Letter

Dissipative time crystals originating from parity-time symmetry

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(Received 18 March 2022; accepted 23 December 2022; published 13 January 2023)

This study aims to provide evidence regarding the emergence of a class of dissipative time crystals when $\mathcal{PT}$ symmetry of the systems is restored in collective spin systems with Lindblad dynamics. First, we show that a standard model of boundary time crystals (BTCs) satisfies the Liouvillian $\mathcal{PT}$ symmetry, and prove that BTC exists only when the stationary state is $\mathcal{PT}$ symmetric in the large-spin limit. Also, a similar statement is confirmed numerically for another BTC model. In addition, the mechanism of the appearance of BTCs is discussed through the development of a perturbation theory for a class of the one-spin models under weak dissipations. Consequently, we show that BTCs appear in the first-order correction when the total gain and loss are balanced. These results strongly suggest that BTCs are time crystals originating from $\mathcal{PT}$ symmetry.

DOI: 10.1103/PhysRevA.107.L010201

Introduction. Crystals are ubiquitous many-body systems wherein continuous *space*-translation symmetry is spontaneously broken. Similarly, dynamic many-body states that spontaneously break continuous *time*-translation symmetry, namely (continuous) time crystals, were proposed by Wilczek in 2012 [1]. However, it has been proven that time crystals do not exist in ground and equilibrium states, at least for long-range interacting systems [2,3]. In nonequilibrium systems such as Floquet systems [4–9] and dissipative systems [10–24], (discrete or dissipative) time crystals have been observed theoretically and experimentally.

Dissipative time crystals are nontrivial states characterized by persistent periodic oscillations at late times induced by coupling with the external environment [20]. In particular, a kind of dissipative time crystal, called a boundary time crystal (BTC), has been often studied recently [12–18]. BTCs were first introduced using a collective spin model with Lindblad dynamics [12], which describes a collection of spin $1/2$ with all-to-all couplings interacting collectively with external Markovian baths. This model could be derived by tracing out the bulk (environment) degrees of freedom while leaving the boundary (system) degrees of freedom. Further, it has been confirmed that persistent oscillatory phenomena at late times emerge only *in the thermodynamic limit.* Even though such phenomena were already noted 40 years earlier as cooperative resonance fluorescence [25], it should be emphasized that there are various novel aspects in recent studies of BTCs. In particular, the importance of Liouvillian eigenvalues has been realized [12,13,18] because the dynamics can be fully understood in terms of their eigenvalues and eigenmodes. In addition, recent developments of the spectral theory of dissipative phase transitions [26,27] and exact solutions of the Liouvillian spectrum [28–34] have also increased the interest in investigating Liouvillian eigenvalues.

The dynamical properties of BTCs are often investigated via numerical calculations of Liouvillian eigenvalues [12,13], the mean-field approximation method [12–14,25], and the quantum trajectory method [35]. In particular, BTCs must satisfy two conditions for the Liouvillian spectrum, which characterize nonstationary periodic oscillations at late times: (i) there exist pure imaginary eigenvalues $i\lambda_j$ and (ii) the quotient of each pure imaginary eigenvalue is a rational number $\lambda_j/\lambda_k \in \mathbb{Q}$ for all j, k. Moreover, BTCs are characterized by static properties such as the existence of a highly mixed and low-entangled eigenmode with a zero eigenvalue [13,17,36–38]. Also, the necessity of Hamiltonians' $\mathbb{Z}_2$ symmetry has been argued recently [13]. However, the physical origin of the emergence of BTCs has not yet been elucidated, and most studies on Liouvillian eigenvalues for BTCs have been numerical.

Phase transitions accompanied by parity-time ($\mathcal{PT}$) symmetry breaking, namely $\mathcal{PT}$ phase transitions [39,40], are also phenomena wherein persistent oscillations emerge at late times in nonequilibrium systems. These are well-known phenomena in the context of non-Hermitian Hamiltonians (NHHs) [41] with exactly balanced gain and loss, and have been widely investigated in a variety of physical experimental systems, such as mechanics [42], photonics [43], plasmonics [44], electronics [45], and open quantum systems without quantum jumps [46]. Mathematically, the Hamiltonian H is considered to be $\mathcal{PT}$ symmetric if it holds that $[H, PT] = 0$, where P is a parity operator and T is a time reversal operator [39,40]. In addition, $\mathcal{PT}$ phase transitions in systems with Lindblad dynamics, hereafter referred to as Liouvillian $\mathcal{PT}$ phase transitions, have been recently discussed, and their understanding has been progressing [47–53].

This study attempts to demonstrate the emergence of a class of dissipative time crystals when $\mathcal{PT}$ symmetry is realized. We focus on a specific class of systems, collective spin systems with Lindblad dynamics, and provide results about the Liouvillian eigenvalues and stationary state for some specific examples. First, we show that the $\mathcal{PT}$ symmetric phase

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of an open two-spin model with Liouvillian $\mathcal{PT}$ symmetry [47–49] is a BTC. Here, an n-spin model is a system with n collective spin operators in the interaction term. Second, we show that an open collective spin model with interaction owing to a transverse magnetic field and excitation decay (hereafter referred to as the one-spin BTC model) satisfies the proposed definition of the Liouvillian $\mathcal{PT}$ symmetry [48,49] if the parity transformation is appropriately chosen. In addition, we prove that the $\mathcal{PT}$ symmetry breaking of the stationary state occurs at the BTC phase transition point in the large-spin limit. Next, we confirm that the generalized one-spin BTC model studied in Ref. [13] also has Liouvillian $\mathcal{PT}$ symmetry. Further, we numerically show that the stationary state exhibits $\mathcal{PT}$ symmetry in the BTC phase. Finally, we perform a perturbative analysis of a class of one-spin models, including the one-spin BTC model under weak dissipation. Consequently, we show that BTCs appear in the first-order correction owing to the balanced total gain and loss. These results strongly suggest that BTCs in collective spin systems are time crystals originating from $\mathcal{PT}$ symmetry.

Liouvillian spectrum and Liouvillian $\mathcal{PT}$ symmetry. In open quantum systems where the evolution of states is completely positive and trace-preserving (CPTP) Markovian, the time evolution of the density matrix $\rho(t)$ is described by the Lindblad master equation (GKSL equation) [54–57] as follows:

$$\frac{d\rho}{dt} = -i[H, \rho(t)] + \sum_i \mathcal{D}[L_i]\rho, \tag{1}$$

where H is a Hamiltonian, L_i is the Lindblad operator, and the dissipation superoperators $\mathcal{D}[L_i]$ are defined as $\mathcal{D}[L_i]\rho = 2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i$. Here, index i labels the Lindblad operators.

The Lindblad master equation (1) is linear in ρ, thus it can be rewritten with a superoperator, which is a linear operator acting on a vector space of linear operators, as follows:

$$\frac{d\rho(t)}{dt} = \hat{\mathcal{L}}\rho(t). \tag{2}$$

Here, $\hat{\mathcal{L}}$ is referred to as the Liouvillian superoperator.

The eigenvalues λ_i and eigenmodes ρ_i of the Liouvillian can be obtained by solving the following equation:

$$\hat{\mathcal{L}}\rho_i = \lambda_i\rho_i. \tag{3}$$

It is generally known that $\mathrm{Re}[\lambda_i] \leqslant 0, \forall i$; if $\hat{\mathcal{L}}\rho_i = \lambda_i\rho_i$, then $\hat{\mathcal{L}}\rho_i^\dagger = \lambda_i^*\rho_i^\dagger$. [56,57] Here, we assume the existence of a unique steady state and set the eigenvalues as $0 = |\mathrm{Re}[\lambda_0]| < |\mathrm{Re}[\lambda_1]| \leqslant |\mathrm{Re}[\lambda_2]| \leqslant \cdots$. The steady state is then written as $\rho_{ss} = \rho_0/\mathrm{Tr}[\rho_0]$. In addition, the absolute value of the real part of the second maximal eigenvalue is referred to as the Liouvillian gap [26,27] and determines the slowest relaxation rate. Closing the Liouvillian gap is necessary for dissipative phase transitions in steady state [26,27].

It should be noted that imaginary eigenvalues emerge only in the thermodynamic limit for Liouvillian $\mathcal{PT}$ phases and BTCs. Therefore, the thermodynamic limit and the long-time limit are not commutative, that is, $\lim_{S\to\infty}\lim_{t\to\infty}\rho(t) \neq \lim_{t\to\infty}\lim_{S\to\infty}\rho(t)$ [58]. In the former case, the steady state is static without oscillation. We refer to the state $\lim_{t\to\infty}\rho(t)$ as the "*stationary state,*" whereas, in the latter case, the state at late times includes oscillating nondecay modes.

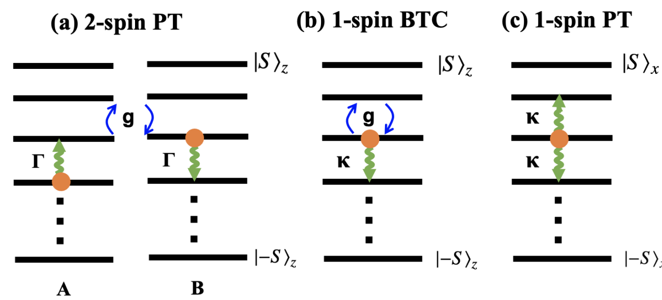

FIG. 1. Illustration of dissipative spin-S models: (a) one-spin $\mathcal{PT}$ model, (b) one-spin BTC model, and (c) one-spin $\mathcal{PT}$ model. These models satisfy Liouvillian $\mathcal{PT}$ symmetry (1) when the parity operator is (a) the exchange of two spins, (b) the reflection of the basis of S_z, (c) the identity operator or the reflection of the basis of S_z.

Many studies on Liouvillian $\mathcal{PT}$ symmetry have been conducted recently [47–53]; however, the definition of Liouvillian $\mathcal{PT}$ symmetry has not been uniquely determined yet. In our arguments, we adopted (a slightly modified version of) the definition proposed in Ref. [49] because similar properties to those of NHH $\mathcal{PT}$ phase transitions have been confirmed in a specific two-spin model that satisfies this definition, as mentioned below. A Liouvillian associated with the Lindblad equation (1) is considered to be $\mathcal{PT}$ symmetric if the following relation holds:

$$\begin{aligned}&\hat{\mathcal{L}}[\mathbb{PT}(H); \mathbb{PT}'(L_\mu), \mu = 1, 2, \dots]\\&\quad= \hat{\mathcal{L}}[H; L_\mu, \mu = 1, 2, \dots],\end{aligned} \tag{4}$$

where $\mathbb{PT}(H) = PTH(PT)^{-1} = P\bar{H}P^{-1}$, $\mathbb{PT}'(L_\mu) = PL_\mu^\dagger P^{-1}$. Here, P and T are parity and time reversal operators, and $\bar{H}$ means the complex conjugation of H. Further, $\mathbb{PT}(H)$ denotes the conventional $\mathcal{PT}$ transformation of Hamiltonian. However, in the $\mathbb{PT}'$ transformation, the time reversal transformation of dissipations represents an exchange of creation and annihilation operators.

Two-spin Liouvillian $\mathcal{PT}$ symmetric model is BTC. To investigate the usefulness of the definition (4), an open two-spin-S model with exactly balanced gain and loss [Fig. 1(a) was actively investigated [47–49]. Here, S denotes the total spin. Denoting the subsystems as A and B, the Lindblad equation is expressed as

$$\frac{\partial}{\partial t}\rho = -ig[H, \rho] + \frac{\Gamma}{2S}\mathcal{D}[S_{+,A}]\rho + \frac{\Gamma}{2S}\mathcal{D}[S_{-,B}]\rho, \tag{5}$$

where $H = (S_{+,A}S_{-,B} + \mathrm{H.c.})/2S$, $S_\pm = S_x \pm iS_y$, and g and Γ are the strengths of the interaction and dissipation, respectively. Note that it decays to a unique steady state for a finite S as spin ladder operators $S_\pm$ exist as one of the dissipation operators [59].

This model satisfies the definition (4) when the parity transformation is the exchange of two spins, and a dissipative phase transition occurs at $\Gamma = g$ when $S = \infty$. Moreover, the symmetry parameter of a stationary state [60], which provides a measure of the parity symmetry of the density operator, changes from 0 to a finite value at the transition point when $S = \infty$ (see Eq. (A.1) in the Supplemental Material [61]). This suggests the occurrence of the $\mathcal{PT}$ symmetry breaking of the stationary state [49].

In addition, the eigenvalue structure and dynamics were obtained when $S = \infty$ [47]. In particular, in the $\mathcal{PT}$ phase, when $S = \infty$, the eigenvalues are expressed as

$$\lambda = in\sqrt{g^2 - \Gamma^2} + r, \tag{6}$$

where $n \in \mathbb{N}$ and $r \in \mathbb{R}_0^-$, and information on degeneracy is omitted. Here, $\mathbb{R}_0^-$ is a set comprising real negative numbers including zero. Equation (6) shows that commensurable pure imaginary eigenvalues exist. Therefore, some physical quantities such as magnetization oscillate periodically at late times. However, in the $\mathcal{PT}$ broken phase, all eigenvalues are real (see Eq. (A.3) in the Supplemental Material [61]); thus, the state decays toward a steady state without oscillation, which is the same as the stationary state. This behavior corresponds to one of the NHH $\mathcal{PT}$ phase transitions.

Moreover, it can be easily confirmed that *the $\mathcal{PT}$ phase is a boundary time crystal* because the eigenvalue structure satisfies the two conditions of nonstationary periodic dynamics only in the thermodynamic limit. (Detailed explanation of this model is provided in the Supplemental Material [61].) In addition, if a model satisfies the definition (4), it has been shown that there exists a stationary state that approaches the identity eigenmode $\rho \propto \mathbb{1}$ in the limit of zero dissipation rate [48].

Boundary time crystals. Among various BTC models, we first focused on the one-spin BTC model investigated in Refs. [12,25,36–38]. The Lindblad equation is expressed as

$$\frac{d}{dt}\rho = -2ig[S_x, \rho] + \frac{\kappa}{S}\mathcal{D}[S_-]\rho, \tag{7}$$

where g and κ are the strengths of interaction and dissipation, respectively [Fig. 1(b)]. In this model, the stationary state is solved exactly for finite S [36,37] as

$$\rho_{ss} = \frac{1}{D}\sum_{n,n'=0}^{2S}\left(i\frac{\kappa}{g}\frac{S_-}{S}\right)^{n'}\left(-i\frac{\kappa}{g}\frac{S_+}{S}\right)^{n}, \tag{8}$$

where D is the normalization constant.

In this model, it was found that various physical quantities, such as magnetization and purity, clearly change; namely, the dissipative phase transition occurs at $\kappa/g = 1$ when $S \to \infty$ [36–38].

Moreover, the eigenvalue structure and dynamics were numerically investigated above and below the transition point [12]. For the BTC phase ($\kappa/g < 1$), the real parts of many eigenvalues approach zero for an enormous S, and the imaginary parts are plotted at regular intervals for any S. This suggests that pure imaginary eigenvalues exist when $S \to \infty$ and that these eigenvalues are commensurable. In other words, the imaginary part of the eigenvalues can be written as $\mathrm{Im}[\lambda] = -icq$ when $S \to \infty$, where $q \in \mathbb{N}$ is the sector and c is a real number dependent on κ/g. Here, the imaginary part is invariant within the same sector. However, for the BTC broken phase ($\kappa/g > 1$), there are no eigenvalues with a nonzero imaginary part that approaches the imaginary axis as S increases.

BTCs are $\mathcal{PT}$ symmetric phases. Here, we first show that the one-spin BTC model (7) has Liouvillian $\mathcal{PT}$ symmetry (4). We choose the parity operator to reflect the basis of S_z, which acts on each spin operator as follows:

$$PS_zP^{-1} = -S_z, \quad PS_\pm P^{-1} = S_\mp. \tag{9}$$

We can verify that $\mathbb{PT}(S_x) = P\overline{S_x}P^{-1} = S_x$ and $\mathbb{PT}'(S_-) = P(S_-)^\dagger P^{-1} = S_-$ hold, that is, the model has Liouvillian $\mathcal{PT}$ symmetry (4).

Then, we analytically show that $\mathcal{PT}$ symmetry breaking of the stationary state (8) occurs at the BTC phase transition point. Using (9), the conventional $\mathcal{PT}$ transformation of the stationary state can be written as

$$PT\rho_{ss}PT = \frac{1}{D}\sum_{n,n'=0}^{2S}\left(-i\frac{\kappa}{g}\frac{S_+}{S}\right)^{n}\left(i\frac{\kappa}{g}\frac{S_-}{S}\right)^{n'}. \tag{10}$$

In the limit $S \to \infty$, we show that $(-i\frac{\kappa}{g}\frac{S_+}{S})^n$ and $(i\frac{\kappa}{g}\frac{S_-}{S})^{n'}$ are commutative only when $\kappa/g < 1$.

This implies that the *stationary state (8) of the one-spin BTC model is $\mathcal{PT}$ symmetric in the BTC phase but not in the BTC broken phase.* The details of the proof are provided in Sec. I.A of the Supplemental Material [61].

Next, we consider the open one-spin model studied in Ref. [13] whose Lindblad equation is given by

$$\frac{d}{dt}\rho = -i[H, \rho] + \frac{\kappa_-}{S}\mathcal{D}[S_-]\rho + \frac{\kappa_+}{S}\mathcal{D}[S_+]\rho, \tag{11}$$

where $H = S(g_z s_z^{p_z} + g_x s_x^{p_x})$, and $p_z, p_x \in \mathbb{N}$. and s_z, s_x are normalized spin operators S_z/S, S_x/S. This model has not been solved analytically; however, it has been numerically observed that BTCs appear when p_z is even [13]. By choosing the parity operator as a reflection of the basis of S_z as before, the model can have Liouvillian $\mathcal{PT}$ symmetry (4) if p_z is even.

To investigate the $\mathcal{PT}$ symmetry breaking of a stationary state, we introduced the $\mathcal{PT}$ symmetry parameter Q_{PT},

$$Q_{PT}(\rho) := \frac{1}{Z}\sum_{i,j}|(\rho - PT\rho PT)_{ij}|, \tag{12}$$

where $Z := \sum_{i,j}|(\rho)_{i,j}| + |(PT\rho PT)_{ij}|$ is a normalization constant, and thus $0 \leqslant Q_{PT} \leqslant 1$. Further, $(\rho)_{i,j}$ is the (i, j) element of matrix ρ. If Q_{PT} is zero, ρ has $\mathcal{PT}$ symmetry.

Figures 2(a) and 2(b) show the purity and $\mathcal{PT}$ symmetry parameter Q_{PT} of the stationary state for $p_z = 2$, $p_x = 1$. In the BTC phase (i.e., the phase with almost zero purity), Q_{PT} is close to 0. Further, Fig. 2(c) shows that Q_{PT} decreases in the BTC phase with increase in S. Therefore, these results suggest that the $\mathcal{PT}$ symmetry of the stationary state is unbroken in the thermodynamic limit, whereas it is broken in the BTC broken phase. In addition, all elements of $|\rho_{ss} - PT\rho_{ss}PT|$ are close to 0 in the BTC phase [Fig. 2(c)], whereas certain elements have finite values in the broken BTC phase [Fig. 2(d)]. Here, $|\rho|$ denotes the matrix that accepts the absolute value of each matrix element ρ. These results indicate that the stationary state exhibits $\mathcal{PT}$ symmetry only in the BTC phase. In Fig. C.2 in the Supplemental Material [61], we numerically investigated the time evolution and quantum trajectory of the normalized magnetization and normalized magnetization of the stationary state.

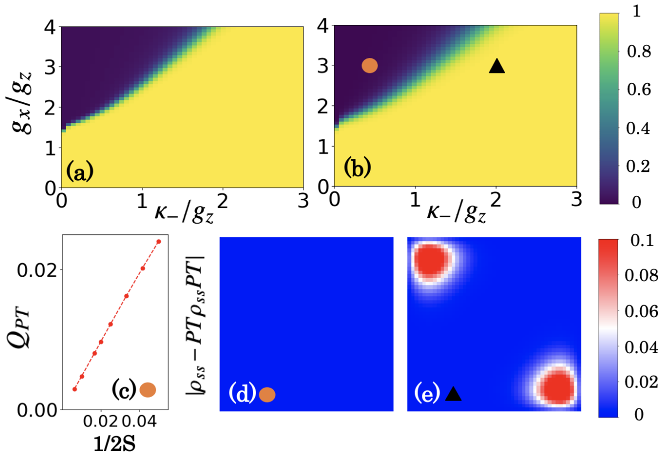

FIG. 2. Numerical analysis of the model (11) for $p_z = 2$, $p_x = 1$, $\kappa_+/g_z = 0$, $S = 23$. These are (a) purity, (b) $\mathcal{PT}$ symmetry parameter Q_{PT}, (c) S dependence of Q_{PT} for $\kappa_-/g_z = 0.5$, $g_x/g_z = 3$ (orange circle), (d) $|\rho_{ss} - PT\rho_{ss}PT|$ for $\kappa_-/g_z = 0.5$, $g_x/g_z = 3$ (orange circle), (e) $|\rho_{ss} - PT\rho_{ss}PT|$ for $\kappa_-/g_z = 2$, $g_x/g_z = 3$ (black triangle), where $|\rho|$ implies the matrix taking the absolute value for each element of the matrix ρ. Here, elements are computed on the basis of the z magnetization. These results indicate that the stationary state has $\mathcal{PT}$ symmetry only in the BTC phase. Here, we have used QuTiP [62] to obtain the stationary state numerically.

We also consider the one-spin model studied in Refs. [31,63,64] whose Liouvillian is expressed as

$$\hat{\mathcal{L}}\rho = -ig[S_x, \rho] + \frac{\kappa(1+p)}{S}\mathcal{D}[S_x^+]\rho + \frac{\kappa(1-p)}{S}\mathcal{D}[S_x^-]\rho, \tag{13}$$

where $-1 \leqslant p \leqslant 1$ and $S_x^{\pm} := S_y \pm iS_z$. The model is BTC only when $p = 0$. When $p = 0$ the model is solvable for any S [31], the eigenvalues $\lambda_{l,q}$ are expressed as

$$\lambda_{l,q} = igq - \frac{2\kappa}{S}[|q| + l(1 + l + 2|q|)], \tag{14}$$

where $q = \{-S, -S+1, \dots, S\}$ is the sector and $l = \{0, 1, \dots, 2S - |q|\}$, which satisfies the two conditions for the emergence of nonstationary oscillating dynamics only in the thermodynamic limit.

We can also discuss the relationship between the BTCs and $\mathcal{PT}$ symmetry for this model. Upon choosing the parity operator as the identity operator or reflection of the basis of S_z, this model (13) satisfies Liouvillian $\mathcal{PT}$ symmetry only when $p = 0$. This implies that *this model is BTC only when it has Liouvillian* $\mathcal{PT}$ *symmetry*. In addition, the stationary state $\rho_{ss} \propto \mathbb{1}$ has $\mathcal{PT}$ symmetry for $p = 0$. In the following, we refer to this model with $p = 0$ as the one-spin $\mathcal{PT}$ model [Fig. 1(c)].

Understanding the mechanism of appearance of BTCs. Let us focus the eigenvalues with the smallest real part for the one-spin $\mathcal{PT}$ model, namely $\lambda_{l=0,q}$ in Eq. (14). The corresponding eigenmodes $\rho_{0,q}$ are proportional to $(S_x^+)^{|q|}$ for $q < 0$ and $(S_x^-)^q$ for $q > 0$ [63]. These exact specific eigenmodes facilitate an understanding of the BTC's mechanism. For example, for $q = 1$ we calculated $-ig[S_x, S_x^-] = igS_x^-$ in the coherent part, and

$$\begin{aligned}\frac{\kappa}{S}(\mathcal{D}[S_x^+] + \mathcal{D}[S_x^-])S_x^- &= \frac{\kappa}{S}([S_x^+, S_x^-]S_x^- + S_x^-[S_x^-, S_x^+]) \\ &= \frac{2\kappa}{S}(S_xS_x^- - S_x^-S_x) = \frac{2\kappa}{S}[S_x, S_x^-] \\ &= -\frac{2\kappa}{S}S_x^-\end{aligned} \tag{15}$$

in dissipative parts. [The calculation for any q is provided in the Supplemental Material [61], Eqs. (A.6) and (A.7).] Moreover, it has a $1/S$ dependence in dissipative parts owing to the cancellation of several terms and the use of commutation relations. Thus, purely imaginary eigenvalues emerge when $p = 0$ and $S \to \infty$, implying that they are BTC. However, when $p \neq 0$, such cancellations of terms are generally unexpected. Indeed, the Liouvillian gap is not closed, even for $S \to \infty$ [31] and no time crystals emerge.

Next, we investigated a class of the one-spin models using perturbation theory [65,66] under weak dissipations, whose Liouvillian is expressed as

$$\hat{\mathcal{L}}\rho = -ig[S_x, \rho] + \frac{\kappa}{S}\sum_{\mu}\mathcal{D}[L_{\mu}]\rho, \tag{16}$$

$$L_{\mu} = \alpha_{\mu}S_x^+ + \beta_{\mu}S_x^- + \gamma_{\mu}S_x, \tag{17}$$

where α_{μ}, β_{μ}, γ_{μ} $\in \mathbb{C}$. This class includes the one-spin BTC model and the model (13). We can show that BTCs appear in first-order perturbation under a weak dissipation rate κ if and only if it holds that

$$\sum_{\mu}|\alpha_{\mu}|^2 = \sum_{\mu}|\beta_{\mu}|^2. \tag{18}$$

Here, this condition can be regarded as exactly balanced total gain and loss on the x basis. Note that BTCs basically appear under weak dissipations. Therefore, if it is not BTC in the first-order correction under weak dissipations, it may not be BTC for all dissipation regimes.

First, we apply the degenerated perturbation theory to the model (13). For the prescription of the n-degenerate case, the first-order eigenvalue correction $\tilde{\lambda}_{n,i}^{(1)}$ can be obtained by solving the following equation:

$$L^{(n)}\psi_{n,i} = \tilde{\lambda}_{n,i}^{(1)}\psi_{n,i} \qquad (i = 1, 2, \dots, n), \tag{19}$$

where $L^{(n)}$ and $\psi_{n,i}$ are the $n$square matrix and n coefficient vector, respectively. (see Eq. (D.10) in the Supplemental Material [61]).

Choosing the nonperturbative Liouvillian $\hat{\mathcal{L}}_0$ as a coherent part of the model (13), namely $\hat{\mathcal{L}}_0 \cdot = -ig[S_x, \cdot]$, the nonperturbative eigenvalues are $(2S + 1 - |q|)$ degenerated for each sector q. Then, $L^{(2S+1-|q|)}$ is a tridiagonal matrix with real-number elements. In particular, for $p = 0$, it becomes a symmetric matrix because of the balanced gain and loss (see Eq. (D.20) in the Supplemental Material [61]). In addition, all high-order corrections are zero because sector q is invariant for the model (13). Thus, perturbative analysis up to the first-order correction yields exact solutions.

Considering these aspects, we also performed perturbation theory on a class of the one-spin models (16). The nonperturbative Liouvillian $\hat{\mathcal{L}}_0$ was again chosen to be a coherent part, and, consequently, the same tridiagonal real matrix $L^{(2S+1-|q|)}$

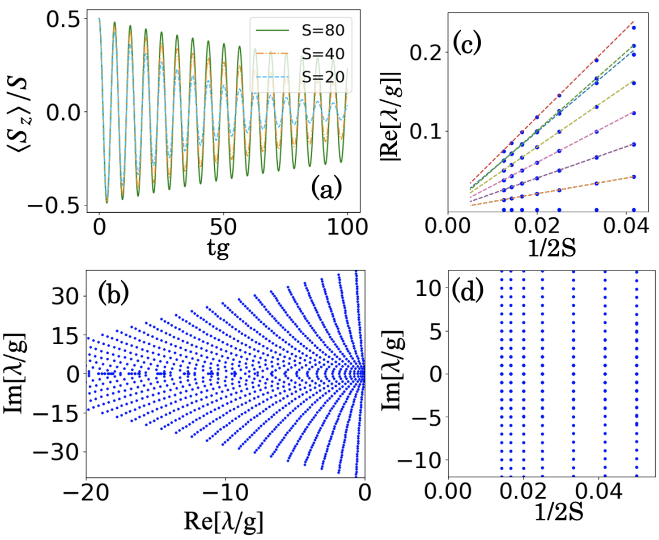

FIG. 3. Numerical analysis of the time evolution of the normalized magnetization and Liouvillian spectrum in the model with $H = S_x$, $L = S_z$. (a) Time evolution of the normalized magnetization $\langle S_z \rangle /S$ for $S = 20$ (dashed light blue), 40 (dashed-dotted orange), and 80 (solid green), and the initial state is set as $\rho(0) = |S/2\rangle_z \langle S/2|_z$. (b) Liouvillian spectrum, (c) 15-minimum absolute values of real parts of the eigenvalues, (d) imaginary parts of the eigenvalues for $\kappa/g = 1$ and $S = 20$. Here, we have used QuTiP [62] to obtain the Lindblad dynamics.

was obtained except for a constant multiplication and a sum of scalar multiplications. Therefore, its eigenvalues have the same properties, and the BTCs emerge for the symmetric case, but not for the nonsymmetric case within the first-order perturbation scheme. In the Supplemental Material [61], we provide details of our proof and numerically indicate that certain properties of the BTC phase transition can be caught up to the second-order corrections for the one-spin BTC model.

Choosing the parity operator as a reflection of the basis of S_z, the Liouvillian $\mathcal{PT}$ symmetry (4) guarantees the condition (18). Therefore, *model (16) is BTC when it exhibits Liouvillian $\mathcal{PT}$ symmetry within the first-order perturbation scheme.* Note that conservation is not always true, that is, the condition (18) does not necessarily imply Liouvillian $\mathcal{PT}$ symmetry. This suggests that the definition of Liouvillian $\mathcal{PT}$ symmetry leaves room for further improvements.

Finally, we provide an example. When $L_1 = -i(S_x^+ - S_x^-)/2 = S_z$ in the model (16) that satisfies the condition (18), the normalized magnetization $\langle S_z \rangle /S$ oscillates and the relaxation time increases with increase in S [Fig. 3(a)]. Furthermore, Figs. 3(c) and 3(d) show that the real parts of the eigenvalues decrease to zero with an increase in S and that the imaginary parts are invariant even as S increases. These results imply that BTC emerges in the thermodynamic limit.

Summary and discussion. In this study, we provided evidence that dissipative time crystals originating from $\mathcal{PT}$ symmetry exist in collective spin systems. In particular, we showed that BTCs are only such examples. In addition, we performed perturbation analysis for a class of one-spin models and showed that BTCs appear in the first-order correction because of the exactly balanced total gain and loss.

Finally, we discuss the robustness of our results. For a class of the one-spin models (16), the BTCs were stable for perturbations of the dissipations satisfying Eq. (18). It was also stable in case of perturbations of Hamiltonian terms that do not break the Liouvillian $\mathcal{PT}$ symmetry (4), such as S_z^{2n} or S_x^n ($n \in \mathbb{N}$) [12,13]. However, the rigidity of the periodic time, which is a property of discrete time crystals, did not appear because the periodic time is generally the variant for the dissipation and interaction strength.

As a natural extension, investigation of the relationship between the Liouvillian $\mathcal{PT}$ symmetry and other time crystals, such as discrete time crystals, dissipative time crystals originating from dynamical symmetry, and boundary time crystals in bosonic systems are expected to yield interesting results.

Acknowledgments. We thank Kohei Yamanaka and Yukiya Yanagihara for fruitful discussions. Y.N. also acknowledges the financial support from JST SPRING, Grant No. JPMJSP2106, and Tokyo Tech Academy for Convergence of Materials and Informatics. The work done by T.S. was supported by JSPS KAKENHI, Grants No. JP18H01141, No. JP18H03672, No. JP19L03665, No. JP21H04432, and No. JP22H01143.

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