Unitary equivalence of twisted quantum states

N. V. Filina and S. S. Baturin*
School of Physics and Engineering, ITMO University, Saint Petersburg 197101, Russia

(Received 10 May 2023; accepted 13 July 2023; published 27 July 2023; corrected 3 August 2023 and 28 September 2023)

We present the time dynamics of twisted quantum states. We find an explicit connection between the well-known stationary Landau state and an evolving twisted state, even when the Hamiltonian accounts for linear energy dissipation. Utilizing this unitary connection, we analyze nonstationary Landau states and unveil some of their properties. The proposed transformation enables simple evaluation of different operator mean values for the evolving twisted state based on the solution to the classical Ermakov equation and matrix elements calculated on the stationary Landau states. The suggested formalism may significantly simplify analysis and become a convenient tool for further theoretical development on the dissipative evolution of the twisted quantum wave packet.

DOI: 10.1103/PhysRevA.108.012219

I. INTRODUCTION

Cylindrical waves are a well-known phenomenon in the field of waveguides and antennas [1]. Such waves naturally appear as a solution to the D'Alembert equation in a cylindrical coordinate system as a consequence of the cylindrical symmetry of the problem.

In Refs. [2,3] Allen *et al.* realized that cylindrical waves could be generated in free space in the optical regime. Moreover, they pointed out that the azimuthal index, l, of the cylindrical electromagnetic wave corresponds to the quantized projection of the angular momentum of light. Indeed, cylindrical symmetry implies invariance of the solution to elementary rotations along the symmetry axis. The generator for cylindrical symmetry is the $-i\partial_\phi$ operator. Thus, the solutions of the corresponding problem must be a superposition of the eigenfunctions of this operator, which is known to form a complete set under periodic boundary conditions. Eigenvalues l, with $l \in \mathbb{Z}$, enumerate basis functions and correspond to the orbital angular momentum (OAM), as, by definition, $\hat{L}_z = -i\partial_\phi$. With the development of the Berry theory, it was pointed out that the orbital angular momentum is just a coefficient in a Berry phase (or geometric phase) defined by [4,5]

$$l = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{d(\arg\psi)}{d\phi}, \tag{1}$$

where ψ is the particle (photon, electron, proton, etc.) wave function normalized to unity, $\langle\psi|\psi\rangle = 1$. On the other hand, for the case of the defined orbital angular momentum (when the wave function factorizes into the radial and angular parts), the formula above could be reinterpreted as

$$l = \lim_{r\to\infty} \frac{1}{4\pi|\psi|^2} \int_0^{2\pi} \mathrm{Im}[\psi^*\partial_\phi\psi]d\phi = \langle\psi|\hat{L}_z|\psi\rangle, \tag{2}$$

unveiling the connection between the strength of the vortex singularity and the eigenvalue of the $\hat{L}_z$ operator. As a result, such waves are often referred to as waves with nontrivial geometric phase, or waves that possess a phase vortex, as well as waves that carry OAM.

Numerous applications of light beams with nonzero OAM, or twisted photons, have been widely discussed [6–9]. It has also been shown that electrons with a phase vortex—twisted electrons—could be a new tool for microscopy, materials science, and high-energy physics [10,11].

While significant understanding was gained on the theoretical side [12–21], there are still several challenges in the complete description of the evolution of the twisted particles. For instance, radiation and the possible consequent loss of angular momentum is an open question, as it relates to the acceleration of twisted massive charged particles to relativistic energies.

In the present paper, inspired by the historical connection of the twisted states to geometry, we apply a geometric transformation of the classical particle phase trajectory [22] and build a corresponding transformation for Schrödinger's equation [23–25]. It is shown that the simple idea of vector field rectification, discussed by Arnold in his famous book [22], has far-reaching consequences [24,25]. In the present paper we take the course of Refs. [23–25] called the quantum Arnold transformation (QAT) as we find it the most intuitive. A comprehensive study of the time-dependant quantum harmonic oscillator can be found in Ref. [26].

The benefit of the QAT approach is twofold. First, it allows mapping any state that satisfies Schrödinger's equation for the case of a temporally constant (or constant in the longitudinal coordinate z with the paraxial approximation) magnetic field (refractive index for the case of the photons) to the case of time-varying potential, including a system with linear dissipation within the Cardirola-Kanai (CK) model [27,28]. The latter, in turn, enables accounting of media absorption for the case of neutral particles, and the radiation friction in the dipole approximation for the case of charged particles. Second, it allows calculation of the mean values of the observable operators, based on the corresponding matrix elements from the solution of the system with stationary (time-independent) potential.

*s.s.baturin@gmail.com

Throughout the paper we use the natural unit system with $\hbar = 1$, $c = 1$, and assume that $e < 0$ is the electron charge.

II. QAT FORMALISM AND 2D ERMAKOV OPERATOR

We analyze the transverse part of the nonrelativistic Schrödinger equation for a massive charged particle in a magnetic field that has the form

$$i\partial_t\psi = \hat{H}_\perp\psi, \quad \hat{H}_\perp = \frac{[\hat{\boldsymbol{p}}_\perp - e\boldsymbol{A}(t)]^2}{2m}. \tag{3}$$

Here $\boldsymbol{A}^T = \{-yB(t)/2, xB(t)/2\}$ is the transverse part of the vector potential, and $B(t)$ is the modulus of the magnetic field directed along the z axis.

Equation (3) is the basic equation for describing the evolution of the transverse part of the twisted particle wave function [16,29], in the paraxial approximation for both photons in free space and fibers [30] (with the proper change in the parameters [16]), and for relativistic electrons (after factoring out the longitudinal part and substitution of $t \to z$) [31]. Using a point particle approximation [32], Eq. (3) describes the transverse dynamics of a nonrelativistic electron traversing a set of solenoid lenses. In all cases, the transverse part of the wave function ψ defines the twisted structure of the corresponding state. In the present paper for definiteness, we consider electrons in a magnetic field. The same approach is valid for other types of particles where Eq. (3) describes transverse dynamics of the wave function.

If we account for the linear friction that may arise, for instance, from radiation friction, then Schrödinger's equation generalizes to the CK model by a canonical transformation [27,28]:

$$i\partial_t\psi = \left[\frac{w(t)\left[\hat{p}_x^2 + \hat{p}_y^2\right]}{2m} + \frac{m\omega^2(t)(\hat{x}^2 + \hat{y}^2)}{2w(t)} + \omega(t)\hat{L}_z\right]\psi \tag{4}$$

where $w(t)$ is the dissipation factor of the form $w(t) = \exp[-\int\gamma(t)dt]$ with $\gamma(t)$ being the classical friction coefficient. We note that Eq. (4) is the sum of the Hamiltonians of two one-dimensional (1D) quantum oscillators coupled through the $\omega\hat{L}_z$ term.

Here

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{5}$$

is the operator of the angular momentum z projection, and

$$\omega(t) = \frac{|e|B(t)}{2m}. \tag{6}$$

A CK system is equivalent to a master equation for the density matrix without fluctuation, assuming there is no stochastic process. However, a full equivalence of the CK model to the standard Lindblad master equation [33,34] could be achieved once the proper stochastic potential is incorporated into Eq. (4) [35]. For the present consideration, we limit ourselves to a simplified CK model but note that the analysis we present is extendable to the case of momentum and coordinate diffusion.

Now, we consider the QAT discussed in Refs. [23–25]. For a one-dimensional system, the transformation is an interchange of the coordinates, time, multiplication of the wave function by a phase factor, and a normalization factor.

Following Ref. [23] we consider the QAT operator that establishes a mapping between the Hilbert space $\mathcal{H}_t$ of solutions $\psi(x,t)$ of the one-dimensional time-dependent Schrödinger equation at time t, on the Hilbert space $\mathcal{H}_\tau$ of solutions $\varphi(\kappa,\tau)$ of the time-dependent Schrödinger's equation for the Galilean free particle at a time τ. The explicit form for the 1D QAT is (see Appendixes A and B)

$$\hat{\mathcal{Q}}: \begin{cases} \kappa = \frac{x}{u_2}, \\ \tau = -\frac{u_1}{u_2}, \\ \varphi(\kappa,\tau) = \psi(x,t)\sqrt{u_2}\exp\left[-\frac{i}{2}\frac{m}{w}\frac{\dot{u}_2}{u_2}x^2\right]. \end{cases} \tag{7}$$

In the equality for the wave function, φ, the arguments x and t on the right, as well as the argument t of the functions u_2 and w, are understood as functions of κ and τ ($x = x(\kappa,\tau)$; $t = t(\tau)$). The dot above u_2 denotes the total derivative by t. The functions u_1 and u_2 are general solutions to the classical Euler-Lagrange equation of the considered quantum oscillator

$$\ddot{u}_{1,2} + \gamma(t)\dot{u}_{1,2} + \omega^2(t)u_{1,2} = 0 \tag{8}$$

and

$$w(t) = u_1\dot{u}_2 - u_2\dot{u}_1 = \exp\left[-\int\gamma dt\right] \tag{9}$$

is the Wronskian built on these solutions. The initial conditions for Eq. (8) are chosen such that linear independence of u_1 and u_2 is guarantied and Eq. (9) holds.

The first part of the transformation is simply the coordinate and time substitution that transforms part of a curved classical phase trajectory in the extended phase space to a straight line [22,23]. The coordinate transformation induces a gauge transformation for the vector potential. Indeed, the classical momentum could be expressed as

$$p_\tau = m\frac{d\kappa}{d\tau} = \frac{dt}{d\tau}\frac{p_x}{u_2} - m\frac{\dot{u}_2}{u_2^2}\frac{dt}{d\tau}x, \tag{10}$$

and consequently, the gauge potential [36] is

$$G = -m\frac{\dot{u}_2}{u_2^2}\frac{dt}{d\tau}x = -m\frac{\dot{u}_2}{w}x. \tag{11}$$

Therefore, the phase of the wave function induced by the transformation of time and coordinate is

$$\theta_\kappa = -m\frac{\dot{u}_2}{w}\int_0^x \tilde{x}d\kappa = -\frac{m}{w}\frac{\dot{u}_2}{u_2}\int_0^x \tilde{x}d\tilde{x} = -\frac{m}{2w}\frac{\dot{u}_2}{u_2}x^2, \tag{12}$$

and we recover the phase multiplier of the wave function in Eq. (7).

Schrödinger's equation is invariant under $\hat{\mathcal{Q}}$, consequently

$$i\frac{\partial\varphi}{\partial\tau} = -\frac{1}{2m}\frac{\partial^2\varphi}{\partial\kappa^2},$$
$$\hat{\mathcal{Q}}^{-1} \Downarrow$$
$$i\frac{\partial\psi}{\partial t} = -\frac{w}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m\omega^2x^2\psi}{2w}. \tag{13}$$

The latter could be checked, for example, by direct substitution (see Appendix B for the details). The transformation

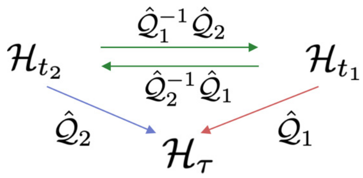

FIG. 1. Schematic diagram of the combined QAT mapping [24,25].

holds up to a common multiplier $\frac{u_2^2\sqrt{u_2}}{w}$ in the second equation and is local in time, as it is valid for $u_2 \neq 0$.

We reiterate, $\hat{\mathcal{Q}}^{-1}$ and $\hat{\mathcal{Q}}$ are operations that transform the free 1D Schrödinger equation to the equation of the 1D CK system and back.

For a given 1D system operator $\hat{\mathcal{Q}}$ is unique. This follows from the fact that $\hat{\mathcal{Q}}^*\hat{\mathcal{Q}} = \hat{I}$, i.e., $\hat{\mathcal{Q}}$ is a unitary operator. The latter follows from the fact that the transformation $\hat{\mathcal{Q}}$ preserves the norm, i.e., $||\phi_x||_{\mathcal{H}_\tau} = ||\psi_x||_{\mathcal{H}_t}$. Indeed, the Jacobian determinant of the transformation for the fixed moment in time, as it follows from the coordinate substitution, is just $1/u_2$, i.e., $d\kappa = dx/u_2$, while the square of the amplitude of the wave function calculates to $|\phi|^2 = |\psi|^2 u_2$. Consequently $\int |\phi|^2 d\kappa = \int |\psi|^2 dx$.

Next, we consider two 1D oscillators that differ by their dissipation factors, $w_{1,2}$, and oscillator frequencies, $\omega_{1,2}$. Using the QAT formalism, both oscillators could be mapped to a free particle (Hilbert space, $\mathcal{H}_\tau$). As a result, the QAT combination (as shown in the diagram Fig. 1) maps the Hilbert space of one oscillator, $\mathcal{H}_{t_1}$, to the Hilbert space of another, $\mathcal{H}_{t_2}$.

From Eq. (7), we derive [25] (see Appendix C)

$$\hat{\mathcal{E}}_{1\mapsto 2}: \begin{cases} x_1 = \frac{x_2}{b(t_2)}, \\ w_1(t_1)dt_1 = \frac{w_2(t_2)}{b^2(t_2)}dt_2, \\ \psi_2(x_2, t_2) = \psi_1(x_1, t_1)\frac{\exp[\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2]}{\sqrt{b}}, \end{cases} \tag{14}$$

where we have introduced the Ermakov mapping operator $\hat{\mathcal{E}}_{1\mapsto 2} \equiv \hat{\mathcal{Q}}_2^{-1}\hat{\mathcal{Q}}_1$, and $b(t_2)$ is a scaling parameter that satisfies the Ermakov-Pinney equation [37,38] with damping:

$$\ddot{b} + \gamma_2(t_2)\dot{b} + \omega_2^2(t_2)b = \frac{w_2^2(t_2)}{w_1^2(t_1)}\frac{\omega_1^2(t_1)}{b^3}. \tag{15}$$

In general, ω_1 and w_1 depend on t_1, expressed through t_2. The dots above b in Eqs. (14) and (15) indicate total derivatives by t_2. A direct check of this transformation, as well as the derivation of Eq. (15), is provided in Appendix C.

The Ermakov operator $\hat{\mathcal{E}}_{1\mapsto 2}$, in full analogy with the QAT operator $\hat{\mathcal{Q}}$, maps the Hilbert space $\mathcal{H}_{t_1}$ of solutions $\psi_1(x_1, t_1)$ of the 1D time-dependent Schrödinger equation of the first oscillator at time t_1, on the Hilbert space $\mathcal{H}_{t_2}$ of solutions $\psi_2(x_2, t_2)$ of the 1D time-dependent Schrödinger equation of the second oscillator at time t_2.

To apply the Ermakov operator to Eq. (4), one needs to extend it to the two-dimensional (2D) case. When the magnetic field is uniform and is directed strictly along the z axis, frequencies of both quantum oscillators in x and y are identical. Consequently, we construct a 2D Ermakov operator as follows:

$$\hat{\mathcal{E}}^{2D}_{1\mapsto 2}: \begin{cases} x_1 = \frac{x_2}{b(t_2)}, \\ y_1 = \frac{y_2}{b(t_2)}, \\ w_1(t_1)dt_1 = \frac{w_2(t_2)}{b^2(t_2)}dt_2, \\ \psi_2(x_2, y_2, t_2) = \frac{1}{b}\psi_1(x_1, y_1, t_1)\chi(x_2, y_2, t_2). \end{cases} \tag{16}$$

Here $\chi(x_2, y_2, t_2)$ is the phase factor given by

$$\chi(x_2, y_2, t_2) = \exp\left[\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2 + y_2^2)\right] \times \exp\left[-il\int\omega_2 dt_2 + il\int\omega_1 dt_1\right], \tag{17}$$

with l being the eigenvalue of the $\hat{L}_z$ operator. Here, as before, $b(t_2)$ satisfies Eq. (15).

Due to the symmetry of the transformation, the ratio, y_1/x_1, is conserved and polar angles

$$\phi_1 = \arctan(y_1/x_1) = \arctan(y_2/x_2) = \phi_2 \tag{18}$$

have a one-to-one correspondence. This fact immediately leads to the conservation of the orbital angular momentum l of the twisted state under the transformation given by the Ermakov operator, i.e., $\langle\psi_2|\hat{L}_{z_2}|\psi_2\rangle = \langle\psi_1|\hat{L}_{z_1}|\psi_1\rangle$.

For the case of twisted states, it is convenient to represent the Ermakov operator in polar coordinates where the transformation becomes essentially 1D, with the only exception of the multiplier in the wave function. Switching from x, y to ρ, ϕ and using $\rho = \sqrt{x^2 + y^2}$ and Eq. (18), we arrive at the expression for the Ermakov operator in polar coordinates:

$$\hat{\mathcal{E}}^{2D}_{1\mapsto 2}: \begin{cases} \rho_1 = \frac{\rho_2}{b(t_2)}, \\ w_1(t_1)dt_1 = \frac{w_2(t_2)}{b^2(t_2)}dt_2, \\ \psi_2(\rho_2, \phi, t_2) = \frac{1}{b}\psi_1(\rho_1, \phi, t_1)\chi(\rho_2, t_2). \end{cases} \tag{19}$$

The operators in Eqs. (16) and (19) establish a unique unitary transformation of the states of one two-dimensional quantum harmonic oscillator into another (see Appendix D), and provide a one-to-one correspondence between twisted states of different types (stationary and dynamic). Thus, we conclude that all dynamic twisted states, including free states, nonstationary states, and states that may appear in different time-dependent fields, are unitarily equivalent to stationary states. This observation opens a convenient way to analyze the properties of dynamic states based on their stationary counterparts.

III. APPLICATIONS

In this section, we consider several examples that illustrate the power of the Ermakov mapping formalism.

We start from a well-known Landau model and recover in a simple fashion nonstationary Landau states discussed in Ref. [31]. We provide formulas for the energy and the mean-square radius of the nonstationary Landau state and reveal nontrivial features of these states.

Next, we consider a representative numerical example that provides further insight into the evolution of the twisted electron in the axisymmetric magnetic field. We point out that the evolution of the twisted electron is essentially classical. To complete the analogy with the classical electron in the magnetic field we introduce a quantum emittance operator that is a quantum analog of the corresponding classical geometric emittance.

We conclude by providing a formula for the current operator for the twisted charged massive particle in the general time-dependent magnetic field that can be readily used to analyze the interaction of these particles with the electromagnetic vacuum and to study various scattering processes.

We note that the simplified model Eq. (4) and its extension to the mean-field interaction with the electromagnetic bath have found a variety of applications in nonlinear optics [39–46]. One of the widely known extensions is the Lugiato-Lefever equation model [39,47] for spontaneous spatial pattern formation [40]. We point out that it is most likely that the formalism and ideas developed in the field of lasers and nonlinear optics can be, with some modifications, applied to the description of the massive charged vortex particles.

A. Nonstationary Landau states

We chose the Landau model [17,48] as a reference system with $w_1 = 1$ and $\omega_1 = \omega_0 = \text{const}$:

$$i\partial_t\psi_1 = \left[\frac{\left[\hat{p}_x^2 + \hat{p}_y^2\right]}{2m} + \frac{m\omega_0^2(\hat{x}^2+\hat{y}^2)}{2} + \omega_0\hat{L}_z\right]\psi_1. \quad (20)$$

The solution is a stationary Landau state given by

$$\psi_1(\rho,\phi,t) = N\left(\frac{\rho}{\rho_H}\right)^{|l|}\mathcal{L}_n^{|l|}\left[\frac{2\rho^2}{\rho_H^2}\right] \times \exp\left[-\frac{\rho^2}{\rho_H^2} + il\phi - i\varepsilon_\perp t\right], \quad (21)$$

where $\mathcal{L}_n^{|l|}$ is the generalized Laguerre polynomial, n is the radial quantum number, l is the orbital angular momentum,

$$\rho_H = \sqrt{\frac{4}{|e|B_0}} = \sqrt{\frac{2}{m\omega_0}} \quad (22)$$

is the characteristic radius of the orbit, and

$$\varepsilon_\perp = \omega_0(2n + |l| + l + 1) \quad (23)$$

is the transverse part of the energy.

Now, we consider a mapping of the Landau system onto itself, and Eq. (15) reduces to

$$\ddot{b} + \omega_0^2 b = \frac{\omega_0^2}{b^3}. \quad (24)$$

With Eqs. (19) and (17) we immediately recover the nonstationary Landau state discussed in detail in Ref. [31]:

$$\psi_2(\rho,\phi,t) = \frac{N}{b}\left(\frac{\rho}{b\rho_H}\right)^{|l|}\mathcal{L}_n^{|l|}\left[\frac{2\rho^2}{b^2\rho_H^2}\right]\exp\left[-\frac{\rho^2}{b^2\rho_H^2}\right] \times \exp\left[-il\omega_0 t + il\phi + \frac{im}{2}\frac{\dot{b}}{b}\rho^2 - i(\varepsilon_\perp - \omega_0 l)\int\frac{dt}{b^2}\right]. \quad (25)$$

The Ermakov operator is a convenient tool for the analysis of different matrix elements based on the known mean values of the stationary Landau state. For instance, $\langle\rho_2^2\rangle_2$ is simply

$$\langle\rho_2^2\rangle_2 = b^2\langle\rho_1^2\rangle_1, \quad (26)$$

where

$$\langle\rho_1^2\rangle_1 = \frac{1}{m\omega_0}(2n + |l| + 1). \quad (27)$$

Indeed, with Eq. (19) and accounting for the fact that $dx_2dy_2 = b^2dx_1dy_1$, we have

$$\langle\rho_2^2\rangle_2 = \int\psi_2^*\rho_2^2\psi_2 dx_2dy_2 = b^2\int\psi_1^*\rho_1^2\psi_1 dx_1dy_1. \quad (28)$$

Under the transformation of Eq. (19), the action of the momentum operator transforms to

$$\hat{\boldsymbol{p}}_2\psi_2 = \chi(\rho_2,t_2)\left[\frac{1}{b^2}\hat{\boldsymbol{p}}_1\psi_1 + \frac{m}{w_2}\frac{\dot{b}}{b}\hat{\boldsymbol{r}}_1\psi_1\right], \quad (29)$$

and, consequently, the mean value of the momentum for a general nonstationary twisted state could be expressed in terms of the mean values of the stationary Landau states:

$$\langle\hat{\boldsymbol{p}}_2\rangle_2 = \frac{\langle\hat{\boldsymbol{p}}_1\rangle_1}{b} + \frac{m}{w_2}\dot{b}\langle\hat{\boldsymbol{r}}_1\rangle_1, \quad (30)$$

with b being a solution to Eq. (15), and $\omega_1 = \omega_0$ and $w_1 = 1$. For the nonstationary Landau state one must set $w_2 = 1$ in the formula above and use Eq. (24) for b.

A more interesting and insightful example is the transformation of the mean energy. To evaluate the time derivative, we again utilize Eq. (19) and get

$$\langle n'|i\frac{\partial}{\partial t_2}|n\rangle_2 = \frac{\dot{b}}{b}\langle n'|\hat{\boldsymbol{p}}_1\hat{\boldsymbol{r}}_1|n\rangle_1 + \frac{1}{b^2}\left[\langle n'|i\frac{\partial}{\partial t_1}|n\rangle_1 - l\omega_0\delta_{n',n}\right] + l\omega_0\delta_{n',n} + \frac{m}{2}\left[\omega_0^2b^2 + \dot{b}^2 - \frac{\omega_0^2}{b^2}\right]\langle n'|\hat{r}_1^2|n\rangle_1. \quad (31)$$

In the expression above, the bra and ket vectors are not necessarily the same. Strikingly, along with the diagonal term (the mean energy of the nonstationary Landau state $\varepsilon_2 \equiv \langle n|i\frac{\partial\psi_2}{\partial t_2}|n\rangle_2$) that reads

$$\varepsilon_2 = \frac{\omega_0(2n+|l|+1)}{2}\left[b^2 + \frac{\dot{b}^2}{\omega_0^2} + \frac{1}{b^2}\right] + l\omega_0, \quad (32)$$

we retrieve a nondiagonal term that reflects the mixing of states with different radial indices. This result is in full agreement with that derived by Lewis and Riesenfeld in Ref. [49]; however, it comes almost at no effort. Mixing terms in the Hamiltonian matrix results in nonzero probabilities for the corresponding transitions and consequent radiation processes.

The mean energy of the nonstationary Landau state itself, Eq. (32), has an interesting structure as well. Although the first term in Eq. (32) explicitly depends on time, it is constant as the factor

$$\frac{1}{2}\left[b^2 + \frac{\dot{b}^2}{\omega_0^2} + \frac{1}{b^2}\right] \geqslant 1 \quad (33)$$

is the first integral of Eq. (24). From the inequality of Eq. (33), it immediately follows that the mean energy of the nonstationary Landau state is always greater than the energy of the stationary Landau state. Along with the off-diagonal terms in the Hamiltonian matrix, and nonvanishing time-dependant quadrupole moment proportional to b^2 according to the Eq. (26), this indicates that nonstationary Landau states should be far less stable than the stationary Landau states.

B. Propagation of the twisted electron through a set of solenoids

We consider the propagation of the twisted electron through two consequent solenoids. We assume the electron is twisted, nonrelativistic, and moving along the z axis at a speed V. We assume localization of the electron in the longitudinal direction [the longitudinal density is $\rho_z \propto \delta(z - Vt)$]. The latter is always possible when the longitudinal momentum dispersion is small $\Delta p_z/\langle p_z\rangle \ll 1$ (see Ref. [32] for details on the approximation). The problem has an exact solution with the transverse motion that factors out. The corresponding setup in the transverse plane corresponds to Eq. (3). After simplifications, the final equation that describes transverse motion reads

$$i\partial_t \psi_2 = \left[\frac{\left[\hat{p}_x^2 + \hat{p}_y^2\right]}{2m} + \frac{m\omega^2(t)(\hat{x}^2 + \hat{y}^2)}{2} + \omega(t)\hat{L}_z\right]\psi_2, \tag{34}$$

with $\omega(t)$ given by Eq. (6). As before, we consider Landau model Eq. (20) as a reference system. With the help of the Ermakov operator Eq. (19) and the expression for the Landau wave function Eq. (21) one may write the solution to Eq. (34) as

$$\psi_2(\rho,\phi,t) = \frac{N}{b}\left(\frac{\rho}{b\rho_H}\right)^{|l|}\mathcal{L}_n^{|l|}\left[\frac{2\rho^2}{b^2\rho_H^2}\right] \times \exp\left[-\frac{\rho^2}{b^2\rho_H^2}\right]\exp\left[-il\int_0^t \omega(t')dt' + il\phi + \frac{im}{2}\frac{\dot{b}}{b}\rho^2 - i(\varepsilon_\perp - \omega_0 l)\int\frac{dt}{b^2}\right]. \tag{35}$$

Scaling parameter b according to the mapping prescription must satisfy Eq. (15) with $\gamma_2 = 0$, $w_1 = w_2 = 1$, and $\omega_1 = \omega_0 = \text{const}$:

$$\ddot{b} + \omega^2(t)b = \frac{\omega_0^2}{b^3}. \tag{36}$$

Equation (36) must be complimented with the initial conditions defined by the initial state of the twisted electron. We assume that at $t = 0$ the transverse mean-square radius of the electron wave packet is $\langle\rho^2\rangle(0) = 0.8^2\rho_H^2(2n + |l| + 1)$ and $\partial_t\langle\rho^2\rangle(0) = 0$. This corresponds to the focal point in a free space. Initial conditions on the mean-square radius translate into the initial conditions for the scaling parameter b as follows:

$$b(0) = 0.8, \quad \dot{b}(0) = 0. \tag{37}$$

To solve Eq. (36) numerically, we will need to use the initial conditions Eq. (37) and the magnetic-field variation shown

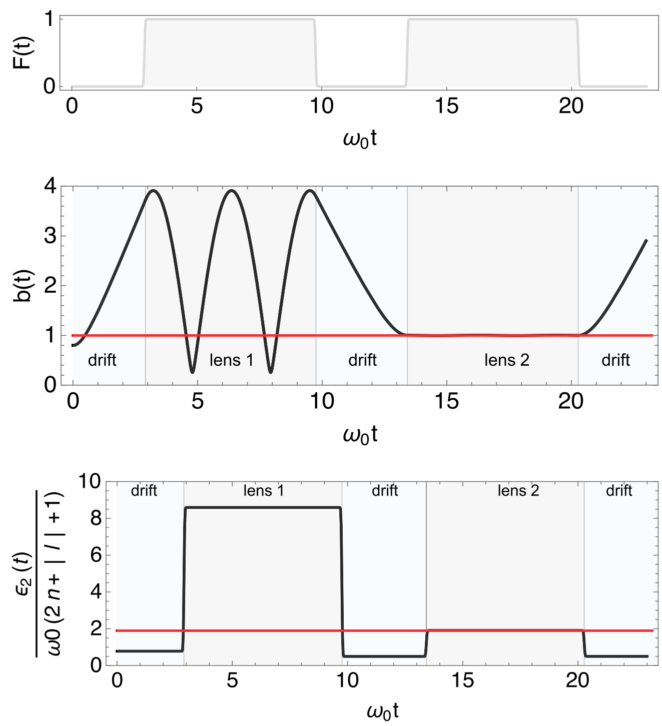

FIG. 2. Time evolution of the scaling parameter $b(t)$ (middle panel) and normalized transverse part of the energy of the twisted electron state (bottom panel) in the time varying magnetic field $F(t)$ (upper panel). The red line in the middle panel and in the bottom panel represents scaling parameter $b = 1$ and scaled energy Eq. (23) $\frac{\varepsilon_\perp}{\omega_0(2n+|l|+1)}$ for the stationary Landau state and serves as a reference. For this example we chose $n = 0$ and $l = 10$.

in Fig. 2 (upper panel). We introduce a dimensionless function $F(t)$ defined as $\omega^2(t)/\omega_0^2$. The magnetic field consists of several segments, starting from free space, a solenoid lens, another free space segment, a second solenoid lens, and a final third free space segment. With these parameters in place, we can proceed with our numerical solution.

As we can see from the middle panel of Fig. 2, the wave packet parabolically expands in free space—a well-known fact in quantum optics. However, inside the first solenoid, the scaling parameter oscillates, leading to oscillations of all wave-packet observables and an increase in state energy, as shown in the bottom panel of Fig. 2. This results in time-dependent nonvanishing multipole moments. The lowest quadrupole moment, which is proportional to the average square distance from the origin, is given by

$$Q \propto \langle\rho^2\rangle \propto b^2(t). \tag{38}$$

This in turn must result in the intensive radiation as

$$\partial_t\varepsilon \propto [\partial_t^3 Q(t)]^2 \propto [\partial_t^3 b^2(t)]^2 \neq 0. \tag{39}$$

The latter is the distinct feature of the charged particles only.

The packet divergence in free space prevents it from being directly captured into the Landau state. The packet size $\sqrt{\langle\rho^2\rangle(t)}$ oscillates in all practical cases, which means that under the assumptions of the model, a charged quantum wave packet will always radiate when entering a magnetic solenoid.

Another observation that follows from Fig. 2 is that the time average of $\langle\rho^2\rangle(t)$ over one period of oscillations $T = 2\pi/\omega_0$ is always greater than the corresponding value $\langle\rho_1^2\rangle_1$ of the Landau state given by Eq. (27), as

$$\frac{1}{2\pi}\int_0^{2\pi} b^2(\tau)d\tau > 1 \tag{40}$$

with $\tau = \omega_0 t$.

However, the geometric average

$$\max[b(t)]\min[b(t)] = 1 \tag{41}$$

always holds inside the lens.

This results in

$$\sqrt{\max[\langle\rho^2\rangle(t)]\min[\langle\rho^2\rangle(t)]} = \frac{1}{m\omega_0}(2n+|l|+1), \tag{42}$$

an invariant geometric size of the packet.

It is worth mentioning that the selection of appropriate parameters plays a crucial role in capturing free twisted electrons into the Landau state. As evident from Fig. 2, placing a second lens at the right spot enables a smooth transition from the vacuum to the Landau level.

It is important to keep in mind that the direct transition to the Landau state is never exact and can only be achieved at one specific point in the parameter space. As a result, $b(t)$ will always oscillate within the lens in any practical setup. However, the amplitude of these oscillations can be minimized, as shown in Fig. 2. Additionally, it is important to note that these oscillations will result in radiation as all the multipole moments oscillate while the electron is in the magnetic field. This radiation induced by the oscillations is different from the common spontaneous emission and is most likely classical. However, further investigation and evidence are needed to fully understand this phenomenon.

One of the immediate consequences of the QAT formalism and the Ermakov mapping is the Ermakov-Lewis invariant [19,49–51] that for the Hamiltonian Eq. (34) after scaling $\omega_0 \to 1$ and $m \to 1$ is just the sum of the 1D invariants:

$$\begin{aligned}\hat{I} &= \hat{I}_x + \hat{I}_y\\ &= \frac{1}{2b^2}[\hat{x}^2 + (b^2\hat{p}_x - b\dot{b}\hat{x})^2] + \frac{1}{2b^2}[\hat{y}^2 + (b^2\hat{p}_y - b\dot{b}\hat{y})^2].\end{aligned} \tag{43}$$

We note that the classical counterpart of the Ermakov-Lewis invariant is the Courant-Snyder invariant [52,53]:

$$\epsilon_x = \frac{1}{\beta}[x^2 + (\beta p_x + \alpha x)^2], \tag{44}$$

a quadratic invariant that can aid interpretation of the dynamics in the classical limit.

The Courant-Snyder invariant up to a π multiplier equals the area of the phase-space ellipse of the system and usually referred to as the geometric emittance. Here β is the so-called envelope function or Twiss beta function; α is the Twiss alpha function that is connected to β as

$$\alpha = -\frac{\dot{\beta}}{2}. \tag{45}$$

Geometric emittance also equals twice the action. We note that the Ermakov-Lewis invariant is the action. To bridge these two quantities we introduce a quantum emittance operator as

$$\begin{aligned}\hat{\epsilon}_{x,y} &= 2\hat{I}_{x,y},\\ \hat{\epsilon}_x &= \frac{1}{b^2}[\hat{x}^2 + (b^2\hat{p}_x - b\dot{b}\hat{x})^2],\\ \hat{\epsilon}_y &= \frac{1}{b^2}[\hat{y}^2 + (b^2\hat{p}_y - b\dot{b}\hat{y})^2].\end{aligned} \tag{46}$$

Dimensionless Twiss beta function β is connected to b as

$$\beta = b^2 \tag{47}$$

and the α parameter is just

$$\alpha = -b\dot{b}. \tag{48}$$

We point out that the quantum emittance as well as its variance are conserved and defined by the initial conditions and the quantum state only.

C. Nonstationary current operator

It is convenient to recast the current operator in terms of the current operator for the stationary Landau state, to analyze the coupling of the nonstationary states to the quantized external electromagnetic field. This is extremely helpful when it comes to the analysis of the possible decay channels of the general nonstationary states to the stationary Landau states as well as to study different scattering processes that involve twisted particles (see Ref. [11] for the details).

One may use known formulas for the corresponding integrals for the stationary Landau states (see, for instance, Ref. [54] and some recent results in Ref. [55]) when calculating the S matrix with nonstationary states.

First, we consider a general case of a nonstationary magnetic field with a dissipation guided by Eq. (4). The transverse part of the vector potential for this model has the form

$$\boldsymbol{A}_2^T = \left\{-\frac{B(t_2)y_2}{2\sqrt{w_2}}, \frac{B(t_2)x_2}{2\sqrt{w_2}}\right\}. \tag{49}$$

As an initial system for the mapping, we chose the Landau model guided by Eq. (20). According to the transformation, Eq. (19), and expression for the transformed momentum, Eq. (29), the potential for the gauge field is

$$\boldsymbol{G} = \frac{m}{w_2}\frac{\dot{b}}{b}\boldsymbol{r}_1. \tag{50}$$

The probability current is given by

$$\hat{\boldsymbol{j}}_2 = \frac{\mathrm{Re}\psi_2^*\sqrt{w_2}\hat{\boldsymbol{p}}_2\psi_2}{m} - \frac{e}{m}\boldsymbol{A}_2|\psi_2|^2. \tag{51}$$

Taking into account that

$$-e\boldsymbol{A}_2|\psi_2|^2 = -e\frac{1}{\sqrt{w_2}}\frac{B(t_2)}{bB_0}\boldsymbol{A}_1|\psi_1|^2, \tag{52}$$

where $\boldsymbol{A}_1^T = (-\frac{B_0y_1}{2}, \frac{B_0x_1}{2})$ is the transverse part of the vector potential for the Landau system, we combine Eqs. (50), (29), and (51) to achieve

$$\hat{\boldsymbol{j}}_2 = \frac{\sqrt{w_2}}{b^2}\left\{\frac{\hat{\boldsymbol{j}}_1}{b} + \left[\left(\frac{1}{b} - \frac{B(t_2)b}{B_0w_2}\right)\frac{e\boldsymbol{A}_1}{m} + \frac{b}{m}\boldsymbol{G}\right]|\psi_1|^2\right\}. \tag{53}$$

For the case of the nonstationary Landau states one should set $w_2 = 1$ and $B(t_2) = B_0$, and b must satisfy Eq. (24). We note that the probability current is not invariant under the Ermakov transformation. However, the structure of $\hat{\boldsymbol{j}}_2$ is simple. Namely, it consists of the rescaled original current, $\hat{\boldsymbol{j}}_1$, and two vector potentials that are linear functions of the coordinates. We highlight that spatial integrals over dx_2dy_2 could be evaluated using stationary states only. This opens a simple and powerful method for the calculation of scattering amplitudes.

IV. CONCLUSION

We have discussed a QAT approach to the time solution of Schrödinger's equation within the Cardirola-Kanai model. This approach takes into account the classical friction in a quantum system. The QAT formalism has been applied to analyze the evolution of the general twisted quantum state based on the corresponding stationary state. We have explicitly derived the 2D Ermakov mapping given by Eqs. (16) and (19), which provides a unique correspondence between the stationary twisted state and its evolving counterpart. We reiterate that the unitary property of the 2D Ermakov operator leads to the fact that all dynamic twisted states, including free states, nonstationary states, and states that may appear in different time-dependent fields, are unitarily equivalent to the corresponding stationary states. We illustrated the method by a trivial derivation of the nonstationary Landau state and provided an explicit formula for the energy of this state. We established a connection between a classical motion of the electron and a quantum electron wave packet in the time-dependent magnetic field and introduced the wave-packet emittance operator. Finally, we derived the current operator for the nonstationary system, which provides a convenient way to analyze the coupling to the quantized external fields of the nonstationary twisted state.

In conclusion, we emphasize that the 2D Ermakov mapping given by Eqs. (16) and (19) is valid for any wave function (not necessarily eigenstates of the Hamiltonian) and applies to the analysis of the dynamics of a wide class of wave-packet solutions, including nonpure states. Once combined with the Foldy-Wouthuysen transformation [56,57], it can become a convenient tool for a relativistic packet state description. The Ermakov mapping naturally accounts for radiation friction, which becomes essential in problems related to the generation of high-energy twisted electrons in linear accelerators.

ACKNOWLEDGMENTS

This work is funded by the Russian Science Foundation and Saint Peterburg Science Foundation, Project No. 22-22-20062 [58]. The authors are grateful to Ivan Terekhov, Igor Chestnov, and Gerard Andonian for useful discussions and suggestions.

APPENDIX A: ARNOLD TRANSFORMATION

We consider two classical systems, a linear oscillator with linear friction and a free system [23]. The Euler-Lagrange equation of the oscillator is

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = 0, \tag{A1}$$

and for the free particle

$$\overset{\circ\circ}{\kappa} = 0. \tag{A2}$$

The dot above denotes the total derivative by the time t,

$$\frac{df}{dt} \equiv \dot{f}, \tag{A3}$$

and the circle above denotes the total derivative by the time τ:

$$\frac{dg}{d\tau} \equiv \mathring{g}\,. \tag{A4}$$

A local map that establishes a connection between these two systems is given by the classical Arnold transformation discussed in Ref. [23]:

$$\mathbb{R} \times T \to \mathbb{R} \times \mathcal{T},$$
$$(x, t) \mapsto (\kappa, \tau),$$

and

$$\kappa = \frac{x}{u_2},$$
$$\tau = -\frac{u_1}{u_2} \tag{A5}$$

where u_1 and u_2 are two linearly independent solutions of Eq. (A1).

We notice the following connection between the time derivatives:

$$\dot{\tau} = -\frac{\dot{u}_1 u_2 - \dot{u}_2 u_1}{u_2^2} = \frac{w}{u_2^2} \;\Rightarrow\; \mathring{t} = \frac{u_2^2}{w} \tag{A6}$$

where $w \equiv u_1\dot{u}_2 - \dot{u}_1 u_2$ is the Wronskian built on u_1 and u_2. For the Wronskian w the following is true:

$$w = e^{-f} \;\Rightarrow\; \dot{w} = -\dot{f}w. \tag{A7}$$

We make a local change of variables explicitly as follows:

$$\begin{aligned}
0 &= \ddot{x} + \dot{f}\dot{x} + \omega^2 x = \ddot{\kappa}u_2 + 2\dot{\kappa}\dot{u}_2 + \kappa\ddot{u}_2 \\
&\quad + \dot{f}\dot{\kappa}u_2 + \dot{f}\kappa\dot{u}_2 + \omega^2\kappa u_2 \\
&= \frac{d}{dt}\left(\mathring{\kappa}\,\frac{w}{u_2^2}\right)u_2 + 2\,\mathring{\kappa}\,\frac{w}{u_2^2}\dot{u}_2 + \dot{f}\,\mathring{\kappa}\,\frac{w}{u_2^2}u_2 \\
&\quad + \kappa(\underline{\ddot{u}_2 + \dot{f}\dot{u}_2 + \omega^2 u_2}) \\
&= \overset{\circ\circ}{\kappa}\,\frac{w^2}{u_2^4}u_2 - \underline{2\,\mathring{\kappa}\,\frac{w}{u_2^3}\dot{u}_2 u_2} \\
&\quad + \mathring{\kappa}\,\frac{\dot{w}}{u_2} + \underline{2\,\mathring{\kappa}\,\frac{w}{u_2^2}\dot{u}_2} + \dot{f}\,\mathring{\kappa}\,\frac{w}{u_2} \\
&= \overset{\circ\circ}{\kappa}\,\frac{w^2}{u_2^3} - \underline{\underline{\mathring{\kappa}\,\frac{\dot{f}w}{u_2}}} + \underline{\underline{\dot{f}\,\mathring{\kappa}\,\frac{w}{u_2}}} = \overset{\circ\circ}{\kappa}\,\frac{w^2}{u_2^3}.
\end{aligned} \tag{A8}$$

Thus we observe that one system is mapped to another under the coordinate substitution Eq. (A5) up to a multiplier w^2/u_2^3. The idea behind the transformation is quite simple and goes back (according to Arnold) to Newton [22]. The locality of the transformation follows from the coordinate substitution, which is inversely proportional to u_2. Consequently, the applicability condition is $u_2 \neq 0$. This implies that if u_2 is periodic, then the transformation holds only within one period.

APPENDIX B: QUANTUM ARNOLD TRANSFORMATION

Along with the classical transformation, one may introduce its quantum analog that maps the Hilbert space of solutions of the Schrödinger's equation for the quantum oscillator $\mathcal{H}_t$ to the Hilbert space of solutions $\mathcal{H}_\tau$ of the Schrödinger's equation for the free particle [23].

It is apparent that coordinate substitution Eq. (A5) generates the change in the momentum given by

$$p_\tau = m\frac{d\kappa}{d\tau} = \frac{\mathring{t}}{u_2}p_x - m\frac{\dot{u}_2}{u_2^2}\mathring{t}\,x, \tag{B1}$$

and consequently the gauge potential [36] is

$$G = -m\frac{\dot{u}_2}{u_2^2}\mathring{t}\,x = -m\frac{\dot{u}_2}{w}x. \tag{B2}$$

On the other hand

$$G = \frac{d\theta_\kappa}{d\kappa}, \tag{B3}$$

where θ_κ is the phase of the wave function.

Therefore, the phase of the wave function induced by the transformation Eq. (A5) is

$$\begin{aligned}\theta_\kappa &= -m\frac{\dot{u}_2}{w}\int_0^x \tilde{x}d\kappa = -\frac{m}{w}\frac{\dot{u}_2}{u_2}\int_0^x \tilde{x}d\tilde{x} \\ &= -\frac{m}{2w}\frac{\dot{u}_2}{u_2}x^2.\end{aligned} \tag{B4}$$

Accounting for the change in norm induced by the coordinate substitution one may introduce the QAT as

$$\hat{\mathcal{Q}} : \begin{cases}\kappa = \frac{x}{u_2}, \\ \tau = -\frac{u_1}{u_2}, \\ \varphi(\kappa,\tau) = \psi(x,t)\sqrt{u_2}e^{-\frac{i}{2}\frac{m}{w}\frac{\dot{u}_2}{u_2}x^2}.\end{cases} \tag{B5}$$

The QAT provides the mapping of one Schrödinger equation to another as

$$i\frac{\partial\varphi}{\partial\tau} = -\frac{1}{2m}\frac{\partial^2\varphi}{\partial\kappa^2} \Rightarrow i\frac{\partial\psi}{\partial t} = -\frac{w}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m\omega^2x^2}{2w}\psi.$$

Explicitly the wave function of the free system $\varphi(\kappa,\tau)$ reads

$$\begin{aligned}\varphi(\kappa,\tau) &= \psi\big(\kappa u_2[t(\tau)], t(\tau)\big)\sqrt{u_2[t(\tau)]} \\ &\quad\times e^{-\frac{i}{2}\frac{m}{w[t(\tau)]}\dot{u}_2[t(\tau)]u_2[t(\tau)]\kappa^2}.\end{aligned}$$

Next, we write left- and right-hand side of the Schrödinger equation in the expanded form and proceed with the substitution (B5). The left-hand side reads

$$\begin{aligned}i\frac{\partial\varphi}{\partial\tau} &= \left[i\frac{\partial\psi}{\partial x}\kappa\dot{u}_2\mathring{t}\sqrt{u_2} + i\frac{\partial\psi}{\partial t}\mathring{t}\sqrt{u_2} + i\psi\frac{\dot{u}_2\mathring{t}}{2\sqrt{u_2}} + i\psi\sqrt{u_2}\left\{-\frac{i}{2}m\kappa^2\mathring{t}\left(\frac{\ddot{u}_2u_2+\dot{u}_2^2}{w} - \frac{\dot{w}}{w^2}\dot{u}_2u_2\right)\right\}\right]e^{-\frac{i}{2}\frac{m}{w}\dot{u}_2u_2\kappa^2} \\ &= \left[\underline{i\frac{\partial\psi}{\partial x}\kappa\frac{\dot{u}_2u_2^2}{w}\sqrt{u_2}} + i\frac{\partial\psi}{\partial t}\frac{u_2^2}{w}\sqrt{u_2} + \underline{\underline{i\psi\frac{\dot{u}_2u_2^2}{2w\sqrt{u_2}}}} + i\psi\sqrt{u_2}\left\{-\frac{i}{2}m\kappa^2\frac{u_2^2}{w^2}\left(-\omega^2u_2^2 + \underset{\sim}{\dot{u}_2^2}\right)\right\}\right]e^{-\frac{i}{2}\frac{m}{w}\dot{u}_2u_2\kappa^2}.\end{aligned}$$

The right-hand side reads

$$-\frac{1}{2m}\frac{\partial^2\varphi}{\partial\kappa^2} = \left[-\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2}u_2^2\sqrt{u_2} \underline{- \frac{1}{m}\frac{\partial\psi}{\partial x}u_2\sqrt{u_2}\left\{-i\frac{m}{w}\dot{u}_2u_2\kappa\right\}} - \frac{1}{2m}\psi\sqrt{u_2}\left\{\underline{\underline{-im\dot{u}_2u_2\frac{1}{w}}} - \underset{\sim}{\frac{m^2}{w^2}\dot{u}_2^2u_2^2\kappa^2}\right\}\right]e^{-\frac{i}{2}\frac{m}{w}\dot{u}_2u_2\kappa^2}.$$

After cancellation of the common terms on both sides we arrive at

$$i\frac{\partial\psi}{\partial t}\frac{u_2^2}{w}\sqrt{u_2} - \frac{m\omega^2x^2}{2w^2}\psi u_2^2\sqrt{u_2} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2}u_2^2\sqrt{u_2}.$$

Thus the free Schrödinger equation under the inverse QAT transforms to the Schrödinger equation of the quantum harmonic oscillator up to a common multiplier $\frac{u_2^2\sqrt{u_2}}{w}$.

APPENDIX C: ONE-DIMENSIONAL ERMAKOV MAPPING

We consider the following mapping diagram [24,25]:

$$\begin{array}{ccc}\mathcal{H}_1 & \xrightarrow{\hat{\mathcal{Q}}_2^{-1}\hat{\mathcal{Q}}_1} & \mathcal{H}_2 \\ \hat{\mathcal{Q}}_1 \searrow & & \swarrow \hat{\mathcal{Q}}_2 \\ & \mathcal{H}_{\text{free}} & \end{array}$$

where $\mathcal{H}_{\text{free}}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are Hilbert spaces that correspond to the free quantum particle and two 1D quantum harmonic oscillators with frequencies ω_1 and ω_2 correspondingly. It is apparent that the mapping

$$\begin{aligned}\hat{\mathcal{Q}}_2^{-1}\hat{\mathcal{Q}}_1 : \qquad &\mathcal{H}_1 \longrightarrow \mathcal{H}_2, \\ &\psi_1(x_1,t_1) \mapsto \psi_2(x_2,t_2)\end{aligned}$$

maps one oscillator onto another. The classical Euler-Lagrange equation for system 1 that has a Hilbert space of solutions $\mathcal{H}_1$ is

$$\mathring{\mathring{x}}_1 + \mathring{f}_1\mathring{x}_1 + \omega_1^2x_1 = 0. \tag{C1}$$

For the two linearly independent solutions of this equation we adopt following notation $u_1^{(1)}$ and $u_2^{(1)}$, and the Wronskian for these solutions is $w_1 = u_1^{(1)}\dot{u}_2^{(1)} - u_2^{(1)}\dot{u}_1^{(1)} = e^{-f_1}$. One may check that the following is true [by analogy to (A7)]: $\mathring{w}_1 = -\mathring{f}_1 w_1$.

The Classical Euler-Lagrange equation for system 2 with the Hilbert space of solutions $\mathcal{H}_2$ is

$$\ddot{x}_2 + \dot{f}_2\dot{x}_2 + \omega_2^2 x_2 = 0. \tag{C2}$$

We denote by $u_1^{(2)}$ and $u_2^{(2)}$ two linear independent solutions of Eq. (C2), and $w_2 = u_1^{(2)}\dot{u}_2^{(2)} - u_2^{(2)}\dot{u}_1^{(2)} = e^{-f_2}$ is the Wronskian. Similarly to the previous case $\dot{w}_2 = -\dot{f}_2 w_2$.

We introduce new function $b(t_2) = \frac{u_2^{(2)}}{u_2^{(1)}} = \frac{x_2 k}{x_1 k} = \frac{x_2}{x_1}$. From Eq. (B5) for $\hat{\mathcal{Q}}_1$ and $\hat{\mathcal{Q}}_2$ one may derive

$$\frac{w_1(t_1)}{\left(u_2^{(1)}\right)^2}dt_1 = d\tau = \frac{w_2(t_2)}{\left(u_2^{(2)}\right)^2}dt_2 \;\Rightarrow\; \dot{t}_1 = \frac{w_2(t_2)}{w_1(t_1)}\frac{1}{b^2(t_2)}.$$

Next we derive the differential equation for $b(t_2)$:

$$\begin{aligned}
0 &= \ddot{x}_2 + \dot{f}_2\dot{x}_2 + \omega_2^2 x_2 = x_1\ddot{b} + 2\dot{x}_1\dot{b} + \ddot{x}_1 b + \dot{f}_2 x_1\dot{b} + \dot{f}_2\dot{x}_1 b + \omega_2^2 x_1 b = x_1\ddot{b} + 2\mathring{x}_1\frac{w_2}{w_1}\frac{1}{b^2}\dot{b} \\
&\quad + \mathring{\mathring{x}}_1\frac{w_2^2}{w_1^2}\frac{b}{b^4} + \mathring{x}_1\frac{\dot{w}_2 b}{w_1 b^2} - 2\mathring{x}_1\frac{w_2\dot{b}b}{w_1 b^3} - \mathring{x}_1\frac{w_2\dot{w}_1 b}{w_1^2 b^2} + \dot{f}_2 x_1\dot{b} + \dot{f}_2\mathring{x}_1\frac{w_2}{w_1}\frac{1}{b} + \omega_2^2 x_1 b \\
&= x_1\ddot{b} - \mathring{f}_1\mathring{x}_1\frac{w_2^2}{w_1^2}\frac{1}{b^3} - \omega_1^2 x_1\frac{w_2^2}{w_1^2}\frac{1}{b^3} - \mathring{x}_1\dot{f}_2\frac{w_2}{w_1 b} + \mathring{x}_1\frac{w_2^2}{w_1^2 b^3}\mathring{f}_1 + \dot{f}_2 x_1\dot{b} + \dot{f}_2\mathring{x}_1\frac{w_2}{w_1}\frac{1}{b} + \omega_2^2 x_1 b \\
&= x_1\ddot{b} - \frac{w_2^2}{w_1^2}\frac{\omega_1^2}{b^3}x_1 + \dot{f}_2 x_1\dot{b} + \omega_2^2 x_1 b \;\Rightarrow\; \boxed{\ddot{b} + \dot{f}_2\dot{b} + \omega_2^2 b = \frac{w_2^2}{w_1^2}\frac{\omega_1^2}{b^3}}.
\end{aligned} \tag{C3}$$

The last equation is a well-known Ermakov-Pinney equation [37,38]. The mapping of one oscillator onto another is possible if and only if the b mapping parameter satisfies the Ermakov-Pinney equation as given above.

Combining direct and inverse QATs for the two systems one can build a mapping that maps one system onto another in a sense of Hilbert spaces. Explicitly this transformation is given by the Ermakov operator and reads

$$\hat{\mathcal{E}}_{1\mapsto 2} = \hat{\mathcal{Q}}_2^{-1}\hat{\mathcal{Q}}_1 : \begin{cases} x_2 = b x_1, \\ dt_2 = \frac{w_1(t_1)}{w_2(t_2)} b^2(t_2) dt_1, \\ \psi_2(x_2, t_2) = \psi_1(x_1, t_1)\frac{1}{\sqrt{b}} e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2}. \end{cases} \tag{C4}$$

It consists of coordinate and time substitution and a phase multiplier due to the gauge transformation.

This mapping enables the following transition:

$$i\frac{\partial\psi_2}{\partial t_2} = -\frac{w_2}{2m}\frac{\partial^2\psi_2}{\partial x_2^2} + \frac{m\omega_2^2 x_2^2}{2w_2}\psi_2 \;\Rightarrow\; i\frac{\partial\psi_1}{\partial t_1} = -\frac{w_1}{2m}\frac{\partial^2\psi_1}{\partial x_1^2} + \frac{m\omega_1^2 x_1^2}{2w_1}\psi_1.$$

Next, we show this transition explicitly. The left-hand side reads

$$\begin{aligned}
i\frac{\partial\psi_2}{\partial t_2} &= \left[-i\frac{\partial\psi_1}{\partial x_1}\frac{x_2\dot{b}}{b^2}\frac{1}{\sqrt{b}} + i\frac{\partial\psi_1}{\partial t_1}\dot{t}_1\frac{1}{\sqrt{b}} - i\psi_1\frac{\dot{b}}{2b\sqrt{b}} - \frac{m}{2}\psi_1\frac{x_2^2}{\sqrt{b}}\left\{-\frac{\dot{w}_2}{w_2^2}\frac{\dot{b}}{b} + \frac{1}{w_2}\frac{\ddot{b}b - \dot{b}^2}{b^2}\right\}\right]e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2} \\
&= \left[-i\frac{\partial\psi_1}{\partial x_1}\frac{x_2\dot{b}}{b^2}\frac{1}{\sqrt{b}} + i\frac{\partial\psi_1}{\partial t_1}\frac{w_2}{w_1 b^2}\frac{1}{\sqrt{b}} - i\psi_1\frac{\dot{b}}{2b\sqrt{b}} - \frac{m}{2}\psi_1\frac{x_2^2}{w_2 b\sqrt{b}}\left\{\frac{\dot{f}_2 w_2\dot{b}}{w_2} - \dot{f}_2\dot{b} - \omega_2^2 b + \frac{w_2^2}{w_1^2}\frac{\omega_1^2}{b^3} - \frac{\dot{b}^2}{b}\right\}\right]e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2}.
\end{aligned}$$

The right-hand side reads

$$-\frac{w_2}{2m}\frac{\partial^2\psi_2}{\partial x_2^2} = \left[-\frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial x_1^2}\frac{1}{b^2\sqrt{b}} - \frac{i}{2}\frac{\partial\psi_1}{\partial x_1}\frac{1}{b\sqrt{b}}\frac{\dot{b}}{b}2x_2 - \frac{i}{2}\psi_1\frac{\dot{b}}{b\sqrt{b}}\left\{1 + \frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}2x_2^2\right\}\right]e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2},$$

$$\frac{m\omega_2^2 x_2^2}{2w_2}\psi_2 = \frac{m\omega_2^2 x_2^2}{2w_2}\psi_1\frac{1}{\sqrt{b}}e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2}.$$

After cancellation of the common terms on both sides we arrive at

$$i\frac{\partial\psi_1}{\partial t_1}\frac{w_2}{w_1 b^2}\frac{1}{\sqrt{b}} - \frac{m}{2}\frac{w_2}{w_1^2}\frac{\omega_1^2 x_1^2}{b^2\sqrt{b}}\psi_1 = -\frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial x_1^2}\frac{1}{b^2\sqrt{b}}.$$

The equation above is exactly the Schrödinger equation for the first system up to a common multiplier $\frac{w_2}{w_1 b^2\sqrt{b}}$.

APPENDIX D: TWO-DIMENSIONAL ERMAKOV MAPPING

We consider a two-dimensional Hamiltonian of the form

$$\hat{H}_2 = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} w_2 + \frac{m\omega_2^2(\hat{x}^2 + \hat{y}^2)}{2w_2} + \omega_2 \hat{L}_z, \tag{D1}$$

along with the 2D Ermakov mapping that reads

$$\hat{\mathcal{E}}^{2D}_{1\mapsto 2} : \begin{cases} x_2 = bx_1, \\ y_2 = by_1, \\ dt_2 = \frac{w_1(t_1)}{w_2(t_2)} b^2(t_2) dt_1, \\ \psi_2(x_2, y_2, t_2) = \frac{1}{b}\psi_1(x_1, y_1, t_1) e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il_2\int \omega_2 dt_2 + il_1 \int \omega_1 dt_1}. \end{cases} \tag{D2}$$

First we prove that under the mapping (D2) the eigenvalue of the $\hat{L}_z$ operator is conserved: $l_1 = l_2$. We consider an action of the operator $\hat{L}_{z_2}$ on the wave function

$$\hat{L}_{z_2}\psi_2 = (\hat{x}_2 \hat{p}_{y_2} - \hat{y}_2 \hat{p}_{x_2})\psi_2$$

with

$$\hat{p}_{x_2}\psi_2 = -i\frac{\partial \psi_2}{\partial x_2} = \left[-i\frac{\partial \psi_1}{\partial x_1}\frac{1}{b^2} + \frac{m}{w_2}\frac{\dot{b}}{b^2} x_2 \psi_1\right] e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il_2\int \omega_2 dt_2 + il_1\int \omega_1 dt_1}$$

and

$$\hat{p}_{y_2}\psi_2 = -i\frac{\partial \psi_2}{\partial y_2} = \left[-i\frac{\partial \psi_1}{\partial y_1}\frac{1}{b^2} + \frac{m}{w_2}\frac{\dot{b}}{b^2} y_2 \psi_1\right] e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il_2\int \omega_2 dt_2 + il_1\int \omega_1 dt_1}$$

and we have

$$\begin{aligned} \hat{L}_{z_2}\psi_2 &= \left[-ix_1\frac{\partial \psi_1}{\partial y_1}\frac{1}{b} + \underbrace{\frac{m}{w_2}\dot{b}x_1 y_1 \psi_1} + iy_1 \frac{\partial \psi_1}{\partial x_1}\frac{1}{b} - \underbrace{\frac{m}{w_2}\dot{b}x_1 y_1 \psi_1}\right] e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il_2\int \omega_2 dt_2 + il_1\int \omega_1 dt_1} \\ &= \frac{1}{b}[\hat{L}_{z_1}\psi_1] e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il_2\int \omega_2 dt_2 + il_1\int \omega_1 dt_1}. \end{aligned}$$

Accounting for the fact that the Jacobian determinant of the coordinate transformation is just b^2 we have

$$l_2 = \langle \psi_2 | \hat{L}_{z_2} | \psi_2 \rangle = \int \psi_2^* \hat{L}_{z_2}\psi_2 dx_2 dy_2 = \int \frac{1}{b}\psi_1^* \frac{1}{b}\hat{L}_{z_1}\psi_1 b^2 dx_1 dy_1 = \langle \psi_1 | \hat{L}_{z_1} | \psi_1 \rangle = l_1.$$

This simplifies 2D Ermakov mapping (D2) to

$$\hat{\mathcal{E}}^{2D}_{1\mapsto 2} : \begin{cases} x_2 = bx_1, \\ y_2 = by_1, \\ dt_2 = \frac{w_1(t_1)}{w_2(t_2)} b^2(t_2) dt_1, \\ \psi_2(x_2, y_2, t_2) = \frac{1}{b}\psi_1(x_1, y_1, t_1) e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il\int \omega_2 dt_2 + il \int \omega_1 dt_1}. \end{cases} \tag{D3}$$

The 2D Ermakov mapping (D3) enables the following transition:

$$\begin{aligned} &i\frac{\partial \psi_2}{\partial t_2} = -\frac{w_2}{2m}\left(\frac{\partial^2 \psi_2}{\partial x_2^2} + \frac{\partial^2 \psi_2}{\partial y_2^2}\right) + \frac{m\omega_2^2(x_2^2 + y_2^2)}{2w_2}\psi_2 + \omega_2 \hat{L}_{z_2}\psi_2 \\ \Rightarrow\ &i\frac{\partial \psi_1}{\partial t_1} = -\frac{w_1}{2m}\left(\frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{\partial^2 \psi_1}{\partial y_1^2}\right) + \frac{m\omega_1^2(x_1^2 + y_1^2)}{2w_1}\psi_1 + \omega_1 \hat{L}_{z_1}\psi_1. \end{aligned}$$

We show this explicitly below.

The left-hand side reads

$$\begin{aligned} i\frac{\partial \psi_2}{\partial t_2} = \Bigg[&- i\frac{\partial \psi_1}{\partial x_1}\frac{\dot{b}}{b^3}x_2 - i\frac{\partial \psi_1}{\partial y_1}\frac{\dot{b}}{b^3}y_2 + i\frac{\partial \psi_1}{\partial t_1}\dot{t}_1\frac{1}{b} - i\psi_1\frac{\dot{b}}{b^2} - \frac{m}{2}\frac{\psi_1}{b}\left\{\frac{1}{w_2}\frac{\ddot{b}b - \dot{b}^2}{b^2} - \frac{\dot{b}}{b}\frac{\dot{w}_2}{w_2^2}\right\}(x_2^2 + y_2^2) \\ &+ l\omega_2\frac{\psi_1}{b} - l\omega_1\frac{\psi_1}{b}\frac{w_2}{w_1}\frac{1}{b^2}\Bigg] e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)} e^{-il\int \omega_2 dt_2 + il\int \omega_1 dt_1} \end{aligned}$$

$$= \left[\underline{-i\frac{\partial\psi_1}{\partial x_1}\frac{\dot{b}}{b^3}x_2 - i\frac{\partial\psi_1}{\partial y_1}\frac{\dot{b}}{b^3}y_2} + i\frac{\partial\psi_1}{\partial t_1}\frac{w_2}{w_1}\frac{1}{b^3}\underline{\underline{-i\psi_1\frac{\dot{b}}{b^2}}} - \frac{m}{2}\psi_1\frac{1}{w_2 b^2}\left\{-\dot{f}_2\dot{b} - \omega_2^2 b + \frac{w_2^2}{w_1^2}\frac{\omega_1^2}{b^3} - \frac{\dot{b}^2}{\underline{\underline{b}}} + \dot{f}_2\dot{b}\right\}\right.$$

$$\left.\times\left(x_2^2+y_2^2\right) + l\omega_2\frac{\psi_1}{b} - l\omega_1\frac{\psi_1}{b}\frac{w_2}{w_1}\frac{1}{b^2}\right]e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)}e^{-il\int\omega_2 dt_2+il\int\omega_1 dt_1}.$$

The right-hand side reads

$$-\frac{w_2}{2m}\frac{\partial^2\psi_2}{\partial x_2^2} + \frac{m\omega_2^2x_2^2}{2w_2}\psi_2 = \left[-\frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial x_1^2}\frac{1}{b^3} \underline{- i\frac{\partial\psi_1}{\partial x_1}\frac{1}{b^2}\frac{\dot{b}}{b}x_2} - \frac{i}{2}\psi_1\frac{\dot{b}}{b^2}\left\{\underline{\underline{1}} + \underline{\underline{i\frac{m}{w_2}\frac{\dot{b}}{b}x_2^2}}\right\} + \frac{m\omega_2^2x_2^2}{2w_2}\frac{\psi_1}{b}\right]$$

$$\times e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)}e^{-il\int\omega_2 dt_2+il\int\omega_1 dt_1},$$

$$-\frac{w_2}{2m}\frac{\partial^2\psi_2}{\partial y_2^2} + \frac{m\omega_2^2y_2^2}{2w_2}\psi_2 = \left[-\frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial y_1^2}\frac{1}{b^3} \underline{- i\frac{\partial\psi_1}{\partial y_1}\frac{1}{b^2}\frac{\dot{b}}{b}y_2} - \frac{i}{2}\psi_1\frac{\dot{b}}{b^2}\left\{\underline{\underline{1}} + \underline{\underline{i\frac{m}{w_2}\frac{\dot{b}}{b}y_2^2}}\right\} + \frac{m\omega_2^2y_2^2}{2w_2}\frac{\psi_1}{b}\right]$$

$$\times e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)}e^{-il\int\omega_2 dt_2+il\int\omega_1 dt_1}.$$

Next, we assume that the wave function is an eigenstate of the $\hat{L}_z$ operator, i.e., $\psi_1, \psi_2 \sim e^{il\phi}$. Then the action of $\hat{L}_z = -i\frac{\partial}{\partial\phi}$ for the second system is

$$\omega_2\hat{L}_{z_2}\psi_2 = \omega_2(-i(il))\psi_2 = \underline{\underline{l\omega_2\frac{\psi_1}{b}e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)}e^{-il\int\omega_2 dt_2+il\int\omega_1 dt_1}}},$$

and for the first it is

$$\omega_1\hat{L}_{z_1}\psi_1 = \omega_1[-i(il)]\psi_1 = l\omega_1\psi_1. \tag{D4}$$

After some simplifications we arrive at

$$i\frac{\partial\psi_1}{\partial t_1}\frac{w_2}{w_1}\frac{1}{b^3} - \frac{m}{2}\frac{w_2}{w_1^2}\frac{\omega_1^2}{b^3}\left(x_1^2+y_1^2\right)\psi_1 - l\omega_1\frac{w_2}{w_1}\frac{1}{b^3}\psi_1 = -\frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial x_1^2}\frac{1}{b^3} - \frac{w_2}{2m}\frac{\partial^2\psi_1}{\partial y_1^2}\frac{1}{b^3}.$$

Dividing by $\frac{w_2}{w_1}\frac{1}{b^3}$, with (D4) we get

$$i\frac{\partial\psi_1}{\partial t_1} = -\frac{w_1}{2m}\left(\frac{\partial^2\psi_1}{\partial x_1^2} + \frac{\partial^2\psi_1}{\partial y_1^2}\right) + \frac{m\omega_1^2\left(x_1^2+y_1^2\right)}{2w_1}\psi_1 + \omega_1\hat{L}_{z_1}\psi_1.$$

APPENDIX E: TRANSFORMATION OF THE PROBABILITY CURRENT

We consider a system $\mathcal{H}_2$ where a probability current is given by

$$\hat{\boldsymbol{j}}_{\boldsymbol{2}} = \frac{1}{m}\left(\sqrt{w_2}\,\mathrm{Re}\{\psi_2^*\hat{\boldsymbol{p}}_{\boldsymbol{2}}\psi_2\} - e\boldsymbol{A}_{\boldsymbol{2}}|\psi_2|^2\right). \tag{E1}$$

First, we consider an action of the operator $\hat{\boldsymbol{p}}_{\boldsymbol{2}} = -i\frac{\partial}{\partial x_2}\boldsymbol{e}_{\boldsymbol{x}} - i\frac{\partial}{\partial y_2}\boldsymbol{e}_{\boldsymbol{y}}$ on ψ_2 and then express the result in terms of system 1 and vectors from $\mathcal{H}_1$:

$$\hat{\boldsymbol{p}}_{\boldsymbol{2}}\psi_2 = \left[-i\left(\frac{\partial\psi_1}{\partial x_1}\boldsymbol{e}_{\boldsymbol{x}} + \frac{\partial\psi_1}{\partial y_1}\boldsymbol{e}_{\boldsymbol{y}}\right)\frac{1}{b^2} + \frac{m}{w_2}\frac{\dot{b}}{b}\psi_1(x_1\boldsymbol{e}_{\boldsymbol{x}} + y_1\boldsymbol{e}_{\boldsymbol{y}})\right]e^{\frac{i}{2}\frac{m}{w_2}\frac{\dot{b}}{b}(x_2^2+y_2^2)}e^{-il\int\omega_2 dt_2+il\int\omega_1 dt_1}.$$

The first term from (E1) reads

$$\frac{\sqrt{w_2}}{m}\mathrm{Re}\{\psi_2^*\hat{\boldsymbol{p}}_{\boldsymbol{2}}\psi_2\} = \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{b^3}\frac{1}{m}\mathrm{Re}\{\sqrt{w_1}\psi_1^*\hat{\boldsymbol{p}}_{\boldsymbol{1}}\psi_1\} + \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{mb}|\psi_1|^2\boldsymbol{G}, \tag{E2}$$

where $\boldsymbol{G} = m\frac{\sqrt{w_1}}{w_2}\frac{\dot{b}}{b}\boldsymbol{r}_{\boldsymbol{1}}$ and $\boldsymbol{r}_{\boldsymbol{1}}^{\boldsymbol{T}} = (x_1,\ y_1,\ 0)$.

Next, we consider the transformation of the second term (51) under the mapping (D3). Under the symmetric gauge we have

$$\boldsymbol{A}_2^T = \left(-\frac{B_2 y_2}{2\sqrt{w_2}}, \frac{B_2 x_2}{2\sqrt{w_2}}, 0\right) = \frac{\sqrt{w_1}}{\sqrt{w_2}}\frac{B_2}{B_1}\left(-\frac{B_1 b y_1}{2\sqrt{w_1}}, \frac{B_1 b x_1}{2\sqrt{w_1}}, 0\right) = \frac{\sqrt{w_1}}{\sqrt{w_2}}\frac{B_2}{B_1} b\boldsymbol{A}_1^T,$$

and consequently

$$-\frac{e}{m}\boldsymbol{A}_2|\psi_2|^2 = -\frac{e}{m}\frac{\sqrt{w_1}}{\sqrt{w_2}}\frac{B_2}{B_1}\frac{1}{b}\boldsymbol{A}_1|\psi_1|^2. \tag{E3}$$

We combine (E2) and (E3) and arrive at

$$\begin{aligned}\hat{\boldsymbol{j}}_2 &= \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{b^3}\frac{1}{m}\mathrm{Re}\{\sqrt{w_1}\psi_1^*\hat{\boldsymbol{p}}_1\psi_1\} + \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{mb}|\psi_1|^2\boldsymbol{G} - \frac{e}{m}\frac{\sqrt{w_1}}{\sqrt{w_2}}\frac{B_2}{B_1}\frac{1}{b}\boldsymbol{A}_1|\psi_1|^2\\ &= \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{b^3}\hat{\boldsymbol{j}}_1 + \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{b^3}\frac{e}{m}\boldsymbol{A}_1|\psi_1|^2 + \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{mb}|\psi_1|^2\boldsymbol{G} - \frac{e}{m}\frac{\sqrt{w_1}}{\sqrt{w_2}}\frac{B_2}{B_1}\frac{1}{b}\boldsymbol{A}_1|\psi_1|^2\\ &= \frac{\sqrt{w_2}}{\sqrt{w_1}}\frac{1}{b^2}\left\{\frac{\hat{\boldsymbol{j}}_1}{b} + \left[\left(\frac{1}{b} - \frac{B_2}{B_1}\frac{w_1}{w_2}b\right)\frac{e\boldsymbol{A}_1}{m} + \frac{b}{m}\boldsymbol{G}\right]|\psi_1|^2\right\}.\end{aligned} \tag{E4}$$

[1] L. A. Vainshtein, *Open Resonators and Open Waveguides* (Golem, Boulder, Colorado, 1969).
[2] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes, Phys. Rev. A **45**, 8185 (1992).
[3] L. Allen, M. Padgett, and M. Babiker, IV the orbital angular momentum of light, *Progress in Optics* (Elsevier, Amsterdam, 1999), Vol. 39, pp. 291–372.
[4] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. A **392**, 45 (1984).
[5] M. V. Berry, Optical vortices evolving from helicoidal integer and fractional phase steps, J. Opt. A **6**, 259 (2004).
[6] *The Angular Momentum of Light*, edited by D. L. Andrews (Cambridge University, New York, 2012).
[7] S. Franke-Arnold, L. Allen, and M. Padgett, Advances in optical angular momentum, Laser Photonics Rev. **2**, 299 (2008).
[8] J. P. Torres and L. Torner, *Twisted Photons: Applications of Light with Orbital Angular Momentum* (Wiley, New York, 2011).
[9] B. Knyazev and V. Serbo, Beams of photons with nonzero projections of orbital angular momenta: New results, Phys. Usp. **61**, 449 (2018).
[10] K. Bliokh, I. Ivanov, G. Guzzinati, L. Clark, R. Van Boxem, A. Béché, R. Juchtmans, M. Alonso, P. Schattschneider, F. Nori, and J. Verbeeck, Theory and applications of free-electron vortex states, Phys. Rep. **690**, 1 (2017).
[11] I. P. Ivanov, Promises and challenges of high-energy vortex states collisions, Prog. Part. Nucl. Phys. **127**, 103987 (2022).
[12] V. V. Dodonov and O. V. Man'ko, Universal invariants of quantum-mechanical and optical systems, J. Opt. Soc. Am. A **17**, 2403 (2000).
[13] R. Jagannathan, R. Simon, E. Sudarshan, and N. Mukunda, Quantum theory of magnetic electron lenses based on the Dirac equation, Phys. Lett. A **134**, 457 (1989).
[14] R. Jagannathan, Quantum theory of electron lenses based on the Dirac equation, Phys. Rev. A **42**, 6674 (1990).
[15] S. A. Khan and R. Jagannathan, Quantum mechanics of charged-particle beam transport through magnetic lenses, Phys. Rev. E **51**, 2510 (1995).
[16] K. Y. Bliokh, Y. P. Bliokh, S. Savel'ev, and F. Nori, Semiclassical Dynamics of Electron Wave Packet States with Phase Vortices, Phys. Rev. Lett. **99**, 190404 (2007).
[17] K. Y. Bliokh, P. Schattschneider, J. Verbeeck, and F. Nori, Electron Vortex Beams in a Magnetic Field: A New Twist on Landau Levels and Aharonov-Bohm States, Phys. Rev. X **2**, 041011 (2012).
[18] C. Greenshields, R. L. Stamps, and S. Franke-Arnold, Vacuum Faraday effect for electrons, New J. Phys. **14**, 103040 (2012).
[19] A. Melkani and S. J. van Enk, Electron vortex beams in nonuniform magnetic fields, Phys. Rev. Res. **3**, 033060 (2021).
[20] D. Karlovets, Relativistic vortex electrons: Paraxial versus nonparaxial regimes, Phys. Rev. A **98**, 012137 (2018).
[21] D. Karlovets, Vortex particles in axially symmetric fields and applications of the quantum Busch theorem, New J. Phys. **23**, 033048 (2021).
[22] V. I. Arnold, *Ordinary Differential Equations* (Springer-Verlag, Berlin, 1992).
[23] V. Aldaya, F. Cossío, J. Guerrero, and F. F. López-Ruiz, The quantum Arnold transformation, J. Phys. A **44**, 065302 (2011).
[24] F. F. López-Ruiz and J. Guerrero, Generalizations of the Ermakov system through the quantum Arnold transformation, J. Phys.: Conf. Ser. **538**, 012015 (2014).
[25] J. Guerrero and F. F. López-Ruiz, The quantum Arnold transformation and the Ermakov-Pinney equation, Phys. Scr. **87**, 038105 (2013).
[26] V. V. Dodonov and V. I. Man'ko, Coherent states and the resonance of a quantum damped oscillator, Phys. Rev. A **20**, 550 (1979).
[27] P. Caldirola, Porze non conservative nella meccanica quantistica, Nuovo cim. **18**, 393 (1941).

[28] E. Kanai, On the quantization of the dissipative systems, Prog. Theor. Phys. **3**, 440 (1948).
[29] A. J. Silenko, Exact quantum-mechanical equations for particle beams, Mod. Phys. Lett. A **37**, 2250097 (2022).
[30] M. Newstein and B. Rudman, LaGuerre-Gaussian periodically focusing beams in a quadratic index medium, IEEE J. Quantum Electron. **23**, 481 (1987).
[31] L. Zou, P. Zhang, and A. J. Silenko, General quantum-mechanical solution for twisted electrons in a uniform magnetic field, Phys. Rev. A **103**, L010201 (2021).
[32] S. S. Baturin, D. V. Grosman, G. K. Sizykh, and D. V. Karlovets, Evolution of an accelerated charged vortex particle in an inhomogeneous magnetic lens, Phys. Rev. A **106**, 042211 (2022).
[33] G. Lindblad, On the generators of quantum dynamical semi-groups, Commun. Math. Phys. **48**, 119 (1976).
[34] D. Manzano, A short introduction to the Lindblad master equation, AIP Adv. **10**, 025106 (2020).
[35] P. Caldirola and L. Lugiato, Connection between the Schrödinger equation for dissipative systems and the master equation, Physica A **116**, 248 (1982).
[36] A. Davidov, *Quantum Mechanics* (Pergamon, New York, 1965).
[37] V. Ermakov, Transformation of differential equations, Univ. Izv. Kiev **20**, 1 (1880).
[38] E. Pinney, The nonlinear differential equation $y + p(x)y + cy^{-3} = 0$, Proc. Amer. Math. Soc. **1**, 681 (1950).
[39] L. A. Lugiato and R. Lefever, Spatial Dissipative Structures in Passive Optical Systems, Phys. Rev. Lett. **58**, 2209 (1987).
[40] M. Brambilla, F. Battipede, L. A. Lugiato, V. Penna, F. Prati, C. Tamm, and C. O. Weiss, Transverse laser patterns. I. phase singularity crystals, Phys. Rev. A **43**, 5090 (1991).
[41] M. Brambilla, L. A. Lugiato, V. Penna, F. Prati, C. Tamm, and C. O. Weiss, Transverse laser patterns. II. Variational principle for pattern selection, spatial multistability, and laser hydrodynamics, Phys. Rev. A **43**, 5114 (1991).
[42] P. Colet, M. San Miguel, M. Brambilla, and L. A. Lugiato, Fluctuations in transverse laser patterns, Phys. Rev. A **43**, 3862 (1991).
[43] A. Scroggie, W. Firth, G. McDonald, M. Tlidi, R. Lefever, and L. Lugiato, Pattern formation in a passive kerr cavity, Chaos Solit. Fractals **4**, 1323 (1994).
[44] P. Ru, L. Narducci, J. Tredicce, D. Bandy, and L. Lugiato, The Gauss-LaGuerre modes of a ring resonator, Opt. Commun. **63**, 310 (1987).
[45] L. Lugiato, Spatio-temporal structures. Part I, Phys. Rep. **219**, 293 (1992).
[46] P. Galatola, L. Lugiato, M. Porreca, and P. Tombesi, Optical switching by variation of the squeezing phase, Opt. Commun. **81**, 175 (1991).
[47] L. Lugiato, F. Prati, and M. Brambilla, *The Lugiato-Lefever Model* (Cambridge University, New York, 2015), pp. 363–377.
[48] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (Butterworth-Heinemann, Burlington, MA, 1981).
[49] H. R. Lewis and W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. **10**, 1458 (1969).
[50] H. R. Lewis, Classical and Quantum Systems with Time-Dependent Harmonic-Oscillator-Type Hamiltonians, Phys. Rev. Lett. **18**, 510 (1967).
[51] H. R. Lewis, Classical and Quantum Systems with Time-Dependent Harmonic-Oscillator-Type Hamiltonians, Phys. Rev. Lett. **18**, 636 (1967).
[52] E. Courant and H. Snyder, Theory of the alternating-gradient synchrotron, Ann. Phys. (NY) **281**, 360 (2000).
[53] S. Y. Lee, *Accelerator Physics* (World Scientific, Singapore, 2012).
[54] A. Sokolov and I. Ternov, *Relativistic Electron* (Nauka, Moscow, 1974).
[55] V. Epp and U. Guselnikova, The angular momentum of electron radiation in a uniform magnetic field, Phys. Lett. A **469**, 128764 (2023).
[56] L. L. Foldy and S. A. Wouthuysen, On the Dirac theory of spin 1/2 particles and its non-relativistic limit, Phys. Rev. **78**, 29 (1950).
[57] A. J. Silenko, Foldy-Wouthuysen transformation for relativistic particles in external fields, J. Math. Phys. **44**, 2952 (2003).
[58] https://rscf.ru/project/22-22-20062/.

Correction: A radical sign was missing from the left-hand side of Eq. (42) and has been inserted.

Second Correction: Errors in an inline equation in the text following Eq. (36), in Eq. (39), in the sentence preceding Eq. (43), and in Eq. (47) have been fixed.