

Catalysis always degrades external quantum correlations

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Catalysts used in quantum resource theories need not be in isolation and, therefore, are possibly correlated with external systems, which the agent does not have access to. Do such correlations help or hinder catalysis, and does the classicality or quantumness of such correlations matter? To answer this question, we first focus on the existence of a noninvasively measurable observable that yields the same outcomes for repeated measurements since this signifies macrorealism, a key property distinguishing classical systems from quantum systems. We show that a system quantumly correlated with an external system so that the joint state is necessarily perturbed by any repeatable quantum measurement also has the same property against general quantum channels. Our full characterization of such systems called *totally quantum systems*, solves the open problem of characterizing tomographically sensitive systems raised in [Phys. Rev. Lett. **130**, 020802 (2023)]. An immediate consequence is that a totally quantum system cannot catalyze any quantum process, even when a measure of correlation with its environment is arbitrarily low. It generalizes to a stronger result, that the mutual information of totally quantum systems cannot be used as a catalyst either. These results culminate in the conclusion that, out of the correlations, that a generic quantum catalyst has with its environment, only classical correlations allow for catalysis, and, therefore, using a correlated catalyst is equivalent to using an ensemble of uncorrelated catalysts.

DOI: [10.1103/PhysRevA.108.012417](https://doi.org/10.1103/PhysRevA.108.012417)**I. INTRODUCTION**

Catalysis in quantum resource theory, a concept inspired by catalysis in chemistry, is a paradigm that utilizes some quantum resources without altering or deteriorating it, whereas expanding the set of accessible quantum states (or channel) transformations [1]. Initially, catalysis was often studied under the condition of being uncorrelated with the final state of the system [2–7]. In recent years, however, a notable trend is the investigation of how the power of catalysis, in enabling state transitions, can be increased by allowing correlations to persist between system and catalyst after the process [8–18]. The relaxation of this constraint simplifies the conditions for state transition significantly, and often leads to a characterization of von Neumann quantities (e.g. entropy)—which have a strong operational significance previously only in the asymptotic independent and identically distributed regime—in one-shot settings. It is argued that in certain scenarios, the resultant correlation can be ignored, which assumes that only the marginal state of catalyst is relevant when catalyst is separated from the system.

However, this line of thought clashes with an often-used concept of “catalyst bank” [19–21], a hypothetical entity that lends quantum resource catalysts to (possibly many) agents and retrieves thereafter. (See Fig. 1.) A catalyst could be only a part of a large collection of quantum systems possessed by the bank. In this case, it is operationally natural for the bank to require the agent who borrows the catalyst to return it in a fashion that the whole quantum system stays in the same state, even though the agent used only a small portion. Even if the catalyst is prepared uncorrelated with other systems after a single round of correlated catalysis by some user, the catalyst will form correlation. When the next user borrows the catalyst,

again, it is natural to require the correlations to be preserved as nothing forbids the same user to borrow the same catalyst twice, and not wasting any resource not possessed by oneself is the prime premise of resource catalysis.

Some studies on correlating catalysis deal with such potential problems, by showing that the amount of correlation formed in catalysis can be made arbitrarily small, e.g., Refs. [9,14]. However, we will show that whenever the correlation between the catalyst and the external systems is of a quantum-mechanical nature (which we formally specify later), even *arbitrarily small* correlations forbid catalysis in the sense that the joint state cannot be left unperturbed (Theorem 2).

In doing so, we fully characterize multipartite states that cannot be used as a catalyst when only local access is allowed. We show that this characterization coincides with the description of quantum states that have no local classical observable (Theorem 1). Here, classical observables mean those that can be measured without perturbing the global quantum state, in other words, they have noninvasive measurability of deterministically distinguishable states, i.e., obey macrorealism [22–25]. It turns out that characterizing catalyst with local access is equivalent to characterizing the property known as (tomographical) sensitivity, which was an open problem in the previous work [26].

The observation that quantum correlation that a catalyst has with other systems only hinders catalysis leads to the conclusion (Theorem 3) that only classical correlation in the correlated catalyst allows for meaningful catalysis. It yields a rather surprising consequence that utilizing a catalyst correlated with an external system is functionally equivalent to using an ensemble of uncorrelated catalysts. In other words, considering a correlated catalyst does not introduce

new types of nontrivial catalytic transformations, but it only induces probabilistic mixtures of conventional catalytic transformations.

II. BACKGROUND

Recall that a bipartite state ρ_{AB} is said to be classical-quantum when there exists an orthonormal basis $\{|i\rangle\}$ of A such that the following expression is possible:

$$\rho_{AB} = \sum_i p_i |i\rangle\langle i|_A \otimes \rho_B^{(i)}, \tag{1}$$

with some probability distribution (p_i) and a set of states $\{\rho_B^{(i)}\}$ on B . It is equivalent to the existence of a rank-1 projective measurement $\{|i\rangle\langle i|\}$ on A that does not disturb the global state ρ_{AB} after the measurement, i.e.,

$$\rho_{AB} = \sum_i (|i\rangle\langle i|_A \otimes \mathbb{1}_B) \rho_{AB} (|i\rangle\langle i|_A \otimes \mathbb{1}_B). \tag{2}$$

One can generalize this definition where the projective measurement need not be rank 1 anymore, leading to the following definition of partial classicality.

Definition 1. A bipartite state ρ_{AB} is said to be partially classical-quantum (PC-Q) when there exists a projective measurement $\{\Pi_k\}_{k=1}^n$ with $n > 1$ on A that preserves ρ_{AB} , i.e.,

$$\rho_{AB} = \sum_{k=1}^n (\Pi_k \otimes \mathbb{1}_B) \rho_{AB} (\Pi_k \otimes \mathbb{1}_B). \tag{3}$$

We will sometimes say that a single system is partially classical (PC) when it is implicitly assumed to be correlated with another system, and they are in a PC-Q state. When a system is not PC, then we will say that it is *totally quantum* (TQ) [18], so a non-PC-Q state is a TQ-Q state. Note that the correlations in a TQ-Q state can be, in general, significantly weaker compared to entanglement. For example, the following evidently separable state is a TQ-Q state:

$$\rho_{AB} = \frac{1}{2} \left(\sum_{i=0}^{d-1} \lambda_i |i\rangle\langle i|_A \right) \otimes |0\rangle\langle 0|_B + \frac{1}{2} |+\rangle\langle +|_A \otimes |1\rangle\langle 1|_B, \tag{4}$$

where $\sum_{i=0}^{d-1} \lambda_i |i\rangle\langle i|_A$ is a nondegenerate quantum state on A and $|+\rangle_A = d^{-1/2} \sum_{i=0}^{d-1} |i\rangle_A$ is a maximally coherent state on A . This definition of classicality can be further generalized to input or output systems of quantum channels or completely positive (CP) maps: we say that the output system of a quantum channel \mathcal{N} is PC when there exists a projective measurement $\mathcal{P} := \sum_k \Pi_k (\cdot) \Pi_k$ that fixes \mathcal{N} , i.e., $\mathcal{P} \circ \mathcal{N} = \mathcal{N}$. Similarly, we say that the input system of \mathcal{N} is PC when a projective measurement \mathcal{Q} exists such that $\mathcal{N} \circ \mathcal{Q} = \mathcal{N}$. Likewise, we say that the input or output system is TQ when it is non-PC.

III. TOTALLY QUANTUMNESS AND SENSITIVITY

One could question the generality of the notion of totally quantumness since allowing weaker measurements, such as

positive operator valued measures (POVM) instead of projective measurements in Definition 1 may give rise to a qualitatively different characterization. Perhaps the most natural definition of totally quantumness could be as follows: A of ρ_{AB} is said to be *totally quantum** (TQ*), or ρ_{AB} is said to be a TQ*-Q state, when any nontrivial quantum measurement on A necessarily perturbs ρ_{AB} . However, by noting that every quantum channel with Kraus operators $\{K_i\}$ can be considered an implementation of the POVM $\{K_i^\dagger K_i\}$, we observe that totally quantumness* defined above is equivalent to the concept of (tomographical) sensitivity introduced in Ref. [26], which characterizes a state’s ability to detect the action of *any* nontrivial local channel.

Definition 2. A bipartite state ρ_{AB} is sensitive on A to a set of quantum operations \mathcal{Q} with $\text{id} \in \mathcal{Q}$ when for every $S \in \mathcal{Q}$ where id denotes the identity channel,

$$(\mathcal{S}_A \otimes \text{id}_B)(\rho_{AB}) = \rho_{AB} \Rightarrow \mathcal{S} = \text{id}_A. \tag{5}$$

When a quantum state is sensitive to the set of all quantum channels, we simply say that it is sensitive, or equivalently, TQ*-Q. Similarly, through the Choi-Jamiołkowski isomorphism, we say that a linear map \mathcal{N} is sensitive to \mathcal{Q} when for every $S \in \mathcal{Q}$,

$$S \circ \mathcal{N} = \mathcal{N} \Rightarrow S = \text{id}. \tag{6}$$

Our first main result shows that actually the more general definition of totally quantumness is equivalent to the weaker one. This result solves the open problem questioned in Ref. [18].

Theorem 1 (TQ=TQ).* A quantum channel is sensitive if and only if its output system is TQ. Similarly, a bipartite state ρ_{AB} is sensitive on A if and only if it is a TQ-Q state.

Theorem 1 can be shown using the structure result of fixed points of quantum channels [27], which, in turn, follows from the Artin-Wedderburn theorem [28,29]. Self-contained elementary proofs of these results are given in the Appendix B. Theorem 1 says that there is no intermediate level of classicality when it comes to noninvasive measurability; in other words, sensitivity to projective measurements automatically implies sensitivity to general quantum channels. This implies that no classical value can be read from a non-PC system without perturbing it globally, even through weak measurements [30–32].

Theorem 1 yields an interesting property of totally quantumness that it is *contagious*; if any system prepared in a pure state unitarily interacts with a totally quantum system, then it also becomes totally quantum.

Proposition 1. Let ρ_{AB} be a TQ-Q state. For any isometry $V: \rightarrow AK$ such that the marginal state τ_E of $\tau_{KAB} = (V \otimes \mathbb{1}_B) \rho_{AB} (V^\dagger \otimes \mathbb{1}_B)$ is full-rank, τ_{KAB} is a TQ-Q state with respect to the bipartition $K|AB$.

An alternative interpretation of Proposition 1 is that any subsystem of a totally quantum system is also totally quantum. This result is analogous to that of Ref. [33] where a local projective measurement on a Q-Q state inevitably forms entanglement with a measurement device. Proposition 1 shows that this holds similarly even when one considers the more general case of POVMs, and TQ-Q states (proof in the Appendix C).

Proposition 1 shows that it is impossible to circumvent Theorem 1 by unitarily extracting a macrorealistic part K from

system A invasively. It means that there could be a nontrivial quantum channel action \mathcal{R} on A as a backaction of the invasive measurement, and one can interpret V as the Stinespring dilation of the quantum channel \mathcal{R} . It provides additional motivation for the nomenclature *totally quantum* system for non-PC systems as it has no classical property even in the weakest sense, i.e., when classicality means macrorealism and noninvasive measurability.

Theorem 1 offers another intuitive explanation of why quantum key distribution is secure. A typical example of TQ-Q state can be found in the BB84 protocol [34]. When Alice wants to send her random bit (say) 0 to Bob through a quantum channel that could be eavesdropped, she encodes that bit in either of two random bases and records it in her memory M ,

$$\rho_{AM} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_M + \frac{1}{2}|+\rangle\langle +|_A \otimes |1\rangle\langle 1|_M. \quad (7)$$

Since ρ_{AM} is a special case of (4), by Theorem 1, any eavesdropper interacting with the qubit A in a nontrivial fashion must alter the global state ρ_{AB} , which results in detectable statistical difference in the later steps of the protocol.

IV. LOCAL CATALYSIS OF THE BIPARTITE STATE

Theorem 1 has a significant consequence about catalysis utilizing correlated states. In resource theories, conventionally catalytic transformations mean processes described as

$$\rho_S \rightarrow \sigma_S := \text{Tr}_C[\Lambda(\rho_S \otimes \tau_C)], \quad (8)$$

with the catalytic constraint requiring that the catalyst remains in its original state in the process,

$$\text{Tr}_S[\Lambda(\rho_S \otimes \tau_C)] = \tau_C, \quad (9)$$

where C is called the catalyst and Λ is a free operation on SC .

However, we do not limit ourselves to state transitions between two fixed quantum states. In this paper, a catalytic transformation means a general quantum channel $\Phi(\rho)$ given as $\Phi(\rho) := \text{Tr}_C[\Lambda(\rho_S \otimes \tau_C)]$ regardless of whether the initial state ρ_S is fixed or not. Note that sometimes it is required that the final state of joint system SC has arbitrarily weak correlation, i.e., $\|\Lambda(\rho_S \otimes \tau_C) - \sigma_S \otimes \tau_C\|_1 < \epsilon$. However, we do not make such an assumption here for generality.

Now, using a correlated system CE in state τ_{CE} as a catalyst when only access to C is given means the transformation of the form in (8) (where τ_S is interpreted as $\text{Tr}_E \tau_{CE}$) with the modified constraint,

$$\text{Tr}_S[\Lambda_{SC} \otimes \text{id}_E(\rho_S \otimes \tau_{CE})] = \tau_{CE}. \quad (10)$$

A typical example of correlated catalyst τ_{CE} is the product of a previous catalysis, i.e., $\tau_{CE} = \Lambda(\rho_E \otimes \tau_C)$, here Λ is same with that in (8) but acts on CE . (See Fig. 1.) In other words, catalyst τ_C is “borrowed” by E for catalysis Λ , formed correlation with E , and returned to be borrowed by S again for another round of catalysis. As discussed in the Introduction, the bipartite state τ_{CE} could be used as a resource whenever two systems C and E are combined again, and any change in τ_{CE} by S can alter its resourceful nature. Our second main result then shows that the catalysis constraint Eq. (10) severely limits the usability of the correlated catalyst.

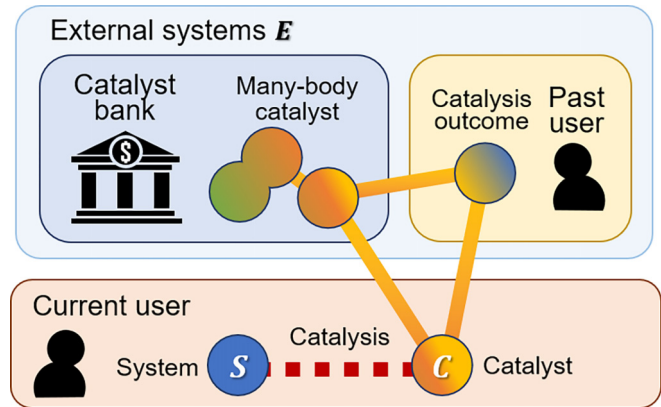


FIG. 1. A quantum resource catalyst could be correlated with external systems inaccessible to a user in many plausible scenarios. First, a catalyst can be a part of multipartite collection of catalysts of the bank. In this case, it is natural for the bank to demand the multipartite state to remain intact after each catalysis. Second, even for an initially uncorrelated catalyst, after a round of correlation-forming catalysis, it remains correlated with its previous user. The same user can borrow the catalyst again, and it is natural to expect the relation with the catalyst to remain the same as the previous round of catalysis.

Theorem 2. A TQ-Q state τ_{CE} cannot be used to catalytically implement a nontrivial transformation when only access to C is given.

Proof. We focus on the fact that once the initial state ρ_S and the interaction channel Λ on SC is fixed as in (8), then the following channel on C is induced.

$$\Gamma(\eta_C) := \text{Tr}_S[\Lambda(\rho_S \otimes \eta_C)]. \quad (11)$$

It follows that a catalyst τ_{CE} must be a fixed point of $\Gamma_C \otimes \text{id}_E$. However, by Theorem 1, the only channel on C that can fix a TQ-Q τ_{CE} is the identity channel id_C . It follows that the channel Λ on SC must be factorized into $\Lambda_{SC} = \Xi_S \otimes \text{id}_C$. It implies that there is no interaction between S and C , therefore, the catalysis is trivial. ■

We remark that our proofs did not assume that Λ is a free operation, hence, the results are applicable to a framework of catalysis much more general than the conventional one where the interaction between system and catalyst should be a free operation. If we assume that Λ_{SC} is a free operation, Ξ_S must be free too, if feeding into Λ_{SC} a free state on C and discarding it are all free operations. Therefore, the whole transformation is of the form $\rho_S \rightarrow \Xi_S(\rho_S)$, which is simply a transformation through a free operation.

In principle, catalysts can be also used for nonfree operations to reduce the cost (or boost the rate) of transformation, but Theorem 2 tells us that not even such generalized catalysis is possible with totally quantum catalysts.

V. MUTUAL INFORMATION CATALYSIS

If preserving the whole state of τ_{CE} is too severe a constraint, one might want to preserve only one measure of its correlation, the mutual information $I(C:E)_\tau$. In other words, one might want to catalytically implement the transformation $\rho_S \rightarrow \sigma_S := \text{Tr}_{CE}[\Lambda_{SC} \otimes \text{id}_E(\rho_S \otimes \tau_{CE})]$ with the constraint

that $I(C:E)_\tau = I(C:E)_\eta$, where

$$\eta_{CE} := \text{Tr}_S[\Lambda_{SC} \otimes \text{id}_E(\rho_S \otimes \tau_{CE})]. \quad (12)$$

If the above holds, then we say that the transformation $\rho \rightarrow \sigma$ is mutual information (MI)-catalytically implemented.

For this purpose, we first prove the following lemma: if one is required to preserve the mutual information of a TQ-Q state ρ_{AB} , then the only actions one can locally apply on system A are unitary operations.

Lemma 1. If ρ_{AB} is a TQ-Q state, then for any quantum channel \mathcal{N} on A with $\sigma_{AB} := (\mathcal{N}_A \otimes \text{id}_B)(\rho_{AB})$ satisfying $I(A:B)_\rho = I(A:B)_\sigma$ must be a unitary operation.

Proof. By the data processing inequality, we have $I(A:B)_\rho \geq I(A:B)_\sigma$. By the saturation condition of the data processing inequality, there exists a recovery channel $\mathcal{R}_\mathcal{N}$ acting on A such that [35]

$$[(\mathcal{R}_\mathcal{N} \circ \mathcal{N})_A \otimes \text{id}_B](\rho_{AB}) = \rho_{AB}. \quad (13)$$

Since ρ_{AB} is a TQ-Q state, it implies that $\mathcal{R}_\mathcal{N} \circ \mathcal{N} = \text{id}_A$. As the dimensions of input and output systems of \mathcal{N}_A are same, it follows that \mathcal{N}_A is a unitary operation [36]. ■

By using a similar proof to that of Theorem 2, but substituting the usage of Theorem 1 with Lemma 1, we can show that this type of catalysis grants us no additional power either.

Corollary 1. A TQ-Q state τ_{CE} cannot be used to MI-catalytically implement a nontrivial transformation when only access to C is given.

This technique provides an answer to the following question: What if different parties try to utilize a multipartite state as a catalyst at the same time? One might wonder if it is possible for two local actions at different sites can cancel each other to enable the recovery of the mutual information. The following result shows that it is nevertheless impossible. In other words, Corollary 1 explicitly shows that indeed quantum correlation would be a hidden resource; whenever a catalyst is quantumly correlated with an environment, no catalysis is possible without destroying such quantum correlations as quantified by the mutual information.

Proposition 2. For any TQ-Q state ρ_{AB} and two channels \mathcal{N}_A and \mathcal{M}_B , if $(\mathcal{N}_A \otimes \mathcal{M}_B)(\rho_{AB}) = \rho_{AB}$, then \mathcal{N}_A is a unitary operation.

Proof. Let $\tau_{AB} := (\mathcal{N}_A \otimes \text{id}_B)(\rho_{AB})$ and $\sigma_{AB} := (\mathcal{N}_A \otimes \mathcal{M}_B)(\rho_{AB})$. By the data processing inequality, we have $I(A:B)_\rho \geq I(A:B)_\tau \geq I(A:B)_\sigma$. However, as $\sigma_{AB} = \rho_{AB}$, we have $I(A:B)_\rho = I(A:B)_\tau$. By Lemma 1, it follows that \mathcal{N} is a unitary operation. ■

VI. GENERAL CORRELATED CATALYSTS

One may get the impression that the results above only have implications for a restricted class of bipartite states that are TQ-Q. However, any non-TQ-Q state τ_{CE} is a PC-Q state. In particular, as a consequence of the Koashi-Imoto theorem [37], any bipartite state τ_{CE} can be decomposed into the following form:

$$\tau_{CE} = \sum_i p_i \tau_{C_i^L} \otimes \tau_{C_i^R E}, \quad (14)$$

where $C = \bigoplus_i C_i$ is a direct sum of subspaces $C_i := C_i^L \otimes C_i^R$ and each $\tau_{C_i^R E}$ is either a TQ-Q state or $|C_i^R| = 1$ (in which

case, $\tau_{C_i^R E}$ is uncorrelated, but we include it for completeness). See the Appendix B for a self-contained and elementary proof of the Koashi-Imoto theorem based on that of Ref. [38].

This observation tells us how restricted the usage of a general correlated catalyst with local access is. The subspaces C_i of the decomposition above can be interpreted as the ‘‘classical degrees of freedom’’ for C that can be read out without disturbing τ_{CE} , and any quantum channel on C preserving τ_{CE} also preserve these sectors. It naturally leads us to the following conclusion. (See the Appendix for a detailed discussion.)

Theorem 3. If a transformation can be catalytically achieved by using the catalyst $\tau_{CE} = \sum_i p_i \tau_{C_i^L} \otimes \tau_{C_i^R E}$ with access to C , such that τ_{CE} is preserved, then the same transformation can be achieved by an ensemble $\{p_i, \tau_{C_i^L}\}$ of local catalysts.

Here, by the ensemble $\{p_i, \tau_{C_i^L}\}$, we mean the probabilistic mixture of quantum states with the classical handle of index i , distinguished from the mixed state $\sum_i p_i \tau_{C_i^L}$. Intuitively, using τ_{CE} on C goes as follows: First, one measures which subspace C_i it is supported on without disturbing τ_{CE} . Since no action can act on C_i^R without disturbing $\tau_{C_i^R E}$, only $\tau_{C_i^L}$ can be utilized as a catalyst. Naturally, it is equivalent to having a local catalyst $\tau_{C_i^L}$ with probability p_i . To summarize, quantum correlations in a correlated catalyst only hinder catalysis, and only the classical part of the correlation allows for catalysis because the property of noninvasive measurability, which is essential for recovering the catalyst state.

VII. CONCLUSION

We inspect the extent to which external correlations that a catalyst might have with its environment (inaccessible to agent) would be affected, when the catalyst is used to facilitate a process. We find that correlations of a quantum-mechanical nature, i.e., contained in TQ-Q states, necessarily degrade in the process of utilizing the catalyst to perform nontrivial transformations. This cautions against potential embezzlement via the consumption of correlations as resources unaccounted for. Alternatively, from a more constructive viewpoint, our paper also shows that there is no advantage of a catalyst bank in creating quantumly correlated states and loaning parts of the states out for catalytic purposes—they might as well prepare classical ensembles of various independent ancillas.

We emphasize again that the ‘‘classicality’’ here means the noninvasive measurability, which means that one can measure a system without altering the state. Hence, our results are not in conflict with previous results on quantum catalysis, where classicality indicates other properties, e.g., nonentangled [39], noncoherent [40], nonimaginary [41], etc. In other words, even when quantum properties of catalyst are utilized, the state should be classically known to its user.

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APPENDIX A: NOTATIONS

In this Appendix, we provide mathematical proofs of the results given in the main text. Throughout the Appendix, we will use the following notations. First, every Hilbert space associated with a quantum system is assumed to be finite dimensional, and $|\mathcal{H}|$ denotes the dimension of a vector space \mathcal{H} . The operator space over a Hilbert space \mathcal{H} is denoted by $B(\mathcal{H})$. We slightly abuse the notation and denote the identity map on $B(\mathcal{H})$ by $\text{id}_{\mathcal{H}}$. Also, we will sometimes say that a linear map Φ defined on $B(\mathcal{H})$ is a linear map on \mathcal{H} , in the sense we identify the operator space associated with a physical system with the system itself. The identity operator in $B(\mathcal{H})$, on the other hand, is denoted by $\mathbb{1}_{\mathcal{H}}$. A linear map Φ on $B(\mathcal{H})$ is called CP when $\Phi \otimes \text{id}_{\mathcal{K}}$ is positive for any Hilbert space \mathcal{K} . A linear map Φ on $B(\mathcal{H})$ is unital if $\Phi(\mathbb{1}_{\mathcal{H}}) = \mathbb{1}_{\mathcal{H}}$.

For any subspace \mathcal{K} of a Hilbert space \mathcal{H} and a linear map Φ on $B(\mathcal{H})$, we define its limitation $\Phi|_{\mathcal{K}}$ as Φ whose domain is limited to $B(\mathcal{K})$ without limiting its image. A ‘‘quantum channel on \mathcal{K} ’’ means a quantum channel defined on $B(\mathcal{K})$ whose image is also in $B(\mathcal{K})$. We will also identify matrices with operators; the term ‘‘matrix’’ will be used to emphasize its algebraic properties. Especially, a ‘‘full matrix algebra’’ is a full operator set over a finite-dimensional Hilbert space, emphasizing that operation composition, i.e., matrix multiplication is well defined. For any $M \in B(\mathcal{H})$, the linear map Ad_M is given as $\text{Ad}_M(\rho) := M\rho M^\dagger$. For any $Q \in B(\mathcal{H})$ such that $Q \geq 0$, $\text{supp}(Q)$ where (supp) means support is used to denote the support of Q , the sum of eigenspaces corresponding to strictly positive eigenvalues of Q .

APPENDIX B: STRUCTURE THEOREM FOR FIXED POINTS OF THE QUANTUM CHANNEL

This Appendix contains the proof of our first main result, i.e., Theorem 1 in the main text. We first provide the proof of a technical result, known in literature [27] as the structure theorem, and use it, subsequently, to prove Theorem 1.

1. Block-diagonal structure

Of central importance for this Appendix is the structure theorem for the fixed point set of a quantum channel, which we state as Theorem 5. Many proofs of this theorem rely on the Artin-Wedderburn theorem, where the proof often requires mathematically advanced tools, such as ring theory [42,43], functional analysis [27,38], or lengthy linear algebraic arguments [37]. In this section, we give an elementary and self-contained proof by focusing on the more concrete case of our interest.

We begin with the following.

Lemma 2. For a unital CP map Φ , if $\Phi(\rho) = \rho$ when ρ is a Hermitian operator with the spectral decomposition $\rho = \sum_{i=1}^n \lambda_i \Pi_i$, we have $\Phi(\Pi_i) = \Pi_i$ for all i .

Proof. Without loss of generality, we let $\lambda_i > \lambda_{i+1}$ for all i . Let $r_i := \text{Tr}[\Pi_i]$ and $\pi_i := \Pi_i/r_i$, then we get that $q_i := \text{Tr}[\pi_n \Phi(\Pi_i)]$ is a probability distribution. With respect to this distribution,

$$\langle \lambda_i \rangle = \sum_i q_i \lambda_i = \text{Tr}[\pi_n \Phi(\rho)] = \text{Tr}[\pi_n \rho] = \lambda_n. \quad (\text{B1})$$

Since the left-hand side is a mean of $\{\lambda_i\}$ and the right-hand side is the smallest element of the set being averaged, it follows that the probability distribution $\{q_i\}$ must satisfy that $q_n = 1$ and $q_i = 0$ for any $1 \leq i \leq n - 1$. It implies that $\Phi(\Pi_n) = \Pi_n$, however, since Φ is unital, we also have that

$$\Phi\left(\sum_{i=1}^{n-1} \Pi_i\right) = \sum_{i=1}^{n-1} \Pi_i. \quad (\text{B2})$$

By limiting Φ onto the support of $\sum_{i=1}^{n-1} \Pi_i$, we can repeat the same argument and conclude that $\Phi(\Pi_{n-1}) = \Pi_{n-1}$. Repeating this process n times gives us the desired result. ■

Lemma 3. For a unital CP map Φ with Kraus operators $\{K_i\}$, Φ fixes ρ if and only if $[K_i, \rho] = [K_i^\dagger, \rho] = 0$ for all i .

Proof. Showing the ifpart is trivial; $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger = \rho \sum_i K_i K_i^\dagger = \rho$. For the other direction, first let us assume that ρ is Hermitian with the spectral decomposition $\rho = \sum_j \lambda_j \Pi_j$. If $\Phi(\rho) = \rho$, then by Lemma 2 we get that $\Phi(\Pi_j) = \Pi_j$ for all j . Next, by conjugating Π_j on $\Phi(\Pi_k) = \sum_i K_i \Pi_k K_i^\dagger = \Pi_k$, we get

$$\sum_i T_{jk}^i T_{jk}^{i\dagger} = \delta_{jk} \Pi_k, \quad (\text{B3})$$

where $T_{jk}^i := \Pi_j K_i \Pi_k$. From the positivity of each $T_{jk}^i T_{jk}^{i\dagger}$, we get that $T_{jk}^i = \Pi_j K_i \Pi_k = 0$ for all i , whenever $j \neq k$. It means that every K_i is block diagonal with respect to $\{\Pi_j\}$,

$$K_i \Pi_j = K_i \left(\mathbb{1} - \sum_{k \neq j} \Pi_k \right) = K_i - \sum_{k \neq j} \Pi_k K_i \Pi_k = \Pi_j K_i. \quad (\text{B4})$$

Hence, $[K_i, \Pi_j] = 0$ for all i and j . Finally, this also then implies that $[K_i, \rho] = 0$. By using the fact that a general matrix can be decomposed into Hermitian and anti-Hermitian parts, and a similar analysis for the anti-Hermitian part, we get the desired result. ■

As a side note to complement Lemma 3, we remark that for any set of matrices $\{K_i\}$, the centralizer of $\{K_i, K_i^\dagger\}$ is the set of fixed points of a unital CP map. This can be shown by down-scaling $K_i \rightarrow cK_i$ by some constant c such that $\sum_i K_i K_i^\dagger \leq \mathbb{1}$ holds, and using $\{K_i\} \cup \{\sqrt{\mathbb{1} - \sum_i K_i K_i^\dagger}\}$ as Kraus operators to construct a unital CP map.

Our main point of interest for this section involves the set of fix points for a unital map Φ for which we denote as F_Φ . By Lemma 3, we know that if $\rho_1, \rho_2 \in F_\Phi$, then their product $\rho_1 \rho_2 \in F_\Phi$ also. We remark that because of Hermitian-preserving property of Φ , if $X \in F_\Phi$ then $X^\dagger \in F_\Phi$. Furthermore, let us also define $\text{min } F_\Phi$ as the set of *minimal, nonzero central projectors* in F_Φ . Here, a minimal projector means that it is not a sum of two nonzero projectors, and being central means that it commutes with every element in F_Φ .

We start with a quick observation on the relation between minimal projectors. Note that for any minimal projector S , $SMS = cS$ for some complex number c (we assume that M is Hermitian without loss of generality). Otherwise SMS has a nontrivial spectral decomposition, and a projector onto an eigenspace of SMS will be smaller than S , which violates that S is minimal.

Lemma 4. If two minimal projectors S and T commute either $S = T$ or $ST = 0$.

Proof. This follows from the fact that if $ST \neq 0$ and $S \neq T$, then ST is a projector smaller than S , which contradicts that S is minimal. ■

The next theorem, known as a variant of the Artin-Wedderburn theorem, states that for unital CP maps Φ , the structure of F_Φ always admits a particular decomposition, i.e., a direct sum according to projections unto $\min F_\Phi$. This particular approach focusing on finding matrix basis elements is inspired by Ref. [43].

Theorem 4 (Artin-Wedderburn [28]). For any unital CP map Φ ,

$$F_\Phi = \bigoplus_{P \in \min F_\Phi} F_\Phi P. \quad (B5)$$

Moreover, there exists a tensor product structure for each $\text{supp}(P) = \mathcal{H}_P \otimes \mathcal{L}_P$ and $F_\Phi P$ factorizes into $B(\mathcal{H}_P) \otimes \mathbb{1}_{\mathcal{L}_P}$, where $B(\mathcal{H}_P)$ is the full matrix algebra on \mathcal{H}_P .

Proof. We start by noting that every two different projectors $P_1, P_2 \in \min F_\Phi$ are orthogonal to each other by Lemma 4. Otherwise, P_1 cannot be minimal since $P_1 = P_1 P_2 + P_1 P_2^\perp$ as both $P_1 P_2$ and $P_1 P_2^\perp$ are central projectors. Next, one can observe that by definition, every central projector in F_Φ is a sum of minimal central projections. Especially, the identity operator, which is in F_Φ since Φ , if it is nonminimal, is then also the sum of all projectors in $\min F_\Phi$. Finally, each $F_\Phi P$ is an algebra with P as its unity.

Now, let us focus on each algebra $F_P := F_\Phi P$. Although P is a minimal central projector, there could in general be minimal projectors Q in F_P such that $Q < P$, even though they are not central, otherwise P would not be in $\min F_\Phi$. In particular, there exists a decomposition of P into minimal projectors

$$P = \sum_i Q_i. \quad (B6)$$

By Lemma 4, we also know that all the Q_i 's in the decomposition of P are mutually orthogonal.

Consider the following relation between Q_i : $Q_i \sim Q_j$ if there exists $X \in F_P$ such that $Q_i X Q_j \neq 0$. We claim that this relation is an equivalence relation—the fact that it is reflexive and symmetric is straightforward, whereas, transitivity is also true: if there exists X and Y in F_P such that $Q_i X Q_j \neq 0$ and $Q_j Y Q_k \neq 0$, then

$$Z := Q_i X Q_j Y Q_k \neq 0. \quad (B7)$$

This follows from the fact that

$$Q_j Y Q_k Y^\dagger Q_j = c_{jk} Q_j, \quad (B8)$$

with $c_{jk} = \|Q_j Y Q_k\|_2^2 / \text{Tr}[Q_j]$, which is nonzero and similarly, $Q_i X Q_j X^\dagger Q_i = c_{ij} Q_i$ with $c_{ij} = \|Q_i X Q_j\|_2^2 / \text{Tr}[Q_i] \neq 0$ so that $ZZ^\dagger = c_{ij} c_{jk} Q_i \neq 0$. Therefore, the equivalence relation \sim splits $\{Q_i\}$ into equivalence classes. Moreover, there exists only one equivalence class; Suppose we have two distinct equivalent classes \mathcal{I} and \mathcal{J} with $Q_{\mathcal{I}} := \sum \mathcal{I} = \sum_{i: Q_i \in \mathcal{I}} Q_i$ (similarly for \mathcal{J}), we then have

$$Q_{\mathcal{I}} F_P Q_{\mathcal{J}} = 0. \quad (B9)$$

It follows that for any $X \in F_P$, $Q_{\mathcal{I}} X (P - Q_{\mathcal{I}}) = 0$, hence, $Q_{\mathcal{I}}$ becomes central in F_P , which contradicts that P is a minimal central projector. As a result, for any two Q_i and Q_j , we have $Q_i \sim Q_j$, i.e., there exists $X \in F_P$ such that $Q_i X Q_j \neq 0$.

Now, we let $E_{11} := Q_1$ and $E_{1j} := Q_1 X_j Q_j$ for some X_j , which is guaranteed to exist, such that $\|Q_{1j}\|_2 = \text{Tr}[Q_1 X_j Q_j X_j^\dagger] = \text{Tr}[Q_1]$. By letting $X_1 := Q_1$, we can interpret E_{11} as a special case of E_{1i} . Then, we define $E_{i1} := E_{1i}^\dagger = Q_i X_j^\dagger Q_1$ and $E_{ij} := E_{i1} E_{1j} = Q_i X_{ij} Q_j$ where $X_{ij} := X_i^\dagger Q_1 X_j$ for all $i, j > 1$. Because of the property,

$$Q_i X Q_i = (\text{Tr}[Q_i X] / \text{Tr}[Q_i]) Q_i, \quad (B10)$$

for all i and $X \in F_P$, we have $E_{1i} E_{j1} = Q_1 X_i Q_i Q_j X_j^\dagger Q_1 = \delta_{ij} (\text{Tr}[Q_1 X_i Q_i X_i^\dagger] / \text{Tr}[Q_1]) Q_1 = \delta_{ij} Q_1$. Therefore,

$$E_{ij} E_{kl} = E_{i1} (E_{1j} E_{k1}) E_{1l} = \delta_{jk} E_{il}, \quad (B11)$$

and that $\text{Tr}[E_{ij}^\dagger E_{kl}] = \delta_{ik} \delta_{jl}$ so that $\|E_{ij}\|_2^2 = \text{Tr}[Q_1]$ for all i and j . Especially, $E_{ii}^2 = E_{ii}$. Because $E_{ii} = Q_i (X_i^\dagger Q_1 X_i) Q_i = r Q_i$ for some positive number r , we have $r = 1$ so that $E_{ii} = Q_i$. It follows that $\text{Tr}[Q_i] = \text{Tr}[E_{i1} E_{1i}] = \text{Tr}[E_{1i} E_{i1}] = \text{Tr}[Q_1]$ for all i , i.e., every minimal projector Q_i has the same rank.

Next, note that for any $Z \in F_P$, we have $Q_i Z Q_j E_{ji} = Q_i (Z Q_j X_{ji}) Q_i = c Q_i$ for some complex number c , and by taking the trace of both hands we get $c = (\text{Tr}[E_{ji} Z] / \text{Tr}[Q_1])$ because $Q_j E_{ji} Q_i = E_{ji}$ and $\text{Tr}[Q_i] = \text{Tr}[Q_1]$. Therefore,

$$Q_i Z Q_j = (\text{Tr}[E_{ij}^\dagger Z] / \text{Tr}[Q_1]) E_{ij}, \quad (B12)$$

and it follows that for any $Z \in F_P$, $Z = P Z P = \sum_{i,j} Q_i Z Q_j = \sum_{i,j} (\text{Tr}[E_{ij}^\dagger Z] / \text{Tr}[Q_1]) E_{ij}$, so $\{E_{ij}\}$ is an orthonormal basis of F_P .

Now, we define a linear map Ψ given as for the basis elements $\{E_{ij}\}$,

$$\Psi(E_{ij}) := |i\rangle\langle j|_{\mathcal{H}_P} \otimes \mathbb{1}_{\mathcal{L}_P}, \quad (B13)$$

with some Hilbert spaces \mathcal{H}_P and \mathcal{L}_P such that $|\mathcal{L}_P| = \text{Tr}[Q_1]$. We can see that it is an isomorphism from the fact that $\Psi(E_{ij})\Psi(E_{kl}) = \delta_{jk} \Psi(E_{il})$ and $\text{Tr}[\Psi(E_{kl})^\dagger \Psi(E_{ij})] = \delta_{ik} \delta_{jl} \text{Tr}[Q_1] = \text{Tr}[E_{kl}^\dagger E_{ij}]$. Therefore, one can see that $F_P = F_\Phi P$ is isomorphic to $B(\mathcal{H}_P) \otimes \mathbb{1}_{\mathcal{L}_P}$. ■

For any $\rho_{AB} \geq 0$, if $\text{Tr}_B(\rho_{AB}) = c|\psi\rangle\langle\psi|_A$ for some pure state $|\psi\rangle_A$ on A , then we necessarily have $\rho_{AB} = |\psi\rangle\langle\psi|_A \otimes \rho_B$. Thus, by considering the Choi matrix of each limitation $\Phi|_{\text{supp}(P)}$, we have the following result.

Corollary 2. For a unital CP map Φ with the fixed point set F_Φ with the decomposition (B5), each limitation $\Phi|_{\text{supp}(P)}$ decomposes into

$$\Phi|_{\text{supp}(P)} = \text{id}_{\mathcal{H}_P} \otimes \Phi_{\mathcal{L}_P}, \quad (B14)$$

for some unital CP map $\Phi_{\mathcal{L}_P}$ on \mathcal{L}_P with $\mathbb{1}_{\mathcal{L}_P}$ as the unique fixed point.

Proof. The set of fixed points of the limitation $\Phi|_{\text{supp}(P)}$ is $M(\mathcal{H}_P) \otimes \mathbb{1}_{\mathcal{L}_P}$. Therefore, every Kraus operator of $\Phi|_{\text{supp}(P)}$ commutes with every matrix of the form $A_{\mathcal{H}_P} \otimes \mathbb{1}_{\mathcal{L}_P}$, which implies that every Kraus operator is in the form of $\mathbb{1}_{\mathcal{H}_P} \otimes K_i$, hence, $\Phi|_{\text{supp}(P)} = \text{id}_{\mathcal{H}_P} \otimes \Phi_{\mathcal{L}_P}$. ■

Corollary 2 is a key tool that is necessary for our goal of proving the structure theorem. Additionally, we need a few technical lemmata. We present Lemma 6 and 8 from Ref. [44]

with an alternative proof based on elementary linear algebra. The proofs of Lemma 5 and 7 are taken from Ref. [27] and presented here for completeness.

Lemma 5 (Proposition 6.8, [27]). If a quantum channel preserves an operator, then it also preserves the Hermitian and anti-Hermitian part of the operator. Moreover, the channel also preserves the positive and negative parts of the (anti-)Hermitian part.

Proof. Let Φ be a quantum channel and $\Phi(X) = X$. If $X = H + iA$ where both H and A are Hermitian, then $\Phi(H) - H = i[A - \Phi(A)]$. Since the only operator that is both Hermitian and anti-Hermitian is 0, we have $\Phi(H) = H$ and $\Phi(A) = A$. Moreover, if $H = P - N$ where $P, N \geq 0$ and $PN = 0$, then by letting Π_P be the projector onto $\text{supp}(P)$ and similarly for N , we have

$$\begin{aligned} \text{Tr}[P] &= \text{Tr}[\Pi_P(P - N)] = \text{Tr}[\Pi_P\Phi(P - N)] \\ &\leq \text{Tr}[\Phi(P)] = \text{Tr}[P]. \end{aligned} \tag{B15}$$

Lemma 6. If λ is an eigenvalue of $A \in B(\mathcal{H})$, then the complex conjugate λ^* is an eigenvalue of A^\dagger . Moreover, the geometric multiplicity of λ for A is same with that of λ^* for A^\dagger .

Proof. The first part immediately follows from that $\det[A - \lambda\mathbb{1}] = 0$ is equivalent to $\det[A^\dagger - \lambda^*\mathbb{1}] = 0$. For the second part, recall that the geometric multiplicity of λ for A is equal to $|\mathcal{H}| - r(A - \lambda\mathbb{1})$ where $r(X)$ is the rank of an operator X . Because rank is invariant under the adjoint transformation, we have

$$|\mathcal{H}| - r(A - \lambda\mathbb{1}) = |\mathcal{H}| - r(A^\dagger - \lambda^*\mathbb{1}), \tag{B16}$$

and, thus, we get the wanted result. ■

Lemma 7 (Proposition 6.10, [27]). For any fixed point $\rho \geq 0$ of a quantum channel Φ , if Q is the projector onto the support of ρ , then $\text{Tr}[(\mathbb{1} - Q)\Phi(Q)] = 0$ and

$$\sigma \leq Q \Rightarrow \Phi(\sigma) \leq Q. \tag{B17}$$

Moreover, for projectors Q , the condition (B17) is equivalent to

$$\Phi^\dagger(Q) \geq Q. \tag{B18}$$

A similar argument can be given for N , too.

Proof. For minimal and maximal positive eigenvalues λ_m and λ_M of ρ , we have $\lambda_m Q \leq \rho \leq \lambda_M Q$, hence,

$$\begin{aligned} 0 &\leq \lambda_m \text{Tr}[(\mathbb{1} - Q)\Phi(Q)] \\ &\leq \text{Tr}[(\mathbb{1} - Q)\Phi(\rho)] \\ &= \text{Tr}[(\mathbb{1} - Q)\rho] \\ &\leq \lambda_M \text{Tr}[(\mathbb{1} - Q)Q] = 0. \end{aligned}$$

For the second part,

(\Rightarrow) Let $Q^\perp := \mathbb{1} - Q$. Then,

$$\text{Tr}[Q^\perp\Phi(Q)] = \text{Tr}[\Phi^\dagger(Q^\perp)Q] = 0, \tag{B19}$$

hence, $Q^\perp\Phi^\dagger(Q^\perp)Q^\perp = \Phi^\dagger(Q^\perp)$. Using $\Phi(\mathbb{1}) = \mathbb{1}$, we get

$$\Phi^\dagger(Q) = Q + Q^\perp\Phi^\dagger(Q)Q^\perp \geq Q. \tag{B20}$$

(\Leftarrow) For any $\sigma \leq Q$, we have that

$$\begin{aligned} \text{Tr}[\sigma] &= \text{Tr}[\sigma Q] = \text{Tr}[\sigma\Phi^\dagger(Q)] \\ &= \text{Tr}[\Phi(\sigma)Q] \\ &\leq \text{Tr}[\Phi(\sigma)] = \text{Tr}[\sigma]. \end{aligned}$$

Therefore, $\text{Tr}[\Phi(\sigma)Q] \leq \text{Tr}[\Phi(\sigma)]$, which implies that $\Phi(\sigma) \leq Q$. ■

Lemma 8 (Theorem 2, [44]). For any quantum channel Φ on A , if every fixed point of Φ^\dagger is proportional to $\mathbb{1}_A$, then, Φ also has a unique fixed density matrix ρ .

Proof. As a linear map on $B(A)$ (by identifying A with its associated Hilbert space), fixed points of Φ are equivalent to eigenvectors corresponding to the eigenvalue 1. If every fixed point of Φ^\dagger is proportional to $\mathbb{1}_A$, then, it means that the geometric multiplicity, or the dimension of the eigenspace of eigenvalue 1 of Φ^\dagger is 1. By Lemma 6, a linear map and its adjoint have the same eigenvalues and the same geometric multiplicities, the geometric multiplicity of 1 as an eigenvalue of Φ is also 1. It means that there is a unique operator ρ in $B(A)$ such that $\Phi(\rho) = \rho$. This ρ has to be Hermitian because of the Hermitian-preserving property of Φ and, moreover, $\rho \geq 0$ because if a Hermitian operator is fixed by a quantum channel, then both of its positive and negative parts should be fixed by the channel by Lemma 5, which makes the geometric multiplicity of 1 larger than 1. After the normalization, it follows that there is only a single quantum state fixed by Φ . ■

Theorem 5 (Structure theorem for fixed points of quantum channel, [27]). For any quantum channel Φ on A , the set F_Φ of all fixed points of Φ have the decomposition of the form

$$F_\Phi = \bigoplus_i B(\mathcal{H}_i) \otimes \rho_i. \tag{B21}$$

Here, for any vector space \mathcal{K} , $\mathcal{K} \otimes \rho_i := \{v \otimes \rho_i : v \in \mathcal{K}\}$.

Proof. By the previous lemmata, F_{Φ^\dagger} has the decomposition Eq. (B5) and for any $P \in \min F_{\Phi^\dagger}$, $\Phi^\dagger|_{\text{supp}(P)} = \text{id}_{\mathcal{H}_P} \otimes \Phi_{\mathcal{L}_P}^\dagger$. Moreover, since $\Phi^\dagger(P) = P$, by Lemma 7, the image of $\Phi|_{\text{supp}(P)}$ is also contained in $B[\text{supp}(P)]$, so that $(\Phi|_{\text{supp}(P)})^\dagger = \Phi^\dagger|_{\text{supp}(P)}$. It follows that each $\Phi_{\mathcal{L}_P} := \Phi_{\mathcal{L}_P}^{\dagger\dagger}$ is a quantum channel on \mathcal{L}_P and has the unique fixed point, say ρ_P by Corollary 2 and Lemma 8. It follows that, for any $X \in B(\mathcal{H}_P)$,

$$\begin{aligned} \Phi(X_{\mathcal{H}_P} \otimes \rho_P) &= \Phi|_{\mathcal{H}_P \otimes \mathcal{L}_P}(X_{\mathcal{H}_P} \otimes \rho_P) \\ &= \text{id}_{\mathcal{H}_P}(X) \otimes \Phi_{\mathcal{L}_P}(\rho_P) = X_{\mathcal{H}_P} \otimes \rho_P. \end{aligned}$$

In other words, $B(\mathcal{H}_P) \otimes \rho_P \subseteq F_\Phi$. Therefore, the following direct sum,

$$G_\Phi := \bigoplus_{P \in \min F_\Phi^\dagger} B(\mathcal{H}_P) \otimes \rho_P \tag{B22}$$

is a subspace of F_Φ as a vector space over the complex number field because each summand is fixed by Φ .

Now, we count the dimension of each space. Recall that the dimension of a direct sum of vector spaces is the sum of the dimension of all the individual summand. Because $|B(\mathcal{H}_P) \otimes \rho_P| = |B(\mathcal{H})| |\text{span}\{\rho_P\}| = |B(\mathcal{H})|$ as ρ_P is understood as a

single-point set, and the span of it is one dimensional, we have

$$|G_\Phi| = \sum_{P \in \min F_{\Phi^\dagger}} |B(\mathcal{H}_P)| = \sum_{P \in \min F_{\Phi^\dagger}} |\mathcal{H}_P|^2. \quad (\text{B23})$$

On the other hand, similarly, $|B(\mathcal{H}_P) \otimes \mathbb{1}_{\mathcal{L}_P}| = |B(\mathcal{H}_P)|$, therefore, it follows that

$$|F_{\Phi^\dagger}| = \sum_{P \in \min F_{\Phi^\dagger}} |B(\mathcal{H}_P)| = \sum_{P \in \min F_{\Phi^\dagger}} |\mathcal{H}_P|^2. \quad (\text{B24})$$

Finally, note that $|F_\Phi| = |F_{\Phi^\dagger}|$ by Lemma 6 because they correspond to the geometric multiplicities of 1 for Φ and Φ^\dagger , respectively. It follows that $G_\Phi = F_\Phi$ because their dimensions are the same. ■

2. Proof of Theorem 1

Proof. The first part of the theorem is straightforward by looking at the contrapositive: if a quantum channel \mathcal{N} has an output system that is not totally quantum, i.e., PC, then by Eq. (6) there exists a nontrivial projective measurement, which is a quantum channel that fixes \mathcal{N} , hence, \mathcal{N} cannot be sensitive.

We show the other direction. Let the output system of \mathcal{N} be TQ, and let us say that a quantum channel \mathcal{S} fixes \mathcal{N} , i.e., $\mathcal{S} \circ \mathcal{N} = \mathcal{N}$. This means that the image of \mathcal{N} is a subset of fixed points of \mathcal{S} . By Theorem 5, the space of all fixed points F_S of \mathcal{S} must have the following unique decomposition:

$$F_S = \bigoplus_i \mathcal{M}_{d_i} \otimes \sigma_i, \quad (\text{B25})$$

with respect to an appropriate basis and for some fixed full-rank quantum states σ_i and the full matrix algebras \mathcal{M}_{d_i} .

If there is more than one term in the direct sum of Eq. (B25), we will call the set of projectors onto their supports $\{\Pi_i\}$. Note that the projective measurement described by $\{\Pi_i\}$ fixes all the states in F_S , and, therefore, all the output states of \mathcal{N} . This contradicts the assumption that the output system of \mathcal{N} is TQ. Hence, there should be only one term in the direct sum of Eq. (B25) so that F_S should have the form of $F_S = \mathcal{M}_d \otimes \sigma$ for some full matrix algebra \mathcal{M}_d and a fixed quantum state σ . However, suppose that there are more than one distinct eigenvalue for σ so that there are, at least, two terms in the spectral decomposition of $\sigma = \sum_j \lambda_j P_j$, where P_j is the projector onto the eigenspace corresponding to the eigenvalue λ_j . Then, we can now identify a projective measurement that fixes F_S , i.e., $\{\mathbb{1}_d \otimes P_j\}$ where $\mathbb{1}_d$ is the identity matrix in \mathcal{M}_d . It also violates the output system of \mathcal{N} being non-PC, hence, σ should be a one-dimensional state so that F_S is isomorphic to a full matrix algebra, i.e., \mathcal{S} fixes every operator. This means that $\mathcal{S} = \text{id}$ and, hence, \mathcal{N} is sensitive.

The statement for bipartite states follows the proof above by applying the Choi-Jamiołkowski isomorphism to \mathcal{S} . ■

APPENDIX C: PROOF OF PROPOSITION 1

Proof. Let us start by noting that for any isometry $V : A \rightarrow AK$, there exists a unitary operator U on AK such that $V = U(|\phi\rangle_K \otimes \mathbb{1}_A)$ for some state $|\phi\rangle_K$. Additionally, consider a measurement operation $\mathcal{P}_K(\cdot) = \sum_i \Pi_i \cdot \Pi_i$ with more than one mutually orthogonal projector $\{\Pi_i\}$ on K that fixes τ_{KAB} .

Consider an induced quantum channel on A defined as

$$\mathcal{Q}(\cdot) := \text{Tr}_K[U^\dagger \circ \mathcal{P}_K \circ U(|\phi\rangle\langle\phi|_K \otimes \cdot)_A], \quad (\text{C1})$$

where U is the unitary channel describing the action of U on AK . Since \mathcal{P}_E fixes τ_{KAB} , \mathcal{Q}_A fixes ρ_{AB} . Since it is a TQ-Q state, it implies that \mathcal{Q} is the identity channel. Since unitary operations cannot form correlation with other systems, (or by the Schrödinger-Hughston-Jozsa-Wootters theorem [45,46]) that $U^\dagger \circ \mathcal{P}_K \circ U(|\phi\rangle\langle\phi|_K \otimes \text{id}_A) = \sigma_K \otimes \text{id}_A$ for some state σ_K . Applying this channel to ρ_{AB} on A , we get

$$\mathcal{U}_{KA}^\dagger \circ \mathcal{P}_E \circ \mathcal{U}_{KA}(|\phi\rangle\langle\phi|_K \otimes \rho_{AB}) = \sigma_K \otimes \rho_{AB}. \quad (\text{C2})$$

Note that $\mathcal{U}_{KA}(|\phi\rangle\langle\phi|_K \otimes \rho_{AB}) = \tau_{KAB}$ and \mathcal{P}_K does not alter τ_{KAB} . Hence, the left-hand side is $\mathcal{U}_{KA}^\dagger \circ \mathcal{U}_{KA}(|\phi\rangle\langle\phi|_K \otimes \rho_{AB}) = |\phi\rangle\langle\phi|_K \otimes \rho_{AB}$. It follows that $\sigma_K = |\phi\rangle\langle\phi|_K$.

Now, a quantum state η is fixed by measurement operation $\mathcal{P}(\cdot) = \sum_i \Pi_i \cdot \Pi_i$ if and only if $\Pi_i \eta \Pi_i = \eta$ for only one i . Moreover, $\mathcal{P}_K \otimes \text{id}_A$ is also a measurement operation. Considering the Choi matrix, it follows that there exists a unique Π_i such that $(\Pi_i \otimes \mathbb{1}_A)U(|\phi\rangle_K \otimes \mathbb{1}_A) = U(|\phi\rangle_K \otimes \mathbb{1}_A)$ and $(\Pi_j \otimes \mathbb{1}_A)U(|\phi\rangle_K \otimes \mathbb{1}_A) = 0$ for any other Π_j . By conjugating the operator above to ρ_{AB} on A and tracing out B , we get that $\Pi_j \tau_K \Pi_j = 0$, which contradicts τ_A being full rank. ■

APPENDIX D: PROOF OF THEOREM 3

1. The Koashi-Imoto theorem and decomposition of the PC-Q state

Here, we prove that arbitrary bipartite state τ_{CE} can be decomposed into the form

$$\tau_{CE} = \bigoplus_i p_i \tau_{C_i^L} \otimes \tau_{C_i^R E}, \quad (\text{D1})$$

where $C = \bigoplus_i C_i^L \otimes C_i^R$ and each $\tau_{C_i^R E}$ is a TQ-Q state. Equivalently, through the Choi-Jamiołkowski isomorphism for any quantum channel $\mathcal{T}: E \rightarrow C$, one has the following unique decomposition.

$$\mathcal{T}(\rho) = \bigoplus_i \tau_{C_i^L} \otimes \mathcal{T}_{C_i^R}(\rho). \quad (\text{D2})$$

with respect to the same C above for any input state ρ , where $\tau_{C_i^L}$ is a quantum state on C_i^L and $\mathcal{T}_{C_i^R}: E \rightarrow C_i^R$ is a sensitive, trace nonincreasing [47]. CP map (or a trivial map with $|C_i^R| = 1$, which we simply count as a special case of sensitive map for simplicity). This result can be attained by using the Koashi-Imoto theorem [37] as a lemma. Here, we provide a concise statement and proof of the Koashi-Imoto theorem that mainly follows that of Ref. [38] without using the result of Ref. [48] directly by using the tools that facilitated the structure theorem (Theorem 5) we developed earlier.

Lemma 9 (Koashi-Imoto [37,38]). For any set of quantum states $\{\rho_k\}$ on a Hilbert space \mathcal{H} , there exists a unique decomposition of $\mathcal{H} = \bigoplus_i \mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}$ that satisfies the following:

- (1) Each ρ_k decomposes as

$$\rho_k = \bigoplus_i q_{i|k} \omega_{i^L} \otimes \rho_{i^R|k}, \quad (\text{D3})$$

where (q_{ik}) is a probability distribution over i and $\rho_{i^R|k}$ is a quantum state on \mathcal{H}_{i^R} depending on k , whereas, ω_{i^L} is a quantum state on \mathcal{H}_{i^L} independent of k .

(2) Any quantum channel \mathcal{C} on \mathcal{H} that fixes all ρ_k is a quantum channel on each subspace $\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}$ and

$$\mathcal{C}|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}} = \mathcal{C}_{\mathcal{H}_{i^L}} \otimes \text{id}_{\mathcal{H}_{i^R}}, \quad \forall i, \quad (\text{D4})$$

where each $\mathcal{C}_{\mathcal{H}_{i^L}}$ fixes ω_{i^L} , i.e., $\mathcal{C}_{\mathcal{H}_{i^L}}(\omega_{i^L}) = \omega_{i^L}$.

Proof. Let $\mathbf{F} := \{\mathcal{F} : \mathcal{F}(\rho_k) = \rho_k, \forall k\}$ be the set of all quantum channels that fixes each $\{\rho_k\}$. Then, when $F_{\mathcal{C}}$ is the set of all fixed points of a linear map \mathcal{C} , we let

$$F_0 := \bigcap_{\mathcal{F} \in \mathbf{F}} F_{\mathcal{F}^\dagger}. \quad (\text{D5})$$

Since every $F_{\mathcal{F}^\dagger}$ is finite dimensional, F_0 can be actually expressed as a finite intersection,

$$F_0 = F_{\mathcal{F}_1^\dagger} \cap F_{\mathcal{F}_2^\dagger} \cap \dots \cap F_{\mathcal{F}_M^\dagger}, \quad (\text{D6})$$

for some $\mathcal{F}_1, \dots, \mathcal{F}_M \in \mathbf{F}$. To see this, observe that intersecting one more $F_{\mathcal{F}}$ can never increase the dimension of the intersection, hence, because of the finite dimensionality, only a finite number of $F_{\mathcal{F}_i^\dagger}$ nontrivially affect the intersection $\bigcap_{\mathcal{F} \in \mathbf{F}} F_{\mathcal{F}^\dagger}$. Let us consider the quantum channel \mathcal{F}_0 given as

$$\mathcal{F}_0 := \frac{1}{M} \sum_{n=1}^M \mathcal{F}_n. \quad (\text{D7})$$

The Kraus operators of \mathcal{F}_0^\dagger are simply the union of scalar multiples of those of \mathcal{F}_i^\dagger . Suppose now, that we have a state $\rho \in F_{\mathcal{F}_0^\dagger}$. Then, by Lemma 3, ρ commutes with all the Kraus operators of each \mathcal{F}_i^\dagger . Therefore, again by Lemma 3, ρ is fixed by all of \mathcal{F}_i^\dagger , i.e., $\rho \in F_{\mathcal{F}_1^\dagger} \cap F_{\mathcal{F}_2^\dagger} \cap \dots \cap F_{\mathcal{F}_M^\dagger}$. Hence, $\rho \in F_0$. It shows that $F_{\mathcal{F}_0^\dagger} \subseteq F_0$, and, therefore, $F_0 = F_{\mathcal{F}_0^\dagger}$. It follows that there exists the decomposition of the form (B5),

$$F_{\mathcal{F}_0^\dagger} = \bigoplus_i \mathbb{1}_{\mathcal{H}_{i^L}} \otimes B(\mathcal{H}_{i^R}), \quad (\text{D8})$$

with respect to a decomposition $\mathcal{H} = \bigoplus_i \mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}$. Define $Q_i := \mathbb{1}_{\mathcal{H}_{i^L}} \otimes \mathbb{1}_{\mathcal{H}_{i^R}}$ to be the projector onto the subspace $\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}$ of \mathcal{H} . Then, we can observe from the block-diagonal structure (D8) that

$$Q_i \in F_{\mathcal{F}_0^\dagger} = \bigcap_{\mathcal{G} \in \mathbf{F}} F_{\mathcal{G}^\dagger} \subseteq F_{\mathcal{F}^\dagger} \quad (\text{D9})$$

for any $\mathcal{F} \in \mathbf{F}$. It means that $\mathcal{F}^\dagger(Q_i) = Q_i$ and, thus, $\mathcal{F}^\dagger(Q_i) \geq Q_i$ and $\mathcal{F}^\dagger(\mathbb{1} - Q_i) \geq \mathbb{1} - Q_i$ for any $\mathcal{F} \in \mathbf{F}$ and i because $\mathcal{F}^\dagger(\mathbb{1}) = \mathbb{1}$. By Lemma 7, it follows that for all i and $\mathcal{F} \in \mathbf{F}$, $\mathcal{F}|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}}$ is a quantum channel on $\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}$ and $(\mathcal{F}|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}})^\dagger = \mathcal{F}^\dagger|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}}$.

Especially for \mathcal{F}_0 , from (D8) it follows that

$$\mathcal{F}_0^\dagger|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}} = \mathcal{G}_{0\mathcal{H}_{i^L}} \otimes \text{id}_{\mathcal{H}_{i^R}}, \quad (\text{D10})$$

with some unital CP map $\mathcal{G}_{0\mathcal{H}_{i^L}}$ on \mathcal{H}_{i^L} with the one-dimensional fixed point set $\{c\mathbb{1}\}$. By Lemma 8,

$$\mathcal{F}_0|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}} = \mathcal{F}_{0\mathcal{H}_{i^L}} \otimes \text{id}_{\mathcal{H}_{i^R}}, \quad (\text{D11})$$

with some quantum channel $\mathcal{F}_{0\mathcal{H}_{i^L}}$ on \mathcal{H}_{i^L} with a unique fixed quantum state, say, ω_{i^L} . Therefore,

$$F_{\mathcal{F}_0} = \bigoplus_i \omega_{i^L} \otimes B(\mathcal{H}_{i^R}). \quad (\text{D12})$$

It follows that every ρ_k as a fixed point of \mathcal{F}_0 indeed has the expression of the form (D3).

Now we prove (ii). For any $\mathcal{F} \in \mathbf{F}$, again, $\mathcal{F}^\dagger|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}} = \mathcal{F}_{\mathcal{H}_{i^L}}^\dagger \otimes \text{id}_{\mathcal{H}_{i^R}}$ for some quantum channel $\mathcal{F}_{\mathcal{H}_{i^L}}$ on \mathcal{H}_{i^L} because $F_0 = F_{\mathcal{F}_0^\dagger} \subseteq F_{\mathcal{F}^\dagger}$, so $\mathcal{F}|_{\mathcal{H}_{i^L} \otimes \mathcal{H}_{i^R}} = \mathcal{F}_{\mathcal{H}_{i^L}} \otimes \text{id}_{\mathcal{H}_{i^R}}$. Since $\{\rho_k\} \subseteq F_{\mathcal{F}}$, it must be that $\omega_{i^L} \otimes \rho_{i^R|k}$ is a fixed point of $\mathcal{F}_{\mathcal{H}_{i^L}} \otimes \text{id}_{\mathcal{H}_{i^R}}$ for all i . Thus, every $\mathcal{F}_{\mathcal{H}_{i^L}}$ must have ω_{i^L} as a fixed state. It proves (ii). ■

We apply this result on the image of the quantum channel \mathcal{T} , $\{\mathcal{T}(\rho)\}$ so that the input state ρ functions as the index k in the statement of the Koashi-Imoto theorem above. Hence, according to (i) above for any ρ , $\mathcal{T}(\rho)$ decomposes as

$$\mathcal{T}(\rho) = \bigoplus_i \omega_{i^L} \otimes \mathcal{T}_{i^R}(\rho). \quad (\text{D13})$$

Since ω_{i^L} is independent of ρ , it follows that each \mathcal{T}_{i^R} is linear in ρ . Moreover, it is immediate that they are CP. Now we claim that each \mathcal{T}_{i^R} is sensitive because otherwise, there exists a nonidentity quantum channel \mathcal{N}_{i^R} on \mathcal{H}_{i^R} such that $\mathcal{N}_{i^R} \circ \mathcal{T}_{i^R} = \mathcal{T}_{i^R}$, and it contradicts (ii) of Lemma 9.

Once we have the decomposition of arbitrary quantum channel (D2), by using the Choi-Jamiołkowski isomorphism again for each CP map \mathcal{T}_{i^R} , one can find the corresponding bipartite quantum state $\rho_{C_i^R E}$ and normalization factor $q_i := \text{Tr}[\mathcal{T}_{i^R}(\mathbb{1}_{\mathcal{H}})]/|\mathcal{H}|$ to get (D1), which should sum up to 1 due to the trace-preserving property of \mathcal{T} . In this case, for any quantum channel \mathcal{C} on C such that $\mathcal{C}_C \otimes \text{id}_E(\rho_{CE}) = \rho_{CE}$, \mathcal{C} should satisfy (D4).

2. Proof of Theorem 3

We follow the logic of the proof of Theorem 1 above. Consider any interaction Λ on SC that implements the catalytic transformation of a state ρ_S on S using τ_{CE} with access to C . We consider the channel \mathcal{S} on S , defined as

$$\mathcal{S}(\rho) := \text{Tr}_{CE}[\Lambda_{SC} \otimes \text{id}_E(\rho_S \otimes \tau_{CE})]. \quad (\text{D14})$$

Recall the statement of the theorem presumes the following decomposition of the catalyst with its environment,

$$\tau_{CE} = \sum_i p_i \tau_{C_i^L} \otimes \tau_{C_i^R E}. \quad (\text{D15})$$

Let $\mathcal{P} = \sum_j \Pi_j \Pi_j$ be the pinching map on C where each Π_j is the projector onto the subspace $C_j^L \otimes C_j^R$. This models the process that acts locally on the system after a projective measurement process. Note that $\mathcal{P}_C \otimes \text{id}_E$ fixes τ_{CE} , i.e.,

$$\mathcal{P}_C \otimes \text{id}_E(\tau_{CE}) = \tau_{CE}. \quad (\text{D16})$$

Thus, one can perform the pinching operation before applying Λ_{SC} without changing \mathcal{S} ,

$$\begin{aligned} \text{Tr}_{CE}[\Lambda_{SC} \otimes \text{id}_E(\rho_S \otimes \tau_{CE})] \\ &= \text{Tr}_{CE}[\Lambda_{SC} \otimes (\text{id}_{SE} \otimes \mathcal{P}_C)(\rho_S \otimes \tau_{CE})] \\ &= \text{Tr}_C[\Lambda_{SC} \circ (\text{id}_S \otimes \mathcal{P}_C)(\rho_S \otimes \tau_C)]. \end{aligned} \quad (\text{D17})$$

Hence, from the discussion of the previous section, we get the factorization,

$$\Lambda_{SC}(\rho_S \otimes \Pi_j \tau_C \Pi_j) = p_j \Lambda_{SC_j^L}(\rho_S \otimes \tau_{C_j^L}) \otimes \tau_{C_j^R}, \quad (\text{D18})$$

with the limitation $\Lambda_{SC_j^L}$ of Λ_{SC} onto SC_j^L , which is by itself a quantum channel on SC_j^L . The resultant transformation of ρ_S

is, therefore, of the form

$$\mathcal{S}(\rho) = \sum_j p_j \text{Tr}_C[\Lambda_{SC_j^L}(\rho_S \otimes \tau_{C_j^L})]. \quad (\text{D19})$$

Therefore, one can see that \mathcal{S} can be implemented with the ensemble $\{p_i, \tau_{C_i^L}\}$.

Conversely, any catalysis possible with the ensemble $\{p_i, \tau_{C_i^L}\}$ is also possible with τ_{CE} through the protocol explained in the main text after Theorem 3.

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