are all zeros.

For $1 \leqslant i \leqslant d-1$, considering Eq. (4) for the states $|S\rangle$ and $|\phi_i\rangle$, we get $m^1_{0,0} = m^1_{i,i}$ by Lemma 2. Therefore, E_1 is proportional to the identity matrix. Hence Alice can only start with a trivial measurement.

Suppose that Bob starts with the first orthogonality-preserving measurement whose elements are represented as $E_2 = (m^2_{a,b})_{a,b\in\mathbb{Z}_d}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes E_2|\phi\rangle = 0. \tag{5}$$

In the same way, considering Eq. (5) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$, we obtain $m^2_{i,j} = m^2_{j,i} = 0$ directly from Lemma 1 for $1 \leqslant i \neq j \leqslant d-1$. Now we consider Eq. (5) for the states $|\phi_0\rangle$ and $|\phi_i\rangle$ for $1 \leqslant i \leqslant d-1$. If $2 \leqslant i \leqslant d-1$, we have $m^2_{0,i} = m^2_{i,0} = 0$ by Lemma 1. If $i = 1$, we have $(\langle 00| - \langle 12|)I_1 \otimes E_2(|10\rangle - |01\rangle) = 0$, i.e., $\langle 0|I_1|1\rangle\langle 0|E_2|0\rangle - \langle 0|I_1|0\rangle\langle 0|E_2|1\rangle - \langle 1|I_1|1\rangle\langle 2|E_2|0\rangle + \langle 1|I_1|0\rangle\langle 2|E_2|1\rangle = 0$, which gives rise to $m^2_{0,1} + m^2_{2,0} = 0$. Since $m^2_{0,2} = m^2_{2,0} = 0$, we can get $m^2_{0,1} = 0$. Thus $m^2_{0,i} = m^2_{i,0} = 0$ for $1 \leqslant i \leqslant d-1$. Therefore, the off-diagonal entries of E_2 are all zeros.

For $1 \leqslant i \leqslant d-1$, considering Eq. (5) for the states $|S\rangle$ and $|\phi_i\rangle$, we get $m^2_{0,0} = m^2_{i,i}$ by Lemma 2. Therefore, E_2 is proportional to the identity matrix. Bob can only implement a trivial orthogonality-preserving measurement also.

Thus the above $d+1$ states are locally stable by definition. This completes the proof. ■

Specifically, the construction is not unique, where $|\phi_0\rangle = |00\rangle_{AB} - |12\rangle_{AB}$ can be $|\phi_0\rangle = |00\rangle_{AB} - |1k\rangle_{AB}$ $(2 \leqslant k \leqslant d-1)$. This is true for other examples presented.

B. Locally stable set in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$

Theorem 2. Let $3 \leqslant d_1 \leqslant d_2$. The following set $\mathcal{S}$ of $d_2 + 1$ orthogonal states is locally stable in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ [see Fig. 1(b) for an intuition of the example where $d_1 = 5$ and $d_2 = 9$]:

$$\begin{aligned}
&|\phi_0\rangle = |00\rangle_{AB} - |12\rangle_{AB},\\
&|\phi_i\rangle = |i0\rangle_{AB} - |0i\rangle_{AB}, \quad 1 \leqslant i \leqslant d_1 - 1,\\
&|\phi_j\rangle = |0j\rangle_{AB} - |2(j-1)\rangle_{AB}, \quad d_1 \leqslant j \leqslant d_2 - 1,\\
&|S\rangle = |0 + \cdots + (d_1 - 1)\rangle_A |0 + \cdots + (d_2 - 1)\rangle_B.
\end{aligned} \tag{6}$$

Proof. Obviously, Alice could only start with trivial orthogonality-preserving measurement by the same argument as case $\mathbb{C}^d \otimes \mathbb{C}^d$. We only need to show that the orthogonality-preserving measurement Bob could perform is the trivial one. Suppose that Bob starts with the first orthogonality-preserving measurement whose elements are represented as $E_2 = (m^2_{a,b})_{a,b\in\mathbb{Z}_{d_2}}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes E_2|\phi\rangle = 0. \tag{7}$$

With a similar argument as the case $\mathbb{C}^d \otimes \mathbb{C}^d$, we could obtain that $m^2_{i,i'} = m^2_{i',i} = 0$ for all $0 \leqslant i \neq i' \leqslant d_1 - 1$. Considering Eq. (7) for the states $|\phi_0\rangle$ and $|\phi_j\rangle$, we directly get $m^2_{0,j} = m^2_{j,0} = 0$ for $d_1 \leqslant j \leqslant d_2 - 1$ by Lemma 1.

Now we consider Eq. (7) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$ for $1 \leqslant i \leqslant d_1 - 1$ and $d_1 \leqslant j \leqslant d_2 - 1$. If $i \neq 2$, we get $m^2_{i,j} = m^2_{j,i} = 0$ directly from Lemma 1. If $i = 2$, we have $\langle 2|I_1|0\rangle\langle 0|E_2|j\rangle - \langle 2|I_1|2\rangle\langle 0|E_2|j-1\rangle - \langle 0|I_1|0\rangle\langle 2|E_2|j\rangle + \langle 0|I_1|2\rangle\langle 2|E_2|j-1\rangle = 0$, which deduces that $m^2_{0,j-1} + m^2_{2,j} = 0$. Since $m^2_{0,j-1} = m^2_{j-1,0} = 0$, we have $m^2_{2,j} = 0$. Therefore, we have $m^2_{i,j} = m^2_{j,i} = 0$ for all $1 \leqslant i \leqslant d_1 - 1, d_1 \leqslant j \leqslant d_2 - 1$.

Then we consider Eq. (7) for the states $|\phi_j\rangle$ and $|\phi_{j'}\rangle$ for $d_1 \leqslant j < j' \leqslant d_2 - 1$. That is, we have the equation $\langle 0|I_A|0\rangle\langle j|E_2|j'\rangle - \langle 0|I_1|2\rangle\langle j|E_2|j'-1\rangle - \langle 2|I_1|0\rangle\langle j-1|E_2|j'\rangle + \langle 2|I_1|2\rangle\langle j-1|E_2|j'-1\rangle = 0$, which implies that $m^2_{j,j'} = -m^2_{(j-1),(j'-1)}$. Therefore,

$$m^2_{j,j'} = -m^2_{(j-1),(j'-1)} = \cdots = (-1)^{j-d_1+1} m^2_{d_1-1,(j'-j+d_1-1)},$$

which is equal to zero as the last term $m^2_{d_1-1,(j'-j+d_1-1)} = 0$ has been obtained. Thus we get $m^2_{j',j} = m^2_{j,j'} = 0$ for $d_1 \leqslant j < j' \leqslant d_2 - 1$. Up to now, we have shown that the off-diagonal entries of E_2 are all zeros.

For $1 \leqslant i \leqslant d_1 - 1$, considering Eq. (5) for the states $|S\rangle$ and $|\phi_i\rangle$, we get $m^2_{i,i} = m^2_{0,0}$ by Lemma 2. Similarly, considering the states $|S\rangle$ and $|\phi_j\rangle$ for $d_1 \leqslant j \leqslant d_2 - 1$, we have $m^2_{j,j} = m^2_{j-1,j-1}$, which implies that

$$m^2_{j,j} = m^2_{j-1,j-1} = \cdots = m^2_{d_1-1,d_1-1} = m^2_{0,0}.$$

Therefore, $E_2 \propto \mathbb{I}$. Bob cannot start with a nontrivial measurement either.

In summary, both participants can only start with a trivial orthogonality-preserving measurement. Thus the above $d_2 + 1$ states are locally stable. This completes the proof. ■

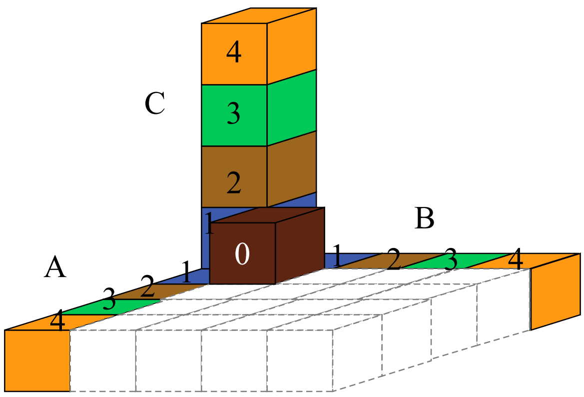


FIG. 2. Intuition of the structure of states we constructed in Eq. (8) for the setting $n = 3$ and $d = 5$. Note that the cubic with coordinate (0,0,0) should be labeled with "0". The squares indicated by the same color represent a unique state and the numbers represent the subscripts of the states. For example, the orange squares (4,0,0), (0,4,0), and (0,0,4) with label "4" correspond to the state $|\phi_4\rangle = |400\rangle_{ABC} + \omega_3|040\rangle_{ABC} + \omega_3^2|004\rangle_{ABC}$.

IV. CONSTRUCTIONS IN MULTIPARTITE QUANTUM SYSTEMS

In this section, we put forward the constructions of the locally stable sets in multipartite quantum systems $(\mathbb{C}^d)^{\otimes n}$ ($d \geqslant 2, n \geqslant 3$) and $\otimes_{i=1}^n \mathbb{C}^{d_i}$ ($3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, $n \geqslant 3$).

A. Locally stable set in $(\mathbb{C}^d)^{\otimes n}$

Theorem 3. In $(\mathbb{C}^d)^{\otimes n}$ ($d \geqslant 2, n \geqslant 3$), the following set $\mathcal{S}$ of $d+1$ orthogonal states are locally stable [see Fig. 2 for an intuition of the example where $n = 3$ and $d = 5$]:

$$\begin{aligned}
|\phi_0\rangle &= |00\cdots 00\rangle_{A_1A_2\cdots A_n} - |11\cdots 11\rangle_{A_1A_2\cdots A_n},\\
|\phi_i\rangle &= |i0\cdots 00\rangle_{A_1A_2\cdots A_n} + \omega_n|0i\cdots 00\rangle_{A_1A_2\cdots A_n} + \cdots\\
&\quad + \omega_n^{n-1}|00\cdots 0i\rangle_{A_1A_2\cdots A_n},\\
|S\rangle &= |0+\cdots+(d-1)\rangle_{A_1}|0+\cdots+(d-1)\rangle_{A_2}\cdots|0\\
&\quad +\cdots+(d-1)\rangle_{A_n},
\end{aligned} \tag{8}$$

where $1 \leqslant i \leqslant d-1$, $d \geqslant 2$.

Proof. Since the states are symmetric, it is sufficient to prove that the first party could only start with a trivial orthogonality-preserving measurement. Let $E_1 = (m_{a,b}^1)_{a,b\in\mathbb{Z}_d}$ represent an element of any orthogonality-preserving measurement performed by Alice. For each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|E_1 \otimes I_2 \otimes \cdots \otimes I_n|\phi\rangle = 0. \tag{9}$$

For $0 \leqslant i \neq j \leqslant d-1$, considering Eq. (9) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$, we obtain $m_{i,j}^1 = m_{j,i}^1 = 0$ by Lemma 1. Therefore, the off-diagonal elements of E_1 are all zeros. Considering Eq. (9) for the states $|S\rangle$ and $|\phi_i\rangle$, we directly get $m_{i,i}^1 = m_{0,0}^1$ for $1 \leqslant i \leqslant d-1$ by Lemma 2. Therefore, E_1 is proportional to the identity matrix. So, the first party cannot start with a nontrivial orthogonality-preserving measurement.

Therefore, the above $d+1$ states are locally stable by definition. This completes the proof. ■

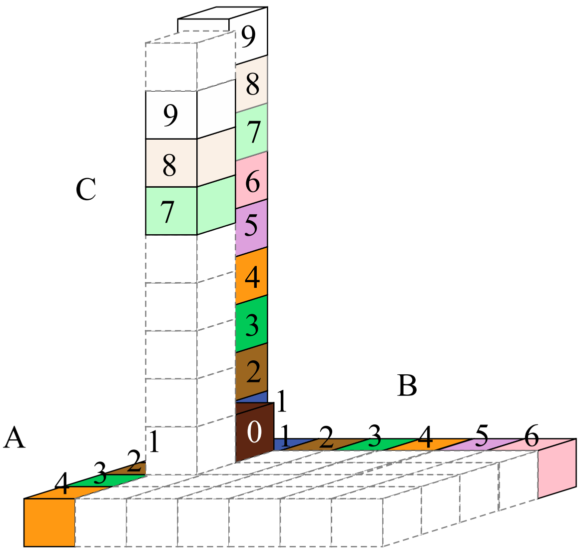


FIG. 3. Intuition of the structure of states we constructed in Eq. (10) for the setting $d_1 = 5$, $d_2 = 7$, and $d_3 = 10$. Note that the cubic with coordinate (0,0,0) should be labeled with "0". The squares indicated by the same color represent a unique state and the numbers represent the subscripts of the states. For example, the two pink squares (0,6,0) and (0,0,6) with label "6" correspond to the state $|\phi_6\rangle = |060\rangle_{ABC} - |006\rangle_{ABC}$; the two white squares (0,0,9) and (2,1,8) with label "9" correspond to the state $|\phi_9\rangle = |009\rangle_{ABC} - |218\rangle_{ABC}$.

Next, we consider the constructions of locally stable sets in the general multipartite quantum systems. In order to be better understood, we first show our construction in arbitrary tripartite quantum systems.

B. Locally stable set in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$

Theorem 4. In $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ ($3 \leqslant d_1 \leqslant d_2 \leqslant d_3$), the following set of $d_3 + 1$ orthogonal states is locally stable [see Fig. 3 for an intuition of the example where $d_1 = 5$, $d_2 = 7$, and $d_3 = 10$]:

$$\begin{aligned}
|\phi_0\rangle &= |000\rangle_{ABC} - |111\rangle_{ABC},\\
|\phi_i\rangle &= |i00\rangle_{ABC} + \omega_3|0i0\rangle_{ABC} + \omega_3^2|00i\rangle_{ABC},\\
&\quad 1 \leqslant i \leqslant d_1 - 1,\\
|\phi_j\rangle &= |0j0\rangle_{ABC} - |00j\rangle_{ABC}, \quad d_1 \leqslant j \leqslant d_2 - 1,\\
|\phi_k\rangle &= |00k\rangle_{ABC} - |21(k-1)\rangle_{ABC}, \quad d_2 \leqslant k \leqslant d_3 - 1,\\
|S\rangle &= |0+\cdots+(d_1-1)\rangle_A|0+\cdots+(d_2-1)\rangle_B|0+\cdots\\
&\quad +(d_3-1)\rangle_C.
\end{aligned} \tag{10}$$

Proof. As far as Alice is concerned, it is the same as the equal dimensional case. We only need to prove that Bob and Charlie have to implement trivial measurement.

Suppose that Bob starts with the first orthogonality-preserving measurement whose elements are represented as $E_2 = (m_{a,b}^2)_{a,b\in\mathbb{Z}_{d_2}}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes E_2 \otimes I_3|\phi\rangle = 0. \tag{11}$$

TABLE I. Zero entries of the matrix $E_3 = (m^3_{a,b})_{a,b\in\mathbb{Z}_{d_3}}$.

Pair of states	Zero entries	Value range
$\lvert\phi_0\rangle, \lvert\phi_i\rangle$	$m^3_{0,i} = m^3_{i,0} = 0$	$1 \leqslant i \leqslant d_1 - 1$
$\lvert\phi_0\rangle, \lvert\phi_j\rangle$	$m^3_{0,j} = m^3_{j,0} = 0$	$d_1 \leqslant j \leqslant d_2 - 1$
$\lvert\phi_0\rangle, \lvert\phi_k\rangle$	$m^3_{0,k} = m^3_{k,0} = 0$	$d_2 \leqslant k \leqslant d_3 - 1$
$\lvert\phi_i\rangle, \lvert\phi_{i'}\rangle$	$m^3_{i,i'} = m^3_{i',i} = 0$	$1 \leqslant i \neq i' \leqslant d_1 - 1$
$\lvert\phi_j\rangle, \lvert\phi_{j'}\rangle$	$m^3_{j,j'} = m^3_{j',j} = 0$	$d_1 \leqslant j \neq j' \leqslant d_2 - 1$
$\lvert\phi_i\rangle, \lvert\phi_j\rangle$	$m^3_{i,j} = m^3_{j,i} = 0$	$1 \leqslant i \leqslant d_1 - 1$, $d_1 \leqslant j \leqslant d_2 - 1$
$\lvert\phi_i\rangle, \lvert\phi_k\rangle$	$m^3_{i,k} = m^3_{k,i} = 0$	$1 \leqslant i \leqslant d_1 - 1$ $d_2 \leqslant k \leqslant d_3 - 1$
$\lvert\phi_j\rangle, \lvert\phi_k\rangle$	$m^3_{j,k} = m^3_{k,j} = 0$	$d_1 \leqslant j \leqslant d_2 - 1$ $d_2 \leqslant k \leqslant d_3 - 1$

Now we consider Eq. (11) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$ for $0 \leqslant i \neq j \leqslant d_2 - 1$. Note that the AC parties of each term of $|\phi_i\rangle$ and $|\phi_j\rangle$ are orthogonal except the terms $|0i0\rangle$ (corresponding to $|\psi_i\rangle$) and $|0j0\rangle$ (corresponding to $|\psi_j\rangle$). Therefore, by Lemma 1, we could obtain that $m^2_{i,j} = m^2_{j,i} = 0$. Therefore, the off-diagonal entries of E_2 are all zeros.

For $1 \leqslant i \leqslant d_2 - 1$, considering Eq. (11) for the states $|S\rangle$ and $|\phi_i\rangle$, we get $m^2_{i,i} = m^2_{0,0}$ by Lemma 2. Therefore, $E_2 \propto \mathbb{I}$. Bob cannot start with a nontrivial measurement either.

Let us consider the third party Charlie. Suppose that Charlie starts with the first orthogonality-preserving measurement whose elements are represented as $E_3 = (m^3_{a,b})_{a,b\in\mathbb{Z}_{d_3}}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes I_2 \otimes E_3|\phi\rangle = 0. \tag{12}$$

Considering Eq. (12) for the pair $|\phi_i\rangle$ and $|\phi_j\rangle$, where $0 \leqslant i \neq j \leqslant d_3 - 1$, by Lemma 1, we could obtain that the off-diagonal entry $m^3_{i,j}$ of the matrix E_3 is zero except $d_2 \leqslant i \neq j \leqslant d_3 - 1$ (see Table I).

TABLE II. Diagonal entries of $E_3 = (m^3_{a,b})_{a,b\in\mathbb{Z}_{d_3}}$.

Pair of states	Diagonal entries	Value range
$\lvert S\rangle, \lvert\phi_i\rangle$	$m^3_{0,0} = m^3_{i,i}$	$1 \leqslant i \leqslant d_1 - 1$
$\lvert S\rangle, \lvert\phi_j\rangle$	$m^3_{0,0} = m^3_{j,j}$	$d_1 \leqslant j \leqslant d_2 - 1$
$\lvert S\rangle, \lvert\phi_k\rangle$	$m^3_{k,k} = m^3_{(k-1),(k-1)}$	$d_2 \leqslant k \leqslant d_3 - 1$

Now we consider Eq. (12) for the pair $|\phi_k\rangle$ and $|\phi_{k'}\rangle$, where $d_2 \leqslant k < k' \leqslant d_3 - 1$. That is,

$$[\langle 00k| - \langle 21(k-1)|]I_1 \otimes I_2 \otimes E_3[|00k'\rangle - |21(k'-1)\rangle] = 0,$$

from which we deduce that $m^3_{k,k'} = -m^3_{(k-1),(k'-1)}$. Therefore, we have

$$m^3_{k,k'} = -m^3_{(k-1),(k'-1)} = \cdots = (-1)^{k-d_2+1} m^3_{d_2-1,(k'-k+d_2-1)},$$

which is equal to zero as the last term $m^3_{d_2-1,(k'-k+d_2-1)} = 0$ has been obtained. Thus we get $m^3_{k,k'} = m^3_{k',k} = 0$.

By Lemma 2, all diagonal entries of the matrix E_3 are equal from Table II, i.e., $E_3 \propto \mathbb{I}$. Charlie cannot start with a nontrivial measurement.

In summary, all the subsystems can only start with a trivial orthogonality-preserving measurement. Therefore, the above $d_3 + 1$ states form a locally stable set. This completes the proof.

C. Locally stable set in $\otimes_{i=1}^n \mathbb{C}^{d_i}$

Theorem 5. The following set $\mathcal{S}$ of $d_n + 1$ orthogonal states are locally stable in $\otimes_{i=1}^n \mathbb{C}^{d_i}$ for $3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ and $n \geqslant 3$:

$$\begin{aligned}
|\phi_0\rangle &= |00\cdots00\rangle_{A_1A_2\cdots A_n} - |11\cdots11\rangle_{A_1A_2\cdots A_n},\\
|\phi_{i_1}\rangle &= |i_1 0\cdots00\rangle_{A_1A_2\cdots A_n} + \omega_n|0i_1 0\cdots00\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_n^{n-2}|00\cdots0i_1 0\rangle_{A_1A_2\cdots A_n}\\
&\quad + \omega_n^{n-1}|00\cdots0i_1\rangle_{A_1A_2\cdots A_n}, \quad 1 \leqslant i_1 \leqslant d_1 - 1,\\
|\phi_{i_2}\rangle &= |0i_2 0\cdots0\rangle_{A_1A_2\cdots A_n} + \omega_{n-1}|00i_2 0\cdots0\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_{n-1}^{n-2}|0\cdots0i_2\rangle_{A_1A_2\cdots A_n}, \quad d_1 \leqslant i_2 \leqslant d_2 - 1,\\
|\phi_{i_3}\rangle &= |00i_3 0\cdots0\rangle_{A_1A_2\cdots A_n} + \omega_{n-2}|000i_3 0\cdots0\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_{n-2}^{n-3}|0\cdots0i_3\rangle_{A_1A_2\cdots A_n}, \quad d_2 \leqslant i_3 \leqslant d_3 - 1\\
&\cdots\cdots\cdots\\
|\phi_{i_{n-1}}\rangle &= |00\cdots0i_{n-1}0\rangle_{A_1A_2\cdots A_n} - |00\cdots0i_{n-1}\rangle_{A_1A_2\cdots A_n}, \quad d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1,\\
|\phi_{i_n}\rangle &= |00\cdots0i_n\rangle_{A_1A_2\cdots A_n} - |21\cdots1(i_n-1)\rangle_{A_1A_2\cdots A_n}, \quad d_{n-1} \leqslant i_n \leqslant d_n - 1,\\
|S\rangle &= |0+\cdots+(d_1-1)\rangle_{A_1}|0+\cdots+(d_2-1)\rangle_{A_2}\cdots|0+\cdots+(d_n-1)\rangle_{A_n}.
\end{aligned} \tag{13}$$

Proof. First, we show that each of the first $(n-1)$ parties could only start with a trivial orthogonality-preserving measurement. Suppose that the kth $(1 \leqslant k \leqslant n-1)$ party starts with the first orthogonality-preserving measurement whose elements are represented as $E_k = (m^k_{a,b})_{a,b\in\mathbb{Z}_{d_k}}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes \cdots \otimes E_k \otimes \cdots \otimes I_n|\phi\rangle = 0. \tag{14}$$

Now we consider Eq. (14) for the states $|\phi_i\rangle$ and $|\phi_j\rangle$ for $0 \leqslant i \neq j \leqslant d_k - 1$. Note that the parties except the kth of each term of $|\phi_i\rangle$ and $|\phi_j\rangle$ are orthogonal except the terms $|0\cdots0i0\cdots0\rangle$ (corresponding to $|\psi_i\rangle$) and $|0\cdots0j0\cdots0\rangle$ (corresponding to $|\psi_j\rangle$), where i, j are in the kth position. Therefore, by Lemma 1, we could obtain that $m^k_{i,j} = m^k_{j,i} = 0$. Therefore, the off-diagonal entries of E_k are all zeros.

For $1 \leqslant i \leqslant d_k - 1$, considering Eq. (14) for the states $|S\rangle$ and $|\phi_i\rangle$, we get $m^k_{i,i} = m^k_{0,0}$ by Lemma 2. Therefore, $E_k \propto \mathbb{I}$.

TABLE III. Zero entries of the matrix $E_n = (m^n_{a,b})_{a,b\in\mathbb{Z}_{d_n}}$.

Pair of states	Zero entries	Value range
$\lvert\phi_0\rangle, \lvert\phi_{i_1}\rangle$	$m^n_{0,i_1} = m^n_{i_1,0} = 0$	$1 \leqslant i_1 \leqslant d_1 - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert\phi_0\rangle, \lvert\phi_{i_{n-1}}\rangle$	$m^n_{0,i_{n-1}} = m^n_{i_{n-1},0} = 0$	$d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1$
$\lvert\phi_0\rangle, \lvert\phi_{i_n}\rangle$	$m^n_{0,i_n} = m^n_{i_n,0} = 0$	$d_{n-1} \leqslant i_n \leqslant d_n - 1$
$\lvert\phi_{i_1}\rangle, \lvert\phi'_{i_1}\rangle$	$m^n_{i_1,i'_1} = m^n_{i'_1,i_1} = 0$	$1 \leqslant i_1 \neq i'_1 \leqslant d_1 - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert\phi_{i_{n-2}}\rangle, \lvert\phi'_{i_{n-2}}\rangle$	$m^n_{i_{n-2},i'_{n-2}} = m^n_{i'_{n-2},i_{n-2}} = 0$	$d_{n-3} \leqslant i_{n-2} \neq i'_{n-2} \leqslant d_{n-2} - 1$
$\lvert\phi_{i_{n-1}}\rangle, \lvert\phi'_{i_{n-1}}\rangle$	$m^n_{i_{n-1},i'_{n-1}} = m^n_{i'_{n-1},i_{n-1}} = 0$	$d_{n-2} \leqslant i_{n-1} \neq i'_{n-1} \leqslant d_{n-1} - 1$
$\lvert\phi_{i_1}\rangle, \lvert\phi_{i_2}\rangle$	$m^n_{i_1,i_2} = m^n_{i_2,i_1} = 0$	$1 \leqslant i_1 \leqslant d_1 - 1, d_1 \leqslant i_2 \leqslant d_2 - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert\phi_{i_1}\rangle, \lvert\phi_{i_{n-1}}\rangle$	$m^n_{i_1,i_{n-1}} = m^n_{i_{n-1},i_1} = 0$	$1 \leqslant i_1 \leqslant d_1 - 1, d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1$
$\lvert\phi_{i_1}\rangle, \lvert\phi_{i_n}\rangle$	$m^n_{i_1,i_n} = m^n_{i_n,i_1} = 0$	$1 \leqslant i_1 \leqslant d_1 - 1,\ d_{n-1} \leqslant i_n \leqslant d_n - 1$
$\lvert\phi_{i_2}\rangle, \lvert\phi_{i_3}\rangle$	$m^n_{i_2,i_3} = m^n_{i_3,i_2} = 0$	$d_1 \leqslant i_2 \leqslant d_2 - 1, d_2 \leqslant i_3 \leqslant d_3 - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert\phi_{i_2}\rangle, \lvert\phi_{i_{n-1}}\rangle$	$m^n_{i_2,i_{n-1}} = m^n_{i_{n-1},i_2} = 0$	$d_1 \leqslant i_2 \leqslant d_2 - 1, d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1$
$\lvert\phi_{i_2}\rangle, \lvert\phi_{i_n}\rangle$	$m^n_{i_2,i_n} = m^n_{i_n,i_2} = 0$	$d_1 \leqslant i_2 \leqslant d_2 - 1,\ d_{n-1} \leqslant i_n \leqslant d_n - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert\phi_{i_{n-1}}\rangle, \lvert\phi_{i_n}\rangle$	$m^n_{i_{n-1},i_n} = m^n_{i_n,i_{n-1}} = 0$	$d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1,\ d_{n-1} \leqslant i_n \leqslant d_n - 1$

So the kth party cannot start with a nontrivial orthogonality-preserving measurement.

Let us consider the last party, i.e., the nth party. Suppose that the nth party starts with the first orthogonality-preserving measurement whose elements are represented as $E_n = (m^n_{a,b})_{a,b\in\mathbb{Z}_{d_n}}$. Then for each pair $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ with $|\psi\rangle \neq |\phi\rangle$, we have

$$\langle\psi|I_1 \otimes I_2 \otimes \cdots \otimes I_{n-1} \otimes E_n|\phi\rangle = 0. \tag{15}$$

Considering Eq. (15) for the pair $|\phi_i\rangle$ and $|\phi_j\rangle$ where $0 \leqslant i \neq j \leqslant d_n - 1$, by Lemma 1, we could obtain that the off-diagonal entries $m^n_{i,j}$ of the matrix E_n are zero except $d_{n-1} \leqslant i \neq j \leqslant d_n - 1$ (see Table III). Now we only consider the remaining off-diagonal entries of the matrix E_n.

For $d_{n-1} \leqslant i_n < i'_n \leqslant d_n - 1$, we consider Eq. (15) for the pair $|\phi_{i_n}\rangle$ and $|\phi_{i'_n}\rangle$. That is, $[\langle 00\cdots 0i_n| - \langle 21\cdots 1(i_n - 1)|]I_1 \otimes I_2 \otimes \cdots \otimes E_n[|00\cdots 0i'_n\rangle - |21\cdots 1(i'_n - 1)\rangle] = 0$ from which we deduce that $m^n_{i_n,i'_n} = -m^n_{i_n-1,i'_n-1}$. Therefore, we have

$$m^n_{i_n,i'_n} = (-1)^{i_n-d_{n-1}+1} m^n_{d_{n-1}-1,(i'_n-i_n+d_{n-1}-1)} = 0,$$

where the last equality has been deduced previously. Thus we get $m^n_{i_n,i'_n} = m^n_{i'_n,i_n} = 0$. Hence we have that the off-diagonal entries of the matrix E_n are zeros.

By Lemma 2, all diagonal entries of the matrix E_n are equal from Table IV.

Therefore, all parties can only start with a trivial orthogonality preserving measurement. The set of $d_n + 1$ orthogonal states is locally stable. ■

Moreover, we put forward a new construction of the locally stable set in $\otimes_{i=1}^n \mathbb{C}^{d_i}$ $(3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n, n \geqslant 3)$, which is composed of genuine entangled states apart from one full product state and also reach the minimum cardinality of the locally stable set proposed in Ref. [46]; see Appendix B for the details.

Many efforts have been made to reduce the cardinality of locally indistinguishable sets. Here we list the cardinalities of locally indistinguishable sets that have been known before (see Table V). As locally stable sets are always locally indistinguishable, there exists some locally indistinguishable sets with cardinality $d_n + 1$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$ (where we assume $d_1 \leqslant \cdots \leqslant d_n$). Thus our work has made a significant improvement towards addressing this issue.

V. CONCLUSION

We studied the construction of locally stable sets for the given multipartite systems. It is interesting to note that the structures reach the lower bound of the cardinality on the locally stable sets. In fact, we presented the constructions of locally stable sets with minimum cardinality in bipartite quantum systems $\mathbb{C}^d \otimes \mathbb{C}^d$ and $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. Then we presented a construction of $d + 1$ orthogonal states in $(\mathbb{C}^d)^{\otimes n}$ and proved that the set is locally stable. Furthermore, we generalized our construction to more general cases and put forward two structures of $d_n + 1$ orthogonal states for arbitrary multipartite quantum systems. Our results give a complete answer

TABLE IV. Diagonal entries of $E_n = (m^n_{a,b})_{a,b\in\mathbb{Z}_{d_k}}$.

Pair of states	Diagonal entries	Value range
$\lvert S\rangle, \lvert\phi_{i_1}\rangle$	$m^n_{0,0} = m^n_{i_1,i_1}$	$1 \leqslant i_1 \leqslant d_1 - 1$
$\lvert S\rangle, \lvert\phi_{i_2}\rangle$	$m^n_{0,0} = m^n_{i_2,i_2}$	$d_1 \leqslant i_2 \leqslant d_2 - 1$
$\vdots$	$\vdots$	$\vdots$
$\lvert S\rangle, \lvert\phi_{i_{n-1}}\rangle$	$m^n_{0,0} = m^n_{i_{n-1},i_{n-1}}$	$d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1} - 1$
$\lvert S\rangle, \lvert\phi_{i_n}\rangle$	$m^n_{i_n,i_n} = m^n_{i_n-1,i_n-1}$	$d_{n-1} \leqslant i_n \leqslant d_n - 1$

TABLE V. Incomplete list of the cardinalities of locally indistinguishable sets that are known before.

Reference	System	Cardinality
[10]	$\mathbb{C}^d\otimes\mathbb{C}^d$	$2d-1$
[24]	$(\mathbb{C}^d)^{\otimes n}$	$2n(d+1)$
[28]	$(\mathbb{C}^d)^{\otimes n}$	$n(d-1)+1$
[16]	$\mathbb{C}^m\otimes\mathbb{C}^n$	$3(m+n)-9$
[20]	$\mathbb{C}^m\otimes\mathbb{C}^n$	$2n-1$
[26]	$\mathbb{C}^m\otimes\mathbb{C}^n$	$2(m+n)-4$
[22]	$\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_3}$	$2(n_2+n_3)-3$
[25]	$\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_n}$	$\sum_{i=1}^{n}(2d_i-3)+1$
[28]	$\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_n}$	$\sum_{i=2}^{n-1}d_i+2d_n-n+1$

to the open problem raised in Ref. [46]. Moreover, all of our constructed locally stable sets are optimal in the sense that removing any state from this set makes it impossible to achieve local stability again.

Here we have considered the constructions of the smallest locally stable sets by utilizing entangled states and a stopper state. However, there are two very important questions that deserve further research. Can we construct the strongest nonlocal sets that reach the corresponding lower bound? How do we quantify the strength of quantum nonlocality?

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APPENDIX A: PROOFS OF LEMMA 1 AND LEMMA 2

Proof of Lemma 1. Substituting the expressions

$$|\phi_i\rangle=\sum_{t=0}^{p_i-1}\omega_{p_i}^t\big|i_1^t\big\rangle_{A_1}\big|i_2^t\big\rangle_{A_2}\cdots\big|i_n^t\big\rangle_{A_n},$$

$$|\phi_j\rangle=\sum_{s=0}^{p_j-1}\omega_{p_j}^s\big|j_1^s\big\rangle_{A_1}\big|j_2^s\big\rangle_{A_2}\cdots\big|j_n^s\big\rangle_{A_n}$$

into

$$\langle\phi_i|I_1\otimes\cdots\otimes E_k\otimes\cdots\otimes I_n|\phi_j\rangle=0,$$

we obtain

$$\left(\sum_{t=0}^{p_i-1}\omega_{p_i}^{-t}\big\langle i_1^t\big|\big\langle i_2^t\big|\cdots\big\langle i_n^t\big|\right)I_1\otimes\cdots\otimes E_k\otimes\cdots$$

$$\otimes I_n\left(\sum_{s=0}^{p_j-1}\omega_{p_j}^s\big|j_1^s\big\rangle\big|j_2^s\big\rangle\cdots\big|j_n^s\big\rangle\right)=0.$$

Further,

$$\sum_{s=0}^{p_j-1}\sum_{t=0}^{p_i-1}\omega_{p_i}^{-t}\omega_{p_j}^s\big\langle i_k^t\big|E_k\big|j_k^s\big\rangle_{A_k}\prod_{\ell\neq k}\big\langle i_\ell^t\big|j_\ell^s\big\rangle_{A_\ell}=0.$$

Since there is only one pair $(t_0,s_0)\in\mathbb{Z}_{p_i}\times\mathbb{Z}_{p_j}$ such that $\prod_{\ell\neq k}\langle i_\ell^{t_0}|j_\ell^{s_0}\rangle_{A_\ell}\neq 0$, therefore we can get $\omega_{p_i}^{-t_0}\omega_{p_j}^{s_0}\langle i_k^{t_0}|E_k|j_k^{s_0}\rangle_{A_k}\prod_{\ell\neq k}\langle i_\ell^{t_0}|j_\ell^{s_0}\rangle_{A_\ell}=0$. Then $\langle i_k^{t_0}|E_k|j_k^{s_0}\rangle=0$, which means that $m^k_{i_k^{t_0},j_k^{s_0}}=0$. ■

Proof of Lemma 2. Substituting the expressions

$$|S\rangle=\otimes_{k=1}^n\left(\sum_{i_k\in\mathbb{Z}_{d_k}}|i_k\rangle_{A_k}\right),$$

$$|\phi_i\rangle=\sum_{t=0}^{p-1}\omega_p^t\big|i_1^t\big\rangle_{A_1}\big|i_2^t\big\rangle_{A_2}\cdots\big|i_n^t\big\rangle_{A_n}$$

into

$$\langle S|I_1\otimes\cdots\otimes E_k\otimes\cdots\otimes I_n|\phi_i\rangle=0,$$

we obtain

$$\left[\otimes_{k=1}^n\left(\sum_{i_k\in Z_{d_k}}\langle i_k|\right)\right]I_1\otimes\cdots\otimes E_k\otimes\cdots$$

$$\otimes I_n\left(\sum_{t=0}^{p-1}\omega_p^t\big|i_1^t\big\rangle\big|i_2^t\big\rangle\cdots\big|i_n^t\big\rangle\right)=0.$$

Further,

$$\sum_{t=0}^{p-1}\omega_p^t\langle 0+\cdots+(d_1-1)\big|i_1^t\big\rangle\cdots\langle 0+\cdots+(d_k-1)|E_k\big|i_k^t\big\rangle$$

$$\cdots\langle 0+\cdots+(d_n-1)\big|i_n^t\big\rangle=0.$$

Moreover,

$$\sum_{t=0}^{p-1}\omega_p^t\langle 0+\cdots+(d_k-1)|E_k\big|i_k^t\big\rangle=0.$$

Since all $m^k_{a,b}=0$ with $0\leqslant a\neq b\leqslant d_k-1$, this means that

$$m^k_{i_k^0,i_k^0}+\omega_p m^k_{i_k^1,i_k^1}+\omega_p^2 m^k_{i_k^2,i_k^2}+\cdots+\omega_p^{p-1}m^k_{i_k^{p-1},i_k^{p-1}}=0.$$

If there exist only two different values $i_k^{t_0}$ and $i_k^{t_1}$ for $i_k^0,i_k^1,\ldots,i_k^{p-1}$, this means that p elements are divided into two groups. There may be $p-1$ elements equal, $p-2$ elements that are equal and other 2 elements are equal, $p-3$ elements that are equal and the remaining 3 elements are equal, etc. Here we only consider the following two cases; the others can be proved in a similar way.

(1) Suppose $i_k^0=i_k^{t_0}$, $i_k^1=\cdots=i_k^{p-1}=i_k^{t_1}$, then

$$m^k_{i_k^{t_0},i_k^{t_0}}=-(\omega_p+\omega_p^2+\cdots+\omega_p^{p-1})m^k_{i_k^{t_1},i_k^{t_1}}.$$

(2) Suppose $i_k^0=i_k^1=i_k^{t_0}$, $i_k^2=\cdots=i_k^{p-1}=i_k^{t_1}$, then

$$(1+\omega_p)m^k_{i_k^{t_0},i_k^{t_0}}=-(\omega_p^2+\cdots+\omega_p^{p-1})m^k_{i_k^{t_1},i_k^{t_1}}.$$

We know that $1+\omega_p+\omega_p^2+\cdots+\omega_p^{p-1}=0$; hence $m^k_{i_k^{t_0},i_k^{t_0}}=m^k_{i_k^{t_1},i_k^{t_1}}$. ■

APPENDIX B: ANOTHER STRUCTURE IN $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$

Theorem 6. The following set $\mathcal{S}$ of d_n orthogonal genuine entangled states and one full product state are locally stable in $\otimes_{i=1}^{n}\mathbb{C}^{d_i}$ for $3 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, $n \geqslant 3$:

$$
\begin{aligned}
|\phi_0\rangle &= |00\cdots00\rangle_{A_1A_2\cdots A_n} - |11\cdots11\rangle_{A_1A_2\cdots A_n},\\
|\phi_{i_1}\rangle &= |i_10\cdots00\rangle_{A_1A_2\cdots A_n} + \omega_n|0i_10\cdots00\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_n^{n-2}|00\cdots0i_10\rangle_{A_1A_2\cdots A_n}\\
&\quad + \omega_n^{n-1}|00\cdots0i_1\rangle_{A_1A_2\cdots A_n}, \quad 1 \leqslant i_1 \leqslant d_1-1,\\
|\phi_{i_2}\rangle &= |0i_20\cdots00\rangle_{A_1A_2\cdots A_n} + \omega_n|00i_20\cdots00\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_n^{n-2}|00\cdots0i_2\rangle_{A_1A_2\cdots A_n}\\
&\quad + \omega_n^{n-1}|10\cdots0i_2\rangle_{A_1A_2\cdots A_n}, \quad d_1 \leqslant i_2 \leqslant d_2-1,\\
|\phi_{i_3}\rangle &= |00i_30\cdots00\rangle_{A_1A_2\cdots A_n} + \omega_{n-1}|000i_30\cdots00\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_{n-1}^{n-3}|00\cdots0i_3\rangle_{A_1A_2\cdots A_n}\\
&\quad + \omega_{n-1}^{n-2}|110\cdots0i_3\rangle_{A_1A_2\cdots A_n}, \quad d_2 \leqslant i_3 \leqslant d_3-1,\\
|\phi_{i_4}\rangle &= |000i_40\cdots0\rangle_{A_1A_2\cdots A_n} + \omega_{n-2}|0000i_40\cdots0\rangle_{A_1A_2\cdots A_n} + \cdots + \omega_{n-2}^{n-4}|00\cdots0i_4\rangle_{A_1A_2\cdots A_n}\\
&\quad + \omega_{n-2}^{n-3}|1110\cdots0i_4\rangle_{A_1A_2\cdots A_n}, \quad d_3 \leqslant i_4 \leqslant d_4-1\\
&\quad \cdots\cdots\cdots\\
|\phi_{i_{n-1}}\rangle &= |0\cdots0i_{n-1}0\rangle_{A_1A_2\cdots A_n} + \omega_3|0\cdots0i_{n-1}\rangle_{A_1A_2\cdots A_n} + \omega_3^2|1\cdots10i_{n-1}\rangle_{A_1A_2\cdots A_n}, \quad d_{n-2} \leqslant i_{n-1} \leqslant d_{n-1}-1,\\
|\phi_{i_n}\rangle &= |00\cdots0i_n\rangle_{A_1A_2\cdots A_n} - |21\cdots1(i_n-1)\rangle_{A_1A_2\cdots A_n}, \quad d_{n-1} \leqslant i_n \leqslant d_n-1,\\
|S\rangle &= |0+\cdots+(d_1-1)\rangle_{A_1}|0+\cdots+(d_2-1)\rangle_{A_2}\cdots|0+\cdots+(d_n-1)\rangle_{A_n}.
\end{aligned} \tag{B1}
$$

Proof. Comparing with Eq. (13), in Eq. (B1), we only made some slight adjustments such that $|\phi_{i_1}\rangle \sim |\phi_{i_n}\rangle$ are genuinely entangled states. Thus we only need to consider some special entries of the matrix E_n.

For the states $|\phi_{i_2}\rangle$ and $|\phi_{i_2'}\rangle$, where $d_1 \leqslant i_2 \neq i_2' \leqslant d_2-1$, we have $(\langle 0i_2\cdots0| + \cdots + \omega_n^{2-n}\langle 0\cdots0i_2| + \omega_n^{1-n}\langle 10\cdots0i_2|)I_1\otimes I_2\otimes\cdots\otimes E_n(|0i_2'\cdots0\rangle + \cdots + \omega_n^{n-2}|00\cdots0i_2'\rangle + \omega_n^{n-1}|10\cdots0i_2'\rangle) = 0$. Because of the fact that only $|00\cdots0i_2\rangle_{A_1\cdots A_n}$ and $|00\cdots0i_2'\rangle_{A_1\cdots A_n}$, $|100\cdots0i_2\rangle_{A_1\cdots A_n}$ and $|100\cdots0i_2'\rangle_{A_1\cdots A_n}$ are not orthogonal on $n-1$ subsystems except the nth subsystem, so $\langle i_2|E_n|i_2'\rangle + \langle i_2|E_n|i_2'\rangle = 0$; thus $m_{i_2,i_2'}^n = m_{i_2',i_2}^n = 0$ for $d_1 \leqslant i_2 \neq i_2' \leqslant d_2-1$.

Similarly, from the states $|\phi_{i_{n-1}}\rangle$ and $|\phi_{i_{n-1}'}\rangle$, we can get $m_{i_{n-1},i_{n-1}'}^n = m_{i_{n-1}',i_{n-1}}^n = 0$ for $d_{n-2} \leqslant i_{n-1} \neq i_{n-1}' \leqslant d_{n-1}-1$.

From the states $|\phi_1\rangle$ and $|\phi_{i_2}\rangle$, we have $(\langle 10\cdots0| + \cdots + \omega_n^{2-n}\langle 00\cdots010| + \omega_n^{1-n}\langle 00\cdots01|)I_1\otimes I_2\otimes\cdots\otimes E_n(|0i_20\cdots0\rangle + \cdots + \omega_n^{n-2}|00\cdots0i_2\rangle + \omega_n^{n-1}|10\cdots0i_2\rangle) = 0$. Because only $|10\cdots0\rangle_{A_1\cdots A_n}$ and $|10\cdots0i_2\rangle_{A_1\cdots A_n}$, $|00\cdots01\rangle_{A_1\cdots A_n}$ and $|00\cdots0i_2\rangle_{A_1\cdots A_n}$ are not orthogonal on $n-1$ subsystems except the nth subsystem, then $\omega_n^{n-1}\langle 0|E_n|i_2\rangle + \omega_n^{-1}\langle 1|E_n|i_2\rangle = 0$; i.e., $\omega_n^{n-1}m_{0,i_2}^n + \omega_n^{-1}m_{1,i_2}^n = 0$, since $m_{0,i_2}^n = m_{i_2,0}^n = 0$, and we can get $m_{1,i_2}^n = m_{i_2,1}^n = 0$ for $d_1 \leqslant i_2 \leqslant d_2-1$.

Therefore, all parties can only start with a trivial orthogonality preserving measurement. The set of d_n+1 orthogonal states is locally stable. ■

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