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Letter

Tailoring dynamical fermionization: Delta-kick cooling of a Tonks-Girardeau gas

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In one spatial dimension, quantum exchange statistics and interactions are inextricably intertwined. As a manifestation, the expansion dynamics of a Tonks-Girardeau gas is characterized by dynamical fermionization (DF), whereby the momentum distribution approaches that of a spin-polarized Fermi gas. Using a phase-space analysis and the unitary evolution of the one-body reduced density matrix, we show that DF can be tailored and reversed, using a generalization of delta-kick cooling (DKC) to interacting systems, establishing a simple protocol to rescale the initial momentum distribution. The protocol applies to both expansions and compressions and can be used for the microscopy of quantum correlations.

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Introduction. In one spatial dimension, the motion of particles gives rise to their scattering, in which the effects of interparticle interactions and quantum statistics are interwoven. This fact makes it possible to describe some strongly correlated quantum systems in terms of noninteracting models with different quantum statistics. This is the basis of the Bose-Fermi duality introduced by Girardeau in 1960 to describe a gas of one-dimensional (1D) hard-core bosons, which is now known as the Tonks-Girardeau (TG) gas [1–3]. The latter can be described in terms of a spin-polarized one-dimensional Fermi gas with no interactions. The Pauli exclusion principle in the Fermi gas makes the wave function vanish at contact, a feature shared by the TG gas due to hard-core interactions. The wave functions of the two systems are identical for a given particle ordering and differ only in their symmetrization. The bosonic TG wave function $\Psi_{\rm TG}$ can be obtained from that of the Fermi gas $\Psi_{\rm F}$ by explicit symmetrization according to the Bose-Fermi mapping $\Psi_{\rm TG} = \prod_{j<k} {\rm sgn}(x_k - x_j)\Psi_{\rm F}$. Similar relations exist in systems governed by strongly attractive p-wave interactions [4–6], general exchange statistics [7–9], long-range [10,11] and finite-strength interactions [12–14], mixtures [15], and spinor degrees of freedom [16], among other examples [3,14].

The TG gas can be considered as the strong-coupling limit of the Lieb-Liniger (LL) gas, which describes one-dimensional bosons subject to contact interactions of finite strength c [17–19]. This model is integrable and solvable by the Bethe ansatz. The relevance of the LL gas to ultracold atom physics was established by Olshanii, who showed that ultracold atoms in tight waveguides are described by the LL model with a tunable coupling constant c [19]. The strongly interacting limit $c \to +\infty$ leads to the TG regime, realized experimentally by making use of an optical lattice [20–22]. The connection between the continuum and lattice version of hard-core bosons is well understood [22,23], and dynamical correlations are in one-to-one correspondence at low densities.

Spatially local correlations such as the density profile are indistinguishable between dual systems [1]. This is not only true at equilibrium but during the dynamics, in which mean-field methods [24] overestimate the phase coherence, e.g., in interference patterns in the density profile [25]. By contrast, correlations depending on the coherences (off-diagonal elements) of the quantum state in the coordinate representation exhibit clear signatures of quantum statistics [26,27], thus distinguishing dual systems related by the Bose-Fermi mapping. A prominent example is the momentum distribution. While that of a 1D Fermi gas in the ground state exhibits a characteristic flat profile [3], that of the TG gas is sharply peaked at $k = 0$ [28]. An analytical approximation for the momentum distribution has been found for a TG gas at equilibrium [29], indicating a $1/|k|^{1/2}$ singularity in a homogeneous system at $k = 0$ [27,30], and a $1/k^4$ power-law decay of the tails [31]. The latter is governed by Tan's constant [32], which depends on the temperature of the gas and the energy density [33–36]. Out of equilibrium, it was predicted that a TG gas, after release from a harmonic trap in a 1D expansion, exhibits dynamical fermionization (DF), with the asymptotic momentum distribution matching that of free fermions in the same initial trap [23,37]. This phenomenon has been recently observed in the laboratory for the first time [38]. DF also governs the asymptotic behavior of an expanding LL gas, which enters the TG regime [39,40]. While it is conveniently described using scale invariance, which makes the density profile at different times self-similar, it does not rely on it, and occurs whether or not the initial confinement is harmonic [8,41–44]. Generalizations of this phenomenon have been reported for a fermionic analog of the TG gas [6], hard-core anyons [8], and spinor quantum gases [45,46]. DF is generally justified as a result of free expansion along the axial direction: As the particle density decreases, the asymptotic momentum distribution is

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that of the rapidities, which are the conserved quantities in a many-body integrable quantum system [8,23,37,41,42,47–49].

In this Letter, we analyze DF in phase space in arbitrary scale-invariant processes, involving both expansions and compressions, that is, whether or not the particle density decreases. While the momentum distribution and density profile of dual systems under DF become equal, the one-body reduced density matrix is shown to evolve unitarily, making the distinguishability of the corresponding quantum states independent of time. As a result, DF can be reversed or induced by a unitary, making use of a generalization of delta-kick cooling (DKC) to interacting systems, pulsing an external potential. This allows us to engineer protocols that rescale the momentum distribution for the microscopy of quantum correlations.

TG gas in a time-dependent trap. Consider a TG gas in a harmonic trap, dual to an ideal Fermi gas in the same confinement [2]. In the ground state, the TG wave function is the absolute value of the fermionic one, which is given by a Slater determinant, e.g., $\Psi_{\rm F}(x_1,\dots,x_{\rm N})=\frac{1}{\sqrt{{\rm N}!}}\det_{n=0,k=1}^{{\rm N}-1,{\rm N}}[\phi_n(x_k)]$ in terms of the single-particle harmonic oscillator eigenstates. Both systems are scale invariant with dimensionality $D=1$, and their time-dependent coherent states take the form [37,47,50,51]

$$\Psi(\vec{x},t)=\frac{1}{b^{\frac{\rm N}{2}}}\exp\left[i\frac{m\dot{b}}{2\hbar b}\sum_{i=1}^{\rm N}{x_i}^2-i\int_0^t\frac{E(0)}{\hbar b(t')^2}dt'\right]\times\Psi\left(\frac{x_1}{b},\dots,\frac{x_{\rm N}}{b},t=0\right),\qquad(1)$$

where $\vec{x}=\{x_1,\dots,x_{\rm N}\}$ and the scaling factor $b(t)>0$ is the solution of the Ermakov equation $\ddot{b}+\omega(t)^2b=\omega_0^2/b^3$ with the initial conditions $b(0)=1$, $\dot{b}(0)=0$. Note that this scaling law is not restricted to the ground state but it is shared by any many-body eigenstate $\Psi(\vec{x},0)$ with energy eigenvalue $E(0)$. We focus on the one-body reduced density matrix (OBRDM), that contains all the information required to analyze one-body observables. It is defined as $\rho_1(x,x',t)={\rm N}\int dx_2\cdots dx_{\rm N}\Psi(x,x_2\cdots x_{\rm N},t)\Psi^*(x',x_2\cdots x_{\rm N},t)$. From it, one can determine the density profile $\rho(x,t)=\rho_1(x,x,t)$, as well as the momentum distribution, making use of the Fourier transform $n(p,t)=\frac{1}{2\pi\hbar}\int dxdx'e^{-ip(x-x')/\hbar}\rho_1(x,x',t)$. Quantities derived from $|\Psi(\vec{x},t)|^2$ are shared by dual systems related by the Bose-Fermi mapping, given that $\prod_{j<k}[{\rm sgn}(x_k-x_j)]^2=1$ as described in Refs. [1,2,25]. Thus, the density profiles are equal for the TG and Fermi gases, $\rho_1^{\rm TG}(x,t)=\rho_1^{\rm F}(x,t)$. By contrast, those dependent on the coherence in real space generally differ. Using (1), the OBRDM evolves according to

$$\rho_1(x,x',t)=\frac{1}{b}\exp\left[i\frac{m\dot{b}}{2\hbar b}(x^2-x'^2)\right]\rho_1\left(\frac{x}{b},\frac{x'}{b},t=0\right).\qquad(2)$$

Similar relations hold for the time evolution of higher-order reduced density matrices [52]. In the limit of adiabatic driving $\dot{b}/b\to 0$, the OBRDM is rescaled as $\rho_1(x,x',t)=\rho_1(x/b,x'/b,0)/b$. The use of controlled expansions involving time-dependent traps and engineered by shortcuts to adiabaticity has been proposed for implementing such scaling without the requirement of slow driving, but generally involves time-dependent traps [51,53]. Such protocols realize in essence a dynamical microscope zooming in on correlations in the OBRDM.

Shared unitary evolution of the OBRDMs and its consequences. Equation (2) indicates that the evolution of OBRDMs for both the TG gas and the spin-polarized ideal Fermi gas is unitary. More precisely, we introduce a generic label $A=\{{\rm TG},{\rm F}\}$ for any of the dual systems and define the corresponding quantum state $\sigma^A=\frac{1}{\rm N}\int dxdx'\rho_1^A(x,x')|x\rangle\langle x'|$ such that ${\rm Tr}(\sigma^A)=1$. Then Eq. (2) implies $\sigma^A(t)=U(t)\sigma^A(0)U^\dagger(t)$, where

$$U(t)=\exp\left[i\frac{m\dot{b}}{2\hbar b}x^2\right]\exp\left[-i\frac{\ln b}{2\hbar}(xp+px)\right],\qquad(3)$$

and the rightmost term is the dilatation operator implementing a scaling transformation in real space by a factor b. Note that $U(t)$ is the same for both $A=\{{\rm TG},{\rm F}\}$. In fact, under scale-invariant dynamics, the evolution of the quantum state associated with any k-body reduced density matrix is also unitary [52]. The identical unitary evolution of the OBRDMs has several consequences:

(i) The spectral decomposition of the OBRDM is of the form $\rho_1(x,x',t)=\sum_\mu\lambda_\mu(0)\phi_\mu(x,t)\phi_\mu(x',t)$, where the eigenvalues $\lambda_\mu(0)$ of the OBRDM, which correspond to the occupation numbers of the instantaneous natural orbitals $\phi_\mu(x,t)$, are constant in time. It follows that the dynamics is isentropic, i.e., it preserves the von Neumann entropy $S(\sigma^A)=-{\rm Tr}[\sigma^A\log\sigma^A]$. The situation in the continuum is thus in contrast with that reported for hard-core bosons in an optical lattice, where Rigol and Muramatsu first pointed out the distinct character of the OBRDMs of dual systems [23]. They reported a small variation of the occupation numbers, consistent with the fact that scale invariance is approximate in the presence of a longitudinal lattice. In addition, we note that the time-dependent natural orbitals fulfill the scaling property $\phi_\mu(x,t)=\exp[i\frac{m\dot{b}}{2\hbar b}x^2]\phi_\mu(x/b,0)/\sqrt{b}$.

(ii) Even if the density profile, and the momentum distribution under DF, are shared by both dual systems, their quantum states remain equally distinct at all times. To see this, consider the Uhlmann fidelity defined as $\mathcal{F}(\sigma,\sigma')={\rm Tr}(\sqrt{\sqrt{\sigma}\sigma'\sqrt{\sigma}})$ [54,55] as a similarity measure between the quantum states σ and σ'. Given that the Uhlmann fidelity is invariant under the action of a unitary, it follows that

$$\mathcal{F}[U(t)\sigma^{\rm TG}(0)U^\dagger(t),U(t)\sigma^{\rm F}(0)U^\dagger(t)]=\mathcal{F}[\sigma^{\rm TG}(0),\sigma^{\rm F}(0)].\qquad(4)$$

Thus, $\sigma^{\rm TG}$ does not approach $\sigma^{\rm F}$ during time evolution.

(iii) The unitary evolution of the OBDRM under scale invariance allows describing DF as the result of a canonical transformation in phase space [56], making it possible to control and reverse DF by pulsing an external potential, as we next discuss.

Phase-space analysis of DF. The Wigner function associated with the OBRDM can be represented as a function of the coordinate x and the canonically conjugated momentum

p [57,58],

$$W(x,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\rho_1(x-y,x+y,t)e^{2ipy/\hbar}dy. \quad (5)$$

The marginals of $W(x,p,t)$ correspond to the density profile $\rho(x,t) = \rho_1(x,x,t) = \int W(x,p,t)dp$ and the momentum distribution $n(p,t) = \rho_1(p,p,t) = \int W(x,p,t)dx$. From the dynamics (2) of the OBRDM, following an arbitrary modulation of the trapping frequency $\omega(t)$, the exact time evolution of the Wigner function reads [52,59]

$$W(x,p,t) = W\left(\frac{x}{b}, bp - m\dot{b}x, t=0\right), \quad (6)$$

where we note that W does not need to be positive, i.e., it can describe a nonclassical state. Note that this evolution is common to all scale-invariant systems, e.g., in harmonic and anharmonic traps [53,60,61]. The Wigner function is rescaled, stretching (compressing) the density profile along the x axis, and compressing (stretching) the momentum distribution along the p axis if $b(t) > 1$ [$b(t) < 1$]. The supplementary term $-m\dot{b}x$ involves a shift in phase space, which induces DF.

Both for the TG and Fermi gas, the density profile exhibits explicitly the scale invariance, while the asymptotic momentum distribution can be related to the initial density profile [52]:

$$\rho(x,t) = \frac{1}{b}\rho\left(\frac{x}{b},0\right), \quad n(p,t) \approx \frac{1}{m\dot{b}}\rho\left(\frac{p}{m\dot{b}},0\right). \quad (7)$$

This relation between $n(p,t)$ and $\rho(x,0)$ is complementary to that under time-of-flight imaging, connecting $\rho(x,t)$ to $n(p,0)$ [62]. The results in (7) are consistent with previous studies limited to sudden expansions [40,42] and require that the initial Wigner function decays over the characteristic spread Δp so that $\Delta p \ll x_0 m\dot{b}b$ during the dynamics, which is equivalent to the condition $\dot{b}b \gg 2\omega_0$ taking $\Delta p \approx 2p_0$. In the special case of a harmonic trap, the initial density profile of the TG gas can be expressed in terms of the rescaled momentum distribution of the Fermi gas as further discussed in the Supplemental Material (SM) [52], making the term "DF" natural in this setting [23,37]. This feature is specific to the harmonic confinement because single-particle eigenstates $\phi_n(x)$ are in this case expressed in terms of Hermite polynomials, which are eigenstates of the Fourier transform. Yet, despite the coincidence of the marginals (density profile and momentum distribution) of the Wigner functions of the TG gas and the Fermi gas at late times, the OBRDMs remain distinct as discussed above. For the sake of demonstration, we analyze the phase-space dynamics in an expansion of a TG gas confined in a harmonic trap with frequency ω_0 that is suddenly released in a wider trap with frequency ω_1, illustrated in Fig. 1. This process leads to a periodic time dependence of the scaling factor $b(t)$ [37,63]; see SM [52] for other protocols. The width of the cloud is controlled by $b(t)$ and oscillates after the release of the TG gas into the wider trap. This behavior induces DF periodically, with the nonequilibrium momentum distribution evolving between that of an equilibrium TG gas and a Fermi gas, as initially predicted [37] and observed experimentally [38]. The DF observed in the momentum distribution results from a linear canonical transformation in

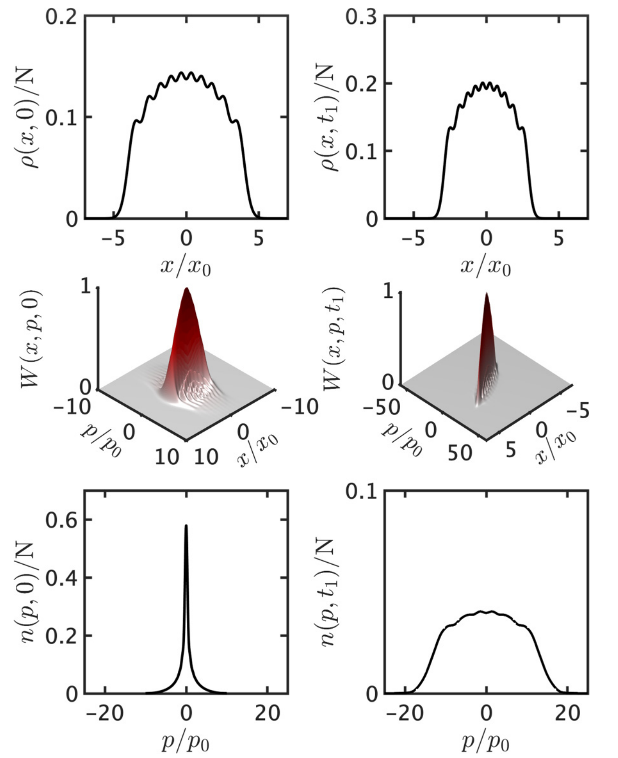

FIG. 1. DF of a TG gas in a sudden expansion in phase space. Under scale-invariant dynamics, DF results from a canonical transformation that describes the evolution of the Wigner function. The density profile is scale invariant at all times. In addition, when the rate of change of the scaling factor is large, the asymptotic momentum distribution can be related to the initial density profile. The initial Wigner function is peaked along the axis $p = 0$, with riddles on both sides that take negative values. At the time $t_1 = \frac{3\pi}{4\omega_1}$, the Wigner function is rotated and dilated in phase space, so that the momentum distribution corresponds to the rescaled density profile ($\omega_0 = 5\omega_1$ and $N = 10$).

phase space, familiar in classical mechanics, involving the rotation and scaling of the Wigner function.

Tailoring and reversing DF with kicks. The momentum shift in (6) is responsible for DF. Classically, one may expect to cancel it by applying a conservative force for a short period of time τ_k inducing a momentum change $\delta p = -\tau_k \partial_x V(x)$, i.e., pulsing an external potential $V(x)$. This argument, limited to classical noninteracting systems, is the basis of delta-kick cooling (DKC) [64–66]. In what follows, we make use of the extension of DKC for scale-invariant interacting systems. Under Eq. (2), excitations encoded in the phase factor proportional to $\dot{b}/b$, that are responsible for DF, can be explicitly canceled in an interacting system by applying a kick potential of appropriate strength. Canceling the phase allows us to tailor the momentum distribution and reverse DF. Given that the phase oscillation in Eq. (1) is quadratic in the coordinates, it can be canceled by pulsing an external harmonic trap with a given frequency ω_k. To this end, consider the Hamiltonian with a δ kick applied at t_k,

$$H_k(t) = H(t) + \delta(t-t_k)\frac{1}{2}m\omega_k^2\sum_{i=1}^{N} {x_i}^2. \quad (8)$$

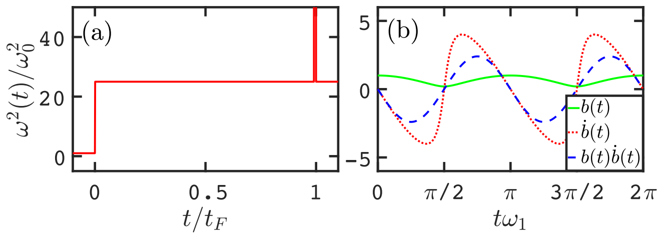

FIG. 2. (a) The frequency modulation leading to the implosion protocol relies on a sudden compression from the initial frequency ω_0 to the final frequency $\omega_1/5 = \omega_0 = 1$. DF is reversed by making use of a pulsed attractive harmonic potential. For illustration, we chose the frequency of the kick $\omega_k^2 = 1000\omega_0^2$, $t_k = 3\pi/4\omega_1$, and the final frequency determined by $b_F = \sqrt{\omega_0/\omega_F} = \sqrt{1/5}$. (b) Evolution of the scaling factor, its derivative, and the product $b\dot{b}$.

The use of a delta function is justified when the duration of the pulse τ_k is short with respect to other timescales [63]. The corresponding time-evolution operator admits the factorization $U_\delta(t_F = t_k + \tau_k, 0) = e^{-i\tau_k \frac{m\omega_k^2}{2\hbar}\sum_{i=1}^{N} {x_i}^2} U(t_k, 0)$, where $U(t, t')$ is the propagator associated with $H(t)$, and τ_k is a small timescale during which the kick is applied. Considering the evolution from $t = 0$ to time $t_k + \tau_k$, one can choose the pulse parameters τ_k and ω_k such that

$$\tau_k\omega_k^2 = \frac{\dot{b}(t_k)}{b(t_k)}. \tag{9}$$

This requires pulsing a harmonic trap with $\omega_k > 0$ in an expansion with $\dot{b}(t_k) > 0$ and an inverted harmonic trap with purely imaginary frequency $i\omega_k$ in a compression with $\dot{b}(t_k) < 0$. In either case, the application of the kick cancels DF, and brings back the OBRDM to the initial one up to a scaling of the coordinates with respect to b,

$$\rho_1(x, x', t_k + \tau_k) = \frac{1}{b(t_k)}\rho_1\left(\frac{x}{b(t_k)}, \frac{x'}{b(t_k)}, t = 0\right). \tag{10}$$

Consequently, the momentum distribution after the kick is $n(p, t) = b\, n(pb, 0)$, and similarly the Wigner function reduces to $W(x, p, t) = W_0(x/b, bp, 0)$ with $b = b(t_k)$. In turn, DKC prepares the same state that would have been obtained under adiabatic dynamics, without the requirement of slow driving.

Imploding TG gas. Intuitively the DF occurs for an expansion as the particle density decreases. This is the case considered in most theoretical studies, in which it is possible to suppress DF by DKC as we show in the SM [52]. Yet, DF is also possible during a compression, as demonstrated experimentally in Ref. [38]. Indeed, the phase-space dynamics (6) yields DF in a compression process with $b(t_F) < b(t = 0) = 1$ if the rate of change of the scaling factor is fast enough so that $\dot{b}b$ is large. We consider a sudden compression protocol, where the trap of initial frequency ω_0 is compressed to a frequency $\omega_1 > \omega_0$, leading to the periodic scaling factor $b(t)$ displayed in Fig. 2. Large values of $\dot{b}b$ induce a high-frequency phase modulation in the coordinate representation. For the prescribed protocol, applying at the time t_k a kick of duration τ_k, the required pulse parameters to reverse DF are set by

$$\tau_k\omega_k^2 = \frac{\omega_1(\omega_0^2 - \omega_1^2)\sin(\omega_1 t_k)\cos(\omega_1 t_k)}{(\omega_0^2 - \omega_1^2)\sin^2(\omega_1 t_k) + \omega_1^2}. \tag{11}$$

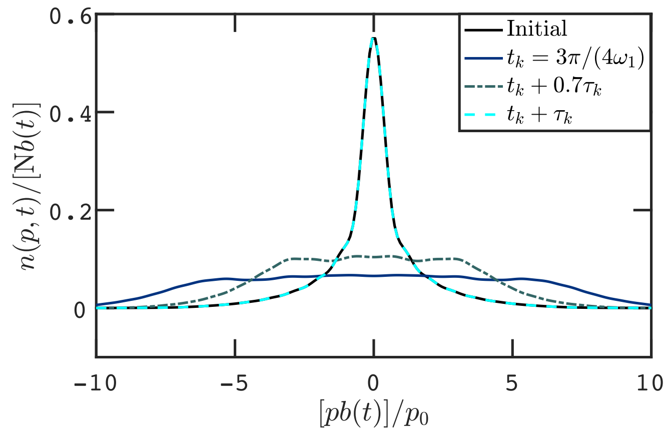

FIG. 3. DKC of an imploding TG gas. (a) Momentum distribution of a trapped TG gas at $t = 0$, after the implosion at $t_k = \frac{3\pi}{4\omega_1}$, after the implosion followed by a pulse of shorten duration $0.7\tau_k$ (green dashed-dotted line), and with the DKC duration τ_k satisfying (11) (turquoise blue dashed line); N = 7. In the dashed green line the kick strength is chosen as $\tau_k\omega_k^2$ in order to cancel DF, with $\omega_k^2 = 1000\omega_0^2$. In choosing the strength of the kick differently one can modulate DF. The implosion protocol corresponds to the sudden compression of the trap from ω_0 to $\omega_1 = 5\omega_0$.

The evolution of the momentum distribution at different stages of the protocol is shown in Fig. 3, for an initially confined TG gas undergoing an implosion engineered by a sudden frequency increase. The oscillatory time dependence of $b(t)$ shown in Fig. 2 yields associated oscillations of the momentum distribution, which exhibits DF exactly at the characteristic times $t_m = \frac{(2m+1)\pi}{4\omega_1}$ with integer m. The required pulse strength takes then the maximum value $|\dot{b}(t_m)b(t_m)| = |\omega_0^2 - \omega_1^2|$. Application of a pulse satisfying the generalized DKC condition (11) gives rise to the reversal of DF, rescaling the initial momentum distribution by a factor $1/b(t_k)$; see Fig. 3. DF can thus be tailored and controlled both in expansions and compressions. We conclude by noting that DF can also be induced, even in a static trap, by applying a DKC pulse, with either an attractive or a repulsive quadratic potential, i.e., keeping the confining harmonic potential constant in time $\omega(t) = \omega_0$.

In summary, we have established the unitary character of the dynamics of the OBRDM of a TG gas in a driven harmonic trap and pointed out its far-reaching consequences. The time evolution is isentropic and preserves the occupation numbers of the natural orbitals. As a result, DF does not affect the distinguishability between the OBRDMs of the dual systems, which is independent of time, e.g., as quantified by the Uhlmann fidelity. For arbitrary driving of the trap frequency, DF can be described by a canonical transformation in phase space, that relates the asymptotic momentum distribution to the initial density profile under rapid acceleration of the width of the atomic cloud. This relation holds for expansions as well as compressions leading to an increase of the interparticle density. DF can thus be further tailored, induced, or completely reversed by applying a kick with a pulsed external potential, generalizing DKC to interacting systems. This allows us to rescale the initial momentum distribution for the microscopy of quantum correlations. Our findings open the way to control nonequilibrium correlations in driven ultracold gases and can be tested in laboratory settings used in recent experiments [38]. They should be generalizable to atomic mixtures, systems with p-wave interactions, finite coupling strength, fractional exchange statistics, and spinor gases, among other examples.

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