Entropy and entanglement in a bipartite quasi-Hermitian system and its Hermitian counterparts

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(Received 27 April 2023; accepted 13 July 2023; published 31 July 2023)

We consider a quantum oscillator coupled to a bath of N other oscillators. The total system evolves with a quasi-Hermitian Hamiltonian. Associated to it is a family of Hermitian systems, parameterized by a unitary map W. Our main goal is to find the influence of W on the entropy and the entanglement in the Hermitian systems. We calculate explicitly the reduced density matrix of the single oscillator for all Hermitian systems and show that, regardless of W, their von Neumann entropy oscillates with a common period which is twice that of the non-Hermitian system. We show that, generically, the oscillator and the bath are entangled for almost all times. While the amount of entanglement depends on the choice of W, the entanglement of the time-averaged density matrix is entirely independent of W. These results describe some universality in the physical properties of all Hermitian systems associated to a given non-Hermitian one.

DOI: 10.1103/PhysRevA.108.012223

I. INTRODUCTION

In recent years there has been much interest in extensions of quantum mechanics that allow for non-Hermitian Hamiltonians. In PT-symmetric quantum theory, for instance, non-Hermitian Hamiltonians with a purely real spectrum are often encountered. In many cases those Hamiltonians are quasi-Hermitian [1–10]. For these one can produce associated Hermitian Hamiltonians, which can then be studied using "standard" methods of quantum theory.

The assignment of a Hermitian to a non-Hermitian system is called the *Dyson map* [11–13]. The Dyson map is determined up to a unitary. To a given non-Hermitian Hamiltonian H are assigned Hermitian Hamiltonians h which are all unitarily equivalent. In this sense, physical properties of the original H are uniquely encoded by any associated h. However, if the system has a subsystem structure, then the unitaries generally reshuffle local degrees of freedom and the choice of the unitary plays a crucial physical role. This is, in particular, the case for open systems, where the system and bath degrees of freedom are reshuffled. One should then ask what relevant information about the original non-Hermitian system is encoded in "the" associated Hermitian one. In fact, if one allows the Dyson map to be time dependent, as is often the approach taken in the literature [13–15], then the above-mentioned nonuniqueness is compounded by additional freedom. It turns out that, given any quasi-Hermitian system H, one can chose a suitable time-dependent Dyson map so that the associated Hermitian system has the trivial Hamiltonian $h = 0$. We explain this in Appendix C.

The ambiguity in the associated Hermitian systems is the subject of the present paper. In the recent literature on PT-symmetric quantum theory, it was suggested that, instead of studying the entropy of an open quasi-Hermitian systems, one may study the entropy of the corresponding associated Hermitian system [16–19]. Concretely, in [17], the authors examined the entropy of an oscillator coupled to an "environment" of N other oscillators via a PT-symmetric Hamiltonian. As a proxy for the von Neumann entropy of the single oscillator in the non-Hermitian system, they studied the corresponding quantity for *one particular choice* of an associated Hermitian system, constructed by a time-dependent Dyson map. The authors found a different qualitative behavior of the entropy of the Hermitian system according to the PT-symmetry phases of the non-Hermitian system. This interesting result was found for one particular choice of the associated Hermitian system. The choice may be regarded as a good one as it discerns the different symmetry phases by the qualitative form of the dynamics of the entropy. For a different choice, however, the entropy dynamics would look entirely different (and in particular, it would be time-independent upon chosing $h = 0$, as explained above).

In the current work we investigate *all* associated systems in the regime of unbroken PT symmetry in which the non-Hermitian Hamiltonian is actually quasi-Hermitian. We vary over all time-independent Dyson maps (excluding the time-dependent ones for the reason mentioned above).

A Hamiltonian H is said to be *quasi-Hermitian* if

$$H^\dagger = \eta H \eta^{-1}$$

for some bounded, positive operator $\eta > 0$, called a *metric operator*. One obtains a Hermitian Hamiltonian by

$$h = SHS^{-1},$$

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where S is any operator with

$$S^{\dagger}S = \eta.$$

The assignment $H \mapsto h$ is called *the Dyson map* [11,12], even though to a single H one can associate different h for the following two reasons.

(1) There are different metric operators η for the same quasi-Hermitian H.

(2) Once a metric η is fixed, the general solution of the equation $S^{\dagger}S = \eta$ is $S = W\sqrt{\eta}$, where $\sqrt{\eta}$ is the unique positive operator that squares to η and W is an arbitrary unitary W.

The metric η is uniquely determined if one fixes a sufficiently rich (irreducible) set of quasi-Hermitian operators (including H) to be *observables*; see [10] and Sec. II. Once this choice is made, all associated Hermitian systems are parameterized by the unitary W. Different choices of W give different Hermitian h, which are unitarily equivalent. However, generally W reshuffles the degrees of freedom and modifies local physical properties, such as entanglement between subsystems.

In this work, we consider a concrete quasi-Hermitian system with Hamiltonian H, describing the interaction of a single oscillator with a bath of N other oscillators. This is an open quantum system of the "system-bath" (SB) type, described by a bipartite Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. We take the metric η to be general but fixed, and then analyze all the associated Hermitian systems resulting from varying over all choices of W. The dynamics of the non-Hermitian system and associated Hermitian systems is given by the time-dependent wave functions

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle \text{ and } |\phi(t)\rangle = S|\psi(t)\rangle = W\sqrt{\eta}|\psi(t)\rangle,$$

respectively. The vector $|\psi(t)\rangle$ is normalized with respect to the metric induced by the inner product $\langle\cdot|\eta|\cdot\rangle$, while $|\phi(t)\rangle$ is normalized relative to the "original' inner product $\langle\cdot|\cdot\rangle$. The *reduced density matrices* for the non-Hermitian and the Hermitian systems are obtained by taking the partial trace over the bath degrees of freedom (see Sec. II B)

$$\bar{\rho}_H(t) = \mathrm{tr}_B(|\psi(t)\rangle\langle\psi(t)|\eta) \text{ and } \bar{\rho}_{h_W}(t) = \mathrm{tr}_B(|\phi(t)\rangle\langle\phi(t)|).$$

Our main findings are summed up as follows.

1. *Explicit evolution.* We obtain explicit formulas for the states $|\psi(t)\rangle$ and $|\phi(t)\rangle$ as well as the reduced states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$. See Secs. III B and III C.

2. *Metric* η. The operator $\bar{\rho}_H(t)$ is a density matrix exactly when η is of product form $\Lambda_S \otimes \Lambda_B$, otherwise $\bar{\rho}_H(t)$ has complex eigenvalues. See Sec. III A. We thus take η of the product form in the further analysis.

3. *Subsystem entropy.* The reduced states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ are periodic[1] in time, both having the same period regardless of the choice of W. The von Neumann entropy $S = -\mathrm{tr}(\rho\ln\rho)$ of $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ is periodic in time as well, but for *generic initial conditions*[2] *and generic* W, the period of the entropy of the Hermitian system is *double* that of the non-Hermitian system. See Sec. IV.

4. *SB entanglement.* The non-Hermitian and the Hermitian SB states $|\psi(t)\rangle$, $|\phi(t)\rangle$ are entangled for all times except at periodically reoccurring single instants.[3] Given any entangled state $|\psi\rangle$, one can find W such that the associated $|\phi\rangle$ is disentangled, and for any disentangled $|\psi\rangle$ there are W such that $|\phi\rangle$ is entangled. However, in an averaged sense, the choice of W does not influence the entanglement at all. Namely, the *concurrence* of the time-averaged density matrix, $\langle\rho\rangle = \frac{1}{T}\int_0^T |\phi(t)\rangle\langle\phi(t)|\,dt$, where T is the period of $|\phi(t)\rangle\langle\phi(t)|$, is *independent* of W. Its value is determined entirely by the initial condition and the choice of the metric. We identify the initial states for which $\langle\rho\rangle$ is separable and for which it is maximally entangled. See Sec. V.

II. QUASI-HERMITIAN SYSTEMS

Let $\mathcal{H}$ be a finite-dimensional Hilbert space with inner product $\langle\cdot|\cdot\rangle$. An operator η is said to be positive, denoted as $\eta > 0$, if $\langle\psi|\eta\psi\rangle > 0$ for all nonzero $|\psi\rangle \in \mathcal{H}$. This is equivalent with saying that $\eta^{\dagger} = \eta$ and all eigenvalues of η are strictly positive. Here $A^{\dagger}$ is the adjoint of the operator A, defined by $\langle\psi|A\phi\rangle = \langle A^{\dagger}\psi|\phi\rangle$ for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. An operator H on $\mathcal{H}$ is called (η-)*quasi-Hermitian* if there exists a positive operator $\eta > 0$ such that

$$H^{\dagger} = \eta H\eta^{-1}. \tag{1}$$

Quasi-Hermiticity is a special case of *pseudo-Hermiticity*, where (1) holds with an invertible (but not necessarily positive) Hermitian operator η. Pseudo- and quasi-Hermitian Hamiltonians arise in PT-symmetric quantum theory, see, for instance [20], and references therein.

A. Hermitian counterparts

Let H be a non-Hermitian operator on $\mathcal{H}$, a candidate for the Hamiltonian of a physical system. To obtain a Hermitian quantum theory, one could do either of the following.

(1) Modify the inner product of $\mathcal{H}$ to $\langle\cdot|\eta\,\cdot\rangle$ for some $\eta > 0$ (called a metric operator), such that H becomes Hermitian in the Hilbert space $\mathcal{H}_\eta$ with this new inner product.

(2) Take a similarity transformation (invertible map) S such that the transformed $h = SHS^{-1}$ is Hermitian in the original Hilbert space $\mathcal{H}$.

If H is quasi-Hermitian, then both options (1) and (2) are possible, but neither the metric nor the similarity transform in options (1) and (2) are unique. To explore this nonuniqueness, we first notice that any quasi-Hermitian H is diagonalizable [6,21,22]. More precisely,

$$H = \sum_{n=1}^{N} E_n|\psi_n\rangle\langle\phi_n|, \tag{2}$$

[1] The entire SB complex consists of $N+1$ oscillators, so the energy spectrum of all the Hamiltonians involved consists of discrete eigenvalues only, without a continuous spectrum. This explains the periodicity.

[2] Only for specially tuned initial states and W's is the situation nongeneric, see Sec. III C.

[3] For a class of exceptional initial conditions the states are entangled for *all* times, see Sec. V.

where the $E_n \in \mathbb{R}$ are the eigenvalues and the $\{|\psi_n\rangle, |\phi_n\rangle\}_{n=1}^N$ form a complete biorthonormal family, meaning that $\langle\psi_k|\phi_l\rangle = \delta_{kl}$ and $\sum |\psi_n\rangle\langle\phi_n| = \mathbb{1}$. Let us consider the case where all eigenvalues E_n are distinct for simplicity (a discussion including degenerate eigenvalues can be done similarly, but this is not our focus here). Then the decomposition (2) is unique, it is the spectral representation of the operator H, and the $P_n \equiv |\psi_n\rangle\langle\phi_n|$ are the uniquely defined (generally not orthogonal) spectral projections. The vectors $|\psi_n\rangle$ and $|\phi_n\rangle$, however, are determined only up to a joint scaling $|\phi_n\rangle \mapsto z_n|\phi_n\rangle$ and $|\psi_n\rangle \mapsto \frac{1}{\bar{z}_n}|\psi_n\rangle$, with $0 \neq z_n \in \mathbb{C}$ arbitrary.

(1) First let us explore option (1). A metric η is called a *metric for H* if H is η-quasi-Hermitian. Let A be a linear operator on $\mathcal{H}$, with adjoint $A^\dagger$ as defined above. If A is viewed as an operator on $\mathcal{H}_\eta$, then $\langle\phi|\eta A\psi\rangle = \langle\eta^{-1}A^\dagger\eta\phi|\eta\psi\rangle$, so the adjoint of A in $\mathcal{H}_\eta$ is $A^\ddagger = \eta^{-1}A^\dagger\eta$. It follows that a given $\eta > 0$ is a metric for H if and only if $H^\ddagger = H$, that is, if and only if H is Hermitian acting on $\mathcal{H}_\eta$. It is well known (see, e.g., [9,22]) that

$$\eta \text{ is a metric for } H$$
$$\iff \eta = \sum_{n=1}^N x_n|\phi_n\rangle\langle\phi_n| \quad \text{for some } x_1, \dots, x_N > 0, \quad (3)$$

where the $|\phi_n\rangle$ are the vectors appearing in (3). The multitude of metrics obtained by varying the x_j in (2) naturally appears due to the fact that $|\phi_n\rangle$ is only determined up to an arbitrary nonzero scaling factor z_n [as explained after (2)], which results in the scaling $x_n \mapsto x_n|z_n|^2$. Given this nonuniqueness of the metric, which one should be chosen to define the physical Hilbert space $\mathcal{H}_\eta$?

One answer is that the metric is fixed provided that instead of just H, one chooses an entire irreducible family of operators to be Hermitian observables. Namely, it is shown in [10] (see also [23] for the two-dimensional case) that if there is a family of operators $\{A_i\}_i$ on $\mathcal{H}$, and positive operators η, η' such that $A_i^\dagger = \eta A_i \eta^{-1}$ and $A_i^\dagger = \eta' A_i (\eta')^{-1}$ for all i, then

$$\eta' \text{ is a scalar multiple of } \eta$$
$$\iff \{A_i\}_i \text{ is an irreducible family of operators on } \mathcal{H}.$$

This means that for an irreducible family of quasi-Hermitian operators, there is exactly one metric (up to a scalar multiple) that makes those operators Hermitian. The chosen family[4] can then be viewed as the physical observables of the theory and the space of pure states is $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle_\eta$.

On the other hand, if interested only in the single observable H (the Hamiltonian), one should keep the x_n in (3) general.

(2) Next, let us investigate option (2) for H of the form (2). Let η be a metric for H, so it is of the form (3). We find all invertible S such that the transformed $h \equiv SHS^{-1}$ is Hermitian

$$h = SHS^{-1} = (SHS^{-1})^\dagger = h^\dagger. \quad (4)$$

One readily sees that (4) is equivalent to $TH = HT$, where $T = \eta^{-1}S^\dagger S$. That T commutes with H, as in (2), is equivalent to T being diagonal in the same biorthonormal system as H, that is, $T = \sum_{n=1}^N t_n|\psi_n\rangle\langle\phi_n|$ for some $t_n \in \mathbb{C}$. Now

$$S^\dagger S = \eta T = \left(\sum_{n=1}^N x_n|\phi_n\rangle\langle\phi_n|\right)\left(\sum_{k=1}^N t_k|\psi_k\rangle\langle\phi_k|\right)$$
$$= \sum_{n=1}^N x_n t_n|\phi_n\rangle\langle\phi_n|, \quad (5)$$

and as $S^\dagger S > 0$ and $x_n > 0$, we have $t_n > 0$ as well. It follows from (3) and (5) that ηT is also a metric for H. In fact, (3) and (5) show that given a fixed metric η for H and varying ηT over all operators

$$T = \sum_{n=1}^N t_n|\psi_n\rangle\langle\phi_n| \quad \text{with} \quad t_n > 0, \quad (6)$$

we obtain all of the metrics for H. We conclude that given η, the S we are looking for are the solutions of $S^\dagger S = \eta T$, where T is an operator of the form (6). The general solution is

$$S = W\sqrt{\eta T}, \quad (7)$$

where W is any unitary and where for a positive operator A, $\sqrt{A}$ is the unique positive operator whose square equals A.

Once W and T are chosen, the associated Hermitian h in (4) becomes

$$h_{W,T} = W\sqrt{\eta T}\,H\frac{1}{\sqrt{\eta T}}W^\dagger. \quad (8)$$

We stress with this notation that h depends on the choice of W and T. The h obtained from two different choices of unitaries, say V and W, are unitarily equivalent, with $h_{V,T} = Uh_{W,T}U^\dagger$ and $U = VW^\dagger$. In this sense, the choice of W is globally immaterial. However, if the Hilbert space has a local structure, say is of bipartite nature $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, then the global unitary U may well change the local properties of the two local subsystems in which case the choice of W will play a physically relevant role. We also point out that the spectrum of $h_{W,T}$ does not depend on either W or T (or η, for that matter).

In this work we take the following approach. We start with a given quasi-Hermitian Hamiltonian H and an arbitrary metric η for H and we view $\mathcal{H}_\eta$ as the physical Hilbert space. We analyze the class of all associated *Hermitian systems* $h_{W,T}$, where W and T vary over all unitaries and all positive operators commuting with H, respectively. As explained above, considering all metrics η is the same as considering all metrics ηT, so varying over T is redundant if η is kept arbitrary. We may then set $T = \mathbb{1}$ and only consider

$$S = W\sqrt{\eta}, \qquad h_W = W\sqrt{\eta}\,H\frac{1}{\sqrt{\eta}}W^\dagger \quad (9)$$

for all W and η.

[4] Examples of irreducible families are the Pauli matrices for a spin, with the Euclidean inner product on $\mathbb{C}^2$, or the position $\hat{x}$ and momentum $\hat{p} = -i\hbar\nabla_x$ for a quantum particle (rather, the bounded Weyl operators generated by them) with the inner product $\langle\psi|\phi\rangle = \int_{\mathbb{R}^3} \bar{\psi}(x)\phi(x)\,d^3x$.

B. States, reduced states, von Neumann entropy

Consider now a fixed metric η, so that the physical Hilbert space is $\mathcal{H}_\eta$ and H is Hermitian on $\mathcal{H}_\eta$, $H^\ddagger = H$. Then e^{-itH} is the unitary Schrödinger dynamics on $\mathcal{H}_\eta$. The average of an observable A on $\mathcal{H}_\eta$ in the state $\psi \in \mathcal{H}_\eta$ is given by

$$\langle\psi|A\psi\rangle_\eta = \langle\psi|\eta A\psi\rangle = \text{tr}(|\psi\rangle\langle\psi|\eta A) = \text{tr}(\widetilde{\rho}A), \tag{10}$$

where

$$\widetilde{\rho} = |\psi\rangle\langle\psi|\eta \tag{11}$$

is a density matrix on $\mathcal{H}_\eta$ (a positive, trace-one operator). This $\widetilde{\rho}$ is called the "generalized density matrix" in [24]. It is important to point out that the trace in (10) is a purely algebraic quantity: it is the sum of the eigenvalues of the operator, and therefore does not depend on the choice of metric.

To arrive at a Hermitian Hamiltonian, it is necessary to make a choice for the unitary W in (9). The associated Hermitian Hamiltonian h_W is then given by (9). Let

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle, \quad |\phi(t)\rangle = e^{-ith_W}|\phi(0)\rangle \tag{12}$$

be the evolution of the initial states $|\psi(0)\rangle$, $|\phi(0)\rangle$ with respect to H and h_W, respectively. The states are related by

$$|\phi(t)\rangle = S|\psi(t)\rangle, \quad S = W\sqrt{\eta}, \tag{13}$$

and the density matrices associated to these vector states for the non-Hermitian [see (11)] and the Hermitian systems are

$$\rho_H(t) = |\psi(t)\rangle\langle\psi(t)|\eta \quad \text{and} \quad \rho_{h_W}(t) = |\phi(t)\rangle\langle\phi(t)|, \tag{14}$$

respectively. (We adopt the notation ρ_{h_W} and ρ_H for the density matrices on the Hermitian and non-Hermitian sides of the problem from [17].) It is clear from (13) that

$$\begin{aligned}\rho_{h_W}(t) &= S|\psi(t)\rangle\langle\psi(t)|S^\dagger = S|\psi(t)\rangle\langle\psi(t)|(S^\dagger S)S^{-1}\\ &= S\rho_H(t)S^{-1}.\end{aligned} \tag{15}$$

It follows that $\rho_{h_W}(t)$ and $\rho_H(t)$ have the same eigenvalues and hence the same von Neumann entropy $\mathcal{E}[\rho_{h_W}(t)] = \mathcal{E}[\rho_H(t)]$, where

$$\mathcal{E}(\rho) = -\text{tr}(\rho\ln\rho) = -\sum_i \lambda_i \ln\lambda_i \tag{16}$$

and $\{\lambda_i\}$ are the eigenvalues of ρ.

Consider now a bipartite system with $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ ("system" and "bath"). We consider the *reduced states* (denoted by an overbar) defined by tracing out the degrees of freedom of the subsystem $\mathcal{H}_B$,

$$\bar{\rho}_H(t) = \text{tr}_{\mathcal{H}_B}[\rho_H(t)], \quad \bar{\rho}_{h_W}(t) = \text{tr}_{\mathcal{H}_B}[\rho_{h_W}(t)]. \tag{17}$$

In some recent works [16–19], the dynamics of a bipartite system generated by a non-Hermitian Hamiltonian H was studied, with particular focus on the von Neumann entropy of the reduced density matrix $\bar{\rho}_H(t)$. The strategy proposed in those works was to examine the entropy of $\bar{\rho}_{h_W}(t)$ as a proxy for that of $\bar{\rho}_H(t)$. In this respect, however, one should observe the following facts.

1. The operator $\bar{\rho}_H(t)$ always satisfies $\text{tr}_{\mathcal{H}_S}[\bar{\rho}_H(t)] = 1$, but for some choices of η the eigenvalues of $\bar{\rho}_H(t)$ can be complex, in which case it is not a valid density matrix.

2. Even if the metric η is chosen such that $\bar{\rho}_H(t)$ is a density matrix, for generic choices of W the von Neumann entropies $\mathcal{E}[\bar{\rho}_H(t)]$ and $\mathcal{E}[\bar{\rho}_{h_W}(t)]$ are not the same. The latter in fact depends on the choice of W.

To understand the normalization of the trace mentioned in fact 1. above, we observe (using $\mathbb{1}_S$ as the system observable) that

$$\begin{aligned}\text{tr}_{\mathcal{H}_S}[\bar{\rho}_H(t)] &= \text{tr}_{\mathcal{H}_S}[\bar{\rho}_H(t)\mathbb{1}_S] = \text{tr}_{\mathcal{H}_S\otimes\mathcal{H}_B}[\rho_H(t)(\mathbb{1}_S\otimes\mathbb{1}_B)]\\ &= \text{tr}_{\mathcal{H}_S\otimes\mathcal{H}_B}[\rho_H(t)] = 1.\end{aligned}$$

If $S = S_S \otimes S_B$, then $\bar{\rho}_h = S_S\bar{\rho}_H S_S^{-1}$ and so the spectra and thus the von Neumann entropies of $\bar{\rho}_h$ and $\bar{\rho}_H$ coincide. However, if S is entangling (not of product form $S_S \otimes S_B$), then the eigenvalues of the two reduced density matrices are not the same in general, and neither are their entropies.

These difficulties are resolved in the next section, where we study the concrete model used in [17]. In particular, we determine for which choices of η the reduced operator $\bar{\rho}_H(t)$ is indeed a density matrix, and then we find the von Neumann entropy of $\bar{\rho}_{h_W}(t)$ *for all possible choices of the unitary* W.

III. MODEL

An oscillator with creation and annihilation operators $a^\dagger$, a is coupled to a "bath" of N-independent oscillators with creation and annihilation operators $q_i^\dagger$, q_i, $i = 1, \dots, N$. The total Hilbert space of the $N+1$ oscillators is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B, \tag{18}$$

where $\mathcal{H}_S$ is the space of a single oscillator and $\mathcal{H}_B$ is that of the other N. As in the previous section, we denote the inner product by $\langle\cdot|\cdot\rangle$ and let $\dagger$ denote the adjoint in this inner product. The commutation relations are $[a, a^\dagger] = 1 = [q_i, q_i^\dagger]$, and all operators belonging to different oscillators commute. This open quantum system, non-Hermitian model was used in [17].

A. Quasi-Hermitian system

The coupled total system–bath Hamiltonian is

$$H = \nu N_{\text{tot}} + (g+\kappa)\sqrt{N}\,a^\dagger Q + (g-\kappa)\sqrt{N}\,aQ^\dagger, \tag{19}$$

where $\nu > 0$ and $g, \kappa \in \mathbb{R}$ are parameters and

$$N_{\text{tot}} = a^\dagger a + \sum_{n=1}^N q_n^\dagger q_n, \quad Q = \frac{1}{\sqrt{N}}\sum_{n=1}^N q_n. \tag{20}$$

Due to the different prefactors of κ in the interaction term of (19), H is $\dagger$-Hermitian if and only if $\kappa = 0$.

The "uncoupled" ($g = \kappa = 0$) Hamiltonian is simply νN_{tot}, a multiple of the total number operator N_{tot}. As H commutes with N_{tot}, each eigenspace of N_{tot}, with a fixed number of excitations (in the system plus the bath) is left invariant. Denote by $|0_S0_B\rangle$ the "vacuum" zero excitation state, where all oscillators are in the ground state. The single excitation space is defined as

$$\mathcal{E}_1 = \text{span}\{|1_S0_B\rangle, |0_S1_1\rangle, |0_S1_2\rangle, \dots, |0_S1_N\rangle\}, \tag{21}$$

where $|1_S0_B\rangle = a^\dagger|0_S0_B\rangle$ and $|0_S1_i\rangle = q_i^\dagger|0_S0_B\rangle$ for $i = 1, \dots, N$. When H is applied to a vector in $\mathcal{E}_1$ the result is again a vector in $\mathcal{E}_1$. Moreover, due to the collective, symmetric nature of the system-bath interaction in (19), H leaves the even smaller space

$$\mathcal{H}_1 = \text{span}\{|e_S\rangle,\ |e_B\rangle\} \tag{22}$$

invariant, where

$$|e_S\rangle = |1_S 0_B\rangle, \quad |e_B\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|0_S 1_n\rangle. \tag{23}$$

Those two vectors describe states in which a single excitation is either in S (the state $|e_S\rangle$) or in B, collectively spread over the N bath oscillators (the state $|e_B\rangle$). Therefore, we may view H as an operator on $\mathcal{H}_1$. When we do this we denote it by H_1, which has the form

$$H_1 = \nu\mathbb{1} + (g-\kappa)\sqrt{N}\,|e_B\rangle\langle e_S| + (g+\kappa)\sqrt{N}\,|e_S\rangle\langle e_B|. \tag{24}$$

The eigenvalues of H_1 are

$$\omega_\pm = \nu \pm \omega, \quad \omega = \sqrt{N}\sqrt{g^2-\kappa^2}, \tag{25}$$

which are real for $\kappa^2 \leqslant g^2$ and (purely imaginary) complex conjugates for $\kappa^2 > g^2$. See [17] for a discussion of the PT symmetry of H. The operator H_1 is diagonalizable except at the transition points defined by $\kappa^2 = g^2 \neq 0$, where H_1 reduces to a Jordan block. Note that increasing the number N of oscillators in the bath simply amounts to speeding up the dynamics (the frequency ω) by a factor $\sqrt{N}$.

Remark. In principle, one might directly introduce the two-level model (24) without deriving it as the single-excitation reduction of the full model (19). However, then there would not be any system-bath (tensor product) structure and the notion of entanglement would not make sense. Furthermore, we also think it would be interesting to extend the analysis to higher excitation sectors. For these reasons we first introduce the underlying model (19).

We consider the "PT-symmetry unbroken regime" $\kappa^2 < g^2$, so that $\omega_\pm \in \mathbb{R}$. For definiteness we take $g > 0$ (the case $g < 0$ can be dealt with in the same fashion), so

$$0 \leqslant |\kappa| < g, \tag{26}$$

which is equivalent to $g+\kappa > 0$ and $g-\kappa > 0$. Then we have $\omega > 0$ and

$$a_1 = \sqrt{g+\kappa} > 0, \quad a_2 = \sqrt{g-\kappa} > 0, \tag{27}$$

where the equalities in (27) define the quantities a_1, a_2. The two linearly independent (not normalized) eigenvectors of H_1 and its adjoint $H_1^\dagger$ are

$$|v_\pm\rangle \propto a_1|e_S\rangle \pm a_2|e_B\rangle \quad \text{and} \quad |v_\pm^*\rangle \propto a_2|e_S\rangle \pm a_1|e_B\rangle,$$

respectively. They satisfy $H_1|v_\pm\rangle = \omega_\pm|v_\pm\rangle$ and $H_1^\dagger|v_\pm^*\rangle = \omega_\pm|v_\pm^*\rangle$. Note that $|v_\pm^*\rangle$ denote the eigenvectors of $H^\dagger$, not to be confused with the complex conjugates of the eigenvectors $|v_\pm\rangle$ of H. We normalize the vectors as

$$|v_\pm\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{a_1}{a_2}}|e_S\rangle \pm \sqrt{\frac{a_2}{a_1}}|e_B\rangle\right) \quad \text{and}$$
$$|v_\pm^*\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{a_2}{a_1}}|e_S\rangle \pm \sqrt{\frac{a_1}{a_2}}|e_B\rangle\right). \tag{28}$$

Then $\{|v_\pm\rangle, |v_\pm^*\rangle\}$ is a biorthonormal basis, satisfying $\langle v_\pm^*|v_\mp\rangle = 0$ and $\langle v_\pm^*|v_\pm\rangle = 1$, and the operator H_1 can be written as

$$H_1 = \omega_+|v_+\rangle\langle v_+^*| + \omega_-|v_-\rangle\langle v_-^*|. \tag{29}$$

Using this, one easily finds

$$\begin{aligned} e^{-itH_1} &= e^{-it\omega_+}|v_+\rangle\langle v_+^*| + e^{-it\omega_-}|v_-\rangle\langle v_-^*| \\ &= e^{-it\nu}\cos(\omega t)\mathbb{1} - ie^{-it\nu}\sin(\omega t) \\ &\quad \times\left(\frac{a_1}{a_2}|e_S\rangle\langle e_B| + \frac{a_2}{a_1}|e_B\rangle\langle e_S|\right). \end{aligned} \tag{30}$$

We consider initial states which are vectors in $\mathcal{H}_1$, as defined in (22), so the dynamics generated by H is entirely given by the operator H_1 from (24). We still consider the regime (26), so that the spectrum of H_1 consists of two distinct real eigenvalues. Comparing (2), (3) and (29), we see that H_1 is quasi-Hermitian and the set of all associated metrics is

$$\mathcal{M}_+ = \{\eta = x_1|v_+^*\rangle\langle v_+^*| + x_2|v_-^*\rangle\langle v_-^*| : x_1, x_2 > 0\}. \tag{31}$$

Written as a matrix in the basis $\{|e_S\rangle, |e_B\rangle\}$, we obtain from (28)

$$\eta = \frac{1}{2}\begin{pmatrix} (x_1+x_2)\,a_2/a_1 & x_1-x_2 \\ x_1-x_2 & (x_1+x_2)\,a_1/a_2 \end{pmatrix}. \tag{32}$$

This is diagonal exactly when $x_1 = x_2$. As we will see in Appendix A, this is equivalent to η being the *restriction to* $\mathcal{H}_1$ *of a product metric* $\Lambda_S \otimes \Lambda_B$ *on* $\mathcal{H}$. In addition, as we discuss below after (40), this is also equivalent to the reduced system state $\bar{\rho}_H(t)$ in (39) being a positive operator.

B. Reduced non-Hermitian system dynamics

Fix an $\eta \in \mathcal{M}_+$ and take an initial state of the form

$$|\psi(0)\rangle = A|e_S\rangle + B|e_B\rangle \tag{33}$$

for some $A, B \in \mathbb{C}$ normalized to have $\|\psi(0)\|_\eta^2 = 1$, that is,

$$1 = \left(\frac{x_1+x_2}{2}\right)\left(\frac{a_2}{a_1}|A|^2 + \frac{a_1}{a_2}|B|^2\right) + (x_1-x_2)\text{Re}(AB^*). \tag{34}$$

The dynamics is given by

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle = e^{-it\nu}A(t)|e_S\rangle + e^{-it\nu}B(t)|e_B\rangle, \tag{35}$$

where

$$\begin{aligned} A(t) &= A\cos(\omega t) - iB\frac{a_1}{a_2}\sin(\omega t), \\ B(t) &= B\cos(\omega t) - iA\frac{a_2}{a_1}\sin(\omega t). \end{aligned} \tag{36}$$

The normalization

$$\|\psi(t)\|_\eta^2 = x_1|\langle v_+^*|\psi(t)\rangle|^2 + x_2|\langle v_-^*|\psi(t)\rangle|^2 = 1 \tag{37}$$

holds for all t, as e^{-itH} acts unitarily on $\mathcal{H}_1$ equipped with the inner product $\langle\cdot|\cdot\rangle_\eta$. The relation (37) is the same as (34) with A and B replaced by $A(t)$ and $B(t)$.

We now introduce the reduction of the system to the single oscillator $(a^\dagger, a)$. The average of a system observable O_S (observable of the single oscillator) in the state $|\psi(t)\rangle$ given by (35) evolves according to

$$\langle\psi(t)|\eta O_S|\psi(t)\rangle = \text{tr}_S[\bar{\rho}_H(t)O_S], \tag{38}$$

where the reduced system state is

$$\bar{\rho}_H(t) = \mathrm{tr}_B\, \rho_H(t) = \mathrm{tr}_B[|\psi(t)\rangle\langle\psi(t)|\eta]. \tag{39}$$

For the partial trace we have the identities $\mathrm{tr}_B|e_S\rangle\langle e_S| = |1_S\rangle\langle 1_S|$, $\mathrm{tr}_B|e_B\rangle\langle e_B| = |0_S\rangle\langle 0_S|$ and $\mathrm{tr}_B|e_S\rangle\langle e_B| = 0 = \mathrm{tr}_B|e_B\rangle\langle e_S|$. Using (35) and η of the form (31), we obtain after a calculation

$$\bar{\rho}_H(t) = \Big(\frac{x_1+x_2}{2}\frac{a_1}{a_2}|B(t)|^2 + \frac{x_1-x_2}{2}A(t)B(t)^*\Big)\,|0_S\rangle\langle 0_S| + \Big(\frac{x_1+x_2}{2}\frac{a_2}{a_1}|A(t)|^2 + \frac{x_1-x_2}{2}A(t)^*B(t)\Big)\,|1_S\rangle\langle 1_S|. \tag{40}$$

This matrix is diagonal in the basis $\{|0_S\rangle, |1_S\rangle\}$ and the two diagonal entries are its eigenvalues. One checks directly that $\mathrm{tr}_S[\bar{\rho}_H(t)] = 1$ (the sum of the diagonal elements equals $\|\psi(t)\|^2_\eta = 1$). However, the eigenvalues of $\bar{\rho}_H(t)$ are complex, in general, unless the metric is chosen to satisfy $x_1 = x_2$. Indeed, the imaginary part of the first eigenvalue is $\frac{x_1-x_2}{2}\mathrm{Im}[A(t)B(t)^*]$. If A, B, the coefficients in the initial state (33), are real then this quantity becomes[5] $-(x_1-x_2)(\frac{1}{x_1+x_2} - \frac{a_2}{a_1}A^2 - \frac{x_1-x_2}{x_1+x_2}AB)\cos(\omega t)\sin(\omega t)$. Unless $x_1 = x_2$ or the initial condition satisfies $(x_1+x_2)\frac{a_2}{a_1}A^2 + (x_1-x_2)AB = 1$, the eigenvalues of $\bar{\rho}_H(t)$ will not be real except at the discrete set of times t when $\sin(\omega t)\cos(\omega t) = 0$.

We require $\bar{\rho}_H(t)$ to be a density matrix (and in particular to have nonnegative eigenvalues) for all times. To do so with a metric that does not depend on the initial conditions we therefore must choose $x_1 = x_2$. We thus take

$$x_1 = x_2 = x > 0$$

for the remainder of the paper. In the basis $\{|e_S\rangle, |e_B\rangle\}$ the metric η is diagonal

$$\eta = x\begin{pmatrix} a_2/a_1 & 0 \\ 0 & a_1/a_2 \end{pmatrix}, \tag{41}$$

see (32). As explained after (32) above, this is equivalent to η being of product form. With this choice, $\bar{\rho}_H(t)$ given by (40) is † Hermitian. According to (40) and (36) we have

$$\bar{\rho}_H(t) = p(t)\,|0_S\rangle\langle 0_S| + (1-p(t))\,|1_S\rangle\langle 1_S|, \tag{42}$$

where

$$\begin{aligned} p(t) &= x\frac{a_1}{a_2}|B(t)|^2 \\ &= x\Big(\frac{a_1}{a_2}|B|^2\cos^2(\omega t) + \frac{a_2}{a_1}|A|^2\sin^2(\omega t) \\ &\quad -2\sin(\omega t)\cos(\omega t)\mathrm{Im}(A^*B)\Big) \\ &= \frac{1}{2} + \Big(\frac{1}{2} - x\frac{a_2}{a_1}|A|^2\Big)\cos(2\omega t) - x\sin(2\omega t)\mathrm{Im}(A^*B). \end{aligned} \tag{43}$$

In the last step, we used the normalization condition (34), resulting in $\frac{a_1}{a_2}|B|^2 = \frac{1}{x} - \frac{a_2}{a_1}|A|^2$, and the trigonometric identities $\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$, $\cos^2(\omega t) = \frac{1}{2}[1+\cos(2\omega t)]$ and $\sin^2(\omega t) = \frac{1}{2}[1-\cos(2\omega t)]$. In view of (43) it is natural to introduce the parameter

$$\alpha \equiv x\frac{a_2}{a_1}|A|^2 \in [0, 1]. \tag{44}$$

Equations (42) and (43) show the following.

Properties of $p(t)$:

1. $p(t)$ and $\bar{\rho}_H(t)$ depend on time unless $\alpha = \frac{1}{2}$ and $A^*B \in \mathbb{R}$, in which case $p(t) = \frac{1}{2}$ and $\bar{\rho}_H(t) = \frac{1}{2}\mathbb{1}$.

2. Otherwise $p(t)$ and $\bar{\rho}_H(t)$ are periodic in time, with period π/ω, and the mean value of $p(t)$ is

$$p_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} p(t)dt = \frac{1}{2}. \tag{45}$$

Remark. The periodicity of the dynamics is a consequence of the simple form of the spectrum of H_1 from (29). In a different setting where the frequencies would be incommensurate, the dynamics would generally be quasiperiodic (sum of periodic functions with different frequencies). However, since the full Hamiltonian H in (19) has purely discrete spectrum, the dynamics will not relax to a stationary state (it will keep oscillating for all times).

C. Reduced Hermitian system dynamics

Next, we turn our attention to the density matrix of the Hermitian system, which according to (15) is

$$\begin{aligned} \rho_{h_W}(t) &= S\rho_H(t)S^{-1} = W\sqrt{\eta}\rho_H(t)\frac{1}{\sqrt{\eta}}W^\dagger \\ &= W\sqrt{\eta}\,|\psi(t)\rangle\langle\psi(t)|\sqrt{\eta}\,W^\dagger. \end{aligned} \tag{46}$$

We keep W in the notation h_W to highlight that the choice of h depends on W, see (9). Again choosing a metric η of the form (31) with $x_1 = x_2 = x > 0$, we use (35) to obtain

$$\sqrt{\eta}|\psi(t)\rangle = e^{-it\nu}\gamma(t)|e_S\rangle + e^{-it\nu}\delta(t)|e_B\rangle, \tag{47}$$

where

$$\gamma(t) = \sqrt{x\frac{a_2}{a_1}}A(t), \quad \delta(t) = \sqrt{x\frac{a_1}{a_2}}B(t). \tag{48}$$

We then obtain

$$\sqrt{\eta}\,|\psi(t)\rangle\langle\psi(t)|\sqrt{\eta} = \begin{pmatrix} |\gamma(t)|^2 & \gamma(t)\delta(t)^* \\ \gamma(t)^*\delta(t) & |\delta(t)|^2 \end{pmatrix}, \tag{49}$$

written in matrix form in the ordered basis $\{|e_S\rangle, |e_B\rangle\}$ of $\mathcal{H}_1$. Next, we take a general (time-independent) unitary on $\mathcal{H}_1$, expressed in the same basis as

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ac^* + bd^* = 0, \qquad |a|^2 + |b|^2 = 1 = |c|^2 + |d|^2. \tag{50}$$

[5]Use equations (36) and write B as a function of A according to the normalization condition (34).

Using (46), (49), (50) and writing momentarily δ, γ for $\delta(t)$, $\gamma(t)$, we get

$$\rho_{h_W}(t) = \begin{pmatrix} |a\gamma + b\delta|^2 & ac^*|\gamma|^2 + bc^*\gamma^*\delta + ad^*\gamma\delta^* + bd^*|\delta|^2 \\ a^*c|\gamma|^2 + b^*c\gamma\delta^* + a^*d\gamma^*\delta + b^*d|\delta|^2 & |c\gamma + d\delta|^2 \end{pmatrix}. \tag{51}$$

We observe that $\rho_{h_W}(t)$ is periodic in time, with period π/ω. This follows from (48) together with (36) since $\cos^2(\omega t)$, $\sin^2(\omega t)$ and $\sin(\omega t)\cos(\omega t)$ all have period π/ω.

Next we calculate the reduced density matrix $\bar{\rho}_{h_W}(t)$ of S by taking the partial trace of $\rho_{h_W}(t)$ over B,

$$\bar{\rho}_{h_W}(t) = q(t)\,|0_S\rangle\langle 0_S| + [1 - q(t)]\,|1_S\rangle\langle 1_S|, \tag{52}$$

where $q(t) = |c\gamma(t) + d\delta(t)|^2$ and $\gamma(t), \delta(t)$ are given in (48). We get the expression

$$q(t) = x\left|c\sqrt{\frac{a_2}{a_1}}A(t) + d\sqrt{\frac{a_1}{a_2}}B(t)\right|^2, \tag{53}$$

where $A(t)$ and $B(t)$ are given in (36). Expanding the square and using $|d|^2 = 1 - |c|^2$ as well as the normalization (34) [which is valid for A, B replaced by $A(t), B(t)$ at any time], we obtain

$$\begin{aligned} q(t) &= |c|^2 + (1 - 2|c|^2)x\frac{a_1}{a_2}|B(t)|^2 + 2x\mathrm{Re}[cd^*A(t)B(t)^*] \\ &= |c|^2 + (1 - 2|c|^2)p(t) + 2x\mathrm{Re}[cd^*A(t)B(t)^*], \end{aligned} \tag{54}$$

where $p(t)$ is the population of $\bar{\rho}_H(t)$ evaluated above in (43). Expanding the real part term in (54) using (36), we arrive at

$$\begin{aligned} q(t) = {}& |c|^2 + (1 - 2|c|^2)p(t) - \sin(2\omega t)\left(1 - 2x\frac{a_2}{a_1}|A|^2\right) \\ & \times \mathrm{Im}(cd^*) - 2x\cos(2\omega t)\mathrm{Im}(AB^*)\mathrm{Im}(cd^*) \\ & + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*). \end{aligned} \tag{55}$$

We take into account (43) to rewrite

$$\begin{aligned} q(t) = {}& \tfrac{1}{2} + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) \\ & + \cos(2\omega t)\left[\left(\tfrac{1}{2} - \alpha\right)(1-2|c|^2) - 2x\mathrm{Im}(AB^*)\mathrm{Im}(cd^*)\right] \\ & - \sin(2\omega t)[(1 - 2\alpha)\mathrm{Im}(cd^*) - x(1-2|c|^2)\mathrm{Im}(AB^*)], \end{aligned} \tag{56}$$

where we recall that α is given in (44). Using the explicit form (56) of $q(t)$, we obtain the following information.

Properties of $q(t)$:

1. $q(t)$ and $\bar{\rho}_{h_W}(t)$ depend on time unless both factors of the cosine and sine terms in (56) vanish. Thus $q(t)$ is time independent and hence equal to

$$q_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} q(t)dt = \frac{1}{2} + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) \tag{57}$$

if and only if:

(i) $\alpha = 1/2$, $|c| = 1/\sqrt{2}$ and $\mathrm{Im}(\mathrm{AB}^*)\mathrm{Im}(\mathrm{cd}^*) = 0$; or
(ii) $\alpha = 1/2$, $|c| \neq 1/\sqrt{2}$ and $\mathrm{Im}(AB^*) = 0$; or
(iii) $\alpha \neq 1/2$, $|c| = 1/\sqrt{2}$ and $\mathrm{Im}(cd^*) = 0$; or
(iv) $\alpha \neq 1/2$, $|c| \neq 1/\sqrt{2}$ and $\mathrm{Im}(A^*B) = \frac{1}{2x}|1 - 2\alpha|$ and $\mathrm{Im}(cd^*) = \frac{1}{2}(1 - 2|c|^2)\mathrm{sgn}(1 - 2\alpha)$, where $\mathrm{sgn}(x) = |x|/x$.

2. Otherwise $q(t)$ and $\bar{\rho}_{h_W}(t)$ are periodic in time, with period π/ω, and the mean value of $q(t)$ is q_0 from (57).

Generic initial states and unitaries

As $x > 0$, the average q_0 equals $1/2$ exactly when $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) = 0$. This is a condition on the initial state (via A, B) and the unitary W (via c, d). We call the initial state and the unitary *generic*, respectively, when

$$AB^* \notin \mathbb{R} \quad \text{and} \quad cd^* \notin \mathbb{R}. \tag{58}$$

In other words, for generic initial states and unitaries, the average q_0 of the population of $\bar{\rho}_{h_W}(t)$ differs from the average $p_0 = 1/2$ of the population of $\bar{\rho}_H(t)$. As we show in the next section, this deviation from the value $1/2$ causes the entropy of $\bar{\rho}_{h_W}(t)$ to oscillate with exactly *half* the frequency of the entropy of $\bar{\rho}_H(t)$.

IV. ENTROPY

Recall that the states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ are given in (42) and (52), with associated populations $p(t)$, $q(t)$ evaluated in (43) and (56). Their von Neumann entropy is given by

$$\begin{aligned} \mathcal{E}[\bar{\rho}_H(t)] &= -p(t)\ln p(t) - [1 - p(t)]\ln[1 - p(t)], \\ \mathcal{E}[\bar{\rho}_{h_W}(t)] &= -q(t)\ln q(t) - [1 - q(t)]\ln[1 - q(t)]. \end{aligned} \tag{59}$$

We show in Appendix B that

$$\mathcal{E}[\bar{\rho}_H(t)] = \mathcal{E}[\bar{\rho}_{h_W}(t)] \text{ for all } t \geqslant 0 \text{ and initial conditions } (A, B) \iff cd = 0.$$

If $\mathrm{Re}(cd^*) \neq 0$, then according to (45) and (57) the averages p_0 and q_0 around which the populations $p(t)$ and $q(t)$ oscillate are *different* for all generic initial conditions, i.e., all coefficients A, B satisfying $\mathrm{Re}(AB^*) \neq 0$. This translates into a modification of the period of the entropy of $\bar{\rho}_{h_W}(t)$ as a function of time t relative to that of $\bar{\rho}_H(t)$, as we explain now.

A. Period doubling of von Neumann entropy in the non-Hermitian versus the Hermitian system

Consider the function

$$\mathcal{E}(Q) = -Q\ln Q - (1 - Q)\ln(1 - Q), \quad Q \in [0, 1].$$

Suppose now that $Q = Q(t)$ depends periodically on time and has average Q_0,

$$Q(t) = Q_0 + \Delta(t) \in [0, 1], \tag{60}$$

with $\Delta(t)$ having period π/ω and zero average. This setup incorporates both cases $p(t)$ and $q(t)$ in one. As Fig. 1 illustrates, if $Q_0 = 1/2$, which is the value where $\mathcal{E}(Q)$ takes its maximum, then as $Q(t)$ moves over one period, the entropy $\mathcal{E}[Q(t)]$ moves over *two* periods.

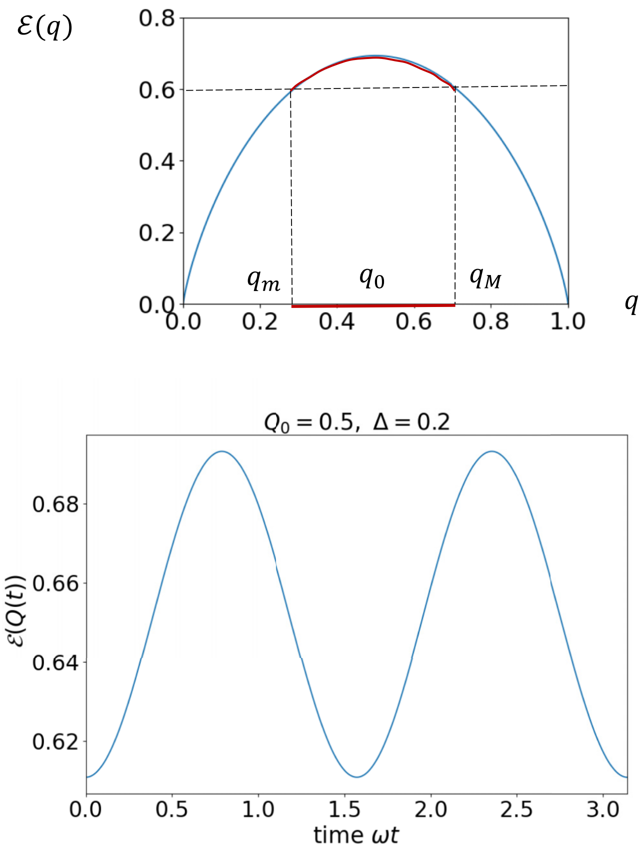

FIG. 1. Parameters $Q_0 = 0.5$, $\Delta \equiv \max_t \Delta(t) = 0.2$. According to (60), $Q(t)$ starts at $Q_m = 0.3$ (at a time we take to be $t = 0$) and moves to $Q_M = 0.7$ at time $\omega t = \pi/2$, and then back to Q_m at time $\omega t = \pi$ (top panel), so the value of the entropy $\mathcal{E}(Q(t))$ evolves through two periods (bottom panel). In each period, the entropy has two local minima (counting minima at the endpoints of the considered intervals once).

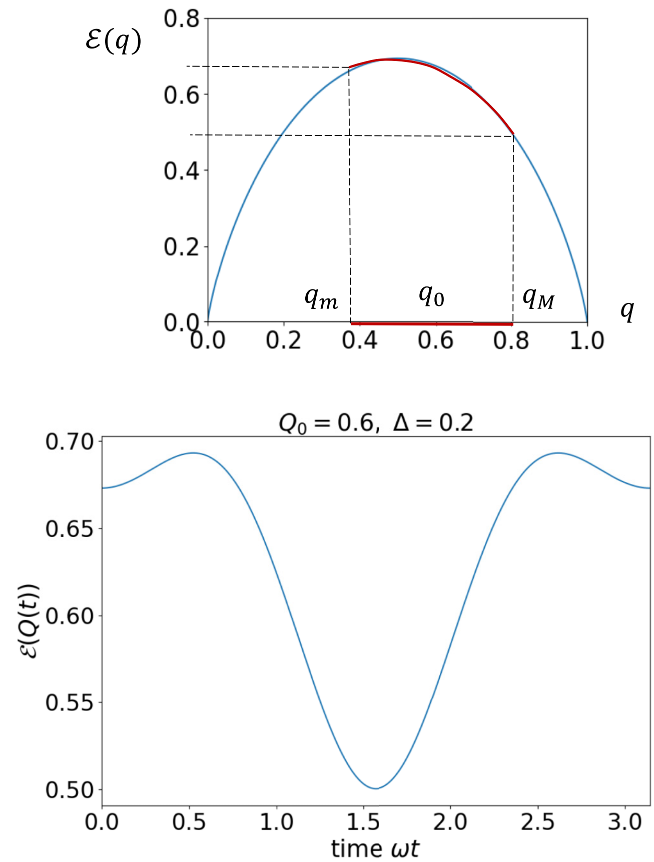

FIG. 2. Parameters $Q_0 = 0.6$, $\Delta = \max_t \Delta(t) = 0.2$. In this case $Q(t)$ starts at $Q_m = 0.4$ when $t = 0$ (upon a possible shift of the time axis) and moves to $Q_M = 0.8$ at time $\omega t = \pi/2$ and back to Q_m at time $\omega t = \pi$ (top panel). The value of the entropy $\mathcal{E}(Q(t))$ evolves through one single period (bottom panel). In each period, the entropy has two local minima (counting minima at the endpoints once).

On the other hand, if $Q_0 \neq 1/2$, then the period of the entropy $\mathcal{E}[Q(t)]$ is *not* doubled relative to that of $Q(t)$, as Figs. 2 and 3 show.

We draw the following conclusions.

1. The period of the von Neumann entropy of the *non-Hermitian* system $\mathcal{E}[\bar{\rho}_H(t)]$ is $\frac{1}{2}\pi/\omega$, regardless of the initial condition (except for the stationary state).

2. Regardless of the metric (parameter x), the period of the von Neumann entropy of the *Hermitian* system $\mathcal{E}[\bar{\rho}_{h_W}(t)]$ is

(a) π/ω, provided $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) \neq 0$ (generic case),

(b) $\frac{1}{2}\pi/\omega$, provided $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) = 0$ (special case).

This means that for generic initial conditions [meaning $\mathrm{Re}(AB^*) \neq 0$] and generic choices of the unitary W [meaning $\mathrm{Re}(cd^*) \neq 0$], *the period of the von Neumann entropy of the Hermitian system is double that of the non-Hermitian system*. That is, the entropy of the non-Hermitian system oscillates faster. This is so even though the populations in both cases have the same frequency π/ω. The change of the period is due to the shift of the average in the population induced by W, as given in (57).

B. Numerical illustration of the period doubling

We plot the populations and entropies for parameters in the regime

$$x = x_1 = x_2 > 0 \quad [\text{metric } \eta, (41)] \tag{61}$$

$$c, d \geqslant 0 \quad [\text{unitary } W, (50)] \tag{62}$$

$$A, B \geqslant 0 \quad [\text{initial state } |\psi(0)\rangle, (33)]. \tag{63}$$

According to (62) and the unitarity of W, we have $d = \sqrt{1-c^2}$. Moreover, $x^2A^2B^2 = \alpha(1-\alpha)$, where $\alpha = x\frac{a_2}{a_1}A^2 \in [0, 1]$. The population $q(t)$ of the Hermitian system reduced density matrix, given in (56), then becomes

$$q(t) = q_0 + \Delta\cos(2\omega t), \tag{64}$$

with

$$q_0 = \tfrac{1}{2} + 2\sqrt{c^2(1-c^2)}\sqrt{\alpha(1-\alpha)}, \tag{65}$$

$$\Delta = \tfrac{1}{2}(1-2c^2)(1-2\alpha). \tag{66}$$

Here, $\alpha = x\frac{a_2}{a_1}A^2 \in [0, 1]$ and $c \in [0, 1]$ can be chosen freely. The population $p(t)$ of the non-Hermitian system, in (43), is simply the expression (64) with $c = 0$.

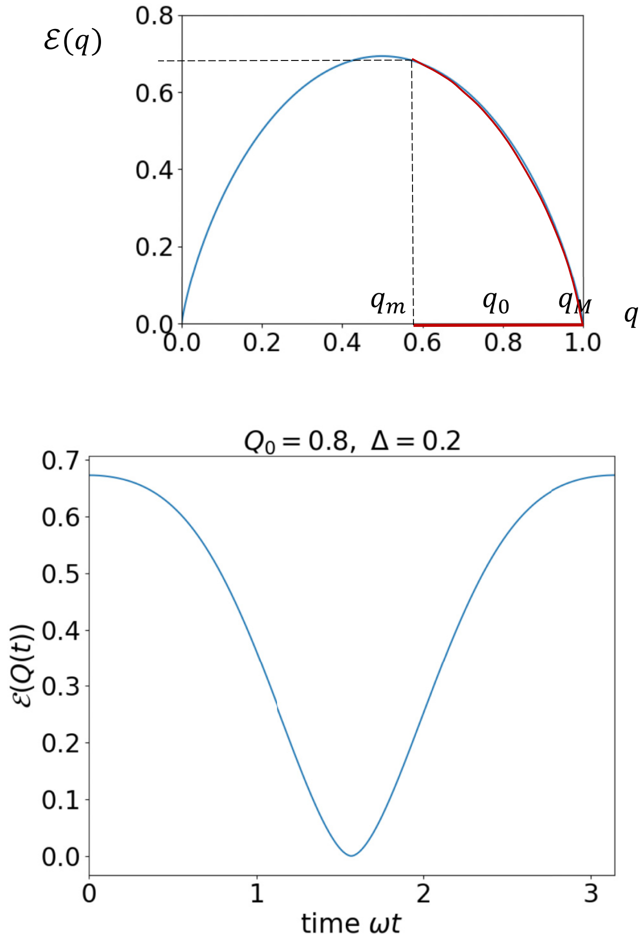

FIG. 3. Parameters $Q_0 = 0.8$, $\Delta = \max_t \Delta(t) = 0.2$. In this case $Q(t)$ starts at $Q_m = 0.6$ when $t = 0$ (after possibly shifting the time axis) and moves to $Q_M = 1.0$ at time $\omega t = \pi/2$ and back to Q_m at time $\omega t = \pi$ (top panel). The value of the entropy $\mathcal{E}(Q(t))$ evolves through one single period (bottom panel). Since 0.5 is not in the interval (Q_m, Q_M), the graph of the entropy has only one local minimum in each period, instead of two when the interval contains the value 0.5 for Q.

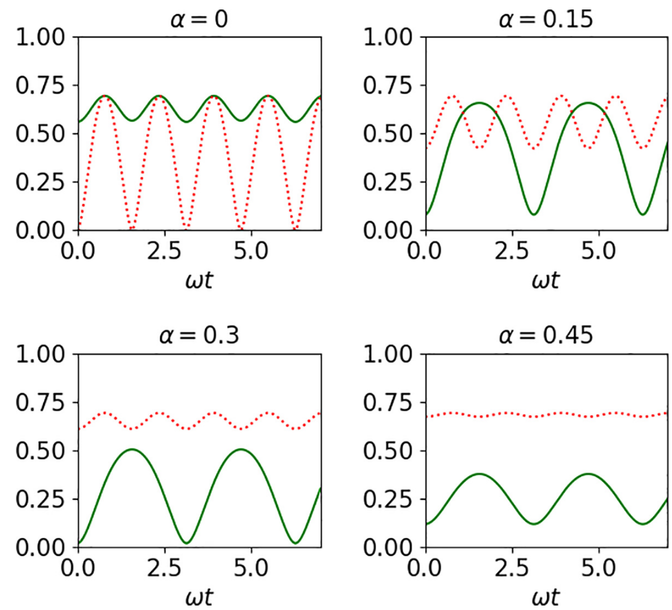

FIG. 4. Comparing the entropies $\mathcal{E}(\bar{\rho}_{h_W}(t))$ (solid line, online in green) and $\mathcal{E}(\bar{\rho}_H(t))$ (dashed line, online in red) for the values $c = 0.5$ and $\alpha = 0, 0.15, 0.3, 0.45$. The doubling of the period for $\alpha \neq 0$ is manifest. The oscillations decrease as α approaches 0.5, which gives the stationary state.

Note that the change $\alpha \mapsto 1 - \alpha$ leaves q_0 invariant and flips the sign of Δ. It then suffices to plot graphs for $\alpha \in [0, 1/2]$. For $\alpha = 1/2$ we get $\Delta = 0$, which gives a stationary state (for all c). The same invariance of q_0 and sign flip of Δ is induced by $c^2 \mapsto 1 - c^2$.

In Fig. 4, we compare the von Neumann entropies of the two density matrices $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$, directly seeing the doubling of the period.

V. ENTANGLEMENT OF SYSTEM AND BATH OSCILLATORS

The total Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ in (18) is bipartite, one part being the singled-out oscillator (system), the other being the remaining N oscillators (bath). We say that a nonzero vector $|\psi\rangle \in \mathcal{H}$ is of *product form*, or *disentangled*, if $|\psi\rangle = |\psi_S\rangle \otimes |\psi_B\rangle$ for some $|\psi_S\rangle \in \mathcal{H}_S$ and some $|\psi_B\rangle \in \mathcal{H}_B$. We call a nonzero $|\psi\rangle \in \mathcal{H}$ *entangled* if it is not of product form. The notion of being entangled or not does not depend on the metric determining the inner product of $\mathcal{H}$. Nevertheless, the physical interpretation of entanglement in terms of independence of the subsystems S and B does depend on the metric. The physical manifestation of disentangled states is the independence of the two subsystems S and B. Namely, if the inner product of $\mathcal{H}$ is given by a metric of the form $\eta = \Lambda_S \otimes \Lambda_B$ (a particular example being $\mathbb{1}_S \otimes \mathbb{1}_B$), then *measurement outcomes* of observables on either of the subsystems are *independent random variables*. This follows because expectation values of observables $O_S \otimes O_B$ in a state $|\psi_S\rangle \otimes |\psi_B\rangle$ split into products

$$\begin{aligned}&\langle \psi_S \otimes \psi_B | \eta (O_S \otimes O_B) \psi_S \otimes \psi_B \rangle \\ &\quad = \langle \psi_S | \Lambda_S O_S \psi_S \rangle \langle \psi_B | \Lambda_B O_B \psi_B \rangle.\end{aligned}$$

However, those random variables become dependent (correlated) if η is not of product form, because then their average will not split into a product of a system term times a bath term.

In the model defined in Sec. III, the metrics η we consider are restrictions to the subspace $\mathcal{H}_1$ of product metrics $\Lambda_S \otimes \Lambda_B$ of $\mathcal{H}$ [cf. (32) and Appendix A]. The physical meaning of SB entanglement in terms of subsystem independence does therefore not depend on the choice of η within this class. In other words, measurement outcomes of system and bath observables in $|\psi\rangle \in \mathcal{H}_1$ are independent or dependent, according to whether $|\psi\rangle \in \mathcal{H}_1$ is disentangled or not, regardless of the choice of η. It is then sensible to investigate the SB entanglement in pure states belonging to the subspace $\mathcal{H}_1$ for all η.

Any vector $|\psi\rangle \in \mathcal{H}_1$ is of the form

$$|\psi\rangle = A|e_S\rangle + B|e_B\rangle,\ A, B \in \mathbb{C}. \tag{67}$$

In accordance with (23), we may write $|e_S\rangle = |10\rangle = |1\rangle \otimes |0\rangle \in \mathcal{H}_S \otimes \mathcal{H}_B$ and $|e_B\rangle = |01\rangle = |0\rangle \otimes |1\rangle \in \mathcal{H}_S \otimes \mathcal{H}_B$.

Explicitly, $|0\rangle, |1\rangle \in \mathcal{H}_S$ are the ground state and first excited state of the system oscillator, and $|0\rangle, |1\rangle \in \mathcal{H}_B$ are the ground state of all the N bath oscillators and the distributed excitation state $\frac{1}{\sqrt{N}}\sum_{n=1}^{N}|1_n\rangle$, respectively; see (23).

We now show that

$$|\psi\rangle \text{ of the form (67) is disentangled} \iff AB = 0. \tag{68}$$

To see that the implication $\Rightarrow$ in (68) holds, let $\rho_S \equiv \mathrm{tr}_B|\psi\rangle\langle\psi|$ (partial trace over B). On the one hand, (67) gives $\rho_S = |A|^2|1\rangle\langle 1| + |B|^2|0\rangle\langle 0|$. On the other hand, if $|\psi\rangle$ is disentangled, then ρ_S must have rank one since $\rho_S = \mathrm{tr}_B(|\psi_S\rangle\langle\psi_S| \otimes |\psi_B\rangle\langle\psi_B|) = |\psi_S\rangle\langle\psi_S|\ \|\psi_B\|^2$. This forces either $A = 0$ or $B = 0$. Conversely, to see the implication $\Leftarrow$ in (68), we note that if either of A or B vanish, then $|\psi\rangle$ is is proportional to $|e_B\rangle$ or $|e_S\rangle$, so $|\psi\rangle$ is disentangled.

1. Entanglement in the non-Hermitian system

An initial state $|\psi(0)\rangle = A|e_S\rangle + B|e_B\rangle$ evolves into $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle = A(t)|e_S\rangle + B(t)|e_B\rangle$, where the time-dependent coefficients are given in (36). According to (68), $|\psi(t)\rangle$ is disentangled exactly if $A(t)B(t) = 0$. Let us first analyze the condition $A(t) = 0$. This equality is equivalent to the two equations

$$\begin{aligned}(\mathrm{Re}A)\cos(\omega t) + (\mathrm{Im}B)\frac{a_1}{a_2}\sin(\omega t) &= 0,\\ (\mathrm{Im}A)\cos(\omega t) - (\mathrm{Re}B)\frac{a_1}{a_2}\sin(\omega t) &= 0.\end{aligned} \tag{69}$$

The condition $B(t) = 0$ is the same as (69) but with $A \leftrightarrow B$ swapped and $a_1 \leftrightarrow a_2$ swapped.

Suppose $|\psi(0)\rangle$ is disentangled, so $AB = 0$. Then exactly one of A or B vanish and the equations (69) are satisfied for $\omega t \in \pi\mathbb{Z}$ (if $A = 0$) or for $\omega t = \frac{\pi}{2}(2\mathbb{Z}+1)$ (if $B = 0$). We conclude that $|\psi(t)\rangle$ is entangled except periodically at discrete moments in time where it is disentangled.

On the other hand, if $|\psi(0)\rangle$ is entangled, then both A and B do not vanish. If A and B are both real or both purely imaginary, then (69) is not satisfied for any t. For all other A and B (69) is satisfied for discrete, periodically repeating values of t.

We conclude with the following.

(a) If the initial state $|\psi(0)\rangle$ is entangled and both A, B are either purely real or purely imaginary, then $|\psi(t)\rangle$ is entangled for all times t.

(b) With the exception of case (a) and regardless of the entanglement in the initial state $|\psi(0)\rangle$, the state $|\psi(t)\rangle$ is entangled except at periodically repeating instants.

2. Entanglement in the Hermitian systems

The Hermitian system pure state vector is given by [see also (13)]

$$|\phi(t)\rangle = S|\psi(t)\rangle = W\sqrt{\eta}|\psi(t)\rangle = \widetilde{A}(t)|e_S\rangle + \widetilde{B}(t)|e_B\rangle,$$

a normalized vector in $\mathcal{H}$ (with the original inner product), where

$$\begin{pmatrix}\widetilde{A}(t)\\ \widetilde{B}(t)\end{pmatrix} = T\begin{pmatrix}A(t)\\ B(t)\end{pmatrix},\quad T = \sqrt{x\frac{a_2}{a_1}}\begin{pmatrix}a & ba_1/a_2\\ c & da_1/a_2\end{pmatrix},$$
$$\det T = x\det W \neq 0 \tag{70}$$

satisfy $|\widetilde{A}(t)|^2 + |\widetilde{B}(t)|^2 = 1$; cf. (48) and (50). It follows from (68) that $|\phi(t)\rangle$ is entangled if and only if $\widetilde{A}(t)\widetilde{B}(t) = 0$. An analysis of the latter equality along the lines of that carried out after (69) shows that $|\phi(t)\rangle$ is entangled except at isolated, periodically reoccurring instants in time, just like the state of the non-Hermitian system.

3. Effect of choice of W on entanglement

Given a state $|\psi\rangle = A|e_S\rangle + B|e_B\rangle$ of the non-Hermitian system, the associated Hermitian system state is $|\phi\rangle = W\sqrt{\eta}|\psi\rangle$. For the choice $W = \mathbb{1}$ we have

$$|\phi\rangle = \sqrt{\eta}|\psi\rangle = \sqrt{xa_2/a_1}A|e_S\rangle + \sqrt{xa_1/a_2}B|e_B\rangle. \tag{71}$$

Hence for $W = \mathbb{1}$, $|\phi\rangle$ is entangled if and only if $|\psi\rangle$ is entangled [recall (68)]. The metric η does not alter the property of being entangled. Choosing a different W to build $|\phi\rangle$ from $|\psi\rangle$, however, changes this. It is not hard to see that the unitaries W that map every product state (that is $|e_S\rangle$ and $|e_B\rangle$) into another product state are exactly the diagonal and the off-diagonal W. Furthermore, given an entangled $\sqrt{\eta}|\psi\rangle$ as in (71), one can always find unitaries W such that $W\sqrt{\eta}|\psi\rangle$ is not entangled. Those W are precisely the ones with $|a| = |d| = \sqrt{a_1/(xa_2)}|B|$ and $|b| = |c| = \sqrt{xa_2/a_1}|A|$.[6]

We now examine the effect of W on the *time-averaged density matrix*

$$\langle\rho\rangle = \frac{\omega}{\pi}\int_0^{\pi/\omega}|\phi(t)\rangle\langle\phi(t)|\,dt, \tag{72}$$

where we integrate $|\phi(t)\rangle\langle\phi(t)| = \rho_{h_W}(t)$ over one period, see (51). A direct calculation yields

$$\langle\rho\rangle = \begin{pmatrix}q_0 & z\\ z^* & 1-q_0\end{pmatrix},\quad z = (bc^* + ad^*)x\mathrm{Re}(AB^*), \tag{73}$$

with $q_0 = \frac{1}{2} + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*)$, cf. (57). The density matrix (73) is written in the basis $\{|e_S\rangle \equiv |10\rangle, |e_B\rangle \equiv |01\rangle\}$ of $\mathcal{H}_1$, using the same notation as after (67). We view $\mathcal{H}_1$ as a subspace of the four dimensional space of two qubits, spanned by the vectors $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In this basis, the density matrix (73) takes the form

$$\langle\rho\rangle = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1-q_0 & z^* & 0\\ 0 & z & q_0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{74}$$

We calculate the *concurrence* [25,26] of $\langle\rho\rangle$ to be[7]

$$\mathcal{C}(\langle\rho\rangle) = 2x|\mathrm{Re}(AB^*)|. \tag{75}$$

The concurrence of any two qubit density matrix is bounded below by 0 (separable state) and above by 1 (maximally entangled state). Both inequalities in

$$2x|\mathrm{Re}(AB^*)| \leqslant 2x|A||B| \leqslant x\left(\frac{a_2}{a_1}|A|^2 + \frac{a_1}{a_2}|B|^2\right) = 1$$

[6]This follows from the characterization (68) of product states and the normalization $\|\sqrt{\eta}|\psi\rangle\|_{\mathcal{H}} = 1$.

[7]In the present case, the square of the concurrence is the difference between the two nonzero eigenvalues of the squared matrix $\langle\rho\rangle^2$.

[the last equality is the normalization (34), with $x = x_1 = x_2 > 0$] are saturated exactly if $AB^* \in \mathbb{R}$ and $|A| = \frac{a_1}{a_2}|B|$.

We conclude that the concurrence of $\langle\rho\rangle$ is the same for all choices of W, so it only depends on the initial state. The state $\langle\rho\rangle$ is separable if and only if $\mathrm{Re}(AB^*) = 0$, and is maximally entangled if and only if $\mathrm{Im}(AB^*) = 0$ and $|A| = \frac{a_1}{a_2}|B|$.

VI. CONCLUSION

The Dyson map assigns to a given quasi-Hermitian quantum system an associated Hermitian system in a nonunique way. We quantify the nonuniqueness by means of a metric operator η and a unitary map W. The physical properties of the Hermitian systems depend on the choice of W, and it is not obvious how to capture the dynamics of the original quasi-Hermitian system in its Hermitian counterparts, unless there happens to be some universality throughout the Hermitian family. We describe an aspect of universality for a quasi-Hermitian open system consisting of a single oscillator coupled to a bath of N oscillators. We show that there is a unique metric operator for which the reduced state of the system (single oscillator) is a well-defined density matrix. Using this metric, we construct all Hermitian systems obtained from the quasi-Hermitian one by varying W. We find that the entropy of the single oscillator in the Hermitian system evolves periodically in time with exactly *double the period* of the corresponding entropy of the quasi-Hermitian system, independently of W. We further show that the oscillator-bath entanglement of the time-averaged state is independent of the choice of W.

ACKNOWLEDGMENTS

The authors are grateful to Andreas Fring for graciously explaining the ideas and results of [17]. The authors also thank two anonymous referees for valuable comments and suggestions. All authors are supported by Discovery Grants from the Natural Sciences and Engineering Research Council of Canada (NSERC) - RGPIN-2017-04259 [GC], RGPIN-2017-05038 [MM].

APPENDIX A: DIAGONAL FORM OF η IS EQUIVALENT TO PRODUCT FORM

We see above in (41) that η must be diagonal in order for the populations of $\bar{\rho}_H(t)$ to be nonnegative and that conversely if η is diagonal, then the populations of $\bar{\rho}_H(t)$ are positive. As it turns out, η being diagonal is also equivalent to η being of product form. More precisely, the following two statements are equivalent:

(1) η is diagonal in the basis $|e_S\rangle, |e_B\rangle$ of $\mathcal{H}_1$;

(2) there are metrics Λ_S and Λ_B on $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively, such that $\Lambda_S\otimes\Lambda_B$ leaves $\mathcal{H}_1$ invariant and $\eta = \Lambda_S\otimes\Lambda_B\restriction_{\mathcal{H}_1}$ is the restriction of this product to $\mathcal{H}_1$.

Given η, the Λ_S and Λ_B are not unique.

Remark. This result on the structure of the metric η, namely that (1) and (2) are equivalent, is entirely independent of the specific form of the form of the Hamiltonian H.

Proof of (1) ⇔ (2). Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_S\otimes\mathcal{H}_B$ with an orthonormal basis $|v_{ij}\rangle = |e_i\rangle\otimes|f_j\rangle$, the $|e_i\rangle$ and $|f_j\rangle$ being orthonormal bases of $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively. Let $\mathcal{H}_1$ be the two-dimensional subspace $\mathcal{H}_1 = \mathrm{span}\{|v_{11}\rangle, |v_{22}\rangle\}$. In this setup, $|v_{11}\rangle$ is identified with $|e_S\rangle$ and $|v_{22}\rangle$ with $|e_B\rangle$. Let $\eta > 0$ be a strictly positive operator on $\mathcal{H}_1$.

We first show (1) ⇒ (2). Assume that η is diagonal, that is, $\eta = a|v_{11}\rangle\langle v_{11}| + b|v_{22}\rangle\langle v_{22}|$, where $a, b > 0$. Set

$$\Lambda_S = \alpha_1|e_1\rangle\langle e_1| + \alpha_2|e_2\rangle\langle e_2| + \Lambda_S^\perp,$$
$$\Lambda_B = \beta_1|f_1\rangle\langle f_1| + \beta_2|f_2\rangle\langle f_2| + \Lambda_B^\perp,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfy $\alpha_1\beta_1 = a$, $\alpha_2\beta_2 = b$ and $\Lambda_S^\perp$, $\Lambda_B^\perp$ are arbitrary positive operators on the orthogonal complements of $\mathrm{span}\{|e_1\rangle, |e_2\rangle\}$ and $\mathrm{span}\{|f_1\rangle, |f_2\rangle\}$ in $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively. Then $\Lambda_S\otimes\Lambda_B$ is a metric on $\mathcal{H}$ which leaves $\mathcal{H}_1$ invariant and satisfies $(\Lambda_S\otimes\Lambda_B)|v_{jj}\rangle = \eta|v_{jj}\rangle$ for $j = 1, 2$.

Next we prove that (2) ⇒ (1). The orthogonal projection onto $\mathcal{H}_1$ is given by

$$\pi = |v_{11}\rangle\langle v_{11}| + |v_{22}\rangle\langle v_{22}| = p_{11}\otimes q_{11} + p_{22}\otimes q_{22},$$

where $p_{ij} = |e_i\rangle\langle e_j|$ and $q_{ij} = |f_i\rangle\langle f_j|$. Since $\Lambda_S\otimes\Lambda_B$ leaves $\mathcal{H}_1$ invariant, we have $(\Lambda_S\otimes\Lambda_B)\pi = \pi(\Lambda_S\otimes\Lambda_B)\pi$. Now

$$(\Lambda_S\otimes\Lambda_B)\pi = \Lambda_S p_{11}\otimes\Lambda_B q_{11} + \Lambda_S p_{22}\otimes\Lambda_B q_{22} \quad \text{(A1)}$$

and, with $[\Lambda_S]_{ij} = \langle e_i, \Lambda_S e_j\rangle$ and similarly for Λ_B,

$$\begin{aligned}\pi(\Lambda_S\otimes\Lambda_B)\pi &= [\Lambda_S]_{11}[\Lambda_B]_{11}\, p_{11}\otimes q_{11} + [\Lambda_S]_{12}[\Lambda_B]_{12}\, p_{12}\otimes q_{12}\\ &= [\Lambda_S]_{21}[\Lambda_B]_{21}\, p_{21}\otimes q_{21} + [\Lambda_S]_{22}[\Lambda_B]_{22}\, p_{22}\otimes q_{22}.\end{aligned} \quad \text{(A2)}$$

Taking the partial trace over B in (A1) and (A2) and equating the two results gives

$$[\Lambda_B]_{11}\Lambda_S p_{11} + [\Lambda_B]_{22}\Lambda_S p_{22} = [\Lambda_S]_{11}[\Lambda_B]_{11}\, p_{11} + [\Lambda_S]_{22}[\Lambda_B]_{22}\, p_{22}. \quad \text{(A3)}$$

Since $[\Lambda_B]_{11}, [\Lambda_B]_{22} > 0$ we get from (A3) that $\Lambda_S p_{jj} = [\Lambda_S]_{jj}p_{jj}$ for $j = 1, 2$, so $\Lambda_S|e_j\rangle = [\Lambda_S]_{jj}|e_j\rangle$. Hence the restriction of Λ_S to $\mathrm{span}\{|e_1\rangle, |e_2\rangle\}$ is diagonal, $\Lambda_S = [\Lambda_1]_{11}p_{11} + [\Lambda_S]_{22}p_{22} + \Lambda_S^\perp$, where $\Lambda_S^\perp$ is the block acting on the orthogonal complement of that span. By taking the partial trace over S in (A1) and (A2) and proceeding analogously, we see that $\Lambda_B = [\Lambda_B]_{11}q_{11} + [\Lambda_B]_{22}q_{22} + \Lambda_B^\perp$. It follows that $(\Lambda_S\otimes\Lambda_B)|v_{jj}\rangle = [\Lambda_S]_{jj}[\Lambda_B]_{jj}\,|v_{jj}\rangle$ for $j = 1, 2$, so

$$\eta = [\Lambda_S]_{11}[\Lambda_B]_{11}\,|v_{11}\rangle\langle v_{11}| + [\Lambda_S]_{22}[\Lambda_B]_{22}\,|v_{22}\rangle\langle v_{22}|$$

is diagonal.

APPENDIX B: CONDITIONS FOR $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ AND $\mathcal{E}[\bar{\rho}_H(t)] = \mathcal{E}[\bar{\rho}_{h_W}(t)]$

In this section we assume the metric η is of the form (31) with $x_1 = x_2 = x > 0$. Recall the formulas (42) and (52) for the quasi-Hermitian and Hermitian density matrices.

First we ask when the two reduced density matrices coincide. We show that the following statements are equivalent:

1. $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all t in an open interval $I\subset\mathbb{R}$ and all $A, B\in\mathbb{C}$;
2. $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all $t\in\mathbb{R}$ and all $A, B\in\mathbb{C}$;

3. there are two real phases Φ_1, Φ_2, such that

$$W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_1} \end{pmatrix}.$$

1. $\Rightarrow$ 3.: Assume that $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for $t \in I$. Then $p(t) = q(t)$ for $t \in I$, where these quantities are given in (43) and (56), respectively. Their equality is equivalent with

$$\xi_1 + \xi_2 \cos(2\omega t) + \xi_3 \sin(2\omega t) = 0 \quad \text{for all } t \in I, \tag{B1}$$

with $\xi_1 = 2x\text{Re}(AB^*)\,\text{Re}(cd^*)$, $\xi_2 = -|c|^2(1-2\alpha) - 2x\text{Im}(AB^*)\text{Im}(cd^*)$, and $\xi_3 = -(1-2\alpha)\text{Im}(cd^*) + 2x(1-|c|^2)\text{Im}(AB^*)$. As the constant function and the sine and cosine are three independent functions, and since (B1) holds for all t in an interval, we conclude that $\xi_1 = \xi_2 = \xi_3 = 0$. Since $\xi_1 = 0$ for all A, B and $x > 0$, we have $\text{Re}(cd^*) = 0$, that is, $\text{Im}(cd^*) = cd^*$. The coefficients A, B and α are related by (34) and (44), resulting in $A = \sqrt{\alpha}\sqrt{\frac{a_1}{xa_2}}e^{if_1}$ and $B = \sqrt{1-\alpha}\sqrt{\frac{a_2}{xa_1}}e^{if_2}$, where $f_1, f_2 \in \mathbb{R}$ are phases. This gives $\text{Im}(AB^*) = \sqrt{\alpha(1-\alpha)}\frac{1}{x}\,\text{Im}\,e^{i(f_1-f_2)}$. Then $\xi_2 = 0$ for all A, B implies that

$$|c|^2(1-2\alpha) = -2\sqrt{\alpha(1-\alpha)}cd^*\,\text{Im}\,e^{i(f_1-f_2)} \quad \text{for all } f_2, f_2 \in \mathbb{R}, \alpha \in [0,1].$$

This forces $|c|^2(1-2\alpha) = 0 = \alpha(1-\alpha)cd^*$ for all $\alpha \in [0,1]$. Hence $c = 0$. Then due to (50), $|d| = 1$ and $bd^* = 0$, so $b = 0$, and statement 3 holds.

3. $\Rightarrow$ 2.: Suppose $c = 0$. Then $|d| = 1$ and from (53) we have $q(t) = x\frac{a_1}{a_2}|B(t)|^2$, which equals the population of $|0_S\rangle$ in $\bar{\rho}_H(t)$, see (40). Therefore statement 2 holds.

2. $\Rightarrow$ 1.: Obvious.

This completes the proof of the equivalence of the three statements 1 to 3.

Next we ask when the entropies of the two density matrices coincide. We show that the following statements 4 to 6 are equivalent:

4. $\mathcal{E}[\bar{\rho}_H(t)] = \mathcal{E}[\bar{\rho}_{h_W}(t)]$ for all t in an open interval $I \subset \mathbb{R}$ and all $A, B \in \mathbb{C}$;

5. $\mathcal{E}[\bar{\rho}_H(t)] = \mathcal{E}[\bar{\rho}_{h_W}(t)]$ for all $t \in \mathbb{R}$ and all $A, B \in \mathbb{C}$;

6. there are two real phases Φ_1, Φ_2 such that W is of either of the two forms

$$W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_1} \end{pmatrix} \quad \text{or} \quad W = \begin{pmatrix} 0 & e^{i\Phi_1} \\ e^{i\Phi_2} & 0 \end{pmatrix}.$$

4. $\Rightarrow$ 6.: Start by looking at the function $\mathcal{E}(q) = -q\ln(q) - (1-q)\ln(1-q)$, for $q \in [0,1]$. It is clear from the graph of $\mathcal{E}(q)$ (see the left panel of Fig. 1) that $\mathcal{E}(q) = \mathcal{E}(q')$ exactly if either $q = q'$ or $q = 1-q'$. Consequently, if $\mathcal{E}[\bar{\rho}_{h_W}(t)] = \mathcal{E}[\bar{\rho}_H(t)]$ for all $t \in I$, then for each $t \in I$ *individually*, we have either $p(t) = q(t)$ or $p(t) = 1 - q(t)$. We now show that the same alternative must happen for *all* $t \in I$.

Suppose first that $p(t_0) \neq 1 - q(t_0)$ for some $t_0 \in I$. Then by the continuity of $p(t)$ and $q(t)$, we have $p(t) \neq 1 - q(t)$ for all t in an open interval $I_0 \subset I$ around t_0, so we must have $p(t) = q(t)$ for $t \in I_0$. However, this means that $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all $t \in I_0$. Hence, as statements 2 and 3 are equivalent, W is of the diagonal form as given in point 3 above. Similarly, if $p(t_0) \neq q(t_0)$ for some $t_0 \in I$, we obtain $p(t) = 1 - q(t)$ as an interval around t_0. Proceeding as in the proof of the implication $1 \Rightarrow 3$ above, this implies that $a = d = 0$, so W is of the off-diagonal form given in statement 6 above.

6. $\Rightarrow$ 5.: If W is of the diagonal form, then we already showed that $p(t) = q(t)$ when we proved $3 \Rightarrow 2$. In the same way, if W is off-diagonal, then one sees that $p(t) = 1 - q(t)$. In either case, $\mathcal{E}[\bar{\rho}_H(t)] = \mathcal{E}[\bar{\rho}_{h_W}(t)]$.

5. $\Rightarrow$ 4.: Obvious.

APPENDIX C: ABOUT THE DYSON MAP

The idea of mapping a non-Hermitian Hamiltonian to a Hermitian one was originally presented by Dyson in the context of the theory of magnetization [11,12]. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and let H be an operator on $\mathcal{H}$ that is not necessarily Hermitian with respect to $\langle\cdot|\cdot\rangle$. Denote by $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$ the solution of the evolution equation

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle. \tag{C1}$$

Next, let $S(t)$ be a differentiable family of operators on $\mathcal{H}$ such that $S(t)$ is invertible for each t, and set

$$|\varphi(t)\rangle = S(t)|\psi(t)\rangle. \tag{C2}$$

Passing from $|\psi(t)\rangle$ to $|\varphi(t)\rangle$ represents a (possibly time-dependent) change of variables. The evolution equation for $|\varphi(t)\rangle$ is

$$i\partial_t|\varphi(t)\rangle = h(t)|\varphi(t)\rangle, \tag{C3}$$

with

$$h(t) = S(t)HS(t)^{-1} + i\dot{S}(t)S(t)^{-1}, \tag{C4}$$

the dot being the time derivative. Conversely, if $|\varphi(t)\rangle$ solves (C3) then $|\psi(t)\rangle$ solves (C1). Equation (C4) is called the *time-dependent Dyson equation* [17]. By means of $S(t)$, one may thus equivalently solve (C1) or (C3). Introducing a time dependence in the transformation $S(t)$ changes the Hamiltonian, and hence the physics of the problem. The time dependence reflects some additional (external) action on the system. It was shown in [27] that it results in forces analogous to the classical Coriolis force. So generally, H and $h(t)$ describe different physical systems. If H is not Hermitian, one can look for $S(t)$ such that the resulting $h(t)$ is Hermitian, hence trading a non-Hermitian problem with constant Hamiltonian H for a Hermitian problem with time-dependent Hamiltonian $h(t)$. One readily sees that

$$\begin{aligned} h(t)^\dagger = h(t) \iff & i\partial_t[S^\dagger(t)S(t)] \\ &= H^\dagger[S^\dagger(t)S(t)] - [S^\dagger(t)S(t)]H. \end{aligned} \tag{C5}$$

The operator $\eta(t) = S(t)^\dagger S(t)$ is automatically positive, so $\eta(t)$ is a family of metrics. The equation for $\eta(t)$, according to (C5), is

$$i\partial_t\eta(t) = H^\dagger\eta(t) - \eta(t)H. \tag{C6}$$

This is called the *quasi-Hermiticity relation* in [14]; note that it simplifies to (1) if η does not depend on time. It is clear that (C6) has a unique solution for any initial condition $\eta(0)$, namely

$$\eta(t) = e^{-itH^\dagger}\eta(0)e^{itH}, \tag{C7}$$

and that $\eta(t)$ is positive for all times if and only if it is positive at some t_0.

A strategy to study the dynamics generated by a non-Hermitian H is to find a transformation $S(t)$ such that $h(t)$, as given by (C4), is Hermitian, and then analyze the dynamics of this Hermitian system using usual quantum theoretical methods. Finding $S(t)$ for a specific Hamiltonian H is not easy, however. It often involves making a judicious ansatz containing parameters that must solve rather complicated differential equations, which are obtained by imposing the self-adjointness of $h(t)$. This can be done explicitly for some models [14–17,28–32].

Given H, we seek *all* possible $S(t)$, and the resulting Hermitian Hamiltonians $h(t)$, with the sole requirement that $\eta(t) = S(t)^\dagger S(t)$ is positive and satisfies the quasi-Hermiticity relation (C6). The solution $\eta(t)$ is uniquely determined by the initial condition $\eta(0)$, which we may choose to be any positive, invertible operator. The most general form of $S(t)$ is thus

$$S(t) = W(t)\sqrt{\eta(0)}e^{itH}, \tag{C8}$$

where $W(t)$ is any unitary family and $\sqrt{\eta(0)}$ denotes the unique positive operator squaring to $\eta(0)$. The $h(t)$ associated to (C8) by (C4) is

$$h(t) = i\dot{W}(t)W(t)^\dagger. \tag{C9}$$

Note that we are entirely free to choose $W(t)$. For instance, given an arbitrary $A = A^\dagger$, the choice $W(t) = e^{-itA}$ yields $h(t) = A$. This means any time-independent Hermitian h can be obtained from a suitable choice of $S(t)$. A particularly simple choice is $S(t) = e^{itH}$, which results from undoing the dynamics e^{-itH} (going backwards in time) and has $h = 0$.

More generally, suppose $A(t)$ is a continuous family of operators, and let $W(t)$ solve the differential equation

$$i\dot{W}(t) = A(t)W(t). \tag{C10}$$

It is easily shown that if $A(t) = A(t)^\dagger$ for all t and the initial condition $W(0)$ is unitary, then the solution $W(t)$ is unitary for all t. Choosing this $W(t)$, we find from (C9) the Hermitian Hamiltonian $h(t) = i\dot{W}(t)W(t)^\dagger = i\dot{W}(t)W(t)^{-1} = A(t)$. This means any time-*dependent* Hermitian $h(t)$ can also be obtained from a suitable choice of $S(t)$.

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