Disfavoring the Schrödinger-Newton equation in explaining the emergence of classicality

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The main goal of this paper is to provide some insight into how promising the Schrödinger-Newton equation would be to explain the emergence of classicality. Based on the similarity of the Newton and Coulomb potentials, we add an electric self-interacting term to the Schrödinger-Newton equation for the hydrogen atom. Our results rule out the possibility that single electrons self-interact through their electromagnetic field. Next, we use the hydrogen atom to get insight into the intrinsic difficulty of testing the Schrödinger-Newton equation itself and conclude that the Planck scale must be approached before sound constraints are established. Although our results cannot be used to rule out the Schrödinger-Newton equation at all, they might be seen as disfavoring it if we base our reasoning on the resemblance between the gravitational and electromagnetic interactions at low energies.

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I. INTRODUCTION

The emergence of the classical world from the quantum paradigm remains elusive [1]. Among the different proposals to solve this issue, let us focus on the "quantum state reduction" mechanism presented by Penrose [2,3]. In such a mechanism, gravitational self-interaction would be responsible for vanishing coherent quantum superpositions. One may think of it as follows. Suppose a Schrödinger-cat state where a particle is set in a coherent superposition at two different locations:

$$|\psi\rangle_0 = \frac{1}{\sqrt{2}}(|^\bullet\rangle + |_\bullet\rangle).$$

Now, let us assume the *unorthodox* view that the two branches gravitationally interact with each other driving the system to some sort of self-entanglement:

$$|\psi\rangle_T = \frac{1}{\sqrt{2}}(|^\bullet\rangle|_\circ\rangle + |_\bullet\rangle|^\circ\rangle),$$

with the transition time T being scaled by the gravitational interaction. Then, by tracing out over the "immaterial" $\{|_\circ\rangle, |^\circ\rangle\}$, one would end up with the mixed state given by the density matrix

$$\hat{\rho}_T = \tfrac{1}{2}|^\bullet\rangle\langle^\bullet| + \tfrac{1}{2}|_\bullet\rangle\langle_\bullet|.$$

In Penrose's view, (if confirmed) this would be a factual self-reduction responsible for the emergence of classicality. By the same token, Penrose propounds that the Schrödinger equation for a particle is replaced by the so-called Schrödinger-Newton equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + [V(\vec{r},t) + U_g(\vec{r},t)]\Psi(\vec{r},t), \tag{1}$$

where m is the particle mass, $V(\vec{r},t)$ and

$$U_g(\vec{r},t) \equiv -Gm^2\int d^3\vec{r}\,'\frac{|\Psi(\vec{r}\,',t)|^2}{|\vec{r}-\vec{r}\,'|} \tag{2}$$

are the external and self-interacting potentials, respectively, and G is the gravitational constant. In this sense, $m|\Psi(\vec{r},t)|^2$ plays the role of *real* matter density, enabling different parts of the wave function to interact with each other. Although Eq. (1) is nonlinear, its results would be indistinguishable from the ones given by the Schrödinger equation for typical laboratory experiments because of the weakness of the gravitational interaction. Nevertheless, continuous efforts have been made to solve it and find its properties (see, e.g., Refs. [4–11] and references therein).

On the other hand, it is well known that, at low energies, the electric and gravitational potentials only differ from each other with respect to the coupling constant, and that they have been successfully used in the Schrödinger equation to describe phenomena from the hydrogen atom [12] to the interference of free-falling neutrons [13]. Based on this, it would be "natural" to expect that if particles self-interact through the gravitational interaction, they should also self-interact through the electromagnetic one [14]. By the same token, it seems interesting to consider the Schrödinger-Newton equation for the hydrogen atom with the inclusion of the self-interacting electric potential

$$U_e(\vec{r},t) \equiv k_e e^2\int d^3\vec{r}\,'\frac{|\Psi(\vec{r}\,',t)|^2}{|\vec{r}-\vec{r}\,'|}, \tag{3}$$

where k_e is the Coulomb constant and e is the fundamental charge.

This paper is organized as follows. In Sec. II, we introduce the Schrödinger-Newton equation for the hydrogen

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atom modified by the addition of the electric self-interacting potential. In Sec. III, we solve it numerically and compare the outputs with experimental data. In Sec. IV, we use the hydrogen atom to drive some conclusions about the intrinsic difficulty of testing the Schrödinger-Newton equation itself. In Sec. V, we discuss the results and present our conclusions.

II. THE HYDROGEN ATOM UNDER SELF-INTERACTION

The Schrödinger equation for the hydrogen atom is

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=-\frac{\hbar^2}{2\mu}\nabla^2\Psi(\vec{r},t)+V_T(r)\Psi(\vec{r},t), \tag{4}$$

where $\mu\equiv m_e m_p/(m_e+m_p)$, m_e and m_p are the electron and proton masses, respectively, and

$$V_T(r)\equiv V_g(r)+V_e(r) \tag{5}$$

with $V_g(r)=-Gm_e m_p/r$ and $V_e(r)=-k_e e^2/r$. (We recall that the corresponding spectroscopy is completely determined by the spherically symmetric eigenfunctions.)

Our main goal is to compare the results given by Eq. (4) with the ones provided by the Schrödinger-Newton equation modified by the addition of the electric self-interacting potential:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=-\frac{\hbar^2}{2\mu}\nabla^2\Psi(\vec{r},t)+[V_T(r)+U_T(\vec{r},t)]\Psi(\vec{r},t), \tag{6}$$

where

$$U_T(\vec{r},t)\equiv U_g(\vec{r},t)+U_e(\vec{r},t). \tag{7}$$

Clearly, the electric terms supersede the gravitational ones by many orders of magnitude, and thus the latter could be omitted at this point. Still, we opt to maintain them for the sake of our further discussion. We shall note that Eq. (6) with $V_g(r)=U_g(\vec{r},t)=0$ was investigated in the past in Ref. [15] but no bound states were found for the hydrogen atom. This is at odds with our results and with the ones obtained in Refs. [16,17], which considered the Dirac equation with electric self-interaction to resolve the hydrogen atom. Interestingly enough, by comparing the results of Refs. [16,17] with ours, we see that the self-interacting term plays a more significant role in the nonrelativistic realm than in the relativistic one. In particular, the electron would be five times less bound in the former case than in the latter case. (This may explain why the authors of Ref. [15] were unable to find bound states for the hydrogen atom.) Of the most importance, however, is the fact that Refs. [15–17] consider electric self-interaction in the hydrogen atom under a quite different mindset with respect to ours; quoting Biguaa and Kassandrov (Ref. [17], p. 965),

> "From the point of view of quantum ideas, the electron in the hydrogen atom represents a spatially distributed system, according to the probability density $\Psi^\dagger\Psi$, whose elements interact both with the field of the nucleus and between each other."

We refer to this quote as statement A. In order to fit their results with experimental data, they appeal to a sort of finite "classical renormalization" procedure. As expected, this is not enough to recover consistency with experiments. We shall emphasize that standard quantum mechanics does not assert statement A. Instead, it states that $|\Psi(\vec{r},t)|^2$ is the probability density of finding the electron *when* its position is measured by some apparatus. Also, allegations that radioactive corrections of quantum field theory would endorse self-interaction of single-particle states are unfounded.

Here we compare the results delivered by Eq. (6) with the ones provided by quantum mechanics taken at its face value. Let us look for stationary spherically symmetric solutions

$$\Psi(\vec{r},t)=\psi(r)e^{-iEt/\hbar}$$

for Eq. (6). In this case, Eq. (6) can be cast as

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2}{dr^2}[r\psi(r)]+[V_T(r)+W_T(r)]\psi(r)=E\psi(r), \tag{8}$$

where

$$W_T(r)=4\pi\alpha\left(\frac{1}{r}\int_0^r dr' r'^2|\psi(r')|^2+\int_r^\infty dr' r'|\psi(r')|^2\right) \tag{9}$$

and $\alpha\equiv -Gm_e m_p+k_e e^2$. In order to write it, we have used (see, e.g., Ref. [18])

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

where $r_<\equiv\min(r,r')$, $r_>\equiv\max(r,r')$, and $Y_{lm}(\theta,\phi)$ are the spherical harmonics. Equation (8) can be simplified by defining $\phi(r)\equiv r\psi(r)$, in which case it becomes

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}\phi(r)+[V_T(r)+W_T(r)]\phi(r)=E\phi(r), \tag{10}$$

where

$$W_T(r)=4\pi\alpha\left(\frac{1}{r}\int_0^r dr'|\phi(r')|^2+\int_r^\infty dr'\frac{|\phi(r')|^2}{r'}\right) \tag{11}$$

and we recall that the probability of finding the electron in a spherical shell with inner and outer radius r_i and r_o, respectively, is

$$P(r_i,r_o)=\int_{r_i}^{r_o}dr F(r)\quad\text{with } F(r)\equiv 4\pi|\phi(r)|^2. \tag{12}$$

Although the quest for stationary spherically symmetric solutions simplifies the problem, Eq. (10) is still nontrivial. In order to deal with it, we shall solve it numerically by adapting the code of Ref. [5].

III. NUMERICAL PROCEDURE AND RESULTS

The code employed to numerically solve Eq. (10) is a user-friendly adaptation of the one developed in Ref. [5]; in this adaptation the external potential $V_T(r)$ is included and $U_g(\vec{r},t)$ is replaced by $U_T(\vec{r},t)$. (Furthermore, the convergence procedure is somewhat improved.) Briefly speaking, we depart from a test function, use it to calculate $W_T(r)$ in Eq. (11), and replace the result in Eq. (10) to numerically evaluate the eigenfunctions $\phi(r)$ (and corresponding eigenvalues E). Then, depending on the eigenvalue E we are interested in, we select the associated eigenfunction $\phi(r)$, using it in the next

TABLE I. Comparison of the output of the three smallest eigenenergies given by Eq. (4) with the experimental data (restricted to five significant figures).

Level	Numerical (eV)	Experimental (eV)
1	−13.593	−13.598
2	−3.3993	−3.3996
3	−1.5109	−1.5109

round as a new test function to recalculate $W_T(r)$ and so on. The process is repeated until the eigenenergy E of interest converges.

As a consistency check, we verify at the end that the obtained eigenfunctions and corresponding eigenenergies do satisfy Eq. (10). We have also confirmed that the code reproduces in good approximation the experimental results for the energy levels of the hydrogen atom in the absence of self-interaction [19]; see Table I. (By "level n," $n = 1, 2, 3, \ldots$, we implicitly mean level $n\,s^{1/2}$, since the solutions are spherically symmetric.)

Now, we are in a position to determine the impact of considering self-interaction on the spectroscopy of the hydrogen atom as ruled by the modified Schrödinger-Newton equation (10). We shall stress here that any deviations from the experimental data must be imputed to the electric self-interaction due to the huge dominance of $k_e e^2$ over $Gm_e m_p$. In Table II, we exhibit the three smallest eigenenergies for the hydrogen atom. It becomes clear, in particular, that the ionization energy is at odds with the experimental result due to the electric self-interaction. Indeed, the whole hydrogen atom spectroscopy is ruined, as can be seen in Table III.

Let us note that the energy levels assuming electric self-interaction shown in Table II do not follow the usual Bohr relationship $E_n = E_1/n^2$. This is expected since the self-interacting term should screen the proton charge differently depending on the quantum state.

In Figs. 1–3, we compare the radial probability densities $F(r)$ [see Eq. (12)] associated with the eigenfunctions of the three smallest eigenenergies given by the Schrödinger equation (4) and modified Schrödinger equation (10). We see that the electric self-interaction pushes the probability density farther from the nucleus, in agreement with the larger eigenenergies found for the corresponding eigenfunctions displayed in Table II.

Although the results above experimentally discard the existence of electromagnetic self-interaction and may be understood as disfavoring the Schrödinger-Newton equation (based on the similarity of the Newton and Coulomb potentials), they cannot discard gravitational self-interaction at all. In order to test it itself, we must analyze situations where the gravitational self-interaction turns out to be dominant. In the next section, we use the hydrogen atom to gain insight into how challenging this can be.

TABLE II. Comparison of the three smallest eigenenergies, numerically obtained through Eq. (10), with the experimental data (restricted to five significant figures).

Level	Numerical (eV)	Experimental (eV)
1	−1.2561	−13.598
2	−0.21601	−3.3996
3	−0.074618	−1.5109

TABLE III. Comparison of the energy gap for three transitions, numerically obtained through Eq. (10), with the experimental data (restricted to five significant figures). By "transition $n_i \to n_f$," we mean the electronic transition from the n_ith to the n_fth energy eigenstate.

Transition	Numerical (eV)	Experimental (eV)
$3 \to 2$	−0.14139	−1.8887
$2 \to 1$	−1.0400	−10.199
$3 \to 1$	−1.1814	−12.087

IV. THE HYDROGEN ATOM RESTRICTED TO GRAVITATIONAL SELF-INTERACTION

Let us start by replacing Eq. (6) by

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2\mu}\nabla^2\Psi(\vec{r},t) + [V_T(r) + U_g(\vec{r},t)]\Psi(\vec{r},t), \tag{13}$$

where the electric self-interaction was promptly discarded in the face of the previous results. It is fair to expect that the presence of $U_g(\vec{r},t)$ should effectively increase the nucleus mass by no more than a factor of the electron mass. This comes from Sec. III, where $U_e(\vec{r},t)$ was shown to effectively screen part of the proton charge. Note that by driving $Z \to 0.3Z$, the ionization energy

$$E_1 = -\frac{Z^2\mu}{2\hbar^2 n^2}(k_e e^2)^2 \quad (Z = 1,\ n = 1) \tag{14}$$

would change about as follows -13.598 eV $\to -1.2561$ eV (see Table II). By the same token, the ionization energy, as

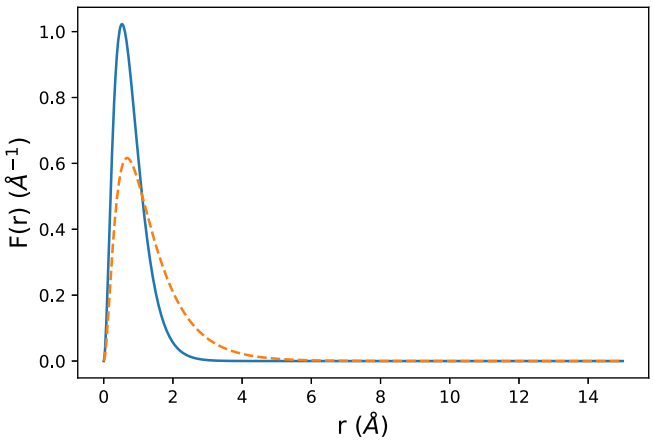

FIG. 1. Plot of the radial probability density $F(r)$ for the ground state obtained from Eq. (4) (solid line) and Eq. (10) (dashed line).

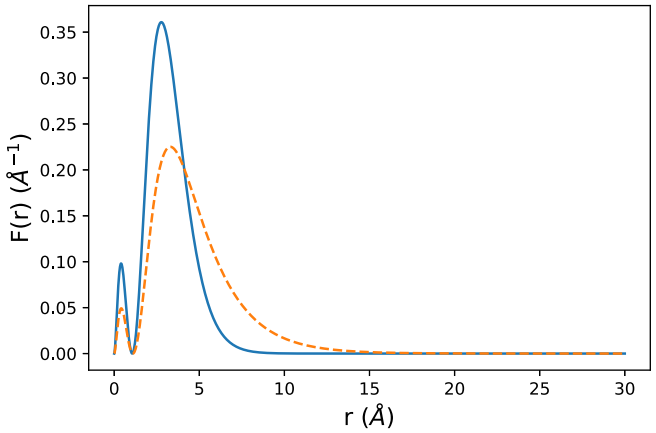

FIG. 2. Plot of the radial probability density $F(r)$ for the first excited state obtained from Eq. (4) (solid line) and Eq. (10) (dashed line).

ruled by Eq. (13), can be estimated to be of the order

$$E_1' \sim -\frac{Z^2\mu}{2\hbar^2 n^2}(k_e e^2 + G m_p m_e)^2 \quad (Z=1,\ n=1), \tag{15}$$

where any enhancement of the proton mass due to the gravitational self-interaction of the electron will be neglected for now since it would only lead to a tiny correction to the $Gm_e m_p$ term. Hence gravity would affect the ionization energy by

$$\frac{E'_1 - E_1}{E_1} \sim 2\left(\frac{G m_p m_e}{k_e e^2}\right), \tag{16}$$

corresponding to 1 part in 10^{39} parts. To put this in context, we recall that the value recommended by the Committee on Data of the International Science Council (CODATA) for the Newtonian constant has only six significant digits: $G = 6.674\,30(15) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ [19]. We note that in order to drive the right-hand side of Eq. (16) to approach the unity, we should have had $m_e m_p \sim (M_P/10)^2$,

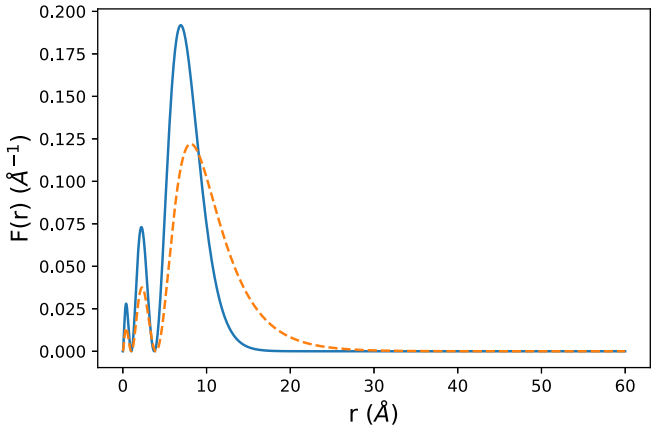

FIG. 3. Plot of the radial probability density $F(r)$ for the second excited state obtained from Eq. (4) (solid line) and Eq. (10) (dashed line).

where

$$M_P \equiv \sqrt{\hbar c/G} \approx 2 \times 10^{-5}\ \mathrm{g}$$

is the Planck mass.

Eventually, the paramount obstacle faced by every attempt to test gravitational self-interaction is that it must approach the quantum gravity regime by combining coherently a large number of particles to act as a single quantum particle with mass $m \lesssim M_P$. It is unclear, however, in what sense the Schrödinger equation and other quantum wave equations would be approximately valid close to the Planck scale. Relativistic,

$$\Box\phi + \lambda\!\!\!^{-}{}^{-2}\phi = 0, \quad i\,\partial\!\!\!/\,\psi - \lambda\!\!\!^{-}{}^{-1}\psi = 0, \tag{17}$$

and nonrelativistic,

$$i\partial_t\phi + (c\lambda\!\!\!^{-}/2)\nabla^2\phi = 0, \tag{18}$$

wave equations are solely characterized by the (reduced) Compton wavelength $\lambda\!\!\!^{-} = \hbar/(mc)$. The closer m approaches to M_P, the closer $\lambda\!\!\!^{-}$ approaches to the Planck length:

$$\lambda\!\!\!^{-} \sim L_P \equiv \sqrt{\hbar G/c^3}\,.$$

Hence, in order to write Eqs. (17) and (18) for Planck-mass particles, one must presuppose that classical space-times are reliable at such distance scales, which does not seem to be the case.

In relativistic space-times, spatial distances can be measured using solely bona fide classical clocks. By *bona fide classical clocks* we mean "pointlike" apparatuses that ascribe the same real number (time interval) to any given arbitrarily close, causally connected pair of events they visit regardless of the state of motion and past history (as defined by relativity). Thus, in order to measure $\lambda\!\!\!^{-} \sim L_P$, there must exist bona fide clocks with size $L_{\mathrm{clock}} < L_P$ and accuracy

$$\delta t \lesssim T_P \equiv \sqrt{\hbar G/c^5}\,,$$

in which case one would end with a ruler with accuracy $\delta l \equiv c\delta t < \lambda\!\!\!^{-}$. On the other hand, quantum mechanics teaches us that clocks with such an accuracy would involve Planck-scale energies: $E_{\mathrm{clock}} \gtrsim \hbar/\delta t$ [20], leading to a clock mass

$$M_{\mathrm{clock}} \gtrsim M_P.$$

It happens, however, that according to Thorne's hoop conjecture, such clocks should collapse into black holes since they would be small enough to fit in spheres with the corresponding Schwarzschild radii R^S_{clock} (see, e.g., Ref. [21]):

$$L_{\mathrm{clock}} < L_P < 2GM_P/c^2 \lesssim 2GM_{\mathrm{clock}}/c^2 \equiv R^S_{\mathrm{clock}}.$$

Not only are black holes not supposed to function as clocks, but also such apparatuses would disturb the background space-time raising doubts as to the validity of Eq. (1) written for the Galilean space-time.

As a result, any proposal to assess the Schrödinger-Newton equation must approach the Planck scale but still not cross the line where bona fide clocks are absent and Galilean space-time cannot be assumed. In Ref. [9], it is suggested that Paul (ionic) traps be employed to confine osmium disks of 10^{-9} g subjected to a harmonic potential. Traces of gravitational self-interaction would be fingerprinted in the corresponding

energy spectrum. For this purpose, it would be necessary to reach temperatures of a few milikelvins. This has already been achieved in Paul traps for masses of 10^{-14} g [22–24] but not for ones of 10^{-9} g yet. (We refer the reader to Ref. [25] for an actual review of particle-trapping technology.)

V. CONCLUSIONS

The Schrödinger-Newton equation was proposed as a trial to recover the classical world from the quantum paradigm. Briefly speaking, particles would decohere due to gravitational self-interaction. On the other hand, at low energies, the gravitational and electric potentials are formally the same (only differing due to the universality of the coupling constant in the gravitational case). This has motivated us to revisit the hydrogen atom, assuming that the electron self-interacts through its proper charge. We quested for spherically symmetric stationary solutions and compared the outputs with the experimental data. We have obtained that, according to the modified Schrödinger equation (10), the ionization energy of the hydrogen atom and the corresponding spectroscopy data are at odds with each other, ruling out the possibility that electrons self-interact through their charge. Although our results only refer to the electromagnetic interaction, they may be seen as disfavoring the Schrödinger-Newton equation in the sense that if a single electron "perceived" itself through its mass distribution, it would be "natural" to argue that the same would be true concerning its charge distribution. Finally, we assessed some intrinsic difficulties of testing the Schrödinger-Newton equation as such.

The code employed to numerically solve Eq. (10) is included in the Supplemental Material [26].

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[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.108.012214 for the code employed to numerically solve Eq. (10).