Chain rules for a mutual information based on Rényi zero-relative entropy

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Quantum Rényi relative entropies play a significant part in characterizing operational tasks in quantum information theory. In this paper, we first give some fundamental properties of the zero-relative entropy, and we find that the geometric Rényi relative entropy reduces to the zero-relative entropy in the limit case. Then, we define a new mutual information via the zero-relative entropy, and it is the so-called zero-mutual information. In particular, the zero-mutual information is a lower bound of the Petz-Rényi and geometric generalized mutual information. We establish the chain rules for the unsmoothed and smoothed versions of the zero-mutual information. As an application, we discuss generalized mutual information with the Petz-Rényi, sandwiched Rényi, and geometric Rényi types, our result gives a uniform chain rule inequality for quantum generalized mutual information.

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I. INTRODUCTION

von Neumann entropy, quantum conditional entropy, and quantum mutual information are the fundamental quantities for analyzing nearly all quantum-information-processing protocols. The von Neumann entropy has an operational interpretation as the optimal rate of quantum data compression [1]. The quantum conditional entropy is of operational significance in the context of randomness extraction and quantum cryptography [2–6], and the quantum mutual information works as a measure for the capacity of communication channels [7–10]. It has also been successfully employed to characterize classical-quantum-channel coding and error exponent analysis [11–19]. These three information measures obey the chain rule (or chain relation), i.e.,

$$I(A;B) = S(A) - S(A|B). \tag{1}$$

It is well known that they can be directly derived from quantum relative entropy [5,6,20].

It is well-known that Rényi entropies [21] are powerful tools in many information-theoretic tasks, and it is desirable to construct quantum versions of these entropic quantities. Since the noncommutative nature of quantum states, there are at least three nonequivalent quantum Rényi relative entropies, i.e., Petz-Rényi relative entropy [22,23], sandwiched Rényi relative entropy [24–26], and geometric Rényi relative entropy [27–29]. They are meaningful for understanding quantum information tasks and have already found many applications. In particular, by taking different values at α, one can obtain some important entropic measures, such as quantum relative entropy [20], collision relative entropy [2], Belavkin-Staszewski relative entropy [30,31], max-relative entropy [32], min-relative entropy, and zero-relative entropy (also named alternative min-relative entropy) [2,32,33], and the ways they relate to each other are depicted in Fig. 1.

The quantum relative entropy has an operational meaning in terms of the task of quantum hypothesis testing in the asymptotic regime of many use of independent and identically distributed (i.i.d.) resources. However, we know that the i.i.d. resources are not available in practical scenarios. This is to say that the resources available are typically finite and operations can be achieved only approximately, and so this makes it necessary to consider the nonasymptotic setting. In particular, we are interested in the amount of resource needed to perfect a task just once if we allow for a small error in cryptography. A key step in this direction is to construct the one-shot scenario; thus, Renner and Wolf introduced the framework of smooth entropy in Ref. [34]. After that, there was a lot of research on smooth and nonsmooth quantum generalized Rényi entropic measures (see Refs. [35–61]). There are four equivalent definitions of the quantum mutual information based on the quantum relative entropy, but for quantum Rényi relative entropies, they will lead to multiple nonequivalent definitions in use. Based on the one-shot setting, Ciganović *et al.* [62] summarized several possible generalized mutual information quantities, which can be defined as

$$\begin{aligned} I^{\varepsilon}_{\text{gen}}(A;B) := {} & H^{\varepsilon}_{\min}(A) - H^{\varepsilon}_{\min}(A|B) \\ \text{or } & H^{\varepsilon}_{\min}(A) - H^{\varepsilon}_{\max}(A|B) \\ \text{or } & H^{\varepsilon}_{\max}(A) - H^{\varepsilon}_{\max}(A|B) \\ \text{or } & H^{\varepsilon}_{\max}(A) - H^{\varepsilon}_{\min}(A|B). \end{aligned} \tag{2}$$

They discussed smooth max-information, a generalization of von Neumann mutual information derived from the max-relative entropy, and gave their lower chain rules [62]. Onorati explored the chain rules for min-information defined by the min-relative entropy [63]. For the zero-relative entropy, Datta and Leditzky claimed that it has a simple operational interpretation in the problem of binary state discrimination; that

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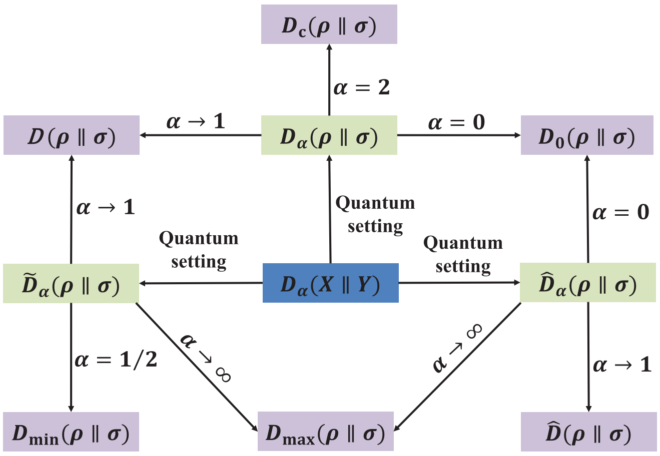

FIG. 1. The diagram of quantum Rényi relative entropies.

is, it is equal to the minimum probability of the type-II error under the condition that the probability of the type-I error is 0 [33]. We also know that in perfect entanglement dilution, the zero-relative entropy provided an explicit characterization of the entanglement cost of a bipartite state in the one-shot setting [64]. In addition, it was also employed in the trade-off between the rate of the success probability and the compression rate in state compression [65]. Therefore, it is necessary to discuss the entropy, conditional entropy and mutual information defined by the zero-relative entropy. The conditional max-entropies (defined by the min-entropies or zero-relative entropies) have been studied in Refs. [40,42–44,47,66], and it has been found that the zero-mutual information of a channel is related to the feedback-assisted capacity with zero error [67]. However, in addition to their applications, it is also very important to study their basic properties and establish the relationship between these generalized entropy measures.

In this paper, we give some fundamental properties of the zero-relative entropy, and we also find that the zero-relative entropy is a special case of the geometric Rényi relative entropy in the limit $\alpha \to 0$. We then focus on a new mutual information based on the zero-relative entropy, and we denote the zero-mutual information. Since the zero-relative entropy can be obtained from the Petz-Rényi and geometric relative entropies in the limit $\alpha \to 0$, from the fact that the Petz-Rényi and geometric relative entropies increase with respect to parameter α, it is easy to get that the zero-mutual information is a lower bound of the Petz-Rényi and geometric generalized mutual information. The chain rule of mutual information is of central importance to the information theory [5,6,20,66,68]; however, it is almost impossible to establish an equal chain rule for the zero-mutual information. Therefore, we provide the upper and lower bounds of chain rules. In addition, we define the smooth version of the zero-mutual information and get its chain rule. Since the zero-mutual information is the smallest measure of the Petz-Rényi and geometric generalized mutual information, its lower chain rule can be used to constrain the latter two. Finally, we discuss generalized mutual information with the Petz-Rényi, sandwiched Rényi, and geometric Rényi types. Our result gives a uniform chain rule bound for quantum generalized mutual information.

The remainder of this paper is organized as follows. In Sec. II, we present formal notations and give some properties of the zero-relative entropy. In Sec. III, we consider the (smooth) zero-mutual information and discuss its chain rules. In Sec. IV, we try to discuss the chain rules for the quantum generalized mutual information. Finally, in Sec. V, we conclude this study and present an outlook for future research.

II. NOTATION AND DEFINITIONS

A quantum system A is associated with a finite-dimensional Hilbert space $\mathcal{H}_A$, with $d_A = \dim(\mathcal{H}_A)$. The composite system AB is associated with the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We use $\mathcal{P}(\mathcal{H})$ to denote the set of positive semidefinite operators on $\mathcal{H}$. We define the set of quantum states by $\mathcal{S}_=(\mathcal{H}) := \{\rho \in \mathcal{P}(\mathcal{H}) : \mathrm{Tr}\rho = 1\}$ and the set of subnormalized states by $\mathcal{S}_\leqslant(\mathcal{H}) := \{\rho \in \mathcal{P}(\mathcal{H}) : 0 < \mathrm{Tr}\rho \leqslant 1\}$. The identity operator on $\mathcal{H}_A$ is denoted by I_A. Given an operator ρ_{AB} on a composite Hilbert space $\mathcal{H}_{AB}$, $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ is a reduced operator on the subsystem. For every linear operator ρ, the trace norm is defined as $\|\rho\|_1 = \mathrm{Tr}\sqrt{\rho^\dagger\rho}$, and the ∞ norm is defined as $\|\rho\|_\infty = \lambda_{\max}$, where $\lambda_{\max}$ is the largest singular value of ρ. For any $\rho \in S_\leqslant(\mathcal{H})$, one can define the ball of ε-close states around ρ as

$$\mathcal{B}^\varepsilon(\rho) = \{\rho' \in S_\leqslant(\mathcal{H}) : P(\rho, \rho') \leqslant \varepsilon\}. \quad (3)$$

We write $\rho \approx_\varepsilon \sigma$ iff $P(\rho, \sigma) \leqslant \varepsilon$. Here, $P(\rho, \sigma)$ is the purified distance [69–72] and is defined as $P(\rho, \sigma) = \sqrt{1 - \bar{F}(\rho, \sigma)^2}$, where $\bar{F}(\rho, \sigma) = F(\rho, \sigma) + \sqrt{(1 - \mathrm{Tr}\rho)(1 - \mathrm{Tr}\sigma)}$ is the generalized fidelity and $F(\rho, \sigma) = \|\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\|_1$.

Then, we introduce three different quantum Rényi relative entropies, that is, the Petz-Rényi, sandwiched Rényi, and geometric Rényi relative entropies, we refer the readers to Refs. [3,5,6] for more discussions. The logarithm is base 2 throughout the paper.

Definition 1. For any $\alpha \in [0, 1) \cup (1, 2]$, $\rho \in \mathcal{S}_=(\mathcal{H})$, and $\sigma \in \mathcal{S}_\leqslant(\mathcal{H})$, if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, the Petz-Rényi relative entropy is defined as

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log_2 \mathrm{Tr}(\rho^\alpha\sigma^{1-\alpha}). \quad (4)$$

Definition 2. For any $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty)$, $\rho \in \mathcal{S}_=(\mathcal{H})$, and $\sigma \in \mathcal{S}_\leqslant(\mathcal{H})$, if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, the sandwiched Rényi relative entropy is defined as

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log_2 \mathrm{Tr}(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha. \quad (5)$$

Definition 3. For any $\alpha \in [0, 1) \cup (1, \infty)$, $\rho \in \mathcal{S}_=(\mathcal{H})$, and $\sigma \in \mathcal{S}_\leqslant(\mathcal{H})$, if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, the geometric Rényi relative entropy is defined as

$$\hat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log_2 \mathrm{Tr}[\sigma(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^\alpha]. \quad (6)$$

For the above three quantum Rényi relative entropies, for any $\alpha \in [\frac{1}{2}, 1) \cup (1, 2]$, they satisfy the following order relations, i.e.,

$$\tilde{D}_\alpha(\rho\|\sigma) \leqslant D_\alpha(\rho\|\sigma) \leqslant \hat{D}_\alpha(\rho\|\sigma). \quad (7)$$

We know that the Petz-Rényi and sandwiched Rényi relative entropies yield the quantum relative entropy in the limit

$\alpha \to 1$, i.e.,

$$\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = \lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma). \quad (8)$$

Here, $D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ is the quantum relative entropy [20,73].

Note that the limit of the geometric Rényi relative entropy as $\alpha \to 1$ is the Belavkin-Staszewski relative entropy [6]. Besides, one can evaluate other values at $\alpha \in \{0, \frac{1}{2}, \infty\}$. The sandwiched and geometric Rényi relative entropies are monotonically increasing in α, by taking the limit $\alpha \to \infty$, one can obtain the max-relative entropy [6,24,56], i.e.,

$$\lim_{\alpha\to\infty} \tilde{D}_\alpha(\rho\|\sigma) = \lim_{\alpha\to\infty} \hat{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma), \quad (9)$$

where $D_{\max}(\rho\|\sigma) = \log_2 \inf\{\lambda : \rho \leqslant \lambda\sigma\}$ is the max-relative entropy.

One evaluates the value at $\alpha = \frac{1}{2}$, then we can obtain three different results for the above three generalized Rényi relative entropies, and Tomamichel gives an example and compares these relative entropies in Ref. [5]. In particular, the sandwiched Rényi relative entropy is related to the fidelity, i.e., $\tilde{D}_{\frac{1}{2}}(\rho\|\sigma) = -2\log_2 F(\rho,\sigma)$. Note that $\tilde{D}_{\frac{1}{2}}(\rho\|\sigma)$ is also called the min-relative entropy, and it is also denoted by $D_{\min}(\rho\|\sigma)$ [2,32,63].

Clearly, for $\alpha = 0$, using the Petz-Rényi relative entropy, we can obtain the zero-relative entropy [32,33], i.e.,

$$D_0(\rho\|\sigma) = -\log_2 \mathrm{Tr}(\Pi_\rho\sigma), \quad (10)$$

where Π_ρ is the projector onto the support of ρ. By taking the limit $\alpha \to 0$ for the sandwiched Rényi relative entropy, if $\mathrm{supp}(\rho) = \mathrm{supp}(\sigma)$, one can obtain

$$\lim_{\alpha\to 0} \tilde{D}_\alpha(\rho\|\sigma) = D_0(\rho\|\sigma). \quad (11)$$

Note that the above identity does not necessarily hold if $\mathrm{supp}(\rho) \subsetneq \mathrm{supp}(\sigma)$ [33]. Another interesting property of the zero-relative entropy is that it is equal to the limit of the geometric Rényi relative entropy as $\alpha \to 0$.

Proposition 1. For any $\rho \in \mathcal{S}_=(\mathcal{H})$ and $\sigma \in \mathcal{S}_\leqslant(\mathcal{H})$, if they are positive definite operators, then the geometric Rényi relative entropy converges to the zero-relative entropy in the limit $\alpha \to 0$, i.e.,

$$D_0(\rho\|\sigma) = \lim_{\alpha\to 0} \hat{D}_\alpha(\rho\|\sigma), \quad (12)$$

Proof. Employing the equality in Eq. (4.6.5) in Ref. [6], for any positive definite operators ρ and σ, and for all $\alpha \in \mathbb{R}$, we have

$$\sigma^{\frac{1}{2}}(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^\alpha\sigma^{\frac{1}{2}} = \rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-\alpha}\rho^{\frac{1}{2}}.$$

Then, taking the limit $\alpha \to 0$, we obtain

$$\mathrm{Tr}[\sigma(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^0] = \mathrm{Tr}(\rho\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}).$$

Therefore, we have

$$\lim_{\alpha\to 0} \hat{D}_\alpha(\rho\|\sigma) = -\log_2 \mathrm{Tr}(\rho^{\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}) = -\log_2 \mathrm{Tr}(\Pi_\rho\sigma),$$

where Π_ρ is the projector onto the support of ρ and the last equality holds from $\Pi_\rho = \rho^{\frac{1}{2}}\rho^{-\frac{1}{2}}$. Thus, we obtain the desired result. ■

Note that if ρ is positive semidefinite, we consider its support space $\mathrm{supp}(\rho)$, then ρ^{-1} is defined to be the inverse of ρ. Thus, if one takes the positive semidefinite σ' which has the same rank as the operator ρ. In particular, if $\rho = |\psi\rangle\langle\psi|$ is pure, using Proposition 4.41 in Ref. [6], we have

$$\lim_{\alpha\to 0} \hat{D}_\alpha(|\psi\rangle\langle\psi|\|\sigma) = -\log_2\langle\psi|\sigma|\psi\rangle = -\log_2 \mathrm{Tr}(\Pi_\rho\sigma).$$

Here, we also give other interesting properties of the zero-relative entropy.

Proposition 2. The zero-relative entropy satisfies the following properties.

(i) For $\sigma, \sigma' \in \mathcal{S}_\leqslant(\mathcal{H})$, if $\sigma \geqslant \sigma'$, then

$$D_0(\rho\|\sigma) \leqslant D_0(\rho\|\sigma'). \quad (13)$$

(ii) For $\rho, \rho' \in \mathcal{S}_=(\mathcal{H})$, if $\Pi_\rho \geqslant \Pi_{\rho'}$, then

$$D_0(\rho\|\sigma) \leqslant D_0(\rho'\|\sigma). \quad (14)$$

(iii) For every $\beta \in (0,\infty)$, then

$$D_0(\rho\|\beta\sigma) = D_0(\rho\|\beta\sigma) - \log_2\beta. \quad (15)$$

(iv) If $\rho \leqslant \sigma$, then

$$D_0(\rho\|\sigma) \leqslant 0. \quad (16)$$

Proof. The first three can directly follow by the definition. To prove the last, consider that $\rho \leqslant \sigma$; it implies that $\sigma - \rho \geqslant 0$. From Proposition 4.33 in Ref. [6], we can take $\hat{\rho} = |0\rangle\langle 0| \otimes \rho$ and $\hat{\sigma} = |0\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes \sigma - \rho$. Since the zero-relative entropy satisfies the data-processing inequality (Lemma 7 in Ref. [32]), then we have

$$D_0(\rho\|\sigma) \leqslant D_0(\hat{\rho}||\hat{\sigma}) = D_0(\rho||\rho) = 0. \quad (17)$$

Thus, we obtain the desired result. ■

For the above three quantum Rényi relative entropies, taking $\sigma = I$, they reduce to the Rényi entropy:

$$H_\alpha(\rho) = \frac{1}{1-\alpha}\log_2 \mathrm{Tr}\rho^\alpha. \quad (18)$$

Specially, from Eq. (18), one can obtain the unconditional min-entropy when $\alpha \to \infty$, i.e.,

$$H_{\min}(A)_\rho = -\log_2\|\rho_A\|_\infty. \quad (19)$$

For the min-entropy $H_{\min}(A)_\rho$, there are many interesting applications in one-shot information theory [19,32,40,42,54,63,74–79], we only list one of the properties for future application.

Lemma 1. (Lemma 5 in Ref. [63]) For any $\rho \in \mathcal{S}_\leqslant(\mathcal{H}_A)$ and $\varepsilon \in (0,1)$, there exists an operator $0 \leqslant \Pi \leqslant I_A$ such that $\rho \approx_{\varepsilon/2} \Pi\rho\Pi$ and

$$H_{\min}^{\varepsilon^2/16}(A)_\rho \leqslant H_{\min}(A)_{\Pi\rho\Pi}. \quad (20)$$

On the other hand, we can also evaluate the Rényi entropy value at $\alpha = 0$ and define the unconditional zero-relative entropy, i.e.,

$$H_0(A)_\rho = \log_2 \mathrm{rank}(\rho_A). \quad (21)$$

Clearly, we have $D_0(\rho\|I) = -H_0(A)_\rho$. In addition, we provide another related entropic quantity [80], i.e.,

$$H_R(A)_\rho = -\sup\log_2\{\lambda \in \mathbb{R} : \rho_A \geqslant 2^\lambda\Pi_A\}, \quad (22)$$

where Π_A is the projector onto the support of ρ_A. Clearly, we have $H_R(A)_\rho \geqslant H_0(A)_\rho$. We also list another property related to this entropy quantity as follows.

Lemma 2 (Lemma B.28 in Ref. [80]). For $\varepsilon \in (0, 1)$ and $\rho_A \in S_\leqslant(\mathcal{H}_A)$. Then there exists $0 \leqslant \Pi_A \leqslant I_A$ such that $\rho_A \approx_{\varepsilon/2} \Pi_A\rho_A\Pi_A$ and

$$H_{\max}^{\varepsilon^2/6}(A)_\rho \geqslant H_{\rm R}(A)_{\Pi\rho\Pi} + 2\log_2\frac{\varepsilon^2}{6}. \tag{23}$$

By the zero-relative entropy, the max-entropy of A conditioned on B is given by Refs. [2,4,45], which is a generalization of conditional entropy.

Definition 4. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$, the smooth conditional max-entropy and the conditional max-entropy are defined as

$$\hat{H}_{\max}^{\varepsilon}(A|B)_\rho = \min_{\tilde{\rho}_{AB}\in\mathcal{B}^\varepsilon(\rho_{AB})} \hat{H}_{\max}(A|B)_\rho \tag{24}$$

and

$$\hat{H}_{\max}(A|B)_\rho = \max_{\sigma_B\in\mathcal{S}_\leqslant(\mathcal{H}_B)} \hat{H}_{\max}(A|B)_{\rho|\sigma}, \tag{25}$$

where $\hat{H}_{\max}(A|B)_{\rho|\sigma} = -D_0(\rho_{AB}\|I_A\otimes\sigma_B)$.

Tomamichel *et al.* showed that the conditional max-entropy can be expressed as a conditional min-entropy of the purified state [45]. The smooth conditional max-entropy gives the amount of entanglement needed in one-shot state merging [32] and it can also characterize source compression with quantum side information [81].

III. LOWER AND UPPER BOUND OF THE ZERO-MUTUAL INFORMATION

Quantum relative entropy is a special quantum generalization of the classical relative entropy, which has operational meaning in quantum hypothesis testing. Using the quantum relative entropy, we can directly give four equivalent definitions of the quantum mutual information, i.e.,

$$\begin{aligned} I(A;B) &= D(\rho_{AB}\|\rho_A\otimes\rho_B) \\ &= \min_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} D(\rho_{AB}\|\rho_A\otimes\sigma_B) \\ &= \min_{\sigma_A\in S_\leqslant(\mathcal{H}_A)} D(\rho_{AB}\|\sigma_A\otimes\rho_B) \\ &= \min_{\substack{\sigma_A\in S_\leqslant(\mathcal{H}_A),\\ \sigma_B\in S_\leqslant(\mathcal{H}_B)}} D(\rho_{AB}\|\sigma_A\otimes\sigma_B). \end{aligned} \tag{26}$$

Similar to the quantum mutual information, we discuss the zero-mutual information based on the zero-relative entropy as follows.

Definition 5. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$, $\rho_A \in S_\leqslant(\mathcal{H}_A)$, and $\rho_B \in S_\leqslant(\mathcal{H}_B)$, four different versions of zero-mutual information are defined as

$$I_0^1(A;B)_\rho = D_0(\rho_{AB}\|\rho_A\otimes\rho_B), \tag{27}$$

$$I_0^2(A;B)_\rho = \min_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} D_0(\rho_{AB}\|\rho_A\otimes\sigma_B), \tag{28}$$

$$I_0^{2'}(A;B)_\rho = \min_{\sigma_A\in S_\leqslant(\mathcal{H}_A)} D_0(\rho_{AB}\|\sigma_A\otimes\rho_B), \tag{29}$$

and

$$I_0^3(A;B)_\rho = \min_{\substack{\sigma_A\in S_\leqslant(\mathcal{H}_A),\\ \sigma_B\in S_\leqslant(\mathcal{H}_B)}} D_0(\rho_{AB}\|\sigma_A\otimes\sigma_B). \tag{30}$$

Since $I_0^2(A;B)_\rho$ and $I_0^{2'}(A;B)_\rho$ are very similar, we just consider one of them. Clearly, if ρ_B is optimal, then $I_0^2(A;B)_\rho$ can be reduced to $I_0^1(A;B)_\rho$. Thus, we immediately give the following inequalities:

$$I_0^3(A;B)_\rho \leqslant I_0^2(A;B)_\rho \leqslant I_0^1(A;B)_\rho. \tag{31}$$

The chain rule of mutual information plays an important role in many applications of information tasks. However, for the generalized mutual information, it is difficult to establish the same relationship with quantum mutual information. For example, Berta *et al.* provided the upper and lower chain rule bounds for max-information in Ref. [80], in which they extend the upper chain rule bound to the smooth entropy framework. Ciganović *et al.* established a lower chain rule bound for smooth max-information in Ref. [62]. Fang *et al.* proved a chain rule inequality for the quantum relative entropy, and it can help to solve an open problem for asymptotic quantum channel discrimination in Ref. [68]. For the zero-mutual information, we discuss its lower and upper chain rules as follows. We only consider $I_0^2(A;B)_\rho$ and $I_0^3(A;B)_\rho$, the rest of the definitions can be established in the same way.

Proposition 3. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\rho_A \in S_\leqslant(\mathcal{H}_A)$, then we have

$$\begin{aligned} H_{\min}(A)_\rho - \hat{H}_{\max}(A|B)_\rho &\leqslant I_0^2(A;B)_\rho \\ &\leqslant H_R(A)_\rho - \hat{H}_{\max}(A|B)_\rho. \end{aligned} \tag{32}$$

Proof. Combining Eq. (10) with Eq. (27), we have

$$2^{-I_0^2(A;B)_\rho} = \max_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} \mathrm{Tr}(\Pi_{\rho_{AB}}\rho_A\otimes\sigma_B).$$

Let $\lambda_{\max} = \|\rho_A\|_\infty$, it implies that $\rho_A \leqslant \lambda_{\max}I_A$, and we have

$$\begin{aligned} 2^{-I_0^2(A;B)_\rho} &= \max_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} \lambda_{\max}\mathrm{Tr}(\Pi_{\rho_{AB}}I_A\otimes\sigma_B) \\ &= 2^{-H_{\min}(A)_\rho + \hat{H}_{\max}(A|B)_\rho}. \end{aligned}$$

Hence, we obtain the lower bound.

For the upper bound, let λ^* be such that it optimizes $H_R(A)_\rho$; i.e., we have $\rho_A \geqslant \lambda^*\Pi_A$. Therefore, we have

$$\begin{aligned} 2^{-I_0^2(A;B)_\rho} &\geqslant \max_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} \lambda^*\mathrm{Tr}(\Pi_{\rho_{AB}}\Pi_A\otimes\sigma_B) \\ &= \max_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} \lambda^*\mathrm{Tr}(\Pi_{\rho_{AB}}I_A\otimes\sigma_B) \\ &= 2^{-H_R(A)_\rho + \hat{H}_{\max}(A|B)_\rho}, \end{aligned}$$

where the second equality comes from the fact that multiplication of $\Pi_A\otimes I_B$ does not affect $\Pi_{\rho_{AB}}$, since $\mathrm{supp}(\rho_{AB}) \subseteq \mathrm{supp}(\rho_A)\otimes\mathcal{H}_B$ [2,63]. Thus, we obtain the desired result. ■

Note that we can extend the above form to zero-mutual information $I_0^3(A;B)_\rho$. If we consider the optimal on A and B, we do not obtain the desired chain rule. Thus, we restrict the optimization by the max-relative entropy, and then we can give a new chain rule as follows.

Proposition 4. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\rho_A \in S_\leqslant(\mathcal{H}_A)$, we have

$$\begin{aligned} I_0^3(A;B)_\rho \geqslant{}& H_{\min}(A)_\rho - \hat{H}_{\max}(A|B)_\rho \\ &- \min_{\sigma_A\in S_\leqslant(\mathcal{H}_A)} D_{\max}(\sigma_A||\rho_A). \end{aligned} \tag{33}$$

Proof. Without loss of generality, for given σ_A, we assume that $D_{\max}(\sigma_A||\rho_A) = \log_2 \lambda_{\sigma_A}$, and this implies that $\sigma_A \leqslant \lambda_{\sigma_A}\rho_A$.

Then, we have

$$\begin{aligned}2^{-I_0^3(A;B)_\rho} &= \max_{\substack{\sigma_A\in S_\leqslant(\mathcal{H}_A),\\ \sigma_B\in S_\leqslant(\mathcal{H}_B)}} \mathrm{Tr}(\Pi_{\rho_{AB}}\sigma_A\otimes\sigma_B)\\ &\leqslant \max_{\substack{\sigma_A\in S_\leqslant(\mathcal{H}_A),\\ \sigma_B\in S_\leqslant(\mathcal{H}_B)}} \mathrm{Tr}(\Pi_{\rho_{AB}}\lambda_{\sigma_A}\rho_A\otimes\sigma_B)\\ &= \max_{\substack{\sigma_A\in S_\leqslant(\mathcal{H}_A),\\ \sigma_B\in S_\leqslant(\mathcal{H}_B)}} \lambda_{\sigma_A}\mathrm{Tr}(\Pi_{\rho_{AB}}\rho_A\otimes\sigma_B)\\ &\leqslant \max_{\sigma_A\in S_\leqslant(\mathcal{H}_A)} \lambda_{\sigma_A}2^{-H_{\min}(A)_\rho+\hat{H}_{\max}(A|B)_\rho}.\end{aligned}$$

Then, by simply algebraic operation, we obtain the desired result. ■

On the other hand, if we restrict the operator σ_A by $D_{\max}(\rho_A||\sigma_A)$, we can give an upper bound as follows.

Corollary 1. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\rho_A \in S_=(\mathcal{H}_A)$, then we have

$$\begin{aligned}I_0^3(A;B)_\rho \leqslant{}& H_R(A)_\rho - \hat{H}_{\max}(A|B)_\rho\\ &+ \min_{\sigma_A\in S_\leqslant(\mathcal{H}_A)} D_{\max}(\rho_A||\sigma_A).\end{aligned} \tag{34}$$

From the above results, we restrict the field for operators ρ_A and σ_A by the max-relative entropy. This just accords with the smooth entropy framework. Therefore, we consider a smoothed version of the zero-mutual information.

Definition 6. For $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\varepsilon \in (0, 1)$, the smooth zero-mutual information is defined as

$$I_0^{i,\varepsilon}(A;B)_\rho = \max_{\rho'\in\mathcal{B}^\varepsilon(\rho)} I_0^i(A;B)_{\rho'}, \tag{35}$$

where $i = 1,\ 2,\ 2'$, and 3.

Similar to the nonsmooth case, we only consider $I_0^{2,\varepsilon}(A;B)_\rho$ and $I_0^{3,\varepsilon}(A;B)_\rho$. Then, the following results will give chain rules for the smooth zero-mutual information.

Proposition 5. Let $\rho_{AB} \in S_=(\mathcal{H}_{AB})$ and $\varepsilon \in (0, 1)$, then

$$H_{\min}^{\varepsilon^2/32}(A)_\rho - \hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho \leqslant I_0^{2,\varepsilon}(A;B)_\rho. \tag{36}$$

Proof. From Proposition 3, we have

$$\begin{aligned}&I_0^{2,\varepsilon}(A;B)_\rho\\ &\quad\geqslant \max_{\rho'\in\mathcal{B}^\varepsilon(\rho)} [H_{\min}(A)_{\rho'} - \hat{H}_{\max}(A|B)_{\rho'}]\\ &\quad\geqslant \max_{\omega\in\mathcal{B}^{\varepsilon^2/32}(\rho)} \{\max_{\Pi_A}[H_{\min}(A)_{\Pi_A\omega\Pi_A} - \hat{H}_{\max}(A|B)_{\Pi_A\omega\Pi_A}]\},\end{aligned}$$

where the maximum ranges run over all $0 \leqslant \Pi_A \leqslant I_A$ with $\Pi_A\omega\Pi_A \approx_{\varepsilon/2} \omega$. Then, combining Proposition 2 and the definition of the smooth conditional max-entropy (24), we have

$$\begin{aligned}\hat{H}_{\max}(A|B)_{\Pi_A\omega\Pi_A} &= -\min_{\sigma_B\in\mathcal{S}_\leqslant(\mathcal{H}_B)} D_0(\Pi_A\omega_{AB}\Pi_A\|I_A\otimes\sigma_B)\\ &\leqslant -\min_{\sigma_B\in\mathcal{S}_\leqslant(\mathcal{H}_B)} D_0(\omega_{AB}\|I_A\otimes\sigma_B)\\ &= \hat{H}_{\max}(A|B)_\omega.\end{aligned}$$

Hence, we have

$$\begin{aligned}&I_0^{2,\varepsilon}(A;B)_\rho\\ &\quad\geqslant \max_{\omega\in\mathcal{B}^{\varepsilon^2/32}(\rho)} [\max_{\Pi_A}[H_{\min}(A)_{\Pi_A\omega\Pi_A}] - \hat{H}_{\max}(A|B)_\omega].\end{aligned}$$

Without loss of generality, let $\tilde{\omega} \in \mathcal{B}^{\varepsilon^2/32}(\rho)$ and $\hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho = \hat{H}_{\max}(A|B)_{\tilde{\omega}}$, we then obtain

$$\begin{aligned}I_0^{2,\varepsilon}(A;B)_\rho &\geqslant \max_{\Pi_A} [H_{\min}(A)_{\Pi_A\tilde{\omega}\Pi_A}] - \hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho\\ &\geqslant H_{\min}^{\varepsilon^2/16}(A)_{\tilde{\omega}} - \hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho\\ &\geqslant H_{\min}^{\varepsilon^2/32}(A)_\rho - \hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho.\end{aligned}$$

Here, the second inequality comes from Lemma 1, and we have

$$\max_{\Pi_A} [H_{\min}(A)_{\Pi_A\tilde{\omega}\Pi_A}] \geqslant H_{\min}(A)_{\Pi_A\tilde{\omega}\Pi_A} \geqslant H_{\min}^{\varepsilon^2/16}(A)_{\tilde{\omega}}.$$

Thus, we obtain the desired result. ■

Proposition 6. Let $\rho_{AB} \in S_=(\mathcal{H}_{AB})$ and $\varepsilon \in (0, 1)$, then

$$I_0^{2,\varepsilon}(A;B)_\rho \leqslant H_{\max}^\varepsilon(A)_\rho - \hat{H}_{\max}^{4\sqrt{3\varepsilon}}(A|B)_\rho - \log_2 4\varepsilon^2. \tag{37}$$

Proof. Without loss of generality, we have

$$\begin{aligned}&\hat{H}_{\max}^\varepsilon(A|B)_\rho\\ &\quad\leqslant \min_{\rho'\in\mathcal{B}^\varepsilon(\rho)} [H_\mathrm{R}(A)_{\rho'} - I_0^2(A;B)_{\rho'}]\\ &\quad\leqslant \min_{\omega\in\mathcal{B}^{\varepsilon^2/48}(\rho)} \left[\min_{\Pi_A}[H_\mathrm{R}(A)_{\Pi_A\omega\Pi_A} - I_0^2(A;B)_{\Pi_A\omega\Pi_A}]\right],\end{aligned}$$

where the minimum ranges over all $0 \leqslant \Pi_A \leqslant I_A$ such that $\Pi_A\omega_{AB}\Pi_A \approx_{\varepsilon/2} \omega_{AB}$.

By the definition of the zero-mutual information, we have

$$\begin{aligned}I_0^2(A;B)_{\Pi_A\omega\Pi_A} &= \min_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} D_0(\Pi_A\omega_{AB}\Pi_A\|\rho_A\otimes\sigma_B)\\ &\geqslant \min_{\sigma_B\in S_\leqslant(\mathcal{H}_B)} D_0(\omega_{AB}\|\rho_A\otimes\sigma_B)\\ &= I_0^2(A;B)_\omega,\end{aligned}$$

where the inequality follows from $\omega \geqslant \Pi_A\omega\Pi_A$.

Therefore, we have

$$\begin{aligned}&\hat{H}_{\max}^\varepsilon(A|B)_\rho\\ &\quad\leqslant \min_{\omega\in\mathcal{B}^{\varepsilon^2/48}(\rho)} \left[\min_{\Pi_A}[H_\mathrm{R}(A)_{\Pi_A\omega\Pi_A}] - I_0^2(A;B)_\omega\right].\end{aligned}$$

We know that there exists a state $\tilde{\omega} \in \mathcal{B}^{\varepsilon^2/48}(\rho)$ such that $I_0^{2,\varepsilon^2/48}(A;B)_\rho = I_0^2(A;B)_{\tilde{\omega}}$, and thus we can obtain

$$\hat{H}_{\max}^\varepsilon(A|B)_\rho \leqslant \min_{\Pi_A} [H_\mathrm{R}(A)_{\Pi_A\tilde{\omega}\Pi_A}] - I_0^{2,\varepsilon^2/48}(A;B)_\rho,$$

where the minimum ranges over all $0 \leqslant \Pi_A \leqslant I_A$ such that $\Pi_A\tilde{\omega}_{AB}\Pi_A \approx_{\varepsilon/2} \tilde{\omega}_{AB}$.

Further, employing Lemma 2, we choose $0 \leqslant \Pi_A \leqslant I_A$ with $\Pi_A \tilde{\omega}_{AB} \Pi_A \approx_{\varepsilon/2} \tilde{\omega}_{AB}$ such that

$$H_{\mathrm{R}}(A)_{\Pi_A \tilde{\omega}_{AB} \Pi_A} \leqslant H_{\max}^{\varepsilon^2/24}(A)_{\tilde{\omega}} - 2\log_2 \frac{\varepsilon^2}{24}.$$

Then, we have

$$\begin{aligned}\hat{H}^{\varepsilon}_{\max}(A|B)_\rho &\leqslant H_{\max}^{\varepsilon^2/24}(A)_{\tilde{\omega}} - I_0^{2,\varepsilon^2/48}(A;B)_\rho - 2\log_2 \frac{\varepsilon^2}{24}\\ &\leqslant H_{\max}^{\varepsilon^2/48}(A)_{\rho} - I_0^{2,\varepsilon^2/48}(A;B)_\rho - 2\log_2 \frac{\varepsilon^2}{24}.\end{aligned}$$

Finally, one can relabel $\varepsilon^2/48 \to \varepsilon$, and then we obtain the desired result. ∎

Due to the fact that the max-relative entropy cannot increase under the projector, we have

$$D_{\max}(\Pi\rho\Pi \| \Pi\sigma\Pi) \leqslant D_{\max}(\rho\|\sigma).$$

Therefore, from Propositions 5 and 6, we can get the upper and lower chain rules of the smooth zero-mutual information $I_0^{3,\varepsilon}(A;B)_\rho$ as follows.

Corollary 2. Let $\rho_{AB} \in S_=(\mathcal{H}_{AB})$ and $\varepsilon \in (0, 1)$, then

$$\begin{aligned}I_0^{3,\varepsilon}(A;B)_\rho \geqslant{}& H_{\min}^{\varepsilon^2/32}(A)_\rho - \hat{H}_{\max}^{\varepsilon^2/32}(A|B)_\rho\\ &- \min_{\sigma_A \in S_\leqslant(\mathcal{H}_A)} D_{\max}^{\varepsilon^2/32}(\sigma_A||\rho_A) \end{aligned}\tag{38}$$

and

$$\begin{aligned}I_0^{3,\varepsilon}(A;B)_\rho \leqslant{}& H_{\max}^{\varepsilon}(A)_\rho - \hat{H}_{\max}^{4\sqrt{3\varepsilon}}(A|B)_\rho\\ &+ \min_{\sigma_A \in S_\leqslant(\mathcal{H}_A)} D_{\max}^{\varepsilon}(\rho_A||\sigma_A) - \log_2 4\varepsilon^2. \end{aligned}\tag{39}$$

IV. QUANTUM GENERALIZED MUTUAL INFORMATION

Similar to the definitions of quantum mutual information, we can consider the generalized mutual information derived from the Petz-Rényi, sandwiched Rényi, and geometric Rényi relative entropies. In this section, we are particularly interested in their chain rule.

Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, then the generalized mutual information is defined as

$$I_\alpha^\sharp(A;B) = \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha^\sharp(\rho_{AB}\|\rho_A\otimes\sigma_B),\tag{40}$$

and the generalized conditional entropy is defined as

$$H_\alpha^\sharp(A|B) = -\min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha^\sharp(\rho_{AB}\|I_A\otimes\sigma_B),\tag{41}$$

where the superscript $\sharp$ takes the Petz-Rényi, sandwiched Rényi, and geometric Rényi types, respectively. We refer the readers to Sec. 4.11 of Ref. [6] and its references for a detailed explanation.

Since these three quantum Rényi relative entropies are non-negative, the generalized mutual information is also non-negative. Now, we begin to build their chain rule.

Proposition 7. Let $\rho_{AB} \in \mathcal{S}_=(\mathcal{H}_{AB})$ and $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, then

$$\begin{aligned}H_{\min}(A)_\rho - H_\alpha^\sharp(A|B)_\rho &\leqslant I_\alpha^\sharp(A;B)\\ &\leqslant H_R(A)_\rho - H_\alpha^\sharp(A|B)_\rho.\end{aligned}\tag{42}$$

Proof. We only prove the Petz-Rényi type, the proofs of the other two are analogous to the Petz-Rényi type. Let $\lambda_{\max} = \|\rho_A\|_\infty$; it implies that $\rho_A \leqslant \lambda_{\max} I_A$. For the Petz-Rényi mutual information, we have

$$\begin{aligned}I_\alpha(A;B) &= \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|\rho_A\otimes\sigma_B)\\ &\geqslant \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|\lambda_{\max} I_A\otimes\sigma_B)\\ &= -\log_2\lambda_{\max} + \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|I_A\otimes\sigma_B)\\ &= H_{\min}(A)_\rho - H_\alpha(A|B)_\rho.\end{aligned}\tag{43}$$

Here, the inequality follows from the monotonicity; i.e., if $\sigma' \geqslant \sigma$, then one can have $D_\alpha(\rho\|\sigma) \geqslant D_\alpha(\rho\|\sigma')$.

On the other hand, let λ^* be optimal for $H_R(A)_\rho$, then we have $\rho_A \geqslant \lambda^*\Pi_A$. Then, we have

$$\begin{aligned}I_\alpha(A;B) &= \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|\rho_A\otimes\sigma_B)\\ &\leqslant \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|\lambda^*\Pi_A\otimes\sigma_B)\\ &= -\log_2\lambda^* + \min_{\sigma_B \in S_\leqslant(\mathcal{H}_B)} D_\alpha(\rho_{AB}\|I_A\otimes\sigma_B)\\ &= H_R(A)_\rho - H_\alpha(A|B)_\rho.\end{aligned}\tag{44}$$

Just as with the Petz-Rényi type, we can give the chain rules for the sandwiched and geometric Rényi types. Thus, we obtain the desired result. ∎

V. CONCLUSION

In this paper, we provided some fundamental properties for the zero-relative entropy and found that the geometric Rényi relative entropy reduces to the zero-relative entropy in the limit $\alpha \to 0$. We gave the lower and upper chain rule bounds of the (smooth and nonsmooth) zero-mutual information. We also gave a uniform chain rule bound for quantum generalized mutual information of the Petz-Rényi, sandwiched Rényi, and geometric Rényi types. As an application, our result can establish a chain rule for Belavkin-Staszewski information for any quantum system in Ref. [82]. Finally, we mainly considered the chain rules of the zero-mutual information and the generalized mutual information, and we will try to discuss the asymptotic relationship between their different definitions in future work. We hope that our results can be applied in error exponent analysis and channel capacity estimation.

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