Non-Markovian open quantum dynamics in squeezed environments: Coherent-state unraveling

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We apply the stochastic Schrödinger equation approach to study the non-Markovian dynamics of quantum systems coupled to a squeezed environment. We derive the non-Markovian quantum-state-diffusion equation using coherent-state unraveling and the associated zeroth-order master equation for general models in a microscopic quantum context. Focused on a dissipative optical cavity coupled with a squeezed vacuum, the numerical simulations demonstrate the time evolution of photon numbers in an optical cavity. We observe that the long-time limits of the non-Markovian dynamics are significantly different from those of the Markovian dynamics for various memory factors and squeezing factors. Additionally, non-Markovian dynamics exhibit a distinct pattern of behavior when parameters change. Our work provides a method to study the impact of multiple parameters on the non-Markovian dynamics and long-time limits in a joint manner. The method can be further extended to squeezed finite-temperature environments.

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I. INTRODUCTION

The squeezed state has attracted much interest in both theory and experiment. On the one hand, squeezed states generated from nonlinear optical processes provide insight into nonlinear quantum optics and quantum materials; on the other hand, squeezed states find a wide range of applications in quantum information and quantum metrology [1–12]. One remarkable example is LIGO, where optical squeezed states of light are utilized to reduce quantum noise and improve the accuracy of gravitational wave detection [13–16]. Spin squeezing is another feasible application related to squeezed states, which is essential for enhancing the sensitivity of atomic interferometers [17–23]. Moreover, squeezed states are crucial to continuous variable information processing, exploring quantum decoherence in a nonequilibrium environment, etc. Recently, nonequilibrium reservoirs, such as squeezed environments, have attracted extensive attention since they are expected to increase the efficiency of work generation and surpass standard thermodynamic bounds [24–26]. Therefore, a better understanding of the dynamics of the quantum system coupled to a squeezed environment will extend our knowledge of finite-size and nonequilibrium quantum effects.

However, estimating the dynamics in a squeezed environment is extremely difficult due to the lack of mathematical tools. In previous works, the master equation approach and the input-output approach have been derived using the Born-Markov approximation, in which we assume the coupling strength between the system and environment is relatively weak and the memory time of the environment is extremely short compared with the relaxation time [27–31]. But when the coupling strength is increased to the strong regime, or the memory effect of the environment is not negligible, the Born-Markov approximation will lead to failure. Moreover, it is believed that the non-Markovian dynamics have significantly different behavior in the short-time range compared with the Markovian counterpart, and its long-time limit or steady states still match the Markovian dynamics.

In addition to the methods mentioned above, there are recent advancements in applying the quantum-state-diffusion (QSD) approach with a novel unraveling method using a series of squeezed coherent states [25]. In this paper our approach is based on the traditional coherent-state unraveling, indicating that the environment will collapse to a coherent state after a shot measurement. Both unravelings can be utilized in various situations in quantum optics, offering an alternative perspective for understanding the dynamics of open quantum systems in a squeezed environment. Here we explore two parameters, the squeezing factor and the memory time, and find out the long-time steady states of the non-Markovian dynamics are way different from those of the Markovian dynamics.

Consider an open quantum system (OQS) with a formal Hamiltonian, in the system-environment framework,

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{S}} + \hat{H}_{\text{int}} + \hat{H}_{\text{E}}, \tag{1}$$

where $\hat{H}_{\text{S}}$ is the Hamiltonian of the system, $\hat{H}_{\text{E}}$ is the Hamiltonian of the environment, and $\hat{H}_{\text{int}}$ represents the interaction between the system and the environment. Without loss of generality, we assume the environment Hamiltonian has the form $\hat{H}_{\text{E}} = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$, where $\hat{b}_k(\hat{b}_k^\dagger)$ is the annihilation (creation) operator of the kth mode in the environment, associated with the eigenfrequency ω_k. In addition, $\hat{L}$, an operator in the

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system's Hilbert space, is the Lindblad operator coupled to the environment. Consequently, the interaction Hamiltonian in the rotating frame with respect to $\hat{H}_{\mathrm{E}}$ can be formally written as

$$\hat{H}_{\mathrm{int}}^{I}=\hat{L}\sum_{k}g_{k}^{*}\hat{b}_{k}^{\dagger}e^{i\omega_{k}t}+\hat{L}^{\dagger}\sum_{k}g_{k}\hat{b}_{k}e^{-i\omega_{k}t}. \tag{2}$$

In the conventional context of the stochastic Schrödinger equation approach [32–34], the environment can be understood as a fluctuated influence on the system. Particularly, the QSD approach unravels the influence in the Bargmann coherent-state space representation [35–42].

In this work we revise the QSD approach for the squeezed environment model to study the exact non-Markovian dynamics and explore the long-time steady states in the parameter space. Comparing to the traditional QSD equation approach for a nonsqueezed zero-temperature environment, we find that one extra stochastic process is naturally introduced when applying the functional derivative chain rule, resulting in the exact QSD equation and the associated master equation for the squeezed environment. Our paper is organized as follows. In Sec. II we extend the conventional QSD approach to study a generic OQS coupled to a squeezed-vacuum environment. Moreover, we focus on deriving the QSD equation and the associated master equation. In Sec. III we study an optical cavity model and demonstrate the numerical simulations of the non-Markovian dynamics of its photon population using the QSD equation and the master equation approach. The impacts of the squeeze factor of the environment's initial state will be explored numerically, from a non-Markovian to Markovian regime. We conclude in Sec. IV and extend our discussion toward broader applications.

II. FORMAL QSD EQUATION FOR SQUEEZED ENVIRONMENTS

In this section we formally discuss how to deal with the squeezed environment in the context of the QSD approach. In the system-environment framework, the dynamics of the state of the total system $|\Psi_{\mathrm{tot}}(t)\rangle$ are governed by a closed Schrödinger equation (setting $\hbar=1$),

$$\partial_{t}|\Psi_{\mathrm{tot}}\rangle=-i\big(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{int}}^{I}\big)|\Psi_{\mathrm{tot}}\rangle. \tag{3}$$

Assuming the generic coupling is linear with respect to the environment annihilation operators and creation operators, shown in Eq. (2), arbitrary environment states can be written in the Bargmann space representation $\|z\rangle$, defined as

$$\|z\rangle\equiv\otimes_{k}\|z_{k}\rangle, \tag{4}$$

where the kth-mode Bargmann coherent state reads

$$\|z_{k}\rangle\equiv\sum_{n_{k}}\frac{(z_{k})^{n_{k}}}{\sqrt{n_{k}!}}|n_{k}\rangle. \tag{5}$$

Based on the identity of resolution,

$$\hat{I}_{\mathrm{E}}=\int d^{2}z\frac{e^{-|z|^{2}}}{\pi}\|z\rangle\langle z\|, \tag{6}$$

the reduced density operator of the system can be obtained by taking the partial trace in the Bargmann space representation:

$$\begin{aligned}\hat{\rho}_{r}(t)&=\int d^{2}z\frac{e^{-|z|^{2}}}{\pi}\langle z\|\hat{\rho}_{\mathrm{tot}}(t)\|z\rangle\\&=\int d^{2}z\frac{e^{-|z|^{2}}}{\pi}\langle z\|\Psi_{\mathrm{tot}}(t)\rangle\langle\Psi_{\mathrm{tot}}(t)\|z\rangle.\end{aligned} \tag{7}$$

If we consider the term $\frac{e^{-|z|^{2}}}{\pi}\langle z\|\Psi_{\mathrm{tot}}(t)\rangle\langle\Psi_{\mathrm{tot}}(t)\|z\rangle$ as the outcome of a single shot measurement, the environmental variables $\{z_k\}$ should be interpreted as random numbers. The pure states of the system, defined as $|\psi_z\rangle=\langle z\|\Psi_{\mathrm{tot}}\rangle$, are turned into stochastic quantum trajectories. Furthermore, we can recover the reduced density operator by taking the ensemble average over all the quantum trajectories: $\hat{\rho}_r=\mathcal{M}(|\psi_z\rangle\langle\psi_z|)$, where $\mathcal{M}(\cdot)$ stands for the ensemble average. The stochastic nature of each quantum trajectory arises from the random collapse after the measurement of the environment. Consequently, every different unraveling is corresponding to a specific measurement process. In our work we describe a quantum trajectory corresponding to a shot measurement that the environment collapses to a coherent state. Therefore we choose the coherent-state unraveling.

Now we can obtain the QSD equation governing the dynamics of trajectories:

$$\partial_{t}|\psi_{z}\rangle=-i\hat{H}_{\mathrm{eff}}|\psi_{z}\rangle. \tag{8}$$

In the Bargmann state representation, the effective Hamiltonian reads

$$\hat{H}_{\mathrm{eff}}=\hat{H}_{\mathrm{S}}+\hat{L}\sum_{k}g_{k}^{*}z_{k}^{*}e^{i\omega_{k}t}+\hat{L}^{\dagger}\sum_{k}g_{k}e^{-i\omega_{k}t}\frac{\partial}{\partial z_{k}^{*}}. \tag{9}$$

Assuming the system and environment are separate at $t=0$, that $|\Psi_{\mathrm{tot}}(0)\rangle=|\psi_{\mathrm{S}}(0)\rangle\otimes|\psi_{\mathrm{E}}(0)\rangle$, the initial value of the trajectory generally reads $\langle z\|\Psi_{\mathrm{tot}}(0)\rangle=\langle z\|\psi_{\mathrm{E}}(0)\rangle\otimes|\psi_{\mathrm{S}}(0)\rangle$. Considering a nonsqueezed zero-temperature environment, the inner product factor $\langle z\|\psi_{E}(0)\rangle=\langle z\|0\rangle=1$ and the initial state of every single trajectory $|\psi_z(0)\rangle$ is the same as the system's initial state $|\psi_{\mathrm{S}}(0)\rangle$, that $|\psi_z(0)\rangle=|\psi_{\mathrm{S}}(0)\rangle$.

However, if the environment is not zero temperature but a squeezed vacuum, the above derivation needs to be refined. The initial state of the environment reads [43]

$$|\psi_{\mathrm{E}}(0)\rangle\equiv\bigotimes_{k}\sum_{n_{k}}\frac{(-\tanh r_{k})^{n_{k}}}{\sqrt{\cosh r_{k}}}\frac{\sqrt{(2n_{k})!}}{2^{n_{k}}n_{k}!}|2n_{k}\rangle. \tag{10}$$

Therefore the inner product term $\langle z\|\psi_{\mathrm{E}}(0)\rangle$ is

$$\begin{aligned}\langle z\|\psi_{\mathrm{E}}(0)\rangle&=\prod_{k}\sum_{n_{k}}\frac{z_{k}^{*2n_{k}}}{\sqrt{(2n_{k})!}}\frac{(-\tanh r_{k})^{n_{k}}}{\sqrt{\cosh r_{k}}}\frac{\sqrt{(2n_{k})!}}{2^{n_{k}}n_{k}!}\\&=\prod_{k}\frac{1}{\sqrt{\cosh r_{k}}}e^{\frac{-\tanh r_{k}}{2}(z_{k}^{*})^{2}}.\end{aligned} \tag{11}$$

Since the inner product $\langle z\|\psi_{\mathrm{E}}(0)\rangle\neq1$, one issue arises that the initial states of trajectories $|\psi_z(0)\rangle$ are noise depended. Thus we introduce a *normalized* trajectory $|\varphi_z\rangle$, defined as

$$|\varphi_{z}\rangle\equiv\frac{|\psi_{z}\rangle}{\prod_{k}\frac{1}{\sqrt{\cosh r_{k}}}e^{\frac{-\tanh r_{k}}{2}(z_{k}^{*})^{2}}}, \tag{12}$$

to satisfy the coincidence of the initial condition $|\varphi_z(0)\rangle = |\psi_S(0)\rangle$. Consequently, we obtain

$$\partial_{z_k^*}|\psi_z\rangle = \prod_{k'} \frac{1}{\sqrt{\cosh r_{k'}}} e^{\frac{-\tanh r_{k'}}{2}(z_{k'}^*)^2} \partial_{z_k^*}|\varphi_z\rangle + \prod_{k'} \frac{-\tanh r_k \; z_k^*}{\sqrt{\cosh r_{k'}}} e^{\frac{-\tanh r_{k'}}{2}(z_{k'}^*)^2} |\varphi_z\rangle. \tag{13}$$

Substituting it into Eqs. (8) and (9), a QSD equation of the *normalized* trajectory $|\varphi_z\rangle$ reads

$$\partial_t|\varphi_z\rangle = (-i\hat{H}_S + z_t^*\hat{L} + w_{-t}^*\hat{L}^\dagger)|\varphi_z\rangle - i\hat{L}^\dagger \sum_k g_k e^{-i\omega_k t}\partial_{z_k^*}|\varphi_z\rangle, \tag{14}$$

where two stochastic processes are defined as, both in terms of z_k^*,

$$z_t^* \equiv -i\sum_k g_k^* z_k^* e^{i\omega_k t}, \tag{15}$$

$$w_t^* \equiv i\sum_k g_k z_k^* e^{i\omega_k t}\tanh r_k, \tag{16}$$

respectively. Equation (14) indicates that the new trajectory $|\varphi_z\rangle$ involves two noises. By taking $r_k = 0$, it is easy to verify that $w_t^* = 0$, which proves that the nonsqueezed zero-temperature model is a specific case of Eq. (14). It is worth pointing out that the two noises can be merged when the condition $\hat{L} = \hat{L}^\dagger$ is satisfied [44–46].

Applying the chain rule [47], the term of $\partial_{z_k^*}|\varphi_z\rangle$ can be explicitly extended as a sum of two integrals for the functional derivatives with respect to the two stochastic processes, that

$$\begin{aligned} &-i\sum_k g_k e^{-i\omega_k t}\partial_{z_k^*} \\ &= -i\sum_k g_k e^{-i\omega_k t}\left(\int_0^t ds\frac{\partial z_s^*}{\partial z_k^*}\delta_{z_s^*} + \int_{-t}^0 ds\frac{\partial w_s^*}{\partial z_k^*}\delta_{w_s^*}\right) \\ &= -\int_0^t ds\sum_k |g_k|^2 e^{-i\omega_k(t-s)}\delta_{z_s^*} \\ &\quad + \int_{-t}^0 ds\sum_k g_k^2\tanh r_k e^{-i\omega_k(t-s)}\delta_{w_s^*} \\ &= -\int_0^t ds\,\alpha(t,s)\delta_{z_s^*} - \int_{-t}^0 ds\,\beta(t,s)\delta_{w_s^*}, \end{aligned} \tag{17}$$

where $\alpha(t,s) = \sum_k |g_k|^2 e^{-i\omega_k(t-s)}$ is the correlation function of the noise z_t^*, and $\beta(t,s) = \mathcal{M}(z_t w_s^*)$ is the cross-correlation function between the two noises. Moreover, we define two to-be-determined operators $\hat{O}_z(t,s)$ and $\hat{O}_w(t,s)$ as

$$\hat{O}_z(t,s)|\varphi_z\rangle \equiv \delta_{z_s^*}|\varphi_z\rangle, \quad \hat{O}_w(t,s)|\varphi_z\rangle \equiv \delta_{w_s^*}|\varphi_z\rangle. \tag{18}$$

With the O operators, the linear QSD Eq. (14) can be formally written as [39,46]

$$\partial_t|\varphi_z\rangle = (-i\hat{H}_S + z_t^*\hat{L} + w_{-t}^*\hat{L}^\dagger)|\varphi_z\rangle - [\hat{L}^\dagger\bar{O}_z^\alpha(t) + \hat{L}^\dagger\bar{O}_w^\beta(t)]|\varphi_z\rangle, \tag{19}$$

where

$$\bar{O}_z^\alpha(t) \equiv \int_0^t ds\alpha(t,s)\hat{O}_z(t,s),$$
$$\bar{O}_w^\beta(t) \equiv \int_{-t}^0 ds\beta(t,s)\hat{O}_w(t,s).$$

Here, using the consistency condition that $\partial_t\delta_{z_s^*(w_s^*)}|\varphi_z\rangle = \delta_{z_s^*(w_s^*)}\partial_t|\varphi_z\rangle$, the two operators $\hat{O}_z(t,s)$ and $\hat{O}_w(t,s)$ can be determined by two evolution equations, respectively:

$$\begin{aligned}\partial_t\hat{O}_z(t,s) &= [-i\hat{H}_S + z_t^*\hat{L} + w_{-t}^*\hat{L}^\dagger, \hat{O}_z(t,s)] \\ &\quad + [-\hat{L}^\dagger\bar{O}_z^\alpha(t) - \hat{L}^\dagger\bar{O}_w^\beta(t), \hat{O}_z(t,s)] \\ &\quad - \delta_{z_s^*}\big(\hat{L}^\dagger\bar{O}_z^\alpha(t) + \hat{L}^\dagger\bar{O}_w^\beta(t)\big),\end{aligned} \tag{20}$$

$$\begin{aligned}\partial_t\hat{O}_w(t,s) &= [-i\hat{H}_S + z_t^*\hat{L} + w_{-t}^*\hat{L}^\dagger, \hat{O}_w(t,s)] \\ &\quad + [-\hat{L}^\dagger\bar{O}_z^\alpha(t) - \hat{L}^\dagger\bar{O}_w^\beta(t), \hat{O}_w(t,s)] \\ &\quad - \delta_{w_s^*}\big(\hat{L}^\dagger\bar{O}_z^\alpha(t) + \hat{L}^\dagger\bar{O}_w^\beta(t)\big).\end{aligned} \tag{21}$$

The initial conditions read

$$\hat{O}_z(t, s=t) = \hat{L}, \quad \hat{O}_w(t, s=-t) = \hat{L}^\dagger. \tag{22}$$

Moreover, it is worth noting that the initial condition $\hat{O}_w(t, s=-t) = \hat{L}^\dagger$ in the above discussion indicates the nontrivial dynamics of the system.

III. MODELS AND NUMERICAL RESULTS

A. Non-Markovian QSD equations

In this section we will take the optical cavity as an example to demonstrate the impacts of the squeezed-vacuum environment. The system's Hamiltonian reads

$$\hat{H}_S = \omega\hat{a}^\dagger\hat{a}, \tag{23}$$

and the coupling operator is $\hat{L} = \hat{a}$. For simplicity, we assume the squeezing factors are identical $r_k = r$ for every mode in the environment. As a result, the second noise takes a compact form that $w_{-t}^* = i(\tanh r)\sum_k g_k e^{-i\omega_k t}z_k^*$. Considering the condition $g_k = g_k^* > 0$, the cross-correlation function satisfies $\beta(t,s) = -\tanh r\,\alpha(t,s)$. Consequently, the O operators satisfying Eqs. (20) and (21) and the associated initial condition in Eq. (22) must be formulated as [46]

$$\begin{aligned}\hat{O}_z(t,s) \equiv\; & f_{z1}(t,s)\hat{a} + f_{z2}(t,s)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* j_{z1}(t,s,s') \\ & + \int_{-t}^0 ds' w_{s'}^* j_{z2}(t,s,s'), \\ \hat{O}_w(t,s) \equiv\; & f_{w1}(t,s)\hat{a} + f_{w2}(t,s)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* j_{w1}(t,s,s') \\ & + \int_{-t}^0 ds' w_{s'}^* j_{w2}(t,s,s'), \quad (s<0).\end{aligned} \tag{24}$$

The eight coefficients $f_{z1(2)}$, $f_{w1(2)}$, $j_{z1(2)}$, and $j_{w1(2)}$ can be numerically determined by solving a set of evolution equations (Appendix A).

Moreover, with the ansatzes of $\hat{O}_z$ and $\hat{O}_w$ in Eq. (24), the other two operators $\bar{O}_z^{\alpha}$ and $\bar{O}_w^{\beta}$ can be written explicitly as

$$\bar{O}_z^{\alpha}(t) \equiv F_{z1}^{\alpha}(t)\hat{a} + F_{z2}^{\alpha}(t)\hat{a}^{\dagger} + \int_0^t ds' z_{s'}^* J_{z1}^{\alpha}(t,s') + \int_{-t}^0 ds' w_{s'}^* J_{z2}^{\alpha}(t,s'),$$
$$\bar{O}_w^{\beta}(t) \equiv F_{w1}^{\beta}(t)\hat{a} + F_{w2}^{\beta}(t)\hat{a}^{\dagger} + \int_0^t ds' z_{s'}^* J_{w1}^{\beta}(t,s') + \int_{-t}^0 ds' w_{s'}^* J_{w2}^{\beta}(t,s'), \quad (s<0), \quad (25)$$

where

$$F_{zi}^{\alpha}(t) \equiv \int_0^t ds\alpha(t,s) f_{zi}(t,s),$$
$$F_{wi}^{\beta}(t) \equiv \int_{-t}^0 ds\beta(t,s) f_{wi}(t,s) = -\tanh r \int_{-t}^0 ds\alpha(t,s) f_{wi}(t,s),$$
$$J_{zi}^{\alpha}(t,s') \equiv \int_0^t ds\alpha(t,s) j_{zi}(t,s,s'),$$
$$J_{wi}^{\beta}(t,s') \equiv \int_{-t}^0 ds\beta(t,s) j_{wi}(t,s,s') = -\tanh r \int_{-t}^0 ds\alpha(t,s) j_{wi}(t,s,s'), \times (i=1,2). \quad (26)$$

Similarly, the above eight coefficients $F_{z1(2)}^{\alpha}$, $F_{w1(2)}^{\beta}$, $J_{z1(2)}^{\alpha}$, and $J_{w1(2)}^{\beta}$ can be numerically determined by a set of integro-differential equations (see Appendix B). Now, the QSD equation (19) can be explicitly shown as

$$\partial_t|\varphi_z\rangle = \Big\{(-i\omega - F_{z1}^{\alpha} - F_{w1}^{\beta})\hat{a}^{\dagger}\hat{a} - (F_{z2}^{\alpha} + F_{w2}^{\beta})\hat{a}^{\dagger 2} + z_t^*\hat{a} + w_{-t}^*\hat{a}^{\dagger} - \Big[\int_0^t dsz_s^*[J_{z1}^{\alpha}(t,s) + J_{w1}^{\beta}(t,s)] + \int_{-t}^0 dsw_s^*[J_{z2}^{\alpha}(t,s) + J_{w2}^{\beta}(t,s)]\Big]\hat{a}^{\dagger}\Big\}|\varphi_z\rangle. \quad (27)$$

To compare the dynamics in Markovian and non-Markovian regimes, we choose the Ornstein-Uhlenbeck process whose correlation function $\alpha(t,s) = \frac{\Gamma\gamma}{2}e^{-\gamma|t-s|}e^{-i\Omega(t-s)}$. Here Γ controls the coupling strength, and $1/\gamma$ scales the memory time of the environment. As a result, the evolution equations of the eight coefficients defined in Eq. (26) can be further simplified to a set of differential equations (Appendix B).

B. Non-Markovian master equations

In this section we work on the formal master equation. With the numerically determined quantum trajectories governed by Eq. (27), the reduced density operator $\hat{\rho}_r$ can be obtained by taking the ensemble average over all quantum trajectories. First, we define two operators $\hat{P}_{\psi} \equiv |\psi_z\rangle\langle\psi_z|$ and $\hat{P}_{\varphi} \equiv |\varphi_z\rangle\langle\varphi_z|$ for the two types of quantum trajectories, respectively. These two types of trajectories obey the relation $|\varphi_z\rangle = \frac{1}{K}|\psi_z\rangle$, where $K = \prod_k \frac{1}{\sqrt{\cosh r_k}}\exp(\frac{-\tanh r_k}{2}z_k^{*2})$, by definition. As a result, the reduced density operator reads

$$\hat{\rho}_r \equiv \mathcal{M}(|\psi_z\rangle\langle\psi_z|) \equiv \mathcal{M}(|K|^2|\varphi_z\rangle\langle\varphi_z|). \quad (28)$$

It is easy to prove that

$$\mathcal{M}(|K|^2) = \prod_k \int \frac{d^2 z_k}{\pi\cosh r_k} e^{-|z_k|^2 - (x_k^2 - y_k^2)\tanh r_k} = \prod_k \int \frac{dx_k dy_k}{\pi\cosh r_k} e^{-(1+\tanh r_k)x_k^2 - (1-\tanh r_k)y_k^2} = 1, \quad (29)$$

where x_k and y_k are the real and imaginary parts of z_k. The equation reveals that $\frac{|K|^2 e^{-|z|^2}}{\pi}$ is also a probability distribution, and the squeezing factors $\{r_k\}$ break the symmetry between x_k and y_k in the original complex Gaussian distribution.

Because the evolution equations of the coefficients in $\hat{O}_z$ and $\hat{O}_w$ have similar structures (see Appendix B), we define a new set of coefficients to further simplify the QSD equation,

$$\partial_t|\varphi_z\rangle = \Big[\big(-i\omega - F_1^z\big)\hat{a}^{\dagger}\hat{a} - F_2^z\hat{a}^{\dagger 2} + z_t^*\hat{a} + w_{-t}^*\hat{a}^{\dagger} - \Big(\int_0^t dsz_s^* J_1^z(t,s) + \int_{-t}^0 dsw_s^* J_2^z(t,s)\Big)\hat{a}^{\dagger}\Big]|\varphi_z\rangle, \quad (30)$$

where

$$F_1^z \equiv F_{z1}^{\alpha} + F_{w1}^{\beta}, \qquad F_2^z \equiv F_{z2}^{\alpha} + F_{w2}^{\beta},$$
$$J_1^z \equiv J_{z1}^{\alpha} + J_{w1}^{\beta}, \qquad J_2^z \equiv J_{z2}^{\alpha} + J_{w2}^{\beta}. \quad (31)$$

The formal non-Markovian master equation reads

$$\partial_t\hat{\rho}_r = \mathcal{M}(|K|^2\partial_t|\varphi_z\rangle\langle\varphi_z| + |K|^2|\varphi_z\rangle\partial_t\langle\varphi_z|) = -i[\hat{H}_S, \hat{\rho}_r] + \big\{[\hat{a}\cosh^2 r, \hat{R}_1 + \hat{R}_2] + [\hat{a}^{\dagger}\cosh^2 r, \hat{R}_3 + \hat{R}_4] + \text{H.c.}\big\}, \quad (32)$$

where

$$\hat{R}_1 = \int_0^t ds\mathcal{M}(z_s z_t^*)\mathcal{M}\big(\hat{P}_{\psi}\hat{O}_z^{\dagger}(t,s)\big),$$
$$\hat{R}_2 = \int_{-t}^0 ds\mathcal{M}(w_s z_t^*)\mathcal{M}\big(\hat{P}_{\psi}\hat{O}_w^{\dagger}(t,s)\big),$$
$$\hat{R}_3 = \int_0^t ds\mathcal{M}(z_s w_{-t}^*)\mathcal{M}\big(\hat{P}_{\psi}\hat{O}_z^{\dagger}(t,s)\big),$$
$$\hat{R}_4 = \int_{-t}^0 ds\mathcal{M}(w_s w_{-t}^*)\mathcal{M}\big(\hat{P}_{\psi}\hat{O}_w^{\dagger}(t,s)\big).$$

The derivation of Eq. (32) can be found in Appendix C. For some weak-coupling conditions, the noisy operator in Eq. (30), $[\int_0^t dsz_s^* J_1^z(t,s) + \int_{-t}^0 dsw_s^* J_2^z(t,s)]\hat{a}^{\dagger}$, can be dropped off. (To provide a brief justification, we introduce a small multiplier λ, that $g_k \rightarrow \lambda g_k$, and evaluate the scale of each quantity in terms of λ: $z_t^*, w_t^* \sim \lambda$; the correlation or cross-correlation functions $\alpha(t,s), \beta(t,s), \cdots \sim \lambda^2$; the coefficients $F_1^z, F_2^z, J_1^z, J_2^z, \cdots \sim \lambda^2$; and the dropped-off term $[\int_0^t ds z_s^* J_1^z(t,s) + \int_{-t}^0 ds w_s^* J_2^z(t,s)]\hat{a}^\dagger \sim \lambda^3$.) Consequently, the zeroth-order master equation reads

$$\partial_t \hat{\rho}_r = -i[\hat{H}_\mathrm{S}, \hat{\rho}_r] + \cosh^2 r \sum_{j=1}^{4} \mathcal{L}_j[\hat{\rho}_r], \tag{33}$$

where

$$\begin{aligned}
\mathcal{L}_1[\hat{\rho}_r] &\equiv 2\mathrm{Re}\big(F_1^z\big)\hat{a}\hat{\rho}_r\hat{a}^\dagger - \big(F_1^{z*}\hat{\rho}_r\hat{a}^\dagger\hat{a} + F_1^z\hat{a}^\dagger\hat{a}\hat{\rho}_r\big),\\
\mathcal{L}_2[\hat{\rho}_r] &\equiv F_2^{z*}\hat{a}\hat{\rho}_r\hat{a} + F_2^z\hat{a}^\dagger\hat{\rho}_r\hat{a}^\dagger - \big(F_2^{z*}\hat{\rho}_r\hat{a}^2 + F_2^z\hat{a}^{\dagger 2}\hat{\rho}_r\big),\\
\mathcal{L}_3[\hat{\rho}_r] &\equiv 2\mathrm{Re}\big(F_2^w\big)\hat{a}^\dagger\hat{\rho}_r\hat{a} - \big(F_2^{w*}\hat{\rho}_r\hat{a}\hat{a}^\dagger + F_2^w\hat{a}\hat{a}^\dagger\hat{\rho}_r\big),\\
\mathcal{L}_4[\hat{\rho}_r] &\equiv F_1^{w*}\hat{a}^\dagger\hat{\rho}_r\hat{a}^\dagger + F_1^w\hat{a}\hat{\rho}_r\hat{a} - \big(F_1^{w*}\hat{\rho}_r\hat{a}^{\dagger 2} + F_1^w\hat{a}^2\hat{\rho}_r\big).
\end{aligned}$$

These coefficients can be numerically determined by a group of evolution equations, as shown in Appendix B.

C. Master equations in the Markov limit

When the memory factor $\gamma \rightarrow \infty$, the dynamics of the open system approaches Markovian. The correlation functions turn into time-local functions: $\alpha(t,s) = \mathcal{M}(z_t z_s^*) \rightarrow \Gamma\delta(t,s)$, $\epsilon(t,s) = \mathcal{M}(w_t w_s^*) \rightarrow \tanh^2 r\, \Gamma\delta(t,s)$, and the cross-correlation function satisfies $\mathcal{M}(z_t w_s^*) = 0$ for $t > 0, s < 0$. Consequently, the QSD Eq. (19) can be simplified to

$$\partial_t|\varphi_z\rangle = \left(-i\hat{H}_\mathrm{S} + z_t^*\hat{L} + w_{-t}^*\hat{L}^\dagger - \frac{\Gamma}{2}\hat{L}^\dagger\hat{L}\right)|\varphi_z\rangle. \tag{34}$$

According to the above-simplified QSD equation, the formal Markovian master equation reads

$$\begin{aligned}
\partial_t\hat{\rho}_r = &-i[\hat{H}_\mathrm{S}, \hat{\rho}_r] - \frac{\Gamma}{2}(\hat{L}^\dagger\hat{L}\hat{\rho}_r + \hat{\rho}_r\hat{L}^\dagger\hat{L}) + \hat{L}\mathcal{M}(z_t^*\hat{P}_\psi)\\
&+ \hat{L}^\dagger\mathcal{M}(w_{-t}^*\hat{P}_\psi) + \mathcal{M}(\hat{P}_\psi z_t)\hat{L}^\dagger + \mathcal{M}(\hat{P}_\psi w_{-t})\hat{L}.
\end{aligned} \tag{35}$$

We apply the Novikov theorem and obtain the following relations (see Appendix C):

$$\begin{aligned}
\mathcal{M}(w_{-t}^*\hat{P}_\psi) &= -\tanh^2 r\mathcal{M}(z_t\hat{P}_\psi) + \tanh^2 r\frac{\Gamma}{2}\hat{\rho}_r\hat{L},\\
\mathcal{M}(z_t^*\hat{P}_\psi) &= -\mathcal{M}(w_{-t}\hat{P}_\psi) + \frac{\Gamma}{2}\hat{\rho}_r\hat{L}^\dagger.
\end{aligned} \tag{36}$$

The solution reads

$$\begin{aligned}
\mathcal{M}(z_t^*\hat{P}_\psi) &= -\sinh^2 r\tfrac{\Gamma}{2}\hat{L}^\dagger\hat{\rho}_r + \cosh^2 r\tfrac{\Gamma}{2}\hat{\rho}_r\hat{L}^\dagger,\\
\mathcal{M}(w_{-t}\hat{P}_\psi) &= -\sinh^2 r\tfrac{\Gamma}{2}\hat{\rho}_r\hat{L}^\dagger + \sinh^2 r\tfrac{\Gamma}{2}\hat{L}^\dagger\hat{\rho}_r.
\end{aligned} \tag{37}$$

By substituting the above solution into Eq. (35), the Markovian master equation can be explicitly written as

$$\begin{aligned}
\partial_t\hat{\rho}_r = &-i[\hat{H}_\mathrm{S}, \hat{\rho}_r] + \frac{\Gamma\cosh^2 r}{2}(2\hat{L}\hat{\rho}_r\hat{L}^\dagger - \hat{L}^\dagger\hat{L}\hat{\rho}_r - \hat{\rho}_r\hat{L}^\dagger\hat{L})\\
&+ \frac{\Gamma\sinh^2 r}{2}(2\hat{L}^\dagger\hat{\rho}_r\hat{L} - \hat{L}\hat{L}^\dagger\hat{\rho}_r - \hat{\rho}_r\hat{L}\hat{L}^\dagger).
\end{aligned} \tag{38}$$

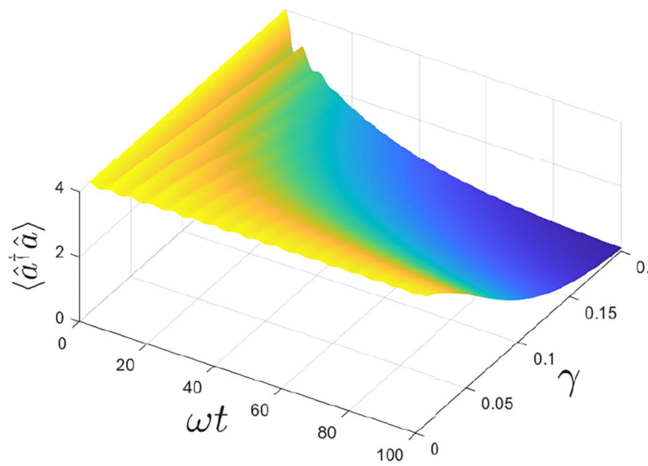

FIG. 1. The dynamics of the population of photons in the optical cavity. Here we choose the coupling strength $\Gamma = 1.2$, the system's frequency $\omega = 1$, and the initial state is a coherent state $|\psi_\mathrm{S}(0)\rangle = |\alpha\rangle = |2\rangle$. The squeezing factor of the environment is set as $r = 0.5$.

In a squeezed-vacuum environment, the average number of photons is $\tilde{n} = \sinh^2(r)$. As a result, the above master equation is consistent with the Lindblad form:

$$\begin{aligned}
\partial_t\hat{\rho}_r = &-i[\hat{H}_\mathrm{S}, \hat{\rho}_r] + \frac{\Gamma}{2}(\tilde{n}+1)(2\hat{L}\hat{\rho}_r\hat{L}^\dagger - \hat{L}^\dagger\hat{L}\hat{\rho}_r - \hat{\rho}_r\hat{L}^\dagger\hat{L})\\
&+ \frac{\Gamma}{2}\tilde{n}(2\hat{L}^\dagger\hat{\rho}_r\hat{L} - \hat{L}\hat{L}^\dagger\hat{\rho}_r - \hat{\rho}_r\hat{L}\hat{L}^\dagger).
\end{aligned} \tag{39}$$

D. Numerical simulations

In this section, we numerically demonstrate the non-Markovian dynamics of the population of photons in the optical cavity $N(t) = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle$ by varying the squeezing factor r for the initial state of the environment and the memory factor γ for correlation functions. When $r = 0$, the squeezed state is turned into a vacuum state; when $\gamma \rightarrow 0$, the memory time is infinite, indicating the dynamics are in a strong non-Markovian regime. In addition, we explore the population of steady states in the optical cavity, from the Markovian to the non-Markovian regime, and from a vacuum to a squeezed environment.

In Fig. 1 we plot the time evolution of the population of the optical cavity interacting with a squeezed environment. We set the coupling strength $\Gamma = 1.2$ and the system's frequency $\omega = 1$. The initial state of the optical cavity is prepared in a coherent state $|\alpha\rangle = |2\rangle$, and the squeezing factor of the environment is set as $r = 0.5$. We explore the dynamics of the population of photons by varying memory factor γ from 0.01 to 0.2. In the simulation we noticed that the memory factor γ changes the dynamics of $N(t)$ significantly when γ is around 0.1. The population gradually decays when $\gamma > 0.1$ but increases to a value larger than the initial photon number when $\gamma < 0.1$. Our simulations show that there are multiple trends in non-Markovian dynamics if we carefully explore the entire parameter space [31].

To single out the impact of the squeezing factor of the environment on the dynamics of the system, we plot the evolution of the population of photons in the optical cavity

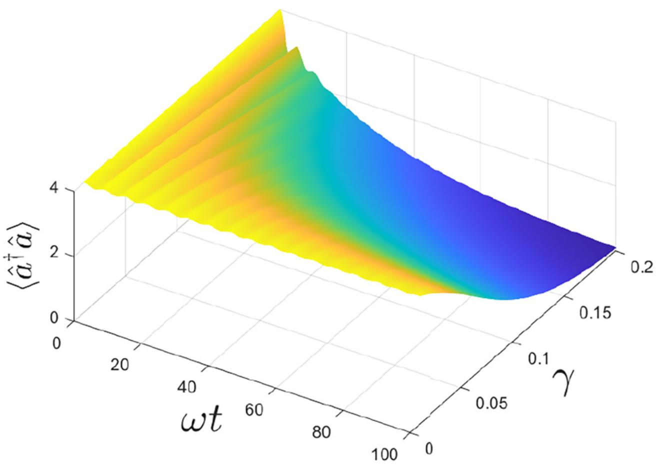

FIG. 2. The dynamics of the population of photons in the optical cavity. Here we choose the coupling strength $\Gamma = 1.2$, the system's frequency $\omega = 1$, and the initial state is a coherent state $|\psi_S(0)\rangle = |\alpha\rangle = |2\rangle$. The squeezing factor of the environment is set as $r = 0$, so the environment is not in a squeezed state initially.

when the environment is in a vacuum state $|\psi_S(0)\rangle = |0\rangle$ in Fig. 2 and compare it with Fig. 1. When the squeezing factor is set to zero, it is observed that the non-Markovian dynamics obey a similar decaying pattern that the decaying rate is solely controlled by the memory factor γ. However, in Fig. 1, when the squeezing factor $r = 0.5$, the population is not always decaying with time but can increase once in a non-Markovian regime with long-time memory. This indicates that the exact dynamical equations for the squeezed environment case are necessary and crucial, since even a slight change in the squeezing factor can bring in a completely different type of dynamics. It is also verified that an approximated Markovian master equation is not appropriate to study the squeezed environment case, even if the squeezing factor is relatively small. Due to the complexity of non-Markovian dynamics, slight differences in the initial states of the environment can make the system evolve along a completely different path and end up at a different steady state.

In addition, we include the central frequency of the environment Ω into the non-Markovian dynamics [48] of the photon population by changing it from $\Omega = 0$ (in Figs. 1 and 2) to $\Omega = 0.5$ (as shown in Fig. 3). We demonstrate that when the environment's central frequency Ω is close to the eigenfrequency of the system ω, the efficiency of the energy injection is increased. As a result, the highest reading of the photon number in the optical cavity is greater compared with the out-of-tune case ($\Omega = 0$).

Finally, we perform numerical simulations to examine the steady population of photons in the cavity, shown in Figs. 4 and 5. Previous works on the dynamics in a squeezed environment usually are performed under the Markovian or near-Markovian approximation, and only a limited amount of work was conducted in the non-Markovian regime. In fact, when the memory factor γ is relatively small, the dynamics, both short time and long time, have not been well understood. At least, it is not proper to characterize

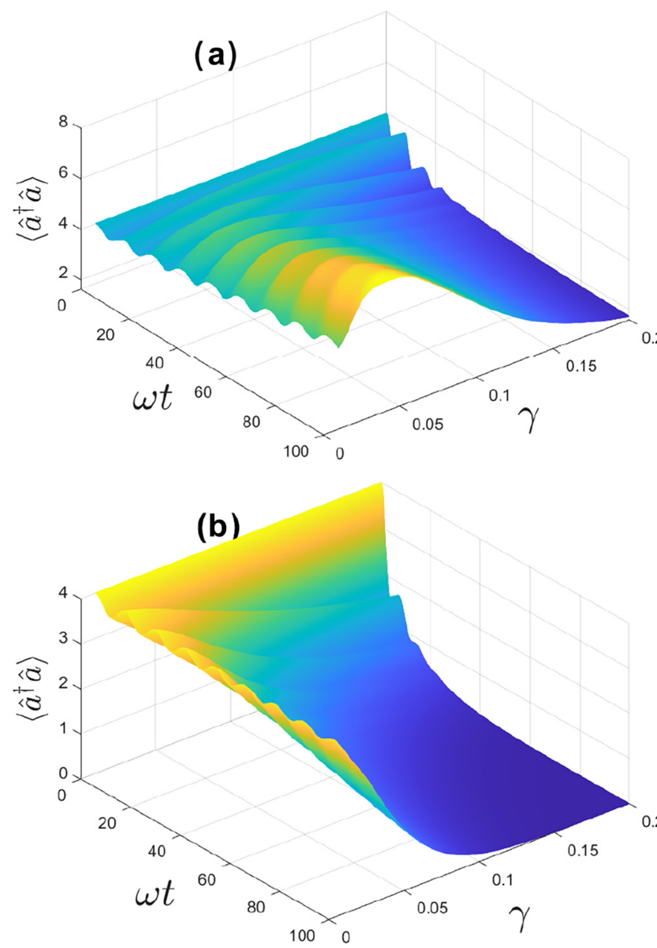

FIG. 3. The dynamics of the population of photons in the optical cavity. Here we choose the coupling strength $\Gamma = 1.2$, the system's frequency $\omega = 1$, the initial state is a coherent $|\psi_S(0)\rangle = |\alpha\rangle = |2\rangle$, and the central frequency of the environment $\Omega = 0.5$. The squeezing factor of the environment is set as (a) $r = 0.5$, (b) $r = 0$, so the environment is not in a squeezed state initially.

the non-Markovian dynamics by the decay rate Γ solely. The numerical simulations in Figs. 4 and 5 reveal that the steady population of photons in the optical cavity is extremely sensitive to the memory factor γ. There is a sharp peak in the steady population when γ is around 0.03, suggesting that the steady population of photons is complicatedly influenced by three factors: the memory factor γ, the squeezing factor r, and the central frequency Ω. Moreover, within the parameter regime around the peak, the dynamics are highly different from the conventional Markovian dynamics, indicating that the cavity can be more efficiently coupled with the environment, allowing it to absorb energy from the environment.

IV. CONCLUSION

We study the non-Markovian dynamics of a quantum system coupled to a bosonic squeezed-vacuum environment. Although several master equations, such as Lindblad or Redfield equations, have provided robust and efficient mathematical tools, these approaches have common shortcomings in

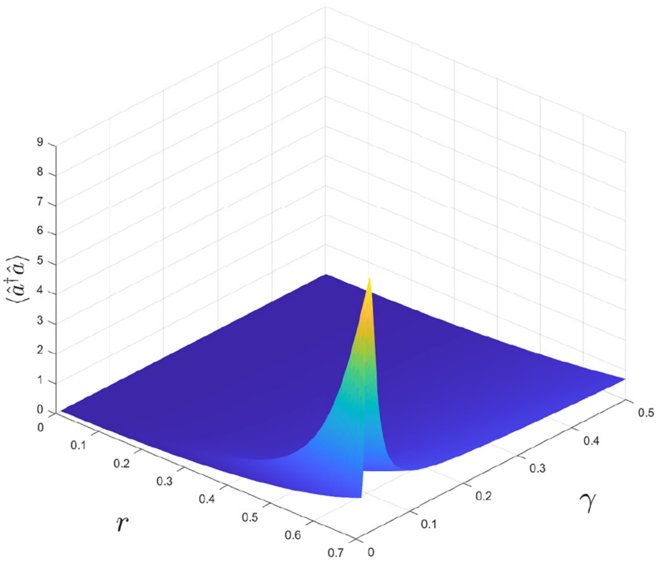

FIG. 4. Long-time limit population of photons. The memory factor γ changes from 0.01 to 0.5. The environmental squeezing factor varies from 0 to 0.7. The frequency of the system is set as $\omega = 1$. The central frequency of the environment is set as $\Omega = 0.5$, and the coupling strength is set as $\Gamma = 0.2$.

that they are rooted in the Born-Markov approximation. In this work we first use the conventional quantum-state-diffusion (QSD) approach to unravel the reduced density operator of the cavity into a set of stochastic pure states—quantum trajectories. There are two significant difficulties in deriving quantum trajectory evolution equations: (1) how to expand a squeezed state in the Bargmann space representation, and (2) how to calculate the derivative term $\frac{\partial}{\partial z_k^*}|\psi_z\rangle$.

For the first issue we modify the conventional QSD approach where the reduced density operator equals the ensemble average over all quantum trajectories, $\hat{\rho}_r = \mathcal{M}(|\psi_z\rangle\langle\psi_z|) = \int \frac{d^2z}{\pi} e^{-|z|^2} |\psi_z\rangle\langle\psi_z|$. By introducing a factor

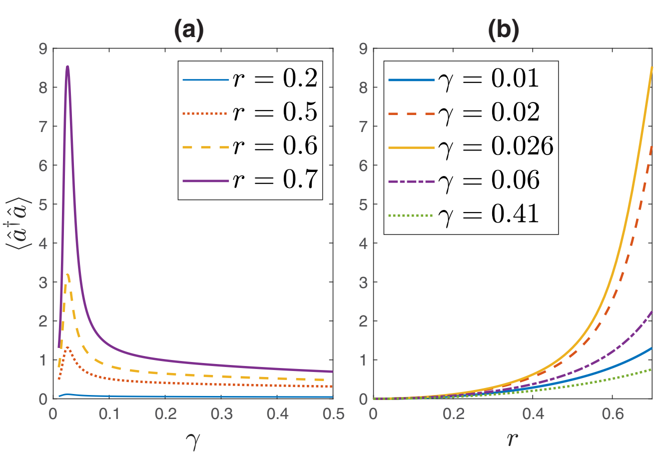

FIG. 5. Long-time limit population of photons. (a) The memory factor γ changes from 0.01 to 0.5. The environmental squeezing factor r is selected as 0.2 (blue solid), 0.5 (dotted), 0.6 (dashed), 0.7 (purple solid). (b) The squeezing factor r varies from 0 to 0.7. The memory factor γ is selected as 0.01 (blue solid), 0.02 (red dashed), 0.026 (orange solid), 0.06 (dash-dotted), 0.41 (dotted).

$K = \prod_k \frac{1}{\sqrt{\cosh r_k}} \exp(\frac{-\tanh r_k}{2} z_k^{*2})$, the reduced density operator can be generated by a new formula $\hat{\rho}_r = \mathcal{M}(|K|^2|\varphi_z\rangle\langle\varphi_z|) = \int \frac{d^2z}{\pi} e^{-|z|^2} |K|^2 |\varphi_z\rangle\langle\varphi_z|$, where $|\varphi_z\rangle = |\psi_z\rangle/K$ is a newly defined quantum trajectory.

For the second issue, due to the factor K, it is natural to introduce two stochastic processes z_t^* and w_{-t}^*. As a result, the chain rule must be extended as $\partial_{z_k^*} = \int_0^t ds \frac{\partial z_s^*}{\partial z_k^*} \delta_{z_s^*} + \int_{-t}^0 ds \frac{\partial w_s^*}{\partial z_k^*} \delta_{w_s^*}$, and the ansatzes of two O operators read $\hat{O}_z(t,s)|\varphi_z\rangle \equiv \delta_{z_s^*}|\varphi_z\rangle$ and $\hat{O}_w(t,s)|\varphi_z\rangle \equiv \delta_{w_s^*}|\varphi_z\rangle$, respectively.

With the above-mentioned solutions, we derive the exact linear QSD equation and the associated zeroth-order master equation for general systems coupled to a squeezed vacuum. In principle, our method is valid for arbitrary correlation functions. In this work we choose the Ornstein-Uhlenbeck process in numerical simulations for simplicity. Furthermore, we prove that the conventional finite-temperature Lindblad master equation is consistent with our general non-Markovian master equations.

At last, we apply this method to estimate the population of photons in the optical cavity coupled to a squeezed vacuum, and we numerically explore the impacts of all factors on the non-Markovian dynamics. The simulation results show a counterintuitive fact that the non-Markovian dynamics are distinct in different parameter regimes. The long-time limit population of photons can increase sharply inside a narrow parameter regime, where the squeezing factor r is greater than 0.3 and the memory factor γ is less than 0.2. Out of the regime, the population of photons of the steady state is very close to zero. Therefore a full exploration of the entire parameter space, specifically inside the strong non-Markovian regime, is necessary. However, the long-time behavior of non-Markovian dynamics is still an open question. In Markovian dynamics, the coupling strength between the system and each mode in the environment remains constant, resulting in white noise. In contrast, non-Markovian dynamics involve mode-dependent coupling strength and colored noise. Therefore some specific non-Markovian phenomena can be missed in the theoretical study if the structured environment is characterized using the decay rate Γ solely. The observed "peak" in the population at the long-time limit in Fig. 4 serves as compelling evidence that in non-Markovian dynamics, the system can form highly efficient coupling with some selected modes.

In summary, we develop a systematic method to obtain the exact linear non-Markovian QSD equation using coherent-state unraveling and the master equation for quantum systems coupled to a squeezed vacuum. With the derived zeroth-order non-Markovian master equation, it is feasible to study the nonequilibrium dynamics of a variety of models, such as quantum optics, optomechanical systems, and spin squeezing.

ACKNOWLEDGMENTS

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APPENDIX A: EVOLUTION EQUATIONS OF COEFFICIENTS IN OPERATORS $\hat{O}_z$ AND $\hat{O}_w$

Here the two O operators, $\hat{O}_z$ and $\hat{O}_w$, read

$$\begin{aligned}\hat{O}_z(t,s) \equiv\ & f_{z1}(t,s)\hat{a} + f_{z2}(t,s)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* j_{z1}(t,s,s') \\ & + \int_{-t}^0 ds' w_{s'}^* j_{z2}(t,s,s'), \\ \hat{O}_w(t,s) \equiv\ & f_{w1}(t,s)\hat{a} + f_{w2}(t,s)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* j_{w1}(t,s,s') \\ & + \int_{-t}^0 ds' w_{s'}^* j_{w2}(t,s,s'), \quad (s<0). \end{aligned} \tag{A1}$$

Substituting them into the consistency condition that $\partial_t \delta_{z_s^*(w_s^*)}|\varphi_z\rangle = \delta_{z_s^*(w_s^*)}\partial_t|\varphi_z\rangle$, the two O operators must satisfy Eqs. (20) and (21). So the coefficients obey the following equations:

$$\begin{aligned}\partial_t f_{z1}(t,s) &= (i\omega + F_{z1}^\alpha + F_{w1}^\beta) f_{z1}, \\ \partial_t f_{z2}(t,s) &= (-i\omega - F_{z1}^\alpha - F_{w1}^\beta) f_{z2} + 2(F_{z2}^\alpha + F_{w2}^\beta) f_{z1} \\ &\quad - J_{z1}^\alpha(t,s) - J_{w1}^\beta(t,s), \\ \partial_t f_{w1}(t,s) &= (i\omega + F_{z1}^\alpha + F_{w1}^\beta) f_{w1}, \\ \partial_t f_{w2}(t,s) &= (-i\omega - F_{z1}^\alpha - F_{w1}^\beta) f_{w2} + 2(F_{z2}^\alpha + F_{w2}^\beta) f_{w1} \\ &\quad - J_{z2}^\alpha(t,s) - J_{w2}^\beta(t,s), \\ \partial_t j_{z1}(t,s,s') &= [J_{z1}^\alpha(t,s') + J_{w1}^\beta(t,s')] f_{z1}(t,s), \\ \partial_t j_{z2}(t,s,s') &= [J_{z2}^\alpha(t,s') + J_{w2}^\beta(t,s')] f_{z1}(t,s), \\ \partial_t j_{w1}(t,s,s') &= [J_{z1}^\alpha(t,s') + J_{w1}^\beta(t,s')] f_{w1}(t,s), \\ \partial_t j_{w2}(t,s,s') &= [J_{z2}^\alpha(t,s') + J_{w2}^\beta(t,s')] f_{w1}(t,s), \end{aligned} \tag{A2}$$

with the initial conditions

$$\begin{aligned} f_{z1}(t=s,s) &= 1, \\ f_{z2}(t=s,s) &= 0, \\ f_{w1}(-t=s, s<0) &= 0, \\ f_{w2}(-t=s, s<0) &= 1, \\ j_{z1}(t,s,s'=t) &= f_{z2}(t,s), \\ j_{z2}(t,s,s'=-t) &= -f_{z1}(t,s), \\ j_{w1}(t,s,s'=t) &= f_{w2}(t,s), \\ j_{w2}(t,s,s'=-t) &= -f_{w1}(t,s). \end{aligned} \tag{A3}$$

As shown in Eqs. (20) and (21), the evolution of O operators depends on two new operators $\bar{O}_z^\alpha$ and $\bar{O}_z^\beta$, which correspond to the coefficients $F_{z1(2)}^\alpha$, $F_{w1(2)}^\beta$, $J_{z1(2)}^\alpha$, and $J_{w1(2)}^\beta$ in Eq. (A2). According to the relations between $f_{z(w)1(2)}$, $j_{z(w)1(2)}$ and $F_{z1(2)}^\alpha$, $F_{w1(2)}^\beta$, $J_{z1(2)}^\alpha$, $J_{w1(2)}^\beta$ demonstrated in Eq. (26), we recognize that Eq. (A2) is essentially a set of integro-differential equations of $f_{z(w)1(2)}$, $j_{z(w)1(2)}$. More tricks frequently used in solving the equations can be found in Appendix B.

APPENDIX B: EVOLUTION EQUATIONS OF COEFFICIENTS IN OPERATORS $\bar{O}_z^\alpha$ AND $\bar{O}_w^\beta$

Due to the definition,

$$\begin{aligned}\bar{O}_z^\alpha(t) &\equiv \int_0^t ds\,\alpha(t,s)\hat{O}_z(t,s), \\ \bar{O}_w^\beta(t) &\equiv \int_{-t}^0 ds\,\beta(t,s)\hat{O}_w(t,s),\end{aligned}$$

the ansatzes of $\bar{O}_z^\alpha$ and $\bar{O}_w^\beta$ read

$$\begin{aligned}\bar{O}_z^\alpha(t) \equiv\ & F_{z1}^\alpha(t)\hat{a} + F_{z2}^\alpha(t)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* J_{z1}^\alpha(t,s') \\ & + \int_{-t}^0 ds' w_{s'}^* J_{z2}^\alpha(t,s'), \\ \bar{O}_w^\beta(t) \equiv\ & F_{w1}^\beta(t)\hat{a} + F_{w2}^\beta(t)\hat{a}^\dagger + \int_0^t ds' z_{s'}^* J_{w1}^\beta(t,s') \\ & + \int_{-t}^0 ds' w_{s'}^* J_{w2}^\beta(t,s'), \quad (s<0), \end{aligned} \tag{B1}$$

where

$$\begin{aligned} F_{zi}^\alpha(t) &\equiv \int_0^t ds\,\alpha(t,s) f_{zi}(t,s), \\ F_{wi}^\beta(t) &\equiv \int_{-t}^0 ds\,\beta(t,s) f_{wi}(t,s) \\ &= -\tanh r \int_{-t}^0 ds\,\alpha(t,s) f_{wi}(t,s), \\ J_{zi}^\alpha(t,s') &\equiv \int_0^t ds\,\alpha(t,s) j_{zi}(t,s,s'), \\ J_{wi}^\beta(t,s') &\equiv \int_{-t}^0 ds\,\beta(t,s) j_{wi}(t,s,s') \\ &\equiv -\tanh r \int_{-t}^0 ds\,\alpha(t,s) j_{wi}(t,s,s'), (i=1,2). \end{aligned} \tag{B2}$$

The dynamics of coefficients $f_{z(w)1(2)}$ and $j_{z(w)1(2)}$ are governed by a set of integro-differential equations. However, if the correlation functions satisfy an exponential form, we can avoid solving the integro-differential equations numerically. For example, suppose $f(t,s)$ satisfies the equation

$$\begin{aligned} &\partial_t f(t,s) = F(t) f(t,s), \quad f(t,s=t) = 1, \\ &F(t) = \int_0^t ds\,\alpha(t,s) f(t,s), \\ &\alpha(t,s) = e^{-\gamma(t-s)}. \end{aligned} \tag{B3}$$

With the given partial differential equation of $f(t,s)$, we can obtain the differential equation of $F(t)$ using the formula

$$\begin{aligned}\frac{d}{dt}F(t) &= \alpha(t,s=t) f(t,s=t) + \int_0^t ds \frac{\partial\alpha(t,s)}{\partial t} f(t,s) \\ &\quad + \int_0^t ds\,\alpha(t,s)\frac{\partial f(t,s)}{\partial t} \\ &= 1 - \gamma F + F^2. \end{aligned} \tag{B4}$$

Similarly, in our QSD equation, the coefficients $F^{\alpha}_{z1(2)}$, $F^{\beta}_{w1(2)}$, $J^{\alpha}_{z1(2)}$, and $J^{\beta}_{w1(2)}$ can be determined by a group of ordinary differential equations. Noting that this method only works for exponential form correlation functions.

In our work we consider the correlation function of Ornstein-Uhlenbeck process

$$\alpha(t,s)=\frac{\Gamma\gamma}{2}e^{-\gamma|t-s|-i\Omega(t-s)}. \tag{B5}$$

Consequently, the coefficients of the operators $\bar{O}^{\alpha}_{z}$ and $\bar{O}^{\beta}_{w}$ satisfy the following equations:

$$\begin{aligned}
\partial_t F^{\alpha}_{z1}(t) &= (i\omega - i\Omega - \gamma + F^{\alpha}_{z1} + F^{\beta}_{w1})F^{\alpha}_{z1} + \frac{\Gamma\gamma}{2},\\
\partial_t F^{\alpha}_{z2}(t) &= (-i\omega - i\Omega - \gamma - F^{\alpha}_{z1} - F^{\beta}_{w1})F^{\alpha}_{z2}\\
&\quad + 2(F^{\alpha}_{z2} + F^{\beta}_{w2})F^{\alpha}_{z1} - \tilde{J}^{\alpha\alpha}_{z1}(t) - \tilde{J}^{\beta\alpha}_{w1}(t),\\
\partial_t F^{\beta}_{w1}(t) &= (i\omega - i\Omega - \gamma + F^{\alpha}_{z1} + F^{\beta}_{w1})F^{\beta}_{w1},\\
\partial_t F^{\beta}_{w2}(t) &= (-i\omega - i\Omega - \gamma - F^{\alpha}_{z1} - F^{\beta}_{w1})F^{\beta}_{w2}\\
&\quad + 2(F^{\alpha}_{z2} + F^{\beta}_{w2})F^{\beta}_{w1} - \tilde{J}^{\alpha\beta}_{z2}(t) - \tilde{J}^{\beta\beta}_{w2}(t)\\
&\quad - \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t}e^{-2i\Omega t},\\
\partial_t J^{\alpha}_{z1}(t,s) &= (-\gamma - i\Omega)J^{\alpha}_{z1} + (J^{\alpha}_{z1} + J^{\beta}_{w1})F^{\alpha}_{z1},\\
\partial_t J^{\alpha}_{z2}(t,s) &= (-\gamma - i\Omega)J^{\alpha}_{z2} + (J^{\alpha}_{z2} + J^{\beta}_{w2})F^{\alpha}_{z1},\\
\partial_t J^{\beta}_{w1}(t,s) &= (-\gamma - i\Omega)J^{\beta}_{w1} + (J^{\alpha}_{z1} + J^{\beta}_{w1})F^{\beta}_{w1},\\
\partial_t J^{\beta}_{w2}(t,s) &= (-\gamma - i\Omega)J^{\beta}_{w2} + (J^{\alpha}_{z2} + J^{\beta}_{w2})F^{\beta}_{w1},
\end{aligned} \tag{B6}$$

with the initial conditions

$$\begin{aligned}
J^{\alpha}_{z1}(t,s=t) &= F^{\alpha}_{z2}(t),\\
J^{\alpha}_{z2}(t,s=-t<0) &= -F^{\alpha}_{z1}(t),\\
J^{\beta}_{w1}(t,s=t) &= F^{\beta}_{w2}(t),\\
J^{\beta}_{w2}(t,s=-t<0) &= -F^{\beta}_{w1}(t).
\end{aligned} \tag{B7}$$

Here, to make the above group of differential equations complete, we define two new sets of to-be-determined coefficients:

$$\begin{aligned}
\tilde{J}^{\alpha\alpha}_{z1}(t) &\equiv \int_0^t ds\,\alpha(t,s)J^{\alpha}_{z1}(t,s),\\
\tilde{J}^{\beta\alpha}_{w1}(t) &\equiv \int_0^t ds\,\alpha(t,s)J^{\beta}_{w1}(t,s),\\
\tilde{J}^{\alpha\beta}_{z2}(t) &\equiv \int_{-t}^0 ds\,\beta(t,s)J^{\alpha}_{z2}(t,s),\\
\tilde{J}^{\beta\beta}_{w2}(t) &\equiv \int_{-t}^0 ds\,\beta(t,s)J^{\beta}_{w2}(t,s),
\end{aligned} \tag{B8}$$

and

$$\begin{aligned}
\tilde{J}^{\alpha\gamma}_{z1}(t) &\equiv \int_0^t ds\,\gamma(-t,s)J^{\alpha}_{z1}(t,s),\\
\tilde{J}^{\beta\gamma}_{w1}(t) &\equiv \int_0^t ds\,\gamma(-t,s)J^{\beta}_{w1}(t,s),\\
\tilde{J}^{\alpha\epsilon}_{z2}(t) &\equiv \int_{-t}^0 ds\,\epsilon(-t,s)J^{\alpha}_{z2}(t,s),\\
\tilde{J}^{\beta\epsilon}_{w2}(t) &\equiv \int_{-t}^0 ds\,\epsilon(-t,s)J^{\beta}_{w2}(t,s),
\end{aligned} \tag{B9}$$

where $\gamma(t,s)\equiv\mathcal{M}(w_t z_s^*)=-\tanh r\alpha(t,s)$ is the cross-correlation function and $\epsilon(t,s)\equiv\mathcal{M}(w_t w_s^*)=\tanh^2 r\alpha(t,s)$ is the correlation function. They are governed by the following evolution equations:

$$\begin{aligned}
\partial_t\tilde{J}^{\alpha\alpha}_{z1} &= (-2\gamma - 2i\Omega + F^{\alpha}_{z1})\tilde{J}^{\alpha\alpha}_{z1} + F^{\alpha}_{z1}\tilde{J}^{\beta\alpha}_{w1} + \frac{\Gamma\gamma}{2}F^{\alpha}_{z2},\\
\partial_t\tilde{J}^{\alpha\beta}_{z2} &= (-2\gamma - 2i\Omega + F^{\alpha}_{z1})\tilde{J}^{\alpha\beta}_{z2} + F^{\alpha}_{z1}\tilde{J}^{\beta\beta}_{w2}\\
&\quad + \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t-2i\Omega t}F^{\alpha}_{z1},\\
\partial_t\tilde{J}^{\beta\alpha}_{w1} &= (-2\gamma - 2i\Omega + F^{\beta}_{w1})\tilde{J}^{\beta\alpha}_{w1} + F^{\beta}_{w1}\tilde{J}^{\alpha\alpha}_{z1} + \frac{\Gamma\gamma}{2}F^{\beta}_{w2},\\
\partial_t\tilde{J}^{\beta\beta}_{w2} &= (-2\gamma - 2i\Omega + F^{\beta}_{w1})\tilde{J}^{\beta\beta}_{w2} + F^{\beta}_{w1}\tilde{J}^{\alpha\beta}_{z2}\\
&\quad + \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t-2i\Omega t}F^{\beta}_{w1},\\
\partial_t\tilde{J}^{\alpha\gamma}_{z1} &= (-2\gamma + F^{\alpha}_{z1})\tilde{J}^{\alpha\gamma}_{z1} + F^{\alpha}_{z1}\tilde{J}^{\beta\gamma}_{w1}\\
&\quad - \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t+2i\Omega t}F^{\alpha}_{z2}\\
\partial_t\tilde{J}^{\alpha\epsilon}_{z2} &= (-2\gamma + F^{\alpha}_{z1})\tilde{J}^{\alpha\epsilon}_{z2} + F^{\alpha}_{z1}\tilde{J}^{\beta\epsilon}_{w2} - \frac{\Gamma\gamma}{2}\tanh^2 rF^{\alpha}_{z1},\\
\partial_t\tilde{J}^{\beta\gamma}_{w1} &= (-2\gamma + F^{\beta}_{w1})\tilde{J}^{\beta\gamma}_{w1} + F^{\beta}_{w1}\tilde{J}^{\alpha\gamma}_{z1}\\
&\quad - \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t+2i\Omega t}F^{\beta}_{w2},\\
\partial_t\tilde{J}^{\beta\epsilon}_{w2} &= (-2\gamma + F^{\beta}_{w1})\tilde{J}^{\beta\epsilon}_{w2} + F^{\beta}_{w1}\tilde{J}^{\alpha\epsilon}_{z2} - \frac{\Gamma\gamma}{2}\tanh^2 rF^{\beta}_{w1}.
\end{aligned} \tag{B10}$$

As we proceed with the derivation of the master equation, we encounter the emergence of several new terms resulting from the application of the Novikov theorem in Eq. (32):

$$\begin{aligned}
F^{\gamma}_{zi}(t) &\equiv \int_0^t ds\,\gamma(-t,s)f_{z1}(t,s)\\
&= -\tanh r\int_0^t ds\,\alpha(-t,s)f_{z1}(t,s),\\
F^{\epsilon}_{wi}(t) &\equiv \int_{-t}^0 ds\,\epsilon(-t,s)f_{wi}(t,s)\\
&= \tanh^2 r\int_{-t}^0 ds\,\alpha(-t,s)f_{wi}(t,s), \quad (i=1,2).
\end{aligned} \tag{B11}$$

The above four coefficients can be numerically determined using the following evolution equations:

$$\begin{aligned}
\partial_t F^{\gamma}_{z1}(t) &= (i\omega + i\Omega - \gamma + F^{\alpha}_{z1} + F^{\beta}_{w1})F^{\gamma}_{z1}(t)\\
&\quad - \frac{\Gamma\gamma}{2}\tanh r e^{-2\gamma t}e^{2i\Omega t},\\
\partial_t F^{\gamma}_{z2}(t) &= [-i\omega + i\Omega - \gamma - F^{\alpha}_{z1}(t) - F^{\beta}_{w1}(t)]F^{\gamma}_{z2}(t)\\
&\quad + 2[F^{\alpha}_{z2}(t) + F^{\beta}_{w2}(t)]F^{\gamma}_{z1}(t) - \tilde{J}^{\alpha\gamma}_{z1}(t) - \tilde{J}^{\beta\gamma}_{w1}(t),
\end{aligned}$$

$$\partial_t F^{\epsilon}_{w1}(t) = (i\omega + i\Omega - \gamma + F^{\alpha}_{z1} + F^{\beta}_{w1})F^{\epsilon}_{w1}(t),$$
$$\partial_t F^{\epsilon}_{w2}(t) = [-i\omega + i\Omega - \gamma - F^{\alpha}_{z1}(t) - F^{\beta}_{w1}(t)]F^{\epsilon}_{w2}(t) + 2[F^{\alpha}_{z2}(t) + F^{\beta}_{w2}(t)]F^{\epsilon}_{w1}(t) - \tilde{J}^{\alpha\epsilon}_{z2}(t) - \tilde{J}^{\beta\epsilon}_{w2}(t) + \frac{\Gamma\gamma}{2}\tanh^2 r. \quad \text{(B12)}$$

Because the evolution equations of the coefficients have a similar structure, we identify F^z_1 and F^z_2 in Eq. (31) and similarly F^w_1 and F^w_2:

$$F^z_1 \equiv F^{\alpha}_{z1} + F^{\beta}_{w1}, \quad F^z_2 \equiv F^{\alpha}_{z2} + F^{\beta}_{w2},$$
$$F^w_1 \equiv F^{\gamma}_{z1} + F^{\epsilon}_{w1}, \quad F^w_2 \equiv F^{\gamma}_{z2} + F^{\epsilon}_{w2}. \quad \text{(B13)}$$

We now briefly demonstrate the Markov limit of these coefficients. Firstly, when the memory factor $\gamma \to \infty$, the correlation function $\alpha(t,s) = \mathcal{M}(z_t z^*_s) \to \Gamma\delta(t,s)$, $\epsilon(t,s) = \mathcal{M}(w_t w^*_s) \to \tanh^2 r\, \Gamma\delta(t,s)$, and the cross-correlation function satisfies $\mathcal{M}(z_t w^*_s) = 0$ for $t > 0, s < 0$ (setting $\Omega = 0$ for simplicity). Consequently, $F^{\beta}_{w1} = F^{\beta}_{w2} = F^{\gamma}_{z1} = F^{\gamma}_{z2} = 0$. Using the initial conditions in Eq. (A3), we can explicitly calculate the coefficients in Eq. (33),

$$F^z_1 = F^{\alpha}_{z1} = \frac{\Gamma}{2} f_{z1}(t,t) = \frac{\Gamma}{2},$$
$$F^z_2 = F^{\alpha}_{z2} = \frac{\Gamma}{2} f_{z2}(t,t) = 0,$$
$$F^w_1 = F^{\epsilon}_{w1} = \frac{\Gamma}{2}\tanh^2 r f_{w1}(t,t) = 0,$$
$$F^w_2 = F^{\epsilon}_{w2} = \frac{\Gamma}{2}\tanh^2 r f_{w2}(t,t) = \frac{\Gamma}{2}\tanh^2 r. \quad \text{(B14)}$$

As a result, the four superoperators in Eq. (33) can be rewritten in the Markov limit as

$$\mathcal{L}_1[\hat\rho_r] \equiv \frac{\Gamma}{2}(2\hat a\hat\rho_r\hat a^\dagger - \hat\rho_r\hat a^\dagger\hat a - \hat a^\dagger\hat a\hat\rho_r),$$
$$\mathcal{L}_2[\hat\rho_r] \equiv 0,$$
$$\mathcal{L}_3[\hat\rho_r] \equiv \frac{\Gamma}{2}\tanh^2 r(2\hat a^\dagger\hat\rho_r\hat a - \hat\rho_r\hat a\hat a^\dagger - \hat a\hat a^\dagger\hat\rho_r),$$
$$\mathcal{L}_4[\hat\rho_r] \equiv 0.$$

Finally, Eq. (39) in the main manuscript can be re-produced from Eq. (33) under the Markov limit.

APPENDIX C: DERIVATION OF EQ. (32)

Here we apply the Novikov theorem [37,49] to compute the four terms $\mathcal{M}(z^*_t\hat P_\psi)$, $\mathcal{M}(w^*_{-t}\hat P_\psi)$, $\mathcal{M}(z_t\hat P_\psi)$, and $\mathcal{M}(w_{-t}\hat P_\psi)$, respectively:

$$\begin{aligned}\mathcal{M}(z^*_t\hat P_\psi) &= -i\sum_k g^*_k e^{i\omega_k t}\mathcal{M}(\partial_{z_k}\hat P_\psi)\\ &= -i\sum_k g^*_k e^{i\omega_k t}(-\tanh r_k)\mathcal{M}(z_k\hat P_\psi) - i\sum_k g^*_k e^{i\omega_k t}\mathcal{M}(|K|^2\partial_{z_k}\hat P_\varphi)\\ &= -\mathcal{M}(w_{-t}\hat P_\psi) + \int_0^t ds\mathcal{M}\big(|K|^2\mathcal{M}(z_s z^*_t)\delta_{z_s}\hat P_\varphi\big) + \int_{-t}^0 ds\mathcal{M}\big(|K|^2\mathcal{M}(w_s z^*_t)\delta_{w_s}\hat P_\varphi\big)\\ &= -\mathcal{M}(w_{-t}\hat P_\psi) + \int_0^t ds\mathcal{M}(z_s z^*_t)\mathcal{M}\big(\hat P_\psi\hat O^\dagger_z(t,s)\big) + \int_{-t}^0 ds\mathcal{M}(w_s z^*_t)\mathcal{M}\big(\hat P_\psi\bar O^\dagger_w(t,s)\big),\end{aligned} \quad \text{(C1)}$$

$$\begin{aligned}\mathcal{M}(w^*_{-t}\hat P_\psi) &= i\sum_k g_k e^{-i\omega_k t}\tanh r_k\mathcal{M}(\partial_{z_k}\hat P_\psi)\\ &= i\sum_k g_k e^{-i\omega_k t}(-\tanh^2 r_k)\mathcal{M}(z_k\hat P_\psi) + i\sum_k g_k e^{-i\omega_k t}\tanh r_k\mathcal{M}(|K|^2\partial_{z_k}\hat P_\varphi)\\ &= -\tanh^2 r\mathcal{M}(z_t\hat P_\psi) + \int_0^t ds\mathcal{M}\big(|K|^2\mathcal{M}(z_s w^*_{-t})\delta_{z_s}\hat P_\varphi\big) + \int_{-t}^0 ds\mathcal{M}\big(|K|^2\mathcal{M}(w_s w^*_{-t})\delta_{w_s}\hat P_\varphi\big),\\ &= -\tanh^2 r\mathcal{M}(z_t\hat P_\psi) + \int_0^t ds\mathcal{M}(z_s w^*_{-t})\mathcal{M}\big(\hat P_\psi\hat O^\dagger_z(t,s)\big) + \int_{-t}^0 ds\mathcal{M}(w_s w^*_{-t})\mathcal{M}\big(\hat P_\psi\hat O^\dagger_w(t,s)\big).\end{aligned} \quad \text{(C2)}$$

Combining Eqs. (C1) and (C2), we can obtain the expressions of $\mathcal{M}(z^*_t\hat P_\psi)$ and $\mathcal{M}(w^*_{-t}\hat P_\psi)$:

$$\begin{aligned}\mathcal{M}(z^*_t\hat P_\psi) &= \cosh^2 r\Big[\int_0^t ds\mathcal{M}(z_s z^*_t)\mathcal{M}\big(\hat P_\psi\hat O^\dagger_z(t,s)\big) + \int_{-t}^0 ds\mathcal{M}(w_s z^*_t)\mathcal{M}\big(\hat P_\psi\hat O^\dagger_w(t,s)\big)\\ &\quad - \int_0^t ds\mathcal{M}(z^*_s w_{-t})\mathcal{M}\big(\hat O_z(t,s)\hat P_\psi\big) - \int_{-t}^0 ds\mathcal{M}(w^*_s w_{-t})\mathcal{M}\big(\hat O_w(t,s)\hat P_\psi\big)\Big],\\ \mathcal{M}(w^*_{-t}\hat P_\psi) &= -\sinh^2 r\Big[\int_0^t ds\mathcal{M}(z_t z^*_s)\mathcal{M}\big(\hat O_z(t,s)\hat P_\psi\big) + \int_{-t}^0 ds\mathcal{M}(z_t w^*_s)\mathcal{M}\big(\hat O_w(t,s)\hat P_\psi\big)\Big]\\ &\quad + \cosh^2 r\Big[\int_0^t ds\mathcal{M}(w^*_{-t}z_s)\mathcal{M}\big(\hat P_\psi\hat O^\dagger_z(t,s)\big) + \int_{-t}^0 ds\mathcal{M}(w^*_{-t}w_s)\mathcal{M}\big(\hat P_\psi\hat O^\dagger_w(t,s)\big)\Big].\end{aligned} \quad \text{(C3)}$$

The explicit expressions of $\mathcal{M}(z_t\hat P_\psi)$ and $\mathcal{M}(z_t\hat P_\psi)$ can be easily obtained through their Hermitian conjugate operators: $\mathcal{M}(z_t\hat P_\psi) = [\mathcal{M}(z^*_t\hat P_\psi)]^\dagger$, $\mathcal{M}(w_{-t}\hat P_\psi) = [\mathcal{M}(w^*_{-t}\hat P_\psi)]^\dagger$.

When the processes are Markovian, the correlation functions turn into $\alpha(t,s) = \mathcal{M}(z_t z^*_s) \to \Gamma\delta(t,s)$, $\epsilon(t,s) = \mathcal{M}(w_t w^*_s) \to \tanh^2 r\,\Gamma\delta(t,s)$, and the cross-correlation function satisfies $\mathcal{M}(z_t w_s^*) = 0$ for $t > 0, s < 0$. Equations (C1) and (C2) are reduced to

$$\mathcal{M}(z_t^* \hat{P}_\psi) = -\mathcal{M}(w_{-t}\hat{P}_\psi) + \int_0^t ds \Gamma \delta(t,s) \mathcal{M}\big(\hat{P}_\psi \hat{O}_z^\dagger(t,s)\big)$$

$$\mathcal{M}(w_{-t}^* \hat{P}_\psi) = -\tanh^2 r \mathcal{M}(z_t \hat{P}_\psi) + \int_{-t}^0 ds \Gamma \delta(-t,s) \tanh^2 r \times \mathcal{M}\big(\hat{P}_\psi \hat{O}_w^\dagger(t,s)\big). \tag{C4}$$

Combined with the initial conditions $\hat{O}_z(t, s=t) = \hat{L}$ and $\hat{O}_w(t, s=-t) = \hat{L}^\dagger$, we obtain the conclusion

$$\mathcal{M}(w_{-t}^* \hat{P}_\psi) = -\tanh^2 r \mathcal{M}(z_t \hat{P}_\psi) + \tanh^2 r \frac{\Gamma}{2} \hat{\rho}_r \hat{L},$$

$$\mathcal{M}(z_t^* \hat{P}_\psi) = -\mathcal{M}(w_{-t}\hat{P}_\psi) + \frac{\Gamma}{2} \hat{\rho}_r \hat{L}^\dagger. \tag{C5}$$

[1] L. C. G. Govia, A. Lingenfelter, and A. A. Clerk, Stabilizing two-qubit entanglement by mimicking a squeezed environment, Phys. Rev. Res. **4**, 023010 (2022).

[2] S. Feng, D. He, and B. Xie, Quantum theory of phase-sensitive heterodyne detection, J. Opt. Soc. Am. B **33**, 1365 (2016).

[3] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Quantum interface between light and atomic ensembles, Rev. Mod. Phys. **82**, 1041 (2010).

[4] J.-T. Hsiang and B.-L. Hu, Fluctuation-dissipation relation for a quantum brownian oscillator in a parametrically squeezed thermal field, Ann. Phys. **433**, 168594 (2021).

[5] A. I. Lvovsky, Squeezed light, in *Photonics* (John Wiley & Sons, Ltd., 2015), Chap. 5, pp. 121–163.

[6] Y.-H. Ma, Q.-Z. Ding, and E. Wu, Entanglement of two nitrogen-vacancy ensembles via a nanotube, Phys. Rev. A **101**, 022311 (2020).

[7] L. Magazzù, P. Talkner, and P. Hänggi, Quantum Brownian motion under generalized position measurements: A converse Zeno scenario, New J. Phys. **20**, 033001 (2018).

[8] A. M. Marino, C. R. Stroud, Jr., V. Wong, R. S. Bennink, and R. W. Boyd, Bichromatic local oscillator for detection of two-mode squeezed states of light, J. Opt. Soc. Am. B **24**, 335 (2007).

[9] W. Song, W. Yang, J. An, and M. Feng, Dissipation-assisted spin squeezing of nitrogen-vacancy centers coupled to a rectangular hollow metallic waveguide, Opt. Express **25**, 19226 (2017).

[10] M. C. Teich and B. E. A. Saleh, Squeezed state of light, Quantum Opt. **1**, 153 (1989).

[11] B. Xie, P. Yang, and S. Feng, Phase-sensitive heterodyne detection of two-mode squeezed light without noise penalty, J. Opt. Soc. Am. B **35**, 2342 (2018).

[12] Y.-C. Zhang, X.-F. Zhou, X. Zhou, G.-C. Guo, and Z.-W. Zhou, Cavity-Assisted Single-Mode and Two-Mode Spin-Squeezed States via Phase-Locked Atom-Photon Coupling, Phys. Rev. Lett. **118**, 083604 (2017).

[13] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A **50**, 67 (1994).

[14] Y. Ma, H. Miao, B. H. Pang, M. Evans, C. Zhao, J. Harms, R. Schnabel, and Y. Chen, Proposal for gravitational-wave detection beyond the standard quantum limit through EPR entanglement, Nat. Phys. **13**, 776 (2017).

[15] S. L. Danilishin, F. Y. Khalili, and H. Miao, Advanced quantum techniques for future gravitational-wave detectors, Living Rev. Relativ. **22**, 2 (2019).

[16] L. McCuller, C. Whittle, D. Ganapathy, K. Komori, M. Tse, A. Fernandez-Galiana, L. Barsotti, P. Fritschel, M. MacInnis, F. Matichard, K. Mason, N. Mavalvala, R. Mittleman, H. Yu, M. E. Zucker, and M. Evans, Frequency-Dependent Squeezing for Advanced LIGO, Phys. Rev. Lett. **124**, 171102 (2020).

[17] J. Ma, X. Wang, C. Sun, and F. Nori, Quantum spin squeezing, Phys. Rep. **509**, 89 (2011).

[18] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Experimental long-lived entanglement of two macroscopic objects, Nature (London) **413**, 400 (2001).

[19] S.-Y. Bai and J.-H. An, Generating Stable Spin Squeezing by Squeezed-Reservoir Engineering, Phys. Rev. Lett. **127**, 083602 (2021).

[20] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Spin Squeezed Atoms: A Macroscopic Entangled Ensemble Created by Light, Phys. Rev. Lett. **83**, 1319 (1999).

[21] Y.-H. Ma, Q.-Z. Ding, and T. Yu, Persistent spin squeezing of a dissipative one-axis twisting model embedded in a general thermal environment, Phys. Rev. A **101**, 022327 (2020).

[22] Z. Zhang, L. Shao, W. Lu, Y. Su, Y.-P. Wang, J. Liu, and X. Wang, Single-photon-triggered spin squeezing with decoherence reduction in optomechanics via phase matching, Phys. Rev. A **104**, 053517 (2021).

[23] Y.-H. Ma, Y. Xu, Q.-Z. Ding, and Y.-S. Chen, Preparation of spin squeezed state in SiV centers coupled by diamond waveguide, Chin. Phys. B **30**, 100311 (2021).

[24] S. Zippilli and F. Illuminati, Non-Markovian dynamics and steady-state entanglement of cavity arrays in finite-bandwidth squeezed reservoirs, Phys. Rev. A **89**, 033803 (2014).

[25] N. Megier, W. T. Strunz, C. Viviescas, and K. Luoma, Parametrization and Optimization of Gaussian Non-Markovian Unravelings for Open Quantum Dynamics, Phys. Rev. Lett. **120**, 150402 (2018).

[26] V. Link, W. T. Strunz, and K. Luoma, Non-Markovian quantum dynamics in a squeezed reservoir, Entropy **24**, 352 (2022).

[27] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, New York, 2002).

[28] C. W. Gardiner and P. Zoller, *Quantum Noise*, 2nd ed., edited by H. Haken (Springer, New York, 2000).

[29] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[30] A. G. Redfield, On the theory of relaxation processes, IBM J. Res. Dev. **1**, 19 (1957).

[31] T. Ma, Y. Chen, T. Chen, S. R. Hedemann, and T. Yu, Crossover between non-Markovian and Markovian dynamics induced by a hierarchical environment, Phys. Rev. A **90**, 042108 (2014).

[32] K. Mølmer, Y. Castin, and J. Dalibard, Monte Carlo wavefunction method in quantum optics, J. Opt. Soc. Am. B **10**, 524 (1993).

[33] J. Dalibard, Y. Castin, and K. Mølmer, Wave-Function Approach to Dissipative Processes in Quantum Optics, Phys. Rev. Lett. **68**, 580 (1992).
[34] H. Carmichael, *An Open Systems Approach to Quantum Optics: Lectures Presented at the Université Libre de Bruxelles, October 28 to November 4, 1991*, v. 18 (Springer Berlin Heidelberg, 1993).
[35] N. Gisin and I. C. Percival, The quantum-state diffusion model applied to open systems, J. Phys. A: Math. Gen. **25**, 5677 (1992).
[36] W. T. Strunz, Linear quantum state diffusion for non-Markovian open quantum systems, Phys. Lett. A **224**, 25 (1996).
[37] T. Yu, L. Diósi, N. Gisin, and W. T. Strunz, Non-Markovian quantum-state diffusion: Perturbation approach, Phys. Rev. A **60**, 91 (1999).
[38] W. T. Strunz, L. Diósi, and N. Gisin, Open System Dynamics with Non-Markovian Quantum Trajectories, Phys. Rev. Lett. **82**, 1801 (1999).
[39] J. Jing and T. Yu, Non-Markovian Relaxation of a Three-Level System: Quantum Trajectory Approach, Phys. Rev. Lett. **105**, 240403 (2010).
[40] Y. Chen, J. Q. You, and T. Yu, Exact non-Markovian master equations for multiple qubit systems: Quantum-trajectory approach, Phys. Rev. A **90**, 052104 (2014).
[41] Y. Chen, J. Q. You, and T. Yu, Non-Markovian quantum interference in multilevel quantum systems: Exact master equation approach, Quantum Inf. Comput. **18**, 1261 (2018).
[42] Z.-M. Wang, F.-H. Ren, D.-W. Luo, Z.-Y. Yan, and L.-A. Wu, Almost-exact state transfer by leakage-elimination-operator control in a non-Markovian environment, Phys. Rev. A **102**, 042406 (2020).
[43] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 2012).
[44] W. T. Strunz and T. Yu, Convolutionless non-Markovian master equations and quantum trajectories: Brownian motion, Phys. Rev. A **69**, 052115 (2004).
[45] T. Yu, Non-markovian quantum trajectories versus master equations: Finite-temperature heat bath, Phys. Rev. A **69**, 062107 (2004).
[46] L. Diósi, N. Gisin, and W. T. Strunz, Non-Markovian quantum state diffusion, Phys. Rev. A **58**, 1699 (1998).
[47] C. J. Broadbent, J. Jing, T. Yu, and J. H. Eberly, Solving non-Markovian open quantum systems with multi-channel reservoir coupling, Ann. Phys. **327**, 1962 (2012).
[48] Q. Ding, P. Zhao, Y. Ma, and Y. Chen, Impact of the central frequency of environment on non-Markovian dynamics in piezoelectric optomechanical devices, Sci. Rep. **11**, 1814 (2021).
[49] E. A. Novikov, Functionals and the random-force method in turbulence theory, Sov. J. Exp. Theor. Phys. **20**, 1290 (1965).