Letter

Inner products of pure states and their antidistinguishability

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(Received 12 September 2022; accepted 6 March 2023; published 27 March 2023)

We study the antidistinguishability problem, which is a fundamental task in quantum computing. A set of d quantum states is said to be antidistinguishable if there exists a d-outcome positive-operator-valued measure that can perfectly identify which state was not measured. We revisit a conjecture by Havlíçek and Barrett which states that if a set of d pure states has small pairwise inner products, then the set must be antidistinguishable. We develop a certificate of antidistinguishability via semidefinite programming duality and use it to provide a counterexample to this conjecture when $d = 4$. Our work thus opens up again the investigation into which sets of pure states are antidistinguishable.

DOI: 10.1103/PhysRevA.107.L030202

I. INTRODUCTION

The distinguishability of a set of quantum states is of central importance in the study of quantum computing. Indeed, many fundamental problems can be cast in terms of how well one can infer the identity of which quantum state one might be holding. Formally, suppose we fix a set of quantum states $\{\rho_1, \ldots, \rho_n\}$ and we set up a game where Alice selects one of them, hands it to Bob, and his task is to determine which state it is. To quantify *how well* Bob can play this game, it often depends on how Alice selects the state (e.g., randomly, adversarially, etc.). However, if we put the strict condition on Bob *having to always give the right answer*, then we get the necessary and sufficient condition that the states must be pairwise orthogonal. To argue this, we note that by Born's rule, when measuring a quantum state ρ with a positive operator-valued measure (POVM) $\{M_1, \ldots, M_n\}$, the probability of the outcome i is given by $\mathrm{Tr}(M_i\rho)$. Therefore, if the states in the set are pair-wise orthogonal, then it is easy to find a measurement which never fails (simply use a projective measurement which includes the projections onto their supports). On the other hand, suppose we have $(M_1, \ldots, M_n)$ being a perfectly distinguishing POVM, i.e., $\mathrm{Tr}(M_i\rho_j) = 0$, or equivalently, $M_i\rho_j = 0$, for all $i \neq j$. Then, for $i \neq j$, $\mathrm{Tr}(\rho_i\rho_j) = \mathrm{Tr}[\rho_i(\sum_k M_k)\rho_j] = 0$, thus the states must be pairwise orthogonal. Therefore, in order to certify that a set of states is *not* perfectly distinguishable, it is sufficient to find two nonorthogonal states in the set. Certificates are convenient proof tools since they show the nonexistence of something, which can sometimes be a challenging task. We explore (more involved) certificates for a different distinguishing task in this note.

Suppose we change the game above and instead of tasking Bob to guess which state he is given, he has to produce a guess for a state he is *not* given. For example, if he is given the state ρ_1 and he responds "the state is *not* ρ_2," then this would correspond to a correct guess. It is worth mentioning that the point here is not trying to "be wrong" in guessing the state (which might be an interpretation after the previously discussed game), but rather to *exclude* a state which was not given. If Bob is able to play this game and win perfectly, we say that the set of states is *antidistinguishable*. Mathematically, this requires an antidistinguishing POVM $\{N_1, \ldots, N_n\}$ satisfying $\mathrm{Tr}(N_i\rho_i) = 0$, for all i. Again, the interpretation of the outcome i is "the state is not ρ_i" (which is why we chose the letter N for the notation of such a POVM). It is worth noting that we *must* exclude a state which is in the given set; we cannot have an extra measurement operator which outputs "I do not know," which is sometimes allowable in state discrimination tasks.

Finding nontrivial necessary and/or sufficient conditions governing when a set of quantum states is antidistinguishable or not is tricky. This is in stark contrast to the simple condition of pairwise orthogonality for the case of perfect distinguishability. Of course, in the case of antidistinguishability, we can always exhibit a measurement and check that it satisfies the defining conditions above. However, in the case of not being antidistinguishable, this is more challenging since this implies the nonexistence of a particular measurement. We soon discuss how to find such a certificate (which we put to use in a later discussion).

Antidistinguishability is an interesting property a set of states may have. Relaxing the notion of *perfect* antidistinguishability to the task of "how antidistinguishable are the states?" was studied in [1] in which they drew connections to the Pusey-Barrett-Rudolph theorem [2]. In [3], the work that inspired this note, the authors used this concept to study communication complexity separations. Moreover, they posed an *antidistinguishability conjecture* as a means to prove the

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existence of a two-player communication game that can be won with $\log(d)$ qubits but would require a one-way communication of $\Omega(d \log d)$ classical bits, thereby providing a (stronger) exponential separation between classical and quantum communication complexities. Their antidistinguishability conjecture is as follows.

Conjecture 1. [3] If a set of d pure states

$$\{|\psi_1\rangle, \dots, |\psi_d\rangle\} \subset \mathbb{C}^d \tag{1}$$

satisfies $|\langle\psi_i|\psi_j\rangle| \leqslant (d-2)/(d-1)$ for all $i \neq j$, then the set is antidistinguishable.

The conjecture holds for the case of $d = 2$ (trivially) and also $d = 3$ (from the work of [4]), but was previously not known to be true for $d > 3$. Numerical approaches to search for counterexamples for $d \in \{4, 5, 6\}$ in [3] did not produce any.

In this Letter, we provide an explicit counterexample to Conjecture 1 when $d = 4$. We do this by presenting four four-dimensional pure states that are deemed to *not* be antidistinguishable via a particular semidefinite program from [1] (which we detail below), and yet, do have small pairwise inner products. We obtained this counterexample by first randomly generating a set of pure states according to the Haar measure, then determining whether this set is antidistinguishable via semidefinite programming along with a check to determine if their pairwise inner products satisfy the bound in the conjecture. The specific counterexample presented in this Letter was found after running more than 1×10^6 random examples. Other counterexamples of this dimension were found, but the one presented here has the greatest optimal value of the semidefinite program that was found via our computational search (and thus is the *least* antidistiguishable, in a sense).

We also provide numerical tools that can be used to study different aspects of the antidistinguishability conjecture for higher dimensions as well as the general principle of antidistinguishability on its own [5].

II. A CERTIFICATE OF NONANTIDISTINGUISHABILITY

We first discuss a semidefinite program (SDP) which is mostly identical to the one in [1]. The only difference is that we do not need to be concerned with any probabilities with which each state is chosen.

Suppose we fix a set of quantum states $\{\rho_1, \dots, \rho_n\}$ and we consider the following SDP:

$$\alpha := \min \left\{ \sum_{i=1}^{n} \text{Tr}(N_i \rho_i) : \sum_{i=1}^{n} N_i = \mathbb{I}, N_1, \dots, N_n \succeq 0 \right\}. \tag{2}$$

Note that the optimal value is indeed attained, hence the use of "min," since the feasible region is compact. We see that $\alpha \geqslant 0$ and, moreover, $\alpha = 0$ if and only if the set is antidistinguishable. The dual SDP is given by

$$\beta := \max \{\text{Tr}(Y) : Y \preceq \rho_i, \ \forall i \in \{1, \dots, n\}\}, \tag{3}$$

where Y is understood to be Hermitian. Strong duality was proven in [1], namely, that $\alpha = \beta$ and that the dual attains its optimal value (and hence our use of "max" above is justified). Therefore, we have the following lemma.

Lemma 2. A set of states $\{\rho_1, \dots, \rho_n\}$ is not antidistinguishable if and only if there exists a Hermitian matrix Y such that $\text{Tr}(Y) > 0$ and $Y \preceq \rho_i$, for all $i \in \{1, \dots, n\}$.

Now it is straightforward to prove a set of states is *not* antidistinguishable; one must only exhibit a certificate Y satisfying the conditions above. Being able to find this certificate is easy in theory; one can solve the dual SDP given in Eq. (3), and for reasonably small examples (say, d up to 1000) this can be done quickly in practice.

III. OUR COUNTEREXAMPLE (WHEN $d = 4$)

Define the following four pure states:

$$\begin{aligned}
|\psi_1\rangle &= \begin{bmatrix} +0.501\,271\,98 - 0.037\,607i \\ -0.006\,981\,52 - 0.590\,973i \\ +0.081\,865\,14 - 0.449\,754\,8i \\ -0.012\,998\,83 + 0.434\,584\,91i \end{bmatrix}, \\
|\psi_2\rangle &= \begin{bmatrix} -0.071\,153\,45 - 0.270\,803\,26i \\ +0.820\,477\,12 + 0.263\,208\,23i \\ +0.221\,050\,89 - 0.209\,199\,6i \\ -0.235\,755\,91 - 0.175\,876\,9i \end{bmatrix}, \\
|\psi_3\rangle &= \begin{bmatrix} +0.313\,609\,06 + 0.463\,393\,13i \\ -0.046\,582\,5 - 0.478\,250\,17i \\ -0.104\,703\,94 - 0.117\,764\,04i \\ +0.602\,315\,15 + 0.261\,549\,59i \end{bmatrix}, \\
|\psi_4\rangle &= \begin{bmatrix} -0.535\,321\,22 - 0.036\,546\,32i \\ +0.409\,559\,41 - 0.151\,505\,76i \\ -0.057\,413\,86 + 0.238\,739\,85i \\ -0.473\,711\,3 - 0.486\,525\,64i \end{bmatrix}.
\end{aligned} \tag{4}$$

We can easily verify that

$$\max_{i \neq j}\{|\langle\psi_i|\psi_j\rangle|\} \approx 0.645\,142\,35 < \frac{d-2}{d-1} = \frac{2}{3}. \tag{5}$$

By solving the dual SDP from Eq. (3) with respect to these four pure states, we can ascertain that $\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|, |\psi_4\rangle\langle\psi_4|\}$ is not antidistinguishable. We now use its numerically found optimal solution and Lemma 2 to provide a certificate of its nonantidistinguishability.

Define the Hermitian operator Y on the following page [see Eq. (6)].

$$Y = \begin{pmatrix} -0.002\,352\,578\,004\,032 & -0.006\,139\,429\,568\,647+ 0.002\,253\,370\,306\,853i & -0.004\,431\,710\,991\,48\,5- 0.000\,778\,124\,769\,934i & 0.004\,045\,982\,033\,136- -0.002\,181\,583\,048\,532i \\ -0.006\,139\,429\,568\,647- 0.002\,253\,370\,306\,853i & 0.003\,589\,384\,258\,236 & 0.002\,517\,710\,068\,163- 0.002\,392\,391795\,840i & -0.009\,308\,704\,240\,406- 0.000\,168\,259\,372\,307i \\ -0.004\,431\,710\,991\,485+ 0.000\,778\,124\,769\,934i & 0.002\,517\,710\,068\,163+ 0.002\,392\,391\,795\,840i & -0.002\,123\,263\,811\,620 & -0.001\,232\,775\,598\,439+ 0.000\,491\,834\,467\,627i \\ 0.004\,045\,982\,033\,136+ 0.002\,181\,583\,048\,532i & -0.009\,308\,704\,240\,406+ 0.000\,168\,259\,372\,307i & -0.001\,232\,775\,598\,439- 0.000\,491\,834\,467\,627i & 0.001\,280\,27\,058\,627\,9 \end{pmatrix} \quad (6)$$

Observe that

$$\mathrm{Tr}(Y) \approx 0.000\,393\,813\,028\,863\,019\,4 > 0. \quad (7)$$

We now wish to show that $|\psi_i\rangle\langle\psi_i| - Y \succeq 0$ holds for each $i \in \{1, 2, 3, 4\}$. Below we list the eigenvalues of each matrix of interest:

$$\begin{aligned} \mathrm{eigs}(|\psi_1\rangle\langle\psi_1| - Y) &= \begin{bmatrix} 0.000\,000\,000\,780\,951 \\ 0.000\,159\,290\,602\,031 \\ 0.007\,593\,054\,347\,881 \\ 0.991\,853\,848\,824\,242 \end{bmatrix}, \\ \mathrm{eigs}(|\psi_2\rangle\langle\psi_2| - Y) &= \begin{bmatrix} 0.000\,000\,000\,845\,682 \\ 0.000\,170\,622\,302\,504 \\ 0.006\,501\,501\,274\,832 \\ 0.992\,934\,060\,068\,367 \end{bmatrix}, \\ \mathrm{eigs}(|\psi_3\rangle\langle\psi_3| - Y) &= \begin{bmatrix} 0.000\,000\,000\,751\,231 \\ 0.000\,136\,742\,588\,802 \\ 0.009\,100\,561\,906\,205 \\ 0.990\,368\,883\,698\,794 \end{bmatrix}, \\ \mathrm{eigs}(|\psi_4\rangle\langle\psi_4| - Y) &= \begin{bmatrix} 0.000\,000\,000\,905\,010 \\ 0.000\,186\,792\,438\,756 \\ 0.007\,152\,857\,760\,097 \\ 0.992\,266\,545\,011\,053 \end{bmatrix}. \end{aligned} \quad (8)$$

Therefore, Y satisfies all the conditions in Lemma 2 implying the set is not antidistinguishable and thus a counterexample to Conjecture 1.

IV. SUPPLEMENTARY SOFTWARE

Supplementary software showcasing the counterexample for $d = 4$ may be found at the following software repository [5]. The repository contains PYTHON code that makes use of the PICOS PYTHON package [6] to invoke the CVXOPT solver [7] for the SDP in Eq. (3).

The set of vectors from Eq. (4) was generated randomly according to the Haar distribution. The authors in [3] followed a similar approach; we simply left our search algorithm running for a very long time [8]. The states provided in the counterexample were found after millions of Haar-random states were generated. Indeed, other such examples were found in this search as well, but the set of states provided here yielded the highest value for $\mathrm{Tr}(Y)$ [see Eq. (7)]. The software from [5] also allows the user to generate a random collection of d d-dimensional pure states and check whether they are antidistinguishable by solving the SDP in Eq. (3). These numerical tools may be of interest to further study the notion of antidistinguishability for larger values of d. On this note, we leave it as an open problem to find the optimal threshold on the inner products when $d = 4$ and, in general, for larger values of d.

ACKNOWLEDGMENTS

This research was supported in part by the Canadian SR&ED program. V.R. thanks Vojtěch Havlíček and Jonathan Barrett for enlightening discussions on their paper that inspired this Letter. We also thank Srinivasan Arunachalam, Basil Singer, Ralph Minderhoud, and Abel Molina for interesting conversations about antidistinguishability.

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