Effects of non-Markovianity on daemonic ergotropy in the quantum switch

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The quantum switch implements indefinite causal order via a coherent control of the orderings of multiple quantum operations, leading to various advantages and applications, one of which being in work extraction where enhancements to daemonic ergotropy is possible. Motivated by recent developments in the connections between non-Markovianity and the quantum switch, we construct a non-Markovian process that reduces to the two-party quantum switch in the fully non-Markovian limit. By controlling the amount of non-Markovianity in the process, we identified two operational regimes with differing behaviors. One has its daemonic ergotropy dependent on the presence and amount of non-Markovianity, achieving the maximum in the quantum switch case of full non-Markovianity. The other regime, however, has no advantages from non-Markovianity. We compare this non-Markovian process with the case of a superposition of independent channels, where two channels are placed in a coherent superposition without indefinite causal order, uncovering the advantages of non-Markovianity. Finally, the conditions required for the production of positive daemonic ergotropy are also derived for the case of fully non-Markovian and fully Markovian limits, where we compare against the conditions required for the superposition of independent channels.

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I. INTRODUCTION

The phenomenon of indefinite causal order, where the orderings of multiple quantum operations can be in a state of quantum uncertainty, has seen much attention in recent years [1–3]. This was most apparent in the field of quantum information theory after the characterization of the two-party quantum switch [4], a setup that implements indefinite causal order by controlling the orderings of two quantum channels Φ_A and Φ_B acting on a main quantum system with a control qubit. For example, suppose that the operation order $\Phi_B \circ \Phi_A$ is set to occur if the control qubit is in a particular state $|a\rangle$, and that the alternate order $\Phi_A \circ \Phi_B$ occurs if it is in a state $|b\rangle$ orthogonal to $|a\rangle$; then indefinite causal order can be achieved if the control qubit is in a quantum superposition of the two orthogonal states $|c\rangle = \sqrt{\alpha}|a\rangle + \sqrt{1-\alpha}|b\rangle$ for $\alpha \neq 0, 1$. This is referred to as a controlled superposition.

Combined with its experimental viability [5–9], there have been numerous studies on the quantum switch's application, which covers various quantum information tasks such as enhancements in computation [4,10–12], communication [13–15], and metrology [16–18]. For example, if Φ_A and Φ_B are completely depolarizing channels, such that communication is impossible individually, putting them in a quantum switch setup can result in nonzero Holevo and coherent information of the output state, implying the possibility for both classical and quantum communications, a phenomenon referred to as the perfect activation of quantum and classical capacities [19,20].

Naturally, the quantum switch's advantages were also studied in work extraction and refrigeration tasks [21–23]. Work extraction is an important topic in quantum thermodynamics, where quantum thermal machines facilitate cyclic conversion of heat from a heat bath into work or usable energy, which can be stored (charged) and extracted (discharged) in quantum batteries [24]. For the quantum switch, it was shown that enhancements to the maximum extractable work of the output state are possible. This is possible even if the channels Φ_A and Φ_B are fully thermalizing channels, which returns the thermal state individually with no extractable work.

These various quantum advantages of the quantum switch are usually attributed to its intrinsic indefinite causality. However, some authors have argued that some of these advantages can be replicated or even surpassed in systems without indefinite causal order [25], such as in a superposition of independent channels where Φ_A and Φ_B are placed in the controlled superposition [15,26], although a fair comparison might be difficult as these systems can utilize additional resources not present in the quantum switch [27,28]. Furthermore, there are recent developments in the connections between non-Markovianity and indefinite causal order, where non-Markovianity can replicate the advantages of the quantum switch [29], or that the quantum switch can enhance or activate hidden non-Markovianity in a system [30]. It was also shown in Ref. [31] that the implementation of the quantum switch has intrinsic non-Markovianity, where non-Markovian backflow of information is possible. The communication enhancements of the quantum switch, as well as the perfect activation of capacities, can then be explained from the backflow of information due to non-Markovianity, with the amount of enhancement dependent on the amount of non-Markovianity present.

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Motivated by these developments, we construct a non-Markovian process that has its non-Markovianity controllable, and will reduce to the quantum switch when it is fully non-Markovian. Focusing on input states that are thermal states, we show that the maximum extractable work of the output state, as quantified by the daemonic ergotropy [32], is dependent on the amount of non-Markovianity, achieving the maximum in the quantum switch case where non-Markovianity is maximal. Furthermore, in this non-Markovian process, the case of a superposition of independent channels is but an intermediate time step in the process. The comparison between this intermediate time step and the final time step of the process reveals that the presence of non-Markovianity is required to outperform the superposition of independent channels.

This paper is outlined as follows: In Sec. II, we review the definitions of a non-Markovian operation in the environmental representation (Sec. II A), from which we show the presence of non-Markovianity in the quantum switch (Sec. II B). In Sec. III A, we construct the non-Markovian process that reduces to the quantum switch in the fully non-Markovian limit, before computing the daemonic ergotropy as the maximum extractable work of this process in a three-stroke cycle heat engine in Sec. III B. In Sec. IV A, we present our results where the daemonic ergotropy is enhanced in certain operational regimes in the presence of non-Markovianity, and in Sec. IV B, we find the conditions for positive daemonic ergotropy for arbitrary bath temperatures in the prethermalization and full thermalization regimes. Finally, we conclude the paper in Sec. V.

II. PRELIMINARIES

A. Non-Markovian operation

A quantum operation or channel that acts on a quantum system Q in an interval $t_0 \to t_\tau$ can be described by a completely positive and trace-preserving (CPTP) map $\Phi_{t_0\to t_\tau}$, such that $\rho^Q_{t_\tau} = \Phi_{t_0\to t_\tau}(\rho^Q_{t_0})$. Such a channel admits an environmental representation where it can be represented by a bipartite unitary operation $U^{QA}_{t_0\to t_\tau}$ acting jointly on the system Q and an ancillary environmental system A, before tracing out A [33], i.e.,

$$\Phi_{t_0\to t_\tau}(\rho^Q) = \mathrm{Tr}_A\big[U^{QA}_{t_0\to t_\tau}(\rho^Q\otimes\rho^A)U^{\dagger QA}_{t_0\to t_\tau}\big]. \tag{1}$$

Note that the superscript indicates the subsystems of a quantum state, or the subsystems that a unitary is acting on. If the ancillary environmental system $\rho^A = \sigma^A_{\beta_A}$, where σ_β is the Gibbs or thermal state at inverse temperature $\beta = 1/k_BT$ with Hamiltonian H, i.e.,

$$\sigma_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}, \tag{2}$$

and the unitary operation U^{QA} conserves total energy with $[U^{QA}, H^Q + H^A] = 0$, then we say that $\Phi_{t_0\to t_\tau}$ is a thermal operation and that it is Gibbs preserving with $\Phi_{t_0\to t_\tau}(\sigma_\beta) = \sigma_\beta$ [34]. Furthermore, we say that the operation is Markovian and completely positive (CP)-divisible if it is divisible into other CPTP maps for all times $t_0 < t < t_\tau$ [35,36], i.e.,

$$\Phi_{t_0\to t_\tau} = \Phi_{t\to t_\tau}\circ\Phi_{t_0\to t}, \quad \forall\, t_0 < t < t_\tau. \tag{3}$$

Physically, this Markovian thermal operation describes the evolution of the quantum system Q if it is brought into contact with a Markovian heat bath at inverse temperature β_A.

A crucial requirement for Eq. (1) to hold is for the main and ancillary environmental systems to be in a product state with $\rho^Q\otimes\rho^A$, such that they are uncorrelated [37,38]. Therefore, if an operation $\Phi_{t_0\to t_2}$ is Markovian as in Eq. (3), and acts on an initial state $\rho^Q_{t_0}$ with $\rho^Q_{t_2} = \Phi_{t_0\to t_2}(\rho^Q_{t_0})$, we have

$$\rho^Q_{t_1} = \Phi_{t_0\to t_1}(\rho^Q_{t_0}) = \mathrm{Tr}_A\big[U^{QA}_{t_0\to t_1}(\rho^Q_{t_0}\otimes\sigma^A_{\beta_A})U^{\dagger QA}_{t_0\to t_1}\big], \tag{4}$$

$$\rho^Q_{t_2} = \Phi_{t_1\to t_2}(\rho^Q_{t_1}) = \mathrm{Tr}_{A'}\big[U^{QA'}_{t_1\to t_2}(\rho^Q_{t_1}\otimes\sigma^{A'}_{\beta_A})U^{\dagger QA'}_{t_1\to t_2}\big], \tag{5}$$

where we have $\Phi_{t_0\to t_2} = \Phi_{t_1\to t_2}\circ\Phi_{t_0\to t_1}$. Thus, the Markovian operation $\Phi_{t_0\to t_2}$ requires two independent environmental subsystems A and A' that are uncorrelated with the main system Q at the start of each interaction.

On the other hand, supposing if we have a channel $\Phi'_{t_0\to t_2}$ where the environment A' of the second time step is correlated to the main system Q such that they cannot be expressed as a product state, i.e.,

$$\rho^Q_{t_2} = \Gamma_{t_1\to t_2}(\rho^Q_{t_1}) = \mathrm{Tr}_{A'}\big[U^{QA'}_{t_1\to t_2}(\rho^{QA'}_{t_1})U^{\dagger QA'}_{t_1\to t_2}\big], \tag{6}$$

then we have $\Phi'_{t_0\to t_2} = \Gamma_{t_1\to t_2}\circ\Phi_{t_0\to t_1}$, where $\Gamma_{t_1\to t_2}$ is not a CPTP map in general, implying that $\Phi'_{t_0\to t_2}$ is a non-Markovian process as it does not fulfill the divisibility property in Eq. (3). This can happen if the environmental subsystem $A' = A$, where its correlations with Q in $\rho^{QA}_{t_1}$ comes from its previous interaction in $\Phi_{t_0\to t_1}$ [39]. Note that the presence of system-environment correlations in the environmental representation is a necessary but not sufficient condition for non-Markovianity, as there are cases where Eq. (1), and thus Eq. (3), still holds despite $\rho^{QA}\neq\rho^Q\otimes\rho^A$ [40–42].

Physically, the environmental subsystem A might be a local subsystem of the larger heat bath that the main system Q interacts with. Hence, until subsystem A is equilibrated back to the thermal state by the larger heat bath, it can retain correlations or memory with the main system Q from previous interactions [43]. If we restrict the main system Q to interact only with this local subsystem A, and if the interaction timescale is shorter than the time it takes for subsystem A to equilibrate back to the thermal state, then the effective dynamics on the evolution of the main system Q is non-Markovian in general, where the evolution of Q does not evolve under CPTP maps for later time steps as we now have system-environment correlations between Q and A, i.e., the case in Eq. (6). In the case of qubits, we refer to such local subsystems as the bath qubit. We illustrate such a non-Markovian process for qubits in Fig. 1, where the main system qubit Q interacts with the bath qubit A at two different time steps, $t_0 \to t_1$ and $t_2 \to t_3$, with interval τ_A and some interaction strength J. We allow time τ_T from $t_1 \to t_2$ for the bath qubit ρ^A to equilibrate towards the thermal state $\sigma^A_{\beta_A}$ of the heat bath, governed by some coupling strength J_T. Therefore, by varying τ_T or J_T between the heat bath and the bath qubit, we can control the non-Markovianity of the second interaction from $t_2 \to t_3$. If $J_T \to 0$ or $\tau_T \to 0$, the bath qubit retains all the correlations created from the first interaction from $t_0 \to t_1$, resulting in a fully non-Markovian

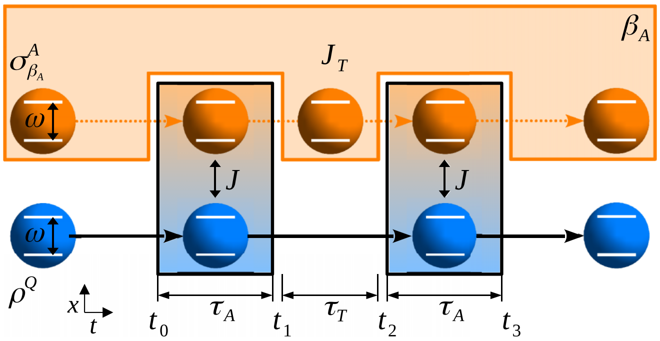

FIG. 1. A two-time-step non-Markovian process. The main quantum system qubit ρ^Q interacts with a Markovian heat bath at temperature T_A via an intermediary bath qubit ρ^A at two different time steps $t_0 \to t_1$ and $t_2 \to t_3$. The bath qubit is initialized in the thermal state of the heat bath with $\rho^A_{t_0} = \sigma^A_{\beta_A}$. The bath qubit ρ^A acts as a memory in the Markovian heat bath, such that the second interaction is correlated to the first, resulting in an overall non-Markovian interaction. In between the interactions at time steps $t_1 \to t_2$, the bath qubit undergoes Markovian interaction with the heat bath, such that for sufficiently large coupling strength J_T or large τ_T, the bath qubit ρ^A can equilibrate back to the thermal state $\sigma^A_{\beta_A}$, losing its memory. Therefore, the non-Markovianity of the process is dependent on J_T and τ_T, with the fully non-Markovian case achieved with $J_T \to 0$ or $\tau_T \to 0$.

second interaction. On the other hand, if J_T or τ_T is sufficiently large, then the bath qubit ρ^A can equilibrate back to the thermal state $\sigma^A_{\beta_A}$, resulting in a fully Markovian second interaction.

Note that physically, the bath qubit interacts with the heat bath continuously. However, in the system under study in Fig. 1, we consider only bipartite interactions where the bath qubit system A decouples from the heat bath when it interacts with the main qubit system Q. This is a reasonable assumption to make for $J \gg J_T$, where the interaction with the main system Q dominates the evolution of bath qubit A.

B. Non-Markovianity in the quantum switch

In the two-party quantum switch, two quantum channels Φ_A and Φ_B that operate on a quantum system Q are placed in a controlled superposition of alternating operation orders $\Phi_B \circ \Phi_A$ and $\Phi_A \circ \Phi_B$, with the superposition coherently controlled by an ancillary control qubit C. Without loss of generality, we set the order $\Phi_B \circ \Phi_A$ when the control qubit is $|0\rangle$, and the order $\Phi_A \circ \Phi_B$ when the control qubit is $|1\rangle$. In the Kraus representation, this quantum switch operation $\Phi^{\rm sw}$ acts as

$$\Phi^{\rm sw}(\rho^C \otimes \rho^Q) = \sum_{i,j} K^{\rm sw}_{ij}(\rho^C \otimes \rho^Q)K^{\rm sw\dagger}_{ij}, \tag{7}$$

with

$$K^{\rm sw}_{ij} = |0\rangle\langle 0| \otimes B_j A_i + |1\rangle\langle 1| \otimes A_i B_j, \tag{8}$$

where $\{A_i\}$ and $\{B_j\}$ are the sets of Kraus operators for Φ_A and Φ_B, i.e., $\Phi_A(\rho^Q) = \sum_i A_i \rho^Q A_i^\dagger$ and $\Phi_B(\rho^Q) = \sum_j B_j \rho^Q B_j^\dagger$. Note that the quantum switch operation $\Phi^{\rm sw}$ remains as a valid CPTP map with $\sum_{i,j} K^{\rm sw\dagger}_{ij} K^{\rm sw}_{ij} = \sum_i A_i^\dagger A_i = \sum_j B_j^\dagger B_j = I$.

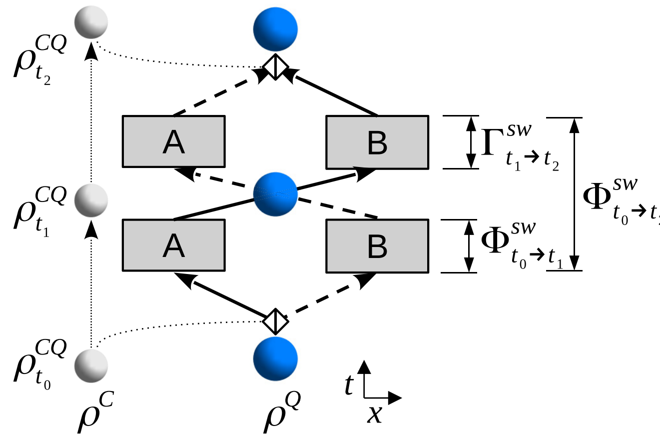

FIG. 2. Schematics of the quantum switch. The control qubit ρ^C controls the operation orders on the main quantum system ρ^Q. If $\rho^C = |0\rangle\langle 0|$, we have $\Phi_B \circ \Phi_A(\rho^Q)$ (solid arrows), and if $\rho^C = |1\rangle\langle 1|$, we have $\Phi_A \circ \Phi_B(\rho^Q)$ (dashed arrows). By breaking the quantum switch operation into two different time steps, i.e., $\Phi^{\rm sw}_{t_0\to t_2} = \Gamma^{\rm sw}_{t_1\to t_2} \circ \Phi^{\rm sw}_{t_0\to t_1}$, it can be shown that $\Gamma^{\rm sw}_{t_1\to t_2}$ is not a CPTP map in general and, thus, is non-Markovian and allows for non-Markovian effects such as backflow of information.

Therefore, it also admits an environmental representation of

$$\Phi^{\rm sw}(\rho^C \otimes \rho^Q) = {\rm Tr}_{A,B}[U^{\rm sw}(\rho^C \otimes \rho^A \otimes \rho^Q \otimes \rho^B)U^{\rm sw\dagger}], \tag{9}$$

where subsystems A and B are the ancillary environmental subsystems corresponding to Φ_A and Φ_B, respectively. It was shown in Ref. [31] that if we split $\Phi^{\rm sw}$ into two different time steps such that

$$\begin{aligned} U^{\rm sw} &= U^{\rm sw}_{t_1\to t_2} U^{\rm sw}_{t_0\to t_1} \\ &= \big[|0\rangle\langle 0|^C \otimes \big(I^A \otimes U^{QB}_B\big) + |1\rangle\langle 1|^C \otimes \big(U^{AQ}_A \otimes I^B\big)\big] \\ &\quad \times \big[|0\rangle\langle 0|^C \otimes \big(U^{AQ}_A \otimes I^B\big) + |1\rangle\langle 1|^C \otimes \big(I^A \otimes U^{QB}_B\big)\big], \end{aligned} \tag{10}$$

we have

$$\begin{aligned} \rho^{CQ}_{t_1} &= \Phi^{\rm sw}_{t_0\to t_1}\big(\rho^C_{t_0} \otimes \rho^Q_{t_0}\big) \\ &= {\rm Tr}_{A,B}\big[U^{\rm sw}_{t_0\to t_1}\big(\rho^C_{t_0} \otimes \rho^A_{t_0} \otimes \rho^Q_{t_0} \otimes \rho^B_{t_0}\big)U^{\rm sw\dagger}_{t_0\to t_1}\big], \end{aligned} \tag{11}$$

$$\begin{aligned} \rho^{CQ}_{t_2} &= \Gamma^{\rm sw}_{t_1\to t_2}\big(\rho^{CQ}_{t_1}\big) \\ &= {\rm Tr}_{A,B}\big[U^{\rm sw}_{t_1\to t_2}\big(\rho^{CAQB}_{t_1}\big)U^{\rm sw\dagger}_{t_1\to t_2}\big]. \end{aligned} \tag{12}$$

That is, we obtained the case of Eq. (6) with $\Phi^{\rm sw}_{t_0\to t_2} = \Gamma^{\rm sw}_{t_1\to t_2} \circ \Phi^{\rm sw}_{t_0\to t_1}$, where $\Gamma^{\rm sw}_{t_1\to t_2}$ is not a CPTP map as the environments A and B are not independent of systems CQ after the first interaction $\Phi^{\rm sw}_{t_0\to t_1}$ with corresponding unitary $U^{\rm sw}_{t_0\to t_1}$. This means that the second half of the quantum switch operation is non-Markovian in general, and can lead to backflow of information that can contribute to enhanced communications. We illustrate this in Fig. 2, and show in Appendix A 1 that Eqs. (9) and (10) indeed correspond to the quantum switch operation of Eqs. (7) and (8).

Suppose if, instead, we dictate that $U^{\rm sw}_{t_1\to t_2}$ acts on new independent environments A' and B' as in Eq. (5), then we have

$$\rho^{CQ}_{t_2} = \Phi^{\rm traj}_{t_1\to t_2}(\rho^{CQ}_1) = \mathrm{Tr}_{A',B'}\big[U^{\rm sw}_{t_1\to t_2}\big(\rho^{A'}\otimes\rho^{CQ}_{t_1}\otimes\rho^{B'}\big)U^{\rm sw\dagger}_{t_1\to t_2}\big], \tag{13}$$

such that $\Phi^{\rm traj}_{t_1\to t_2}$ is a CPTP map, and $\Phi^{\rm traj}\equiv\Phi^{\rm traj}_{t_1\to t_2}\circ\Phi^{\rm sw}_{t_0\to t_1}$ is Markovian; then its corresponding Kraus operator is not the quantum switch:

$$K^{\rm traj}_{ijkl} = \alpha_0|0\rangle\langle 0|\otimes B_lA_i + \alpha_1|1\rangle\langle 1|\otimes A_kB_j, \tag{14}$$

where α_0 and α_1 are complex coefficients to ensure $\sum_{ijkl}K^{\rm traj\dagger}_{ijkl}K^{\rm traj}_{ijkl}=I$. This is referred to as a superposition of trajectories [27], for which we denote its operation as $\Phi^{\rm traj}$, in contrast with the quantum switch's superposition of orders. In other words, the condition of system-environment correlations, i.e., the possibility for a non-Markovian process, is necessary to recover the quantum switch operation $\Phi^{\rm sw}$.

Note that if we were to stop the operation of the quantum switch after $\Phi^{\rm sw}_{t_0\to t_1}$, we have an operation where Φ_A is applied when the control qubit is $|0\rangle$, and Φ_B is applied when the control qubit is $|1\rangle$. This is referred to as a superposition of independent channels [26], which we denote as $\Phi^{\rm indep}$, and has the Kraus operator of

$$K^{\rm indep}_{ij} = \eta_0|0\rangle\langle 0|\otimes A_i + \eta_1|1\rangle\langle 1|\otimes B_j, \tag{15}$$

where η_0 and η_1 are complex coefficients to ensure $\sum_{ij}K^{\rm indep\dagger}_{ij}K^{\rm indep}_{ij}=I$. We show in Appendix A 3 that dictating Eq. (13) indeed leads to Eq. (14), and in Appendix A 2 that terminating the quantum switch after $\Phi^{\rm sw}_{t_0\to t_1}$ indeed leads to Eq. (15).

The case of $\Phi^{\rm indep}$ is often compared to the quantum switch operation $\Phi^{\rm sw}$ in discussions on the origins of the quantum switch's advantages, where the operation $\Phi^{\rm indep}$ was shown to be able to replicate or surpass the communication enhancements of $\Phi^{\rm sw}$ in certain cases [15,26,28]. In this non-Markovian perspective of the quantum switch, the difference in communication enhancements between $\Phi^{\rm sw}$ and $\Phi^{\rm indep}$ can be framed as the presence of non-Markovian effects in the operation of $\Gamma^{\rm sw}_{t_1\to t_2}$, specifically the presence of non-Markovian backflow of information [31]. Likewise, a comparison between $\Phi^{\rm sw}$ and $\Phi^{\rm traj}$ [28] can also be framed as the difference between a system with non-Markovian system-environment correlations and one without it.

III. NON-MARKOVIAN PROCESS THAT REDUCES TO THE QUANTUM SWITCH

A. Extending the quantum switch to control non-Markovianity

As described in Sec. II B, we can break the quantum switch operation $\Phi^{\rm sw}$ into a two-time-step process in the environmental representation of Eq. (9). By having the initial ancillary environmental states to be thermal states with inverse temperatures β_A and β_B, and by ensuring that $U^{\rm sw}=U^{\rm sw}_{t_1\to t_2}U^{\rm sw}_{t_0\to t_1}$ conserves total energy, the quantum switch operation $\Phi^{\rm sw}$ is an indefinite causal order of thermal operations, with environmental representation of

$$\Phi^{\rm sw}(\rho^C\otimes\rho^Q) = \mathrm{Tr}_{A,B}\big[U^{\rm sw}\big(\rho^C\otimes\sigma^A_{\beta_A}\otimes\rho^Q\otimes\sigma^B_{\beta_B}\big)U^{\rm sw\dagger}\big]. \tag{16}$$

Physically, the main system Q interacts with two heat baths $\mathcal{A}$ and $\mathcal{B}$ of inverse temperatures β_A and β_B in a superposition of alternating orders. If the control qubit C is $|0\rangle$, the main system Q interacts with heat bath $\mathcal{A}$ before heat bath $\mathcal{B}$, and if the control qubit is $|1\rangle$, the interaction is with bath $\mathcal{B}$ before bath $\mathcal{A}$. As noted in Sec. II B, the presence of system-environment correlations is necessary to recover the quantum switch operation $\Phi^{\rm sw}$. Therefore, these ancillary environmental subsystems A and B play the role of the local bath subsystems that can retain their memory between the operations of $U^{\rm sw}_{t_0\to t_1}$ and $U^{\rm sw}_{t_1\to t_2}$ (see Sec. II A).

Similar to the case of Fig. 1, we can extend the operation of the quantum switch by allowing control of the non-Markovianity via the equilibration of local bath subsystems A and B, between the operations of $U^{\rm sw}_{t_0\to t_1}$ and $U^{\rm sw}_{t_1\to t_2}$. Since the equilibration takes place between $U^{\rm sw}_{t_0\to t_1}$ and $U^{\rm sw}_{t_1\to t_2}$, we relabel $U^{\rm sw}_{t_1\to t_2}\to U^{\rm sw}_{t_2\to t_3}$, such that the equilibration operation which we denote by $\mathcal{N}_{t_1\to t_2}$ takes place in $t_1\to t_2$.

We refer to this extended quantum switch operation with the additional equilibration process as $\Phi^{\rm ext}$. If this equilibration is absent, we have $\Phi^{\rm ext}=\Phi^{\rm sw}$, and if this equilibration is maximum, such that the bath qubits return to the thermal states, then we have $\Phi^{\rm ext}=\Phi^{\rm traj}$. More specifically, we have

$$\Phi^{\rm ext}(\rho^C\otimes\rho^Q) = \mathrm{Tr}_{A,B}\big[U^{\rm sw}_{t_2\to t_3}\mathcal{N}_{t_1\to t_2}\big(U^{\rm sw}_{t_0\to t_1}\big(\rho^C\otimes\sigma^A_{\beta_A}\otimes\rho^Q\sigma^B_{\beta_B}\big)\times U^{\rm sw\dagger}_{t_0\to t_1}\big)U^{\rm sw\dagger}_{t_2\to t_3}\big], \tag{17}$$

where between the operations of $U^{\rm sw}_{t_0\to t_1}$ and $U^{\rm sw}_{t_2\to t_3}$, we perform a thermal operation $\mathcal{N}_{t_1\to t_2}$ that acts on local bath subsystems A and B.

Let us first define the energy-conserving unitaries $U^{\rm sw}_{t_0\to t_1}$ and $U^{\rm sw}_{t_2\to t_3}$, before returning to the definition of $\mathcal{N}_{t_1\to t_2}$. We consider qubit systems where the free Hamiltonians of the main system Q, and local bath qubits A and B, are

$$H_Q = I^C\otimes I^A\otimes\omega|1\rangle\langle 1|^Q\otimes I^B, \tag{18}$$

$$H_A = I^C\otimes\omega|1\rangle\langle 1|^A\otimes I^Q\otimes I^B, \tag{19}$$

$$H_B = I^C\otimes I^A\otimes I^Q\otimes\omega|1\rangle\langle 1|^B, \tag{20}$$

while the control qubit C has no energy. Next, we define the interaction Hamiltonian between the main system Q and the bath qubits as a simple energy-conserving exchange operation of

$$H^{AQ}_{\rm int} = J(e^{i\varphi}\sigma^A_-\sigma^Q_+ + e^{-i\varphi}\sigma^A_+\sigma^Q_-), \tag{21}$$

$$H^{QB}_{\rm int} = J(e^{i\varphi}\sigma^B_-\sigma^Q_+ + e^{-i\varphi}\sigma^B_+\sigma^Q_-), \tag{22}$$

where J is the interaction strength, φ is an arbitrary phase, and σ_+ and σ_- are the qubit raising and lowering operators for their corresponding subsystems, i.e., $\sigma^Q_+ = I^C\otimes I^A\otimes|1\rangle\langle 0|^Q\otimes I^B$ and $\sigma^Q_- = I^C\otimes I^A\otimes|0\rangle\langle 1|^Q\otimes I^B$, and vice versa for $\sigma^A_\pm$ and $\sigma^B_\pm$. The bipartite unitaries that implement these interactions for a time interval of t can then be derived as

$$U_A^{AQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos Jt & -ie^{i\varphi}\sin Jt & 0 \\ 0 & -ie^{-i\varphi}\sin Jt & \cos Jt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_A & -ie^{i\varphi}\sqrt{1-x_A^2} & 0 \\ 0 & -ie^{-i\varphi}\sqrt{1-x_A^2} & x_A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (23)$$

and likewise,

$$U_B^{QB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_B & -ie^{-i\varphi}\sqrt{1-x_B^2} & 0 \\ 0 & -ie^{i\varphi}\sqrt{1-x_B^2} & x_B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (24)$$

such that

$$e^{-iH_{\text{int}}^{AQ}t/\hbar} = I^C \otimes U_A^{AQ} \otimes I^B, \quad (25)$$

$$e^{-iH_{\text{int}}^{QB}t/\hbar} = I^C \otimes I^A \otimes U_B^{QB}, \quad (26)$$

where for simplicity we have taken $x_A, x_B = \cos Jt$ with $-1 \leqslant x_A, x_B \leqslant 1$. The interactions would then be characterized by the values of x_A and x_B in a discrete time manner, rather than the interaction strength J or time t.

We note that for positive values of x_A and x_B, and for arbitrary φ, U_A^{AQ} and U_B^{QB} are simply the generalized amplitude damping channel (GADC), also referred to as the one-qubit thermal operation [44,45]. Note that we take $\varphi = 3\pi/2$ for the rest of the paper, such that U_A^{AQ} and U_B^{QB} take the same signs as the GADC used in the literature. For negative values of x_A and x_B, the resulting operation is simply the GADC with an additional phase flip of $\sigma_z \otimes \sigma_z$ where σ_z is the Pauli Z matrix. The values of x_A or x_B determine the amount of thermalization on the main system Q. If x_A or x_B is zero, then we have full thermalization where the main system Q becomes a thermal state. Otherwise, as x_A and x_B are distanced from zero, we have prethermalization where the thermalization is weakened, until the values of -1 and 1, where there is no thermalization at all. In the two-time-step quantum switch operation of Fig. 2, we have

$$U_{t_0\to t_1}^{\text{sw}} = |0\rangle\langle 0|^C \otimes U_A^{AQ} \otimes I^B + |1\rangle\langle 1|^C \otimes I^A \otimes U_B^{QB}, \quad (27)$$

$$U_{t_2\to t_3}^{\text{sw}} = |0\rangle\langle 0|^C \otimes I^A \otimes U_B^{QB} + |1\rangle\langle 1|^C \otimes U_A^{AQ} \otimes I^B, \quad (28)$$

which are each a controlled superposition of one-qubit thermal operations.

Finally, for the equilibration operation $\mathcal{N}_{t_1\to t_2}$ of the bath qubits, we can also describe it with the GADC or one-qubit thermal operation. However, since we do not need to keep track of additional ancillaries for the bath qubits, we can simply use its Kraus representation of [46]

$$\begin{aligned} K_1 &= \sqrt{1-N}(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|), \\ K_2 &= \sqrt{\gamma(1-N)}|0\rangle\langle 1|, \\ K_3 &= \sqrt{N}(\sqrt{1-\gamma}|0\rangle\langle 0| + |1\rangle\langle 1|), \\ K_4 &= \sqrt{\gamma N}|1\rangle\langle 0|, \end{aligned} \quad (29)$$

such that

$$\mathcal{N}_{t_1\to t_2}\big(\rho_{t_1}^{CAQB}\big) = \sum_{i,j} W_{ij}\rho_{t_1}^{CAQB}W_{ij}^\dagger, \quad (30)$$

where

$$W_{ij} = I^C \otimes K_i^A \otimes I^Q \otimes K_j^B. \quad (31)$$

$N \in [0, 0.5]$ is the excited state population of a thermal state at the temperature of the heat bath that does the equilibration, i.e.,

$$N_A = \frac{1}{1+e^{\beta_A\omega}}, \quad N_B = \frac{1}{1+e^{\beta_B\omega}}, \quad (32)$$

$$\sigma_{\beta_A} = \begin{pmatrix} 1-N_A & 0 \\ 0 & N_A \end{pmatrix}, \quad \sigma_{\beta_B} = \begin{pmatrix} 1-N_B & 0 \\ 0 & N_B \end{pmatrix}. \quad (33)$$

Note that the equilibration of each individual bath qubit A and B has the canonical master equation of

$$\begin{aligned} \dot{\rho}(t) = {} & J_T n\big[\sigma_-\rho(t)\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\big] \\ & + J_T(n+1)\big[\sigma_+\rho(t)\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\big], \end{aligned} \quad (34)$$

where J_T is the coupling strength with the bath, $n = (e^{\beta\omega} - 1)^{-1}$ is the mean number of excitations in the bath, and $\{\cdot\}$ is the anticommutator. We have [47]

$$\gamma = 1 - e^{J_T(2n+1)(t_2-t_1)}, \quad (35)$$

where $t_2 - t_1$ is the equilibration time, and the parameter $\gamma \in [0, 1]$ controls the strength of the thermalization or equilibration of the bath qubits. For $\gamma = 0$, there is no thermalization and thus the bath qubits retain their correlations with the main system Q, giving us the fully non-Markovian case of $\Phi^{\text{ext}}(\gamma = 0) = \Phi^{\text{sw}}$. If instead we have $\gamma = 1$, then the thermalization is maximum and the bath qubits return to the thermal states, losing their correlations with the main system Q, giving us the fully Markovian case of $\Phi^{\text{ext}}(\gamma = 1) = \Phi^{\text{traj}}$. For $0 < \gamma < 1$, we have the general case that has varying non-Markovianity that is between the extreme cases of the quantum switch Φ^{sw} and the superposition of trajectories Φ^{traj}.

This completes the construction of the extended non-Markovian process Φ^{ext}, and we have illustrated its operation in Fig. 3. This operation Φ^{ext} exemplifies the differences between Φ^{sw}, Φ^{traj}, and Φ^{indep}, as well as covering both full and prethermalization regimes by varying x_A and x_B. We have

$$\rho_{t_1}^{CAQB} = U_{t_0\to t_1}^{\text{sw}}\big(\rho_{t_0}^C \otimes \sigma_{\beta_A}^A \otimes \rho_{t_0}^Q \otimes \sigma_{\beta_B}^B\big)U_{t_0\to t_1}^{\text{sw}\dagger}, \quad (36)$$

$$\rho_{t_2}^{CAQB} = \mathcal{N}_{t_1\to t_2}\big(\rho_{t_1}^{CAQB}\big), \quad (37)$$

$$\rho_{t_3}^{CAQB} = U_{t_2\to t_3}^{\text{sw}}\rho_{t_2}^{CAQB}U_{t_2\to t_3}^{\text{sw}\dagger}, \quad (38)$$

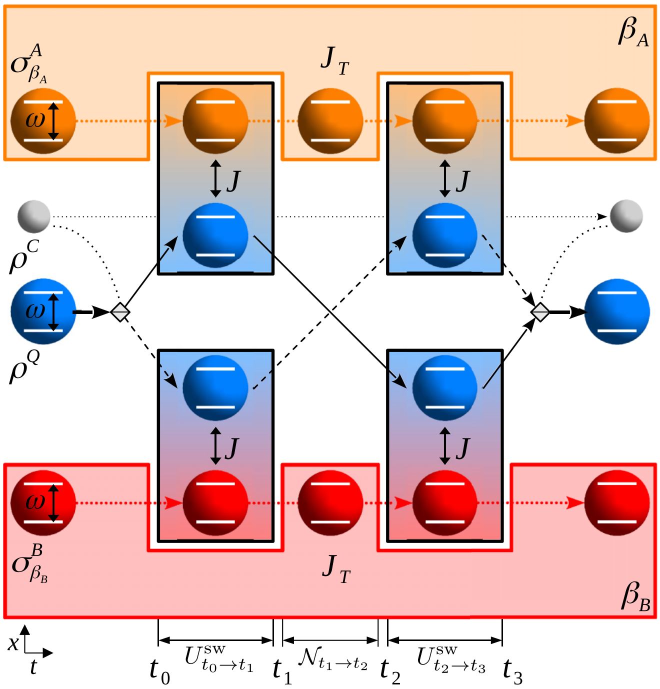

FIG. 3. A non-Markovian operation $\Phi^{\rm ext}$ that reduces to the quantum switch operation $\Phi^{\rm sw}$ when $J_T = 0$ or $t_2 - t_1 = 0$, and to the superposition-of-trajectories operation $\Phi^{\rm traj}$ when $J_T \gg 0$ or $t_2 - t_1 \gg 0$. Equivalently, these correspond to $\gamma = 0$ and $\gamma = 1$, respectively. The main quantum system qubit Q is sent into a superposition of two alternating interaction orders, controlled by a control qubit C. If $\rho^C = |0\rangle\langle 0|^C$, Q interacts with the bath qubit A from $t_0 \to t_1$, followed by bath qubit B from $t_2 \to t_3$ (solid arrows). On the other hand, if $\rho^C = |1\rangle\langle 1|^C$, Q interacts with B first from $t_0 \to t_1$, before A from $t_2 \to t_3$ (dashed arrows). From $t_1 \to t_2$, the bath qubits undergo an equilibration process $\mathcal{N}_{t_1\to t_2}$ by the heat baths they are in, which depends on the parameter γ. For strong equilibration ($\gamma \to 1$), the bath qubits return to the thermal state and lose their memory of the previous interaction, resulting in a Markovian process. On the other hand, for weak equilibration ($\gamma \to 0$), the bath qubits can retain their memory, resulting in a non-Markovian process in general.

and

$$\Phi^{\rm indep}\big(\rho^C_{t_0} \otimes \rho^Q_{t_0}\big) = {\rm Tr}_{A,B}\big[\rho^{CAQB}_{t_1}\big], \tag{39}$$

$$\Phi^{\rm sw}\big(\rho^C_{t_0} \otimes \rho^Q_{t_0}\big) = {\rm Tr}_{A,B}\big[\rho^{CAQB}_{t_3}(\gamma = 0)\big], \tag{40}$$

$$\Phi^{\rm traj}\big(\rho^C_{t_0} \otimes \rho^Q_{t_0}\big) = {\rm Tr}_{A,B}\big[\rho^{CAQB}_{t_3}(\gamma = 1)\big]. \tag{41}$$

B. Work extraction

Now that we have the extended non-Markovian process $\Phi^{\rm ext}$, we are interested in extracting work if it is part of a three-stroke cycle heat engine [24] as shown in Fig. 4. In such a cycle, the working system Q is initialized to a thermal state by contact with a heat bath with inverse temperature β_Q. It is then placed in contact with a typically hotter heat bath, extracting usable and passive energies from it. The extracted usable energy is then stored in a quantum battery which can be used to perform work. Finally, the working body is initialized again by the heat bath of β_Q, dumping its accumulated passive

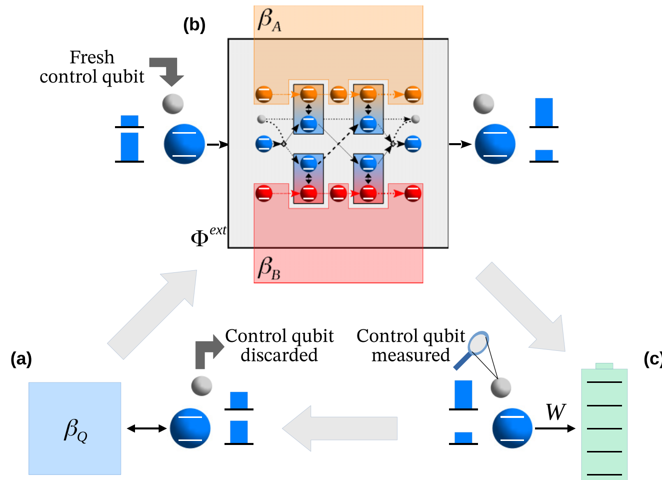

FIG. 4. A three-stroke cycle heat engine that utilizes the extended non-Markovian operation $\Phi^{\rm ext}$ to extract work. (a) The main system Q is the working body, which is initialized to a thermal state with inverse temperature β_Q. (b) The main system Q undergoes the $\Phi^{\rm ext}$ operation, extracting energy from the two heat baths with inverse temperatures β_A and β_B. (c) The extracted energy is stored in a quantum battery which can be used to do work. By considering an ideal weight model for the quantum battery, the maximum extractable work is the ergotropy of the output state in (b). Specifically, we consider the daemonic ergotropy, where the control qubit is measured first before work is extracted from the main system Q. The cycle is then "reset" in (a) when the main system Q is reinitialized, and a fresh control qubit C is provided.

energy and "resetting" the thermodynamic cycle. In our case, the hot baths that the working system extracts energy from are the two heat baths in the $\Phi^{\rm ext}$ operation, with inverse temperatures β_A and β_B, as shown in Fig. 4.

While this construction is general for different bath temperatures, for simplicity we assume $\beta_A = \beta_B = \beta$, or equivalently $N_A = N_B = N$, i.e., the bath qubits A and B are initialized as

$$\sigma^A_\beta = \sigma^B_\beta = (1 - N)|0\rangle\langle 0| + N|1\rangle\langle 1|, \tag{42}$$

and since the main system Q is initialized with a bath with inverse temperature β_Q, we have

$$\rho^Q_0 = \sigma^Q_{\beta_Q} = \big(1 - N_Q\big)|0\rangle\langle 0| + N_Q|1\rangle\langle 1|. \tag{43}$$

For an initializing bath that is at a lower temperature than the heat baths in $\Phi^{\rm ext}$, we have $\beta_Q > \beta$ and $N_Q < N$.

Here, we consider the ideal weight model for the battery [48], such that, in the absence of system-battery correlations and battery coherences in its energy eigenbasis, the maximal extractable work of the cycle is the ergotropy extracted from the hotter heat bath [49–51]. Specifically, taking the same approach as Ref. [22], we consider the daemonic ergotropy [32] as the extractable energy that can be stored in the quantum battery to do useful work, where the control qubit C is measured first, before the ergotropy of the collapsed main system Q is computed, conditioning on the measurement outcomes. But first, we define the ergotropy as follows: For a d-dimensional quantum system Q with system Hamiltonian $H_0 = \sum_{k=1}^d \epsilon_k |\epsilon_k\rangle\langle\epsilon_k|$ such that $\epsilon_k \leqslant \epsilon_{k+1}$, the ergotropy $\mathcal{E}$ of a state $\rho^Q = \sum_{k=1}^d q_k|q_k\rangle\langle q_k|$ such that $q_k \geqslant q_{k+1}$ is defined as

$$\mathcal{E}(\rho^Q) = \max_{U\in\mathcal{U}} \mathrm{Tr}[H_0(\rho^Q - U\rho^Q U^\dagger)], \\ = E(\rho^Q) - E(P_\rho^Q), \tag{44}$$

where the maximization is taken over the set of unitaries $\mathcal{U}$ acting on the space of Q, and we defined the passive state $P_\rho = U_*\rho U_*^\dagger$ where U_* is the unitary that maximizes $\mathcal{E}(\rho)$. Passive states are states that have zero ergotropy, and are diagonal in the energy eigenbasis with descending populations, i.e., $P_\rho = \sum_{k=1}^d q_k|\epsilon_k\rangle\langle\epsilon_k|$. We refer to $E(P_\rho)$ as the passive energy. Furthermore, in the case of qubits, and assuming a Hamiltonian of $H_0 = \omega|1\rangle\langle 1|$, the ergotropy of an arbitrary qubit,

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{pmatrix}, \tag{45}$$

can be separated into incoherent $\mathcal{E}_i$ and coherent $\mathcal{E}_c$ contributions [52], i.e.,

$$\mathcal{E}(\rho) = \mathcal{E}_i(\rho) + \mathcal{E}_c(\rho), \tag{46}$$

with

$$\mathcal{E}_i = \max\{0,\ \omega\delta\rho\}, \tag{47}$$

$$\mathcal{E}_c = \frac{\omega}{2}\left(\sqrt{2P(\rho)-1} - \sqrt{2P(\rho)-1-4|\rho_{01}|^2}\right), \tag{48}$$

where $\delta\rho = \rho_{11} - \rho_{00}$, and $P(\rho) = \mathrm{Tr}[\rho^2]$ is the purity of ρ.

The presence of correlations in bipartite systems can enhance ergotropy in certain cases [53,54], but its effects are nontrivial in general [55]. In our case, the control qubit C and the main system Q are correlated after the operation of Φ^{ext}. By using the daemonic ergotropy, where the control qubit is measured, these correlations can be leveraged for enhanced work production. The information gained from the measurement of the control qubit is used to condition the ergotropy extraction unitary U such that the daemonic ergotropy is

$$\mathcal{E}^D(\rho^{CQ}) = \mathrm{Tr}[H_0\rho^Q] - \min_{U_i\in\mathcal{U}} \sum_i p_i \mathrm{Tr}\left[H_0 U_i \rho_i^Q U_i^\dagger\right] \\ = \sum_i p_i \mathcal{E}\left(\rho_i^Q\right), \tag{49}$$

where the control qubit is measured with the set of orthogonal projectors $\{P_i\}$, with $\rho_i^Q = \mathrm{Tr}_C[(P_i\otimes I)\rho^{CQ}]/p_i$ and probability of measurement $p_i = \mathrm{Tr}[(P_i\otimes I)\rho^{CQ}]$. In general we have $\mathcal{E}^D \geqslant \mathcal{E}$, which saturates when there are no correlations between C and Q.

In this Φ^{ext} process, utilizing the control qubit is necessary for the production of ergotropy. We can see this by explicit derivation of $\rho_{t_1}^Q$, $\rho_{t_2}^Q$, and $\rho_{t_3}^Q$, that is, if we were to discard or trace away the control qubit, such that its measurement outcomes are ignored and not conditioned to perform any further operations. Noting that $\rho_{t_1}^Q = \rho_{t_2}^Q$, we have

$$\rho_{t_1}^Q = \begin{pmatrix} 1-\rho_{t_1,11}^Q & 0 \\ 0 & \rho_{t_1,11}^Q \end{pmatrix}, \tag{50}$$

$$\rho_{t_3}^Q = \begin{pmatrix} 1-\rho_{t_3,11}^Q & 0 \\ 0 & \rho_{t_3,11}^Q \end{pmatrix}, \tag{51}$$

where

$$\rho_{t_1,11}^Q = N\left(1 - \frac{x_A^2 + x_B^2}{2}\right) + N_Q\left(\frac{x_A^2 + x_B^2}{2}\right), \tag{52}$$

$$\rho_{t_3,11}^Q = N\left(1 - x_A^2 x_B^2\right) + N_Q\left(x_A^2 x_B^2\right). \tag{53}$$

Given the constraints $N, N_Q \in [0, 0.5]$, and $x_A, x_B \in [-1, 1]$, such that $(x_A^2 + x_B^2)/2$ and $x_A^2 x_B^2$ has a range of $[0, 1]$, the population of the excited states of both $\rho_{t_1}^Q$ and $\rho_{t_3}^Q$ is a convex combination of N and N_Q, and thus also has a range of $[0, 0.5]$, implying that $\mathcal{E}_i = 0$. Furthermore, since the off-diagonal terms are zero, there is also no coherent contribution to ergotropy with $\mathcal{E}_c = 0$. Therefore, they are passive states that have zero ergotropy, i.e.,

$$\mathcal{E}\left(\rho_{t_1}^Q\right) = \mathcal{E}\left(\rho_{t_2}^Q\right) = \mathcal{E}\left(\rho_{t_3}^Q\right) = 0. \tag{54}$$

This is consistent with works that demonstrate the advantages of the quantum switch, where the loss of the coherence of the superposition means the loss of the quantum switch's advantages [19,20,31].

Additionally, we note that tracing away the control qubit will result in a probabilistic mixture of the two causal orders $\Phi_B \circ \Phi_A$ and $\Phi_A \circ \Phi_B$, with the probability of the two orders depending on the initial state of the control qubit. However, since the two heat baths have the same temperature in our case, the two causal orders commute with $\Phi_B \circ \Phi_A = \Phi_A \circ \Phi_B$, and so the reduced state ρ^Q has no dependence on the initial state of the control qubit.

Here, we further note that the dependence on γ which controls the non-Markovianity is also absent. This is not surprising; as one might notice from Fig. 3, the memory that manifests in the bath qubits after $U^{\mathrm{sw}}_{t_0\to t_1}$ only comes into play in the alternate path of the superposition in $U^{\mathrm{sw}}_{t_2\to t_3}$, e.g., bath qubit A interacts with the main system Q in the $|0\rangle^C$ path in $U^{\mathrm{sw}}_{t_0\to t_1}$, and with Q in the $|1\rangle^C$ path in $U^{\mathrm{sw}}_{t_2\to t_3}$. The non-Markovian effects are thus only present when the superposition remains coherent. We refer to this as coherent non-Markovianity.

Taking the control qubit to be $\rho^C = |+\rangle\langle +|$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, such that the superposition along the two paths is equal, the input joint state of the system is then

$$\rho_{t_0}^{CAQB} = |+\rangle\langle +|^C \otimes \sigma_\beta^A \otimes \sigma_{\beta_Q}^Q \otimes \sigma_\beta^B. \tag{55}$$

To maintain and maximize the coherence of the superposition, we measure the control qubit in the $|\pm\rangle$ basis. The daemonic ergotropy of the Φ^{ext} process at time t is then

$$\mathcal{E}^D_{\mathrm{ext}}\left(\rho_t^{CQ}\right) = p_{+,t}\mathcal{E}\left(\rho_{+,t}^Q\right) + p_{-,t}\mathcal{E}\left(\rho_{-,t}^Q\right), \tag{56}$$

where

$$p_{\pm,t} = \mathrm{Tr}\left[\left(|\pm\rangle\langle\pm|^C \otimes I^Q\right)\rho_t^{CQ}\right], \tag{57}$$

$$\rho_{\pm,t}^Q = \frac{1}{p_{\pm,t}}\mathrm{Tr}_C\left[\left(|\pm\rangle\langle\pm|^C \otimes I^Q\right)\rho_t^{CQ}\right]. \tag{58}$$

We show in Appendix B that $p_{+,t}\mathcal{E}(\rho_{+,t}^Q) = 0$, and so we are left with $p_{-,t}\mathcal{E}(\rho_{-,t}^Q)$. Furthermore, since the main system Q starts in a thermal state, and because thermal operations do not increase coherence, $\rho^{Q}_{\pm,t}$ will be diagonal with no coherent contribution to ergotropy. Therefore, we have

$$\mathcal{E}^{D}_{\text{ext}}\left(\rho^{CQ}_{t_1}\right) = \max\left\{0,\ \omega p_{-,t_1}\delta\rho^{Q}_{-,t_1}\right\}, \tag{59}$$

$$\mathcal{E}^{D}_{\text{ext}}\left(\rho^{CQ}_{t_3}\right) = \max\left\{0,\ \omega p_{-,t_3}\delta\rho^{Q}_{-,t_3}\right\}, \tag{60}$$

where

$$p_{-,t_1}\delta\rho^{Q}_{-,t_1} = -\frac{N-N_Q}{2}\left(x_A^2+x_B^2\right) + \mu(x_A+x_B) + \left(\frac{N-N_Q}{2}-\mu\right)(x_Ax_B+1), \tag{61}$$

$$\begin{aligned} p_{-,t_3}\delta\rho^{Q}_{-,t_3} =\ & \mu N(1-N)\gamma^2(1-x_A)^2(1-x_B)^2 \\ & + (\mu-\nu)(\gamma-\gamma\sqrt{1-\gamma})x_Ax_B(x_A+x_B-x_Ax_B) \\ & + \mu(\gamma+\gamma\sqrt{1-\gamma})(x_A+x_B-1) \\ & + \nu(\gamma-\gamma\sqrt{1-\gamma})(x_A+x_B-1) \\ & + \mu(1-\gamma)\left(x_A^2+x_B^2\right) \\ & + \frac{N-N_Q}{2}\sqrt{1-\gamma}(x_A+x_B)(x_Ax_B-1) \\ & + \left(-\frac{N-N_Q}{2}-\mu\right)x_A^2x_B^2 + \frac{N-N_Q}{2}-\mu, \end{aligned} \tag{62}$$

and

$$\mu = \frac{N}{2}(1-N)(1-2N_Q), \quad \nu = \frac{N}{2}(1-N)(1-2N). \tag{63}$$

Note that we only consider t_1 and t_3 since $\rho^{CQ}_{t_2} = \rho^{CQ}_{t_1}$. Recall that we have the case of superposition of independent channels Φ^{indep} at time step t_1, the case of superposition of trajectories Φ^{traj} at time step t_3 when $\gamma = 1$, and the quantum switch case Φ^{sw} at time step t_3 when $\gamma = 0$. Therefore,

$$\mathcal{E}^{D}_{\text{indep}}\left(\rho^{CQ}_{t_1}\right) = \mathcal{E}^{D}_{\text{ext}}\left(\rho^{CQ}_{t_1}\right), \tag{64}$$

$$\mathcal{E}^{D}_{\text{traj}}\left(\rho^{CQ}_{t_3}\right) = \mathcal{E}^{D}_{\text{ext}}\left(\rho^{CQ}_{t_3}(\gamma=1)\right), \tag{65}$$

$$\mathcal{E}^{D}_{\text{sw}}\left(\rho^{CQ}_{t_3}\right) = \mathcal{E}^{D}_{\text{ext}}\left(\rho^{CQ}_{t_3}(\gamma=0)\right). \tag{66}$$

Note that Eqs. (59)–(62) are only true for the following three assumptions: (1) the two heat baths are at the same temperature with $\beta_A = \beta_B = \beta$ or $N_A = N_B = N$, (2) we have equal controlled superposition with $\rho^C = |+\rangle\langle+|$, and (3) we measure the control qubit C in the $|\pm\rangle$ basis to maximize its coherence in the basis of the two paths of the superposition. We will forgo the mention of these assumptions in subsequent discussions.

IV. EFFECTS OF NON-MARKOVIANITY ON DAEMONIC ERGOTROPY

A. Advantages of non-Markovianity

We plot the daemonic ergotropy at t_3 against x_A and x_B, i.e., Eq. (62), as a color map for different values of γ for $N_Q < N$, i.e., $T_Q < T$ or $\beta_Q > \beta$, in Fig. 5. The case of $N_Q \geqslant N$ is omitted as it does not produce positive daemonic ergotropy. We take $\omega = 5\,\text{GHz}$, $T_A = T_B = T = 4$ GHz ≈ 31 mK, and

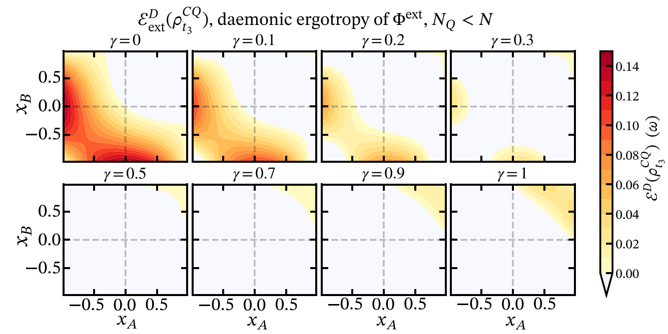

FIG. 5. Daemonic ergotropy $\mathcal{E}^D_{\text{ext}}(\rho^{CQ}_{t_3})$ for the case of the extended non-Markovian operation Φ^{ext} against x_A and x_B for different values of γ, for $N_Q < N$. The case of $N_Q > N$ is not shown as there is no positive daemonic ergotropy.

$T_Q = T/2$, for this example, which can be implemented in superconducting circuits [56]. From Fig. 5, we identify two operational regimes that behave differently with varying γ.

First, in the second, third, and fourth quadrants, i.e., $x_A < 0$ or $x_B < 0$ or both, the daemonic ergotropy is maximum in the fully non-Markovian case of the quantum switch, with $\gamma = 0$. Its magnitude, as well as the possible values of x_A and x_B for positive daemonic ergotropy, decreases with increasing γ, that is, with increasing Markovianity. We refer to this regime as $\Phi^{\text{ext}}_{\text{PF}}$. On the other hand, in the first quadrant, i.e., $x_A, x_B > 0$, the daemonic ergotropy is maximum in the fully Markovian case of the superposition of trajectories, with $\gamma = 1$. Different from $\Phi^{\text{ext}}_{\text{PF}}$, its magnitude and possible values of x_A and x_B increase with increasing γ or Markovianity. We refer to this regime as $\Phi^{\text{ext}}_{\text{no PF}}$. Since trace distance is contractive under CPTP maps, $D(\rho,\sigma) \geqslant D(\Phi(\rho),\Phi(\sigma))$. We can verify the presence of non-Markovianity by computing the Breuer-Laine-Piilo (BLP) measure [57], where the violation of the contractive property of trace distance is indicative of backflow of information, which is a sufficient condition for non-Markovianity. We define this violation as

$$\Delta_{t_2\to t_3}D(\rho^{CQ},\sigma^{CQ}) = D\left(\rho^{CQ}_{t_3},\sigma^{CQ}_{t_3}\right) - D\left(\rho^{CQ}_{t_2},\sigma^{CQ}_{t_2}\right). \tag{67}$$

Therefore, non-Markovianity is present whenever $\Delta_{t_2\to t_3}D(\rho^{CQ},\sigma^{CQ}) > 0$. We plot Eq. (67) in Fig. 6, and note that the non-Markovian backflow of information is only present in the $\Phi^{\text{ext}}_{\text{PF}}$ regime. Furthermore, with the exception of some outliers for small values of γ, areas where there are positive daemonic ergotropy in the $\Phi^{\text{ext}}_{\text{PF}}$ regime also have non-Markovian backflow of information, which decreases with increasing γ.

The presence of different regimes that offer varying performances is not surprising, and can also be observed in works demonstrating the communication advantages of the quantum switch [15,31], where the quantum switch can only offer an advantage for certain choices of channels. Likewise, for daemonic ergotropy, the quantum switch case of $\gamma = 0$ can only grant positive daemonic ergotropy in the $\Phi^{\text{ext}}_{\text{PF}}$ regime, and not in the $\Phi^{\text{ext}}_{\text{no PF}}$ regime as evidenced in Fig. 5.

In addition to the possibility for backflow of information, the difference between the two regimes $\Phi^{\text{ext}}_{\text{PF}}$ and $\Phi^{\text{ext}}_{\text{no PF}}$ is the presence and absence of additional phase-flip operations for negative and positive values of x_A and x_B, respectively (as

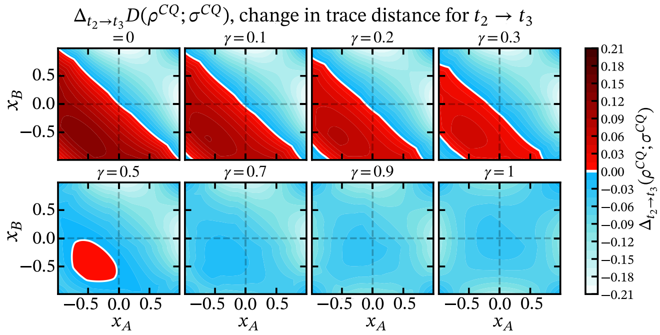

FIG. 6. Violations of the contractive property of trace distance that are indicative of non-Markovian backflow of information. The backflow of information is only present in the $\Phi^{\text{ext}}_{\text{PF}}$ regime where $x_A < 0$ or $x_B < 0$ or both. With the exception of some outliers in small values of γ, areas where there is positive daemonic ergotropy in the $\Phi^{\text{ext}}_{\text{PF}}$ regime also have non-Markovian backflow of information.

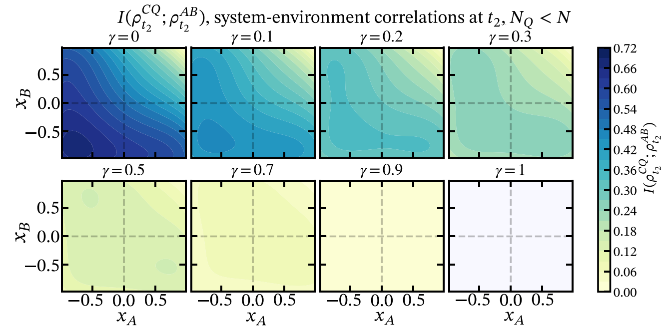

FIG. 7. System-environment correlations or quantum mutual information $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ at time step t_2 after the equilibration operation $\mathcal{N}_{t_1\to t_2}$. It is largest for negative values of x_A and x_B, implying increased non-Markovianity memory at those values. It decreases with increasing γ, which controls the strength of the equilibration, until $\gamma = 1$ where all correlations are destroyed as the bath qubits return back to the thermal state.

discussed in Sec. III A), which simply correspond to different thermal operations or interaction times with the bath qubits. The additional phase-flip operation increases the system-environment correlations between the main subsystems CQ, and the bath qubit subsystems AB at time step t_1 after the first operation of $\Phi^{\text{sw}}_{t_0\to t_1}$. This system-environment correlation is the non-Markovian memory in the system which will decay during the equilibration operation $\mathcal{N}_{t_1\to t_2}$ from $t_1 \to t_2$. We quantify this system-environment correlation as the quantum mutual information $I(\rho^{CQ};\rho^{AB})$, where $I(\rho^X;\rho^Y) = S(\rho^X) + S(\rho^Y) - S(\rho^{XY})$, and $S(\cdot)$ is the von Neumann entropy.

Specifically, since the phase-flip operation commutes with the GADC, it is straightforward to show that we have

$$U^{\text{sw}}_{t_0\to t_1} = U_Z U^{\text{sw}'}_{t_0\to t_1}, \tag{68}$$

where $U^{\text{sw}'}_{t_0\to t_1}$ is simply Eq. (27) with the constraint of positive x_A and x_B, and U_Z is the additional phase-flip operation of

$$U_Z = |0\rangle\langle 0|^C \otimes \sigma_z^A \otimes \sigma_z^Q \otimes I^B + |1\rangle\langle 1|^C \otimes I^A \otimes \sigma_z^Q \otimes \sigma_z^B, \tag{69}$$

for the case where both x_A and x_B are negative, and

$$U_{Z_A} = |0\rangle\langle 0|^C \otimes \sigma_z^A \otimes \sigma_z^Q \otimes I^B + |1\rangle\langle 1|^C \otimes I^A \otimes I^Q \otimes I^B, \tag{70}$$

$$U_{Z_B} = |0\rangle\langle 0|^C \otimes I^A \otimes I^Q \otimes I^B + |1\rangle\langle 1|^C \otimes I^A \otimes \sigma_z^Q \otimes \sigma_z^B, \tag{71}$$

for the cases where only x_A is negative and only x_B is negative, respectively. We observed that the operations of U_Z, U_{Z_A}, and U_{Z_B} always increase the von Neumann entropy $S(\rho^{CQ})$ after the operation of $U^{\text{sw}'}_{t_0\to t_1}$, and since we have

$$\begin{aligned}\Delta_Z I(\rho^{CQ};\rho^{AB}) &= \Delta_Z S(\rho^{CQ}) + \Delta_Z S(\rho^{AB}) - \Delta_Z S(\rho^{CQAB}) \\ &= \Delta_Z S(\rho^{CQ}),\end{aligned} \tag{72}$$

where the difference Δ_Z is taken before and after the operation of U_Z, U_{Z_A}, or U_{Z_B}, they increase the system-environment correlations, or non-Markovian memory, $I(\rho^{CQ}_{t_1};\rho^{AB}_{t_1})$. Note that $\Delta_Z S(\rho^{CQAB}) = 0$ as entropy does not change under unitary operations, and $\Delta_Z S(\rho^{AB}) = 0$ as ρ^{AB} is diagonal without any gain in coherence, and thus does not change under σ_z operations. In other words, the presence of phase-flip operations for negative values of x_A or x_B, or both, in the $\Phi^{\text{ext}}_{\text{PF}}$ regime can grant additional non-Markovian memory to the system as compared to the $\Phi^{\text{ext}}_{\text{no PF}}$ regime. We plot $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ as a color map in Fig. 7, illustrating the increased system-environment correlations in the second, third, and fourth quadrants, as well as its decrease with increasing γ, up until $\gamma = 1$ where all correlations are lost, achieving the fully Markovian case of Φ^{traj}.

While for a bipartite mixed state ρ^{XY}, quantum discord with respect to the measurement on Y was shown to be a lower bound to daemonic ergotropy where X is measured [32], we note that in our case, the quantum discord of ρ^{CQ} with respect to a measurement on Q was found to be zero, rendering it unviable to be a lower bound for daemonic ergotropy with measurement on C. Instead, we note that their mutual information $I(\rho^C_{t_3};\rho^Q_{t_3})$ is a better predictor for daemonic ergotropy, achieving the maximum daemonic ergotropy for values of x_A and x_B that gives maximum $I(\rho^C_{t_3};\rho^Q_{t_3})$ in the $\Phi^{\text{ext}}_{\text{PF}}$ regime, and for most cases in the $\Phi^{\text{ext}}_{\text{no PF}}$ regime, which we show in Fig. 8. The relationship between $I(\rho^C_{t_3};\rho^Q_{t_3})$ and ergotropy are known

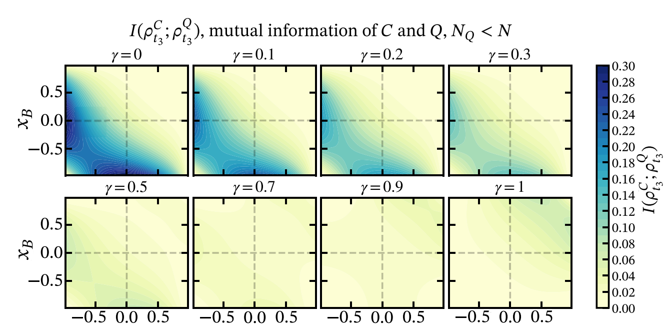

FIG. 8. Mutual information between systems C and Q at t_3 at the end of the Φ^{ext} operation, $I(\rho^C_{t_3};\rho^Q_{t_3})$. It reveals the dependence of daemonic ergotropy on the correlations between C and Q.

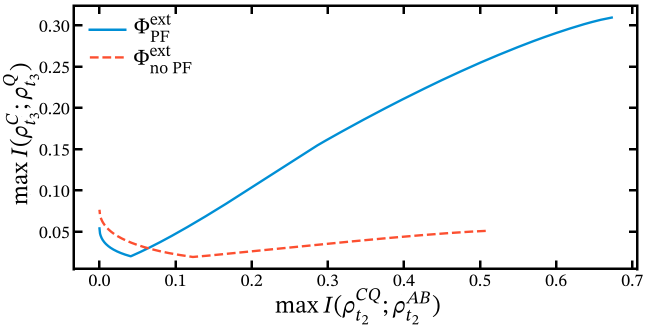

FIG. 9. Maximum correlations between C and Q at t_3 against maximum system-environment correlations, or non-Markovian memory at t_2. The system-environment correlations at t_2 enhance the correlations between C and Q, and the enhancement is stronger in the $\Phi^{\text{ext}}_{\text{PF}}$ regime. However, this is only true if the system-environment correlations are above a certain threshold, below which there is an inverse relation instead.

for locally thermal systems [58], i.e., states that have thermal states as their reduced states, but are nontrivial for general mixed states [55] and for daemonic ergotropy [32]. Here, we will simply note that since

$$\Delta_{t_2\to t_3}I(\rho^C;\rho^Q) = \Delta_{t_2\to t_3}S(\rho^C) + \Delta_{t_2\to t_3}S(\rho^Q) + \Delta_{t_2\to t_3}S(\rho^{AB}) - \Delta_{t_2\to t_3}I(\rho^{CQ};\rho^{AB}), \tag{73}$$

where the difference $\Delta_{t_2\to t_3}$ is taken as the values at t_3 minus the values at t_2, the change in system-environment correlations, or non-Markovian memory, $\Delta_{t_2\to t_3}I(\rho^{CQ};\rho^{AB})$, has a negative contribution to $\Delta_{t_2\to t_3}I(\rho^C;\rho^Q)$. This means that by leveraging the resource of non-Markovian memory $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$, such that it is decreased from $t_2 \to t_3$, i.e., $\Delta_{t_2\to t_3}I(\rho^{CQ};\rho^{AB}) \leqslant 0$, one can grant enhancement to $\Delta_{t_2\to t_3}I(\rho^C;\rho^Q)$, increasing $I(\rho^C_{t_3};\rho^Q_{t_3})$ for increased production of daemonic ergotropy. We show this numerically in Fig. 9, revealing that this effect is stronger for the $\Phi^{\text{ext}}_{\text{PF}}$ regime, and that a threshold has to be reached, before which there is an inverse relation between $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ and $I(\rho^C_{t_3};\rho^Q_{t_3})$ instead.

Finally, we summarize these results of Figs. 5, 7, and 8 in Fig. 10, where we plot the maximum daemonic ergotropy, $\max\mathcal{E}^D$, the range of values of $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ and $I(\rho^C_{t_3};\rho^Q_{t_3})$ (shaded regions), as well as the corresponding values of $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ and $I(\rho^C_{t_3};\rho^Q_{t_3})$ that give maximum daemonic ergotropy (solid and dashed lines), against the parameter γ. Figure 10 makes clear the advantage of the $\Phi^{\text{ext}}_{\text{PF}}$ regime, which decreases with increasing γ until a certain threshold of $\gamma \approx 0.4$, where it can no longer produce positive daemonic ergotropy. More importantly, it is able to surpass the daemonic ergotropy of Φ^{indep}, achieving the maximum daemonic ergotropy in the quantum switch case of $\gamma = 0$. This means that there are operational regimes where the quantum switch can outperform a superposition of independent channels if it is part of the $\Phi^{\text{ext}}_{\text{PF}}$ regime with non-Markovian backflow of information, as well as increased system-environment correlations.

On the other hand, the operational regime of $\Phi^{\text{ext}}_{\text{no PF}}$ has an inverse relation where the upper bound of the daemonic

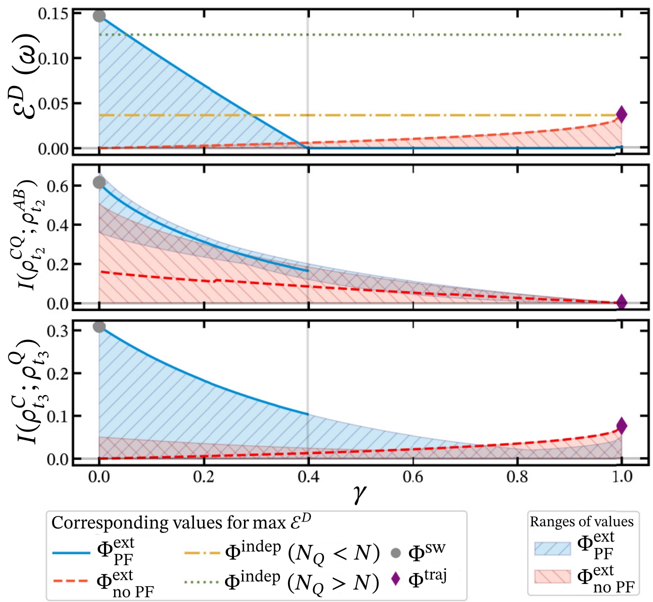

FIG. 10. Maximum daemonic ergotropy, $\max\mathcal{E}^D$, system-environment correlations at t_2, $I(\rho^{CQ}_{t_2}, \rho^{AB}_{t_2})$, and correlations between C and Q at t_3 against γ. The shaded regions are the range of values for each regime as shown in Figs. 5, 7, and 8, while the solid and dashed lines are the corresponding values for points that give maximum daemonic ergotropy. There is a clear advantage of the $\Phi^{\text{ext}}_{\text{PF}}$ operation regime, which decreases with decreasing non-Markovianity until a point where it has no advantage over Φ^{indep}. Furthermore, the fully Markovian case of Φ^{traj} cannot perform better than Φ^{indep}. The difference between $\Phi^{\text{ext}}_{\text{PF}}$ and $\Phi^{\text{ext}}_{\text{no PF}}$ is that $\Phi^{\text{ext}}_{\text{PF}}$ enhances the system-environment correlations, or non-Markovian memory, which in turn enhances the correlations between C and Q at t_3 greatly. Note that in the second and third plots, the corresponding values that give maximum $\mathcal{E}^D$ for $\Phi^{\text{ext}}_{\text{PF}}$ (solid lines) are truncated after $\gamma \approx 0.4$. This is because their daemonic ergotropy is zero above $\gamma \approx 0.4$, and thus there are no corresponding values of $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ and $I(\rho^C_{t_3};\rho^Q_{t_3})$ that give maximum $\mathcal{E}^D$ for $\Phi^{\text{ext}}_{\text{PF}}$.

ergotropy increases with decreasing amount of system-environment correlations. In this regime, the presence of system-environment correlations offers no advantage, which can be attributed to the lack of backflow of information, as well as the decreased system-environment correlations. Additionally, we note that the maximum daemonic ergotropy cannot surpass that of the superposition of independent channels Φ^{indep}, and is only equal to it for the fully Markovian case of Φ^{traj}. Therefore, the superposition of trajectories Φ^{traj} cannot grant any advantages that are not already granted by the superposition of independent channels Φ^{indep}.

In the second plot, we can see that the $\Phi^{\text{ext}}_{\text{PF}}$ regime can grant increased non-Markovian memories $I(\rho^{CQ}_{t_2};\rho^{AB}_{t_2})$ that are unachievable by the $\Phi^{\text{ext}}_{\text{no PF}}$ regime, which can in turn grant increased $I(\rho^C_{t_3};\rho^Q_{t_3})$ in the third plot. The third plot also reveals the possibility of a threshold for $I(\rho^C_{t_3};\rho^Q_{t_3})$ of around 0.1 that is reached when $\gamma \approx 0.4$. Values of $I(\rho^C_{t_3};\rho^Q_{t_3})$ below this threshold cannot grant positive daemonic ergotropy for $\Phi^{\text{ext}}_{\text{PF}}$. We can also see this by comparing Figs. 5 and 8, where the daemonic ergotropy $\mathcal{E}^D_{\text{ext}}(\rho^{CQ}_{t_3})$ for the $\Phi^{\text{ext}}_{\text{PF}}$ regime (negative

x_A or x_B, or both) decreases to zero when $I(\rho^C_{t_3}, \rho^Q_{t_3})$ decreases to below a threshold.

Since the key difference between Φ^{sw} and Φ^{traj} when comparing against Φ^{indep} is the non-Markovian operation of $\Gamma^{\text{sw}}_{t_2\to t_3}$, as opposed to the Markovian $\Phi^{\text{traj}}_{t_2\to t_3}$ (see Sec. II B), Fig. 10 reveals the advantage of the presence of non-Markovianity in the system, especially with the presence of backflow of information for the $\Phi^{\text{ext}}_{\text{PF}}$ regime. Furthermore, this non-Markovianity does not have to be completely preserved as in the case of the quantum switch. That is, we can allow some equilibration of the bath qubits (as parametrized by γ), and still grant advantages that surpass the superposition of independent channels. Experimentally, this translates to some leeway in the short timescales between the interactions of $U^{\text{sw}}_{t_0\to t_1}$ and $U^{\text{sw}}_{t_2\to t_3}$.

We note, however, that these are only true for the case of $N_Q < N$, where the main system Q is initialized in a bath that is colder than the baths used in the Φ^{ext} operation, as Φ^{ext} is unable to produce positive daemonic ergotropy for the case of $N_Q > N$. Conversely, the superposition of independent channels Φ^{indep} can still grant positive daemonic ergotropy even in the case of $N_Q > N$ (see Fig. 10), and is thus one advantage of Φ^{indep} that cannot be achieved by the other operations. In the next section, we will look into these conditions for the production of positive daemonic ergotropy for Φ^{indep}, Φ^{sw}, and Φ^{traj}.

B. Conditions for positive daemonic ergotropy

The choice of thermalizing strength, or regimes, as parametrized by x_A and x_B is important in determining the production of daemonic ergotropy as evidenced in Fig. 5. Furthermore, the choice of bath temperatures can also determine the magnitude of daemonic ergotropy produced. Naturally, we expect the thermalizing regimes to be dependent on the choice of bath temperatures; e.g., if a working body is initialized at a temperature far from the temperature of an interacting heat bath, then we expect a great change to the state of the working body even for short interaction time. On the other hand, if the working body temperature is very close to the temperature of the heat bath, then the converse is true, and a long interaction time is needed to change the state of the working body. In this regard, we attempt to find the conditions that x_A and x_B must fulfill for positive daemonic ergotropy given some arbitrary fixed bath temperatures parametrized by N_Q and N.

1. Prethermalization

The effects of the thermalizing regimes, as parametrized by x_A and x_B, to daemonic ergotropy manifests nontrivially in Eq. (62), but can be gleaned in the special cases of the quantum switch Φ^{sw}, in the superposition of trajectories Φ^{traj}, and in the superposition of independent channels Φ^{indep}, allowing us to derive the conditions for positive daemonic ergotropy production. First, for the case of Φ^{indep}, by setting Eq. (61) to be greater than zero and simplifying, we have

$$\frac{N - N_Q}{N(1-N)(1-2N_Q)} > \frac{(1-x_A)(1-x_B)}{1 + x_A x_B - x_A^2 - x_B^2}, \tag{74}$$

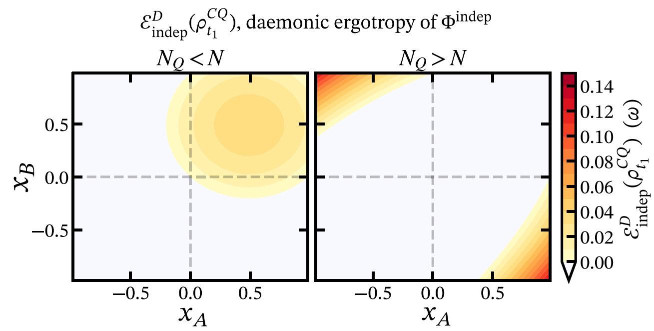

FIG. 11. Daemonic ergotropy $\mathcal{E}^D_{\text{indep}}(\rho^{CQ}_{t_1})$ of the case of superposition of independent channels Φ^{indep} against x_A and x_B, for both $N_Q < N$ and $N_Q > N$. For $N_Q < N$, we take $\beta_Q = 2\beta$, and for $N_Q > N$, we take $\beta_Q = \beta/2$.

where the left-hand term is positive when $N_Q < N$, and negative when $N_Q > N$. The right-hand term can also be positive or negative depending on the values of x_A and x_B; this means that even in the case of $N_Q > N$, there are thermalization regimes, as parametrized by x_A and x_B, where a positive daemonic ergotropy is possible for Φ^{indep} as long as Eq. (74) is fulfilled. In fact, as shown in Fig. 10, the maximum possible daemonic ergotropy can be greater with $N_Q > N$ than with $N_Q < N$. We plot the daemonic ergotropy $\mathcal{E}^D_{\text{indep}}(\rho^{CQ}_{t_1})$ as a color map against x_A and x_B for both $N > N_Q$ and $N < N_Q$ in Fig. 11.

On the other hand, in the case of the quantum switch Φ^{sw}, setting $\gamma = 0$ and simplifying Eq. (62) to be greater than zero gives

$$\frac{N - N_Q}{N(1-N)(1-2N_Q)} > \frac{(1+x_A)(1+x_B)}{1 - x_A x_B}, \tag{75}$$

where the left-hand term is the same as in Eq. (74). However, different from the case of Φ^{indep} in Eq. (74), the right-hand term in Eq. (75) is always positive. This means that it cannot produce positive daemonic ergotropy in the case of $N_Q > N$ where the left-hand term is negative. In other words, the initializing bath for the main system Q has to be at a lower temperature than the baths used in the operation of Φ^{sw} for production of positive daemonic ergotropy. This is also true for the general case of $\gamma > 0$ for Φ^{ext}, which can be verified numerically.

For the fully Markovian case of a superposition of trajectories Φ^{traj}, the bath qubits are completely thermalized by the equilibration operation $\mathcal{N}_{t_1\to t_2}$ with $\gamma = 1$. The necessary conditions for Φ^{traj} to have positive daemonic ergotropy are nontrivial in general. However, by taking the special case of $x_A = 1$ or $x_B = 1$, we can obtain a less general condition instead. Without loss of generality, substituting $x_A = 1$ and $\gamma = 1$ into Eq. (62) and setting it to be greater than zero, we have

$$\frac{N - N_Q}{N(1-N)(1-2N_Q)} > \frac{1 - x_B}{1 + x_B}. \tag{76}$$

Therefore, Φ^{traj} can produce positive daemonic ergotropy if Eq. (76) holds in the case of $x_A = 1$. Similar to the case of the quantum switch in Eq. (75), the right-hand term is always positive, and thus Φ^{traj} can only grant positive daemonic ergotropy for the case of $N_Q < N$ where the left-hand term is also positive.

The case of $x_A = 1$ corresponds to an identity operation, such that we have a controlled superposition of $\Phi_B \circ I$ and $I \circ \Phi_B$, which reduces to the superposition of independent channels of Φ^{indep} where $x_A = x_B$, i.e., a superposition of independent Φ_B operations. We can see this by setting $x_A = x_B$ in Eq. (74) which gives the same condition as Eq. (76).

On the other hand, this is not true for the quantum switch operation Φ^{sw}. That is, if we set $x_A = 1$ for Φ^{sw} in Eq. (75), we obtain $2(1 + x_B)/(1 - x_B)$ on the right-hand side, different from the cases of Φ^{traj} and Φ^{indep}. This is despite the fact that the operation also reduces to a controlled superposition of $\Phi_B \circ I$ and $I \circ \Phi_B$. This is because in the quantum switch case, the Φ_B operations in each path of the superposition are not independent of each other due to the presence of coherent non-Markovian memory in the bath qubit B.

Equations (75) and (76) also hint at the presence of the two operational regimes $\Phi^{\text{ext}}_{\text{PF}}$ and $\Phi^{\text{ext}}_{\text{no PF}}$. For example, to produce positive daemonic ergotropy in Φ^{sw}, we need to minimize the right-hand side of Eq. (75), which is achieved for negative x_A or x_B or both, which is in the $\Phi^{\text{ext}}_{\text{PF}}$ regime. On the other hand, in the case of Φ^{traj}, the right-hand side of Eq. (76) is minimized for large values of x_B, which is in the $\Phi^{\text{ext}}_{\text{no PF}}$ regime.

2. Full thermalization

One might also be interested in the full thermalization case of $x_A = x_B = 0$, where the interaction fully thermalizes the interacting qubit into a thermal state, which is a completely passive state with zero ergotropy. It was shown, however, that the quantum switch can grant nonzero daemonic ergotropy or perform refrigeration tasks even with full thermalization [21,22], which was touted as the advantage granted by the quantum switch's indefinite causality. However, we note that such an advantage can also be achieved with the superposition of independent channels Φ^{indep}, despite the absence of indefinite causality in Φ^{indep}.

Substituting the full thermalization case of $x_A = x_B = 0$ into the Φ^{indep} case of Eq. (74), we have

$$1 - 2N > N^2\left(\frac{1}{N_Q} - 2\right), \tag{77}$$

which after substituting $N = 1/(1 + e^{\beta\omega})$ and $N_Q = 1/(1 + e^{\beta_Q\omega})$, and simplifying, gives

$$\begin{aligned} -\frac{1 - e^{\beta\omega}}{1 + e^{\beta\omega}} &> -\frac{1 - e^{\beta_Q\omega}}{(1 + e^{\beta\omega})^2}, \\ (1 - e^{\beta\omega})(1 + e^{\beta\omega}) &> 1 - e^{\beta_Q\omega}, \\ 1 - e^{2\beta\omega} &> 1 - e^{\beta_Q\omega}, \\ \Rightarrow 2\beta &< \beta_Q. \end{aligned} \tag{78}$$

That is, the superposition of independent channels Φ^{indep} can grant positive daemonic ergotropy even in the full thermalization case, as long as the heat bath in the operation is at least twice that of the initializing heat bath.

Likewise, substituting the full thermalization case of $x_A = x_B = 0$ into the quantum switch Φ^{sw} case of Eq. (75) and simplifying, we have

$$1 - 2N > N^2\left(\frac{1}{N_Q} - 2\right), \tag{79}$$

which is the same as Eq. (77), and hence leads to the same condition of

$$2\beta < \beta_Q. \tag{80}$$

We note that this condition for the quantum switch was also shown in Ref. [22]. Interestingly, Φ^{sw} and Φ^{indep} have the same condition to generate positive daemonic ergotropy with full thermalization. In fact, the amount of daemonic ergotropy generated for both are also the same of $-\mu + (N - N_Q)/2$, which can be shown simply by substituting $x_A = x_B = 0$ into Eqs. (61) and (62). This means that in the case of full thermalization there is no need to implement indefinite causal order with the quantum switch, because a superposition of independent channels, where the working system interacts with two heat baths in a superposition, can achieve the same daemonic ergotropy.

The condition for the superposition of trajectories Φ^{traj} to generate positive daemonic ergotropy with full thermalization is more complicated. We substitute the full thermalization case of $x_A = x_B = 0$ into the Φ^{traj} case of Eq. (62) along with $\gamma = 1$, and set it to be greater than zero, giving us

$$N_Q < \frac{N(1 - N)(3N - N^2 - 3) + N}{2N(1 - N)(N - N^2 - 2) + 1}, \tag{81}$$

which we can rewrite with $N_Q = 1/(1 + e^{\beta_Q\omega})$ and $N = 1/(1 + e^{\beta\omega})$ to obtain

$$\frac{1}{1 + e^{\beta_Q\omega}} < \frac{1 + 2e^{\beta\omega}(1 - e^{\beta\omega})(1 + e^{\beta\omega})}{1 + e^{4\beta\omega}}. \tag{82}$$

Solving for β_Q will then give us

$$\beta_Q > \beta + \frac{1}{\omega}\log\left(\frac{e^{2\beta\omega}(2 + e^{\beta\omega}) - 2}{2e^{\beta\omega}(1 + e^{\beta\omega})(1 - e^{\beta\omega}) + 1}\right). \tag{83}$$

The condition of Eq. (83) is a difficult one. We can see this by plotting the upper bound of N_Q, i.e., the right-hand side of Eq. (81), against N in Fig. 12. The upper bound is only positive for very large values of N. Furthermore, it is much

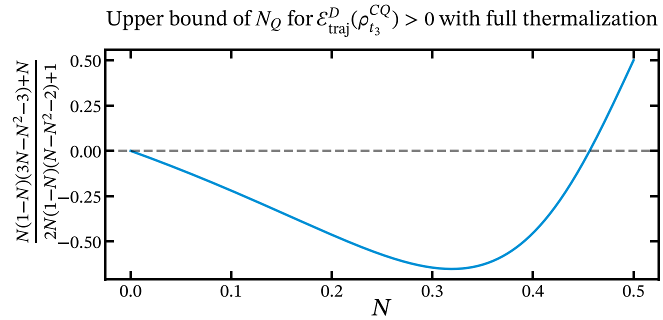

FIG. 12. Upper bound of N_Q against N required for positive daemonic ergotropy in the Φ^{traj} operation for the case of full thermalization. The upper bound is only positive for large values of N, which means that the heat baths used in the operation of Φ^{traj} must be at a very high temperature.

smaller than N until N approaches 0.5, e.g., it is close to zero when $N \approx 0.45$, but approaches 0.5 when $N \rightarrow 0.5$. Since $N \rightarrow 0.5$ implies a $T \rightarrow \infty$ or $\beta \rightarrow 0$, and assuming a fixed ω, these translate to $\beta \ll 1$ and $\beta_Q \gg \beta$; i.e., we require the heat baths to be at a very high temperature.

Ultimately, in the full thermalization case, there are no additional advantages that the quantum switch $\Phi^{\rm sw}$ or the superposition of trajectories $\Phi^{\rm traj}$ can grant that the superposition of independent channels $\Phi^{\rm indep}$ cannot. Therefore, the presence or absence of non-Markovianity plays little role for full thermalization.

V. CONCLUSION

We have demonstrated how the quantum switch operation $\Phi^{\rm sw}$ can have intrinsic non-Markovianity in the second half of its operation, due to the possibility for the violation of CP divisibility which results from the generation of system-environment correlations. As non-Markovianity can be a resource and was recently shown to explain some communication advantages of the quantum switch [31], we asked how non-Markovianity might also play a role in offering advantages in terms of work extraction as quantified by daemonic ergotropy.

By constructing an extended quantum switch process $\Phi^{\rm ext}$, we are able to control the amount of non-Markovianity in terms of system-environment correlations, such that it reduces to the quantum switch operation $\Phi^{\rm sw}$ in the fully non-Markovian case. For the fully Markovian case, the process reduces to the case of a superposition of trajectories $\Phi^{\rm traj}$, which the quantum switch was often shown to triumph over in terms of communication advantages due to the lack of indefinite causality for $\Phi^{\rm traj}$ [27,28]. From this perspective, the difference between $\Phi^{\rm sw}$ and $\Phi^{\rm traj}$ is thus the presence or absence of non-Markovianity, rather than indefinite causal orders.

We identified two operational regimes for $\Phi^{\rm ext}$ where, for some choices of channels or thermalizing regimes, we have the regime $\Phi^{\rm ext}_{\rm PF}$ where non-Markovian backflow of information is present, and exhibits behaviors where daemonic ergotropy is dependent on the amount of system-environment correlations. For other choices of thermalizing regimes, we have the regime $\Phi^{\rm ext}_{\rm no\ PF}$ where there is no non-Markovian backflow of information, and it exhibits behaviors where the amount of system-environment correlations are lesser and does not offer any advantage. The quantum switch operation $\Phi^{\rm sw}$ can only generate positive daemonic ergotropy in the first regime, while the operation $\Phi^{\rm traj}$ can only do so in the second regime.

We compared both regimes to the case of a superposition of independent channels $\Phi^{\rm indep}$, which was often shown to replicate or surpass the quantum switch [15,26]. We showed that $\max \mathcal{E}^D_{\rm indep} \geqslant \max \mathcal{E}^D_{\rm traj} \geqslant \max \mathcal{E}^D_{\rm ext,\ no\ PF}$, while there are possibilities for $\mathcal{E}^D_{\rm ext,\ PF} \geqslant \max \mathcal{E}^D_{\rm indep}$ for small γ where the presence of non-Markovianity is strong. That is, we can only surpass $\Phi^{\rm indep}$ for the operational regime that depends on the presence and amount of non-Markovianity, achieving the best advantage in the quantum switch case of $\Phi^{\rm sw}$ where non-Markovianity is maximum. Finally, we derived the conditions that must be fulfilled to generate positive daemonic ergotropy for $\Phi^{\rm sw}$, $\Phi^{\rm traj}$, and $\Phi^{\rm indep}$ given arbitrary fixed bath temperatures, for the case of prethermalization, as well as for the case of full thermalization, revealing the restriction of $N_Q < N$, i.e., a lower temperature for the initializing bath $\beta_Q > \beta$, in order for $\Phi^{\rm sw}$ and $\Phi^{\rm traj}$ to produce positive daemonic ergotropy, which does not apply for $\Phi^{\rm indep}$. These conditions also restrict $\Phi^{\rm sw}$ and $\Phi^{\rm traj}$ to the $\Phi^{\rm ext}_{\rm PF}$ and $\Phi^{\rm ext}_{\rm no\ PF}$ regimes, respectively, for positive daemonic ergotropy production.

The presence and advantages of non-Markovianity in the quantum switch were shown recently in Ref. [31] in the context of communication capacities, but are still relatively unexplored. The manifestation of non-Markovianity in quantum compositions of channels due to the creation of system-environment correlations in intermediate time steps of the operation is a topic of interest, and its effects on the overall operation of the channel should be explored to uncover untapped quantum resources.

Furthermore, we note that non-Markovianity is only present when the superposition remains coherent, and that the quantum switch is not simply a superposition of different non-Markovian operations. Rather, the non-Markovian memory acts across different paths of the superposition. We suggest that this coherent non-Markovianity plays an important role in the operations of the quantum switch and indefinite causal orders in general, contributing to their various quantum advantages. Therefore, future works can study and quantify the resource of coherent non-Markovianity, generalizing to a multiparty quantum switch or even to systems without indefinite causal orders. Identifying the various quantum resources in different compositions of quantum processes would benefit quantum information and quantum thermodynamical applications in enhanced computation, communication, and work extraction tasks.

APPENDIX A: PROOFS OF THE ENVIRONMENTAL REPRESENTATIONS FOR $\Phi^{\rm SW}$, $\Phi^{\rm TRAJ}$, AND $\Phi^{\rm INDEP}$

1. Quantum switch $\Phi^{\rm sw}$

In Sec. II B, we have the quantum switch operation of

$$\Phi^{\rm sw}(\rho^C \otimes \rho^Q) = \sum_{i,j} K^{\rm sw}_{ij}(\rho^C \otimes \rho^Q) K^{\rm sw^\dagger}_{ij}, \qquad \text{(A1)}$$

where

$$K^{\rm sw}_{ij} = |0\rangle\langle 0| \otimes B_j A_i + |1\rangle\langle 1| \otimes A_i B_j, \qquad \text{(A2)}$$

and we noted that it has an environmental representation of

$$\Phi^{\rm sw}(\rho^C \otimes \rho^Q) = {\rm Tr}_{A,B}[U^{\rm sw}(\rho^C \otimes \rho^A \otimes \rho^Q \otimes \rho^B) U^{\rm sw\dagger}], \qquad \text{(A3)}$$

where

$$\begin{aligned} U^{\rm sw} &= U^{\rm sw}_{t_1\rightarrow t_2} U^{\rm sw}_{t_0\rightarrow t_1} \\ &= \left[|0\rangle\langle 0|^C \otimes \left(I^A \otimes U^{QB}_B\right) + |1\rangle\langle 1|^C \otimes \left(U^{AQ}_A \otimes I^B\right)\right] \\ &\quad \times \left[|0\rangle\langle 0|^C \otimes \left(U^{AQ}_A \otimes I^B\right) + |1\rangle\langle 1|^C \otimes \left(I^A \otimes U^{QB}_B\right)\right] \\ &= |0\rangle\langle 0|^C \otimes \left(I^A \otimes U^{QB}_B\right)\left(U^{AQ}_A \otimes I^B\right) \\ &\quad + |1\rangle\langle 1|^C \otimes \left(U^{AQ}_A \otimes I^B\right)\left(I^A \otimes U^{QB}_B\right). \end{aligned} \qquad \text{(A4)}$$

Note that we consider general U_A^{AQ} and U_B^{QB} here. Taking the environmental states to be pure states of $\rho^A = |a\rangle\langle a|$ and $\rho^B = |b\rangle\langle b|$, the minimum number of dimensions of the environments A and B to describe arbitrary CPTP operations on the main system Q is d^2, where d is the dimension of the main system Q [59]. We show that this environmental representation indeed gives the quantum switch operation:

$$\begin{aligned}
\Phi^{\text{sw}}(\rho^C\otimes\rho^Q) &= \text{Tr}_{A,B}[U^{\text{sw}}(\rho^C\otimes\rho^A\otimes\rho^Q\otimes\rho^B)U^{\text{sw}\dagger}]\\
&= \text{Tr}_{A,B}\big[\big(|0\rangle\langle 0|^C\otimes\big(I^A\otimes U_B^{QB}\big)\big(U_A^{AQ}\otimes I^B\big)+|1\rangle\langle 1|^C\otimes\big(U_A^{AQ}\otimes I^B\big)\big(I^A\otimes U_B^{QB}\big)\big)\\
&\quad\times(\rho^C\otimes|a\rangle\langle a|\otimes\rho^Q\otimes|b\rangle\langle b|)\big(|0\rangle\langle 0|^C\otimes\big(I^A\otimes U_B^{QB}\big)\big(U_A^{AQ}\otimes I^B\big)+|1\rangle\langle 1|^C\otimes\big(U_A^{AQ}\otimes I^B\big)\big(I^A\otimes U_B^{QB}\big)\big)^\dagger\big]\\
&= \sum_{i,j=0}^{d^2-1}\big[|0\rangle\langle 0|^C\otimes(I^Q\otimes\langle j|^B)U_B^{QB}(I^Q\otimes|b\rangle^B)(\langle i|^A\otimes I^Q)U_A^{AQ}(|a\rangle^A\otimes I^Q)\\
&\quad+|1\rangle\langle 1|^C\otimes(\langle i|^A\otimes I^Q)U_A^{AQ}(|a\rangle^A\otimes I^Q)(I^Q\otimes\langle j|^B)U_B^{QB}(I^Q\otimes|b\rangle^B)\big](\rho^C\otimes\rho^Q)\\
&\quad\times\big[|0\rangle\langle 0|^C\otimes(\langle a|^A\otimes I^Q)U_A^{AQ\dagger}(|i\rangle^A\otimes I^Q)(I^Q\otimes\langle b|^B)U_B^{QB\dagger}(I^Q\otimes|j\rangle^B)\\
&\quad+|1\rangle\langle 1|^C\otimes(I^Q\otimes\langle b|^B)U_B^{QB\dagger}(I^Q\otimes|j\rangle^B)(\langle a|^A\otimes I^Q)U_A^{AQ\dagger}(|i\rangle^A\otimes I^Q)\big]\\
&= \sum_{i,j=0}^{d^2-1}(|0\rangle\langle 0|^C\otimes B_jA_i+|1\rangle\langle 1|^C\otimes A_iB_j)(\rho^C\otimes\rho^Q)(|0\rangle\langle 0|^C\otimes A_i^\dagger B_j^\dagger+|1\rangle\langle 1|^C\otimes B_j^\dagger A_i^\dagger)\\
&= \sum_{i,j=0}^{d^2-1}K_{ij}^{\text{sw}}(\rho^C\otimes\rho^Q)K_{ij}^{\text{sw}\dagger},
\end{aligned}\tag{A5}$$

where we obtained the quantum switch operations with Kraus operators of Eq. (A2), with

$$A_i = (\langle i|^A\otimes I^Q)U_A^{AQ}(|a\rangle^A\otimes I^Q),\tag{A6}$$

$$B_j = (I^Q\otimes\langle j|^B)U_B^{QB}(I^Q\otimes|b\rangle^B).\tag{A7}$$

2. Superposition of independent channels Φ^{indep}

Likewise, we noted in the main text that by terminating the quantum switch operation after $\Phi^{\text{sw}}_{t_0\to t_1}$, we will obtain the superposition of independent channels Φ^{indep} with Kraus operator

$$K_{ij}^{\text{indep}} = \eta_0|0\rangle\langle 0|\otimes A_i+\eta_1|1\rangle\langle 1|\otimes B_j.\tag{A8}$$

That is, we have

$$\begin{aligned}
\Phi^{\text{indep}}(\rho^C\otimes\rho^Q) &= \Phi^{\text{sw}}_{t_0\to t_1}(\rho^C\otimes\rho^Q)\\
&= \text{Tr}_{A,B}\big[U^{\text{sw}}_{t_0\to t_1}(\rho^C\otimes\rho^A\otimes\rho^Q\otimes\rho^B)U^{\text{sw}\dagger}_{t_0\to t_1}\big]\\
&= \text{Tr}_{A,B}\big[\big(|0\rangle\langle 0|^C\otimes\big(U_A^{AQ}\otimes I^B\big)+|1\rangle\langle 1|^C\otimes\big(I^A\otimes U_B^{QB}\big)\big)(\rho^C\otimes|a\rangle\langle a|^A\otimes\rho^Q\otimes|b\rangle\langle b|^B)\\
&\quad\times\big(|0\rangle\langle 0|^C\otimes\big(U_A^{AQ}\otimes I^B\big)+|1\rangle\langle 1|^C\otimes\big(I^A\otimes U_B^{QB}\big)\big)^\dagger\big]\\
&= \sum_{i,j=0}^{d^2-1}\big[|0\rangle\langle 0|^C\otimes(\langle i|^A\otimes I^Q)U_A^{AQ}(|a\rangle^A\otimes I^Q)\otimes\langle j|b\rangle^B+|1\rangle\langle 1|^C\otimes\langle i|a\rangle^A\otimes(I^Q\otimes\langle j|^B)U_B^{QB}(I^Q\otimes|b\rangle^B)\big]\\
&\quad\times(\rho^C\otimes\rho^Q)\big[|0\rangle\langle 0|^C\otimes(\langle a|^A\otimes I^Q)U_A^{AQ\dagger}(|i\rangle^A\otimes I^Q)\otimes\langle b|j\rangle^B+|1\rangle\langle 1|^C\otimes\langle a|i\rangle^A\\
&\quad\otimes(I^Q\otimes\langle b|^B)U_B^{QB\dagger}(I^Q\otimes|j\rangle^B)\big]\\
&= \left[|0\rangle\langle 0|^C\otimes\sum_{i=0}^{d^2-1}A_i\otimes\sum_{j=0}^{d^2-1}\langle j|b\rangle^B+|1\rangle\langle 1|^C\otimes\sum_{i=0}^{d^2-1}\langle i|a\rangle^A\otimes\sum_{j=0}^{d^2-1}B_j\right](\rho^C\otimes\rho^Q)\\
&\quad\times[|0\rangle\langle 0|^C\otimes A_i^\dagger\otimes\langle b|j\rangle^B+|1\rangle\langle 1|^C\otimes\langle a|i\rangle^A\otimes B_j^\dagger]\\
&= \left[|0\rangle\langle 0|^C\otimes\sum_{i=0}^{d^2-1}A_i\left(\sum_{j=0}^{d^2-1}\frac{1}{d^2}\right)+|1\rangle\langle 1|^C\otimes\sum_{j=0}^{d^2-1}B_j\left(\sum_{i=0}^{d^2-1}\frac{1}{d^2}\right)\right](\rho^C\otimes\rho^Q)[|0\rangle\langle 0|^C\otimes A_i^\dagger+|1\rangle\langle 1|^C\otimes B_j^\dagger]
\end{aligned}$$

$$= \sum_{i,j=0}^{d^2-1} \frac{1}{d^2}(|0\rangle\langle 0|^C \otimes A_i + |1\rangle\langle 1|^C \otimes B_j)(\rho^C \otimes \rho^Q)(|0\rangle\langle 0|^C \otimes A_i^\dagger + |1\rangle\langle 1|^C \otimes B_j^\dagger)$$

$$= \sum_{i,j=0}^{d^2-1} K_{ij}^{\text{indep}}(\rho^C \otimes \rho^Q)K_{ij}^{\text{indep}\dagger}, \tag{A9}$$

where we obtained the Kraus operators for the Φ^{indep} operation with $\eta_0 = \eta_1 = 1/d$. Note that we have $\sum_i \langle i|a\rangle = \sum_j \langle j|b\rangle = 1$, and we have inserted $\sum_i 1/d^2 = \sum_j 1/d^2 = 1$.

3. Superposition of trajectories Φ^{traj}

For the operation of the superposition of trajectories Φ^{traj}, we noted in the main text that we have $\Phi^{\text{traj}} = \Phi^{\text{traj}}_{t_1\to t_2} \circ \Phi^{\text{sw}}_{t_0\to t_1}$, where

$$\Phi^{\text{sw}}_{t_0\to t_1}\big(\rho^C_{t_0} \otimes \rho^Q_{t_0}\big) = \rho^{CQ}_{t_1} = \text{Tr}_{A,B}\big[U^{\text{sw}}_{t_0\to t_1}\big(\rho^C_{t_0} \otimes \rho^A \otimes \rho^Q_{t_0} \otimes \rho^B\big)U^{\text{sw}\dagger}_{t_0\to t_1}\big], \tag{A10}$$

$$\Phi^{\text{traj}}_{t_1\to t_2}\big(\rho^{CQ}_{t_1}\big) = \text{Tr}_{A,B}\big[U^{\text{sw}}_{t_1\to t_2}\big(\rho^{CQ}_{t_1} \otimes \rho^{A'} \otimes \rho^{B'}\big)U^{\text{sw}\dagger}_{t_1\to t_2}\big], \tag{A11}$$

such that we have additional environmental subsystems $A' \neq A$ and $B' \neq B$, resulting in Kraus operators of

$$K^{\text{traj}}_{ijkl} = \alpha_0|0\rangle\langle 0| \otimes B_l A_i + \alpha_1|1\rangle\langle 1| \otimes A_k B_j. \tag{A12}$$

Since $\Phi^{\text{traj}}_{t_1\to t_2}$ is a CPTP operation similar to Φ^{indep}, with the only difference being the positions of U^{AQ}_A and U^{QB}_B in $U^{\text{sw}}_{t_1\to t_2}$, we can apply Eq. (A9) simply with some relabeling such that

$$\Phi^{\text{traj}}_{t_1\to t_2}\big(\rho^{CQ}_{t_1}\big) = \sum_{k,l=0}^{d^2-1} K'^{\text{indep}}_{kl}\big(\rho^{CQ}_{t_1}\big)K'^{\text{indep}\dagger}_{kl}, \tag{A13}$$

where

$$K'^{\text{indep}}_{kl} = \frac{1}{d}(|0\rangle\langle 0|^C \otimes B_l + |1\rangle\langle 1|^C \otimes A_k). \tag{A14}$$

Since we have $\rho^{CQ}_{t_1} = \Phi^{\text{indep}}(\rho^C \otimes \rho^Q)$ from Eq. (A9), we have

$$\Phi^{\text{traj}}(\rho^C \otimes \rho^Q) = \Phi^{\text{traj}}_{t_1\to t_2}\big(\rho^{CQ}_{t_1}\big) = \sum_{i,j,k,l=0}^{d^2-1} K'^{\text{indep}}_{kl}K^{\text{indep}}_{ij}(\rho^C \otimes \rho^Q)K^{\text{indep}\dagger}_{ij}K'^{\text{indep}\dagger}_{kl}$$

$$= \sum_{i,j,k,l=0}^{d^2-1} K^{\text{traj}}_{ijkl}(\rho^C \otimes \rho^Q)K^{\text{traj}\dagger}_{ij}, \tag{A15}$$

where we obtained the Kraus operators for the Φ^{traj} operation:

$$K^{\text{traj}}_{ijkl} = K'^{\text{indep}}_{kl}K^{\text{indep}}_{ij} = \frac{1}{d^2}(|0\rangle\langle 0|^C \otimes B_l A_i + |1\rangle\langle 1|^C \otimes A_k B_j), \tag{A16}$$

with $\alpha_0 = \alpha_1 = 1/d^2$.

APPENDIX B: NONPOSITIVITY OF $p_{+,t}\mathcal{E}(\rho^Q_{+,t})$

As mentioned in Sec. III B, the daemonic ergotropy is expressed as

$$\mathcal{E}^D_{\text{ext}}\big(\rho^{CQ}_t\big) = p_{+,t}\mathcal{E}\big(\rho^Q_{+,t}\big) + p_{-,t}\mathcal{E}\big(\rho^Q_{-,t}\big), \tag{B1}$$

where

$$p_{+,t}\mathcal{E}\big(\rho^Q_{+,t}\big) = \max\big\{0,\ \omega p_{+,t}\delta\rho^Q_{+,t}\big\}, \tag{B2}$$

$$p_{-,t}\mathcal{E}\big(\rho^Q_{-,t}\big) = \max\big\{0,\ \omega p_{-,t}\delta\rho^Q_{-,t}\big\}, \tag{B3}$$

as there are no coherent contributions. Here, we will show that we have $p_{+,t}\mathcal{E}(\rho^Q_{+,t}) = 0$.

1. At t_1

By explicit computation, we have

$$p_{+,t_1}\delta\rho^Q_{+,t_1} = -\frac{N-N_Q}{2}\big(x_A^2 + x_B^2\big) - \mu(x_A + x_B) - \left(\frac{N-N_Q}{2} - \mu\right)(x_A x_B + 1) + 2N - 1, \tag{B4}$$

where

$$\mu = \frac{N}{2}(1-N)(1-2N_Q). \tag{B5}$$

We will perform a proof by contradiction by first assuming that $p_{+,t_1}\delta\rho^Q_{+,t_1} > 0$. After expanding and rearranging, we have

$$-\frac{(1-x_A)(1-x_B)}{1-x_Ax_B+x_A^2+x_B^2} < \frac{3N+N_Q-2}{N(1-N)(1-2N_Q)}. \tag{B6}$$

We can then check that for $x_A, x_B \in [-1, 1]$ and $N, N_Q \in [0, 0.5]$, the left-hand side has a range of $[-2, 0]$, and the right-hand side has a range of $[-\infty, -2]$. Therefore, Eq. (B6) which implies the condition of $p_{+,t_1}\delta\rho^Q_{+,t_1} > 0$ will never be fulfilled for all the possible values of x_A, x_B, N, and N_Q.

2. At t_3

Similarly, by explicit computation, at t_3 we have

$$\begin{aligned}
p_{+,t_3}\delta\rho^Q_{+,t_3} = &-\mu N(1-N)\gamma^2(1-x_A)^2(1-x_B)^2\\
&-(\mu-\nu)(\gamma-\gamma\sqrt{1-\gamma})x_Ax_B(x_A+x_B-x_Ax_B)\\
&-\mu(\gamma+\gamma\sqrt{1-\gamma})(x_A+x_B-1)\\
&-\nu(\gamma-\gamma\sqrt{1-\gamma})(x_A+x_B-1)\\
&-\mu(1-\gamma)(x_A^2+x_B^2)\\
&-\frac{N-N_Q}{2}\sqrt{1-\gamma}(x_A+x_B)(x_Ax_B-1)\\
&-\left(\frac{3(N-N_Q)}{2}-\mu\right)x_A^2x_B^2-\frac{N-N_Q}{2}+\mu\\
&+2N-1,
\end{aligned} \tag{B7}$$

which we can express in terms of $p_{-,t_3}\delta\rho^Q_{-,t_3}$ as

$$\tfrac{1}{2}p_{+,t_3}\delta\rho^Q_{+,t_3} = N(1-x_A^2x_B^2)+N_Qx_A^2x_B^2-\tfrac{1}{2}-\tfrac{1}{2}p_{-,t_3}\delta\rho^Q_{-,t_3}. \tag{B8}$$

Setting $\frac{1}{2}p_{+,t_3}\delta\rho^Q_{+,t_3} \leqslant 0$, we have

$$N(1-x_A^2x_B^2)+N_Qx_A^2x_B^2-\tfrac{1}{2}p_{-,t_3}\delta\rho^Q_{-,t_3} \leqslant \tfrac{1}{2}. \tag{B9}$$

For $p_{-,t_3}\delta\rho^Q_{-,t_3} \geqslant 0$, it is straightforward to see that the inequality always holds as the first two terms at the left-hand side are a convex sum of N and N_Q, which has a range of $[0, 0.5]$. The case of $p_{-,t_3}\delta\rho^Q_{-,t_3} < 0$ is more difficult, but we note that it can be checked numerically that the left-hand side has a range of $[0, 0.5]$; therefore the inequality, and thus $p_{+,t_3}\delta\rho^Q_{+,t_3} \leqslant 0$, always hold.

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