Third-order corrections to the ground-state energy of the gas of spin-s fermions with arbitrary densities of different spin projections

Piotr H. Chankowski,[*] Jacek Wojtkiewicz,[†] and Szymon Augustynowicz[‡]
Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland

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Recently, we computed third-order corrections to the ground-state energy of an arbitrarily polarized diluted gas of spin-$\frac{1}{2}$ fermions interacting through a spin-independent repulsive two-body potential. Here we extend this result to the gas of spin-s fermions [a system whose Hamiltonian has an accidental SU$(2s+1)$ symmetry] with arbitrary densities of fermions having different spin projections. The corrections are computed semianalytically using the effective-field-theory approach and are parametrized by the s- and p-wave scattering lengths a_0 and a_1 and the s-wave effective radius r_0, measurable in the low-energy fermion-fermion elastic scattering. The result is used to study the impact the higher-order corrections can have on the characteristics of the phase transition (at zero temperature) to the ordered phase (on the emergence of the itinerant ferromagnetism).

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I. INTRODUCTION

Although clear experimental evidence is still lacking [1], there is a strong conviction that in a finite-density system of fermions a transition to the ordered phase, in which the densities of different spin components are not all equal, should be induced by a repulsive spin-independent interaction if the system is sufficiently dense and/or the interaction is strong enough. Such a transition can be conveniently quantified by a nonzero value of an order parameter P, which in the usually considered case of spin $s=\frac{1}{2}$ fermions can be defined so that it has the meaning of the polarization (magnetization) $P=(N_\uparrow-N_\downarrow)/N$, $N=N_\uparrow+N_\downarrow$, of the system. (An analogous order parameter can be defined in the general case of spin $s>\frac{1}{2}$ if a specific pattern of the ordering is assumed.)

Theoretical investigation of this transition at zero temperature reduces to computing the energy density of the system of N fermions enclosed in the volume V as a function of the densities of different spin projections (or as a function of the chosen order parameter) and taking the thermodynamic limit. Such computations are most easily performed using the effective-field-theory approach within which the underlying spatially nonlocal binary interaction of fermions is replaced by an in principle infinite set of local operators of decreasing length dimension and which yields the expansion of the computed energy density in powers (in higher orders modified also by logarithms) of the (overall) Fermi wave vector

$$k_{\rm F}=\left(\frac{6\pi^2}{g_s}\frac{N}{V}\right)^{1/3},\quad g_s=2s+1 \tag{1}$$

of the system. The expansion obtained in this way is naturally parametrized by the scattering lengths a_ℓ, effective radii r_ℓ, etc., $\ell=0,1,\ldots$, which characterize the elastic scattering of low-energy fermions in vacuum and which are taken to specify the underlying interaction potential. This approach, pioneered in [2] (see also [3,4]), recently allowed the completion [5] of the order $k_{\rm F}^4$ (with respect to the energy of the system of completely noninteracting fermions) corrections to the ground state of the system of spin-s fermions with equal densities of different spin projections. Using this approach, it was also easy to reproduce (semianalytically), but in the universal setting, the old (obtained by considering the specific model of hard-sphere interactions) result of Kanno [6], who computed (fully analytically) the order $k_{\rm F}^2$ correction to the ground state of the system of spin-$\frac{1}{2}$ fermions [7] for an arbitrary value of the system's polarization P. The effective-field-theory approach allows us to immediately extend [8] the result of Kanno to the case of spin-s fermions: The complete, up to the order $k_{\rm F}^2$, formula for the ground-state energy density E_Ω/V as a function of the densities of fermions having different spin projections can be now written in the form[1]

$$\frac{E_\Omega}{V}=\frac{k_{\rm F}^3}{6\pi^2}\frac{3}{5}\frac{\hbar^2k_{\rm F}^2}{2m_f}\left(\sum_{\sigma=1}^{g_s}x_\sigma^5+\frac{20}{9\pi}k_{\rm F}a_0\sum_{\sigma'<\sigma}x_{\sigma'}^3x_\sigma^3+\frac{2}{\pi^2}(k_{\rm F}a_0)^2\sum_{\sigma'<\sigma}J_K(x_{\sigma'},x_\sigma)\right), \tag{2}$$

in which

$$x_\sigma=\frac{p_{{\rm F}\sigma}}{k_{\rm F}},\quad p_{{\rm F}\sigma}=\left(6\pi^2\frac{N_\sigma}{V}\right)^{1/3} \tag{3}$$

[*]chank@fuw.edu.pl
[†]wjacek@fuw.edu.pl
[‡]szymon.augustynowicz@student.uw.edu.pl

[1]The order $k_{\rm F}$ correction is the extension (obvious from the effective-field-theory point of view) of the textbook [9,10] mean-field result to the case of spin-s fermions.

and

$$J_K(x,y) = \frac{1}{21}\Bigg[(22x^3y^3(x+y) - 4x^7\ln\left(1+\frac{y}{x}\right) - 4y^7\ln\left(1+\frac{x}{y}\right) + \frac{1}{2}xy(x-y)^2(x+y) \times [15(x^2+y^2)+11xy] - \frac{7}{4}(x-y)^4(x+y) \times [(x+y)^2+xy]\ln\left(\frac{x+y}{x-y}\right)\Bigg] \tag{4}$$

is the function obtained by Kanno in his computation [6]. The extension to $s > \frac{1}{2}$ is of obvious interest in view of the fact that the systems of interacting fermions (and bosons) are nowadays investigated in experiments with cold atoms in which the interaction strength can be tuned by exploiting the physics of the so-called Feshbach resonance.

The emergence of the spontaneous magnetization (the paramagnetic to ferromagnetic transition) that occurs in the system of spin-$\frac{1}{2}$ fermions with the (spin-independent) repulsive interaction constitutes the simple textbook example [9,10] of a continuous phase transition because it exhibits such a character (when it is induced both by decreasing the temperature and by increasing the density and/or the interaction strengths) when the computations are restricted to the mean-field approximation (only the order-k_Fa_0 corrections to the thermodynamic functions are taken into account). However, when the order-$(k_Fa_0)^2$ corrections are included this transition turns at low temperatures into the ordinary first-order one as has been found in [11] and at exactly $T = 0$ can be easily confirmed by using the formula (2) [8,12]. Although in this approximation the transition at zero temperature occurs at $k_Fa_0 = 1.054$, at which value the perturbative expansion cannot most probably be trusted, such a character of the transition seems to be in line with the arguments [13] based on general principles of the Ginzburg-Landau theory.

On the other hand, in a series of papers [14] (overlooked in [7,12]) the ground-state energy of the system of spin-$\frac{1}{2}$ fermions as a function of the polarization P has been computed beyond a fixed order, by performing a resummation of an infinite subclass of the effective-field-theory diagrams contributing to it. As the authors of these papers claim, the transition to the ordered state at $T = 0$ becomes then again of the continuous type. Moreover, the critical value of the parameter k_Fa_0 found by them is in surprisingly good agreement with the estimates based on the quantum Monte Carlo computations reported in [15]. While assessing the reliability of the approximations made in [14] is in general difficult, it might be instructive to see, using the complete order k_F^3 corrections, which we have recently computed in [16] in the case of spin $\frac{1}{2}$, how the third-order terms included in the resummation compare with the ones neglected and how the character of the phase transition changes when it is analyzed using the complete third-order formula for the ground-state energy. In view of the mentioned circumstance that in experimental setups the role of fermions is played by cold atomic gases which can have spins greater than $\frac{1}{2}$, it is also of interest to investigate the transition in the case of higher spins. Finally, according to the power counting rules which organize the expansion within the effective-field theory [2,17], in the order k_F^3 to the energy density for the first time contribute the interaction operators involving, in addition to four-fermion field operators, also two spatial derivatives and introduce the dependence on the p-wave scattering length a_1 and the s-wave effective radius r_0, which parametrize the departure of the result from strict universality. It is therefore natural to check their potential impact on the phase transition.

Therefore, the plan of the paper is as follows. In the next section, to prepare the ground, we recall the effective-field-theory approach allowing us to easily recover the order k_Fa_0 and $(k_Fa_0)^2$ (this one semianalytically) corrections in (2). In Secs. III and IV we determine the next-order correction to the formula (2), adapting appropriately the computation [16] of the third-order correction to the ground-state energy of the arbitrarily polarized gas of spin-$\frac{1}{2}$ fermions. The discussion of the approach of [14] is included in Sec. III. The character of the transition to the ordered state is discussed in Sec. V. We summarize in Sec. VI.

II. SECOND-ORDER CORRECTIONS

As said, the effective-field theory naturally yields an expansion of the computed quantities in powers of the product kR of the wave vector k corresponding to the characteristic energy scale of the problem and of the characteristic length scale R of the fundamental interaction potential. In the case of the corrections to the ground-state energy of the gas of fermions, the role of the characteristic wave vector plays the Fermi wave vector k_F defined in (3) and they are most conveniently computed (using the standard Feynman rules formulated, e.g., in [18]) as the sum of connected vacuum diagrams using the Dyson expansion of the right-hand side of the formula

$$\lim_{T\to\infty}\exp\left(-\frac{iT\left(E_\Omega - E_{\Omega_0}\right)}{\hbar}\right) = \lim_{T\to\infty}\langle\Omega_0|\mathrm{T}\exp\left(-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt\,V_{\mathrm{int}}^I(t)\right)|\Omega_0\rangle, \tag{5}$$

in which $|\Omega_0\rangle$ is the ground state of the free Hamiltonian in the N-fermion subspace of the Fock space, $V_{\mathrm{int}}^I(t)$ is the theory interaction operator V_{int} taken in the interaction picture, and the limits $T \to \infty$ and $V \to \infty$ are implicit (T stands for the chronological product.

If, as assumed, the underlying fundamental interaction of spin-s fermions is spin independent and (in the infinite-volume limit) Galileo invariant, V_{int} in (5) is the effective-field-theory interaction operator[2] of the form [2]

$$V_{\mathrm{int}} = C_0\int d^3\mathbf{x}\sum_{\sigma'<\sigma}\psi_{\sigma'}^\dagger\psi_{\sigma'}\psi_\sigma^\dagger\psi_\sigma + V_{\mathrm{int}}^{(C_2)} + V_{\mathrm{int}}^{(C_2')} + \cdots. \tag{6}$$

The terms $V_{\mathrm{int}}^{(C_2)}$ and $V_{\mathrm{int}}^{(C_2')}$, which will be specified in Sec. IV, involve two spatial derivatives and are of lower length

[2]The absence of the $\sigma' = \sigma$ term in the sum is the immediate consequence of the anticommutativity of the fermionic field operators; in [7], where only the spin $s = \frac{1}{2}$ was treated, the sum in (6) consisted of a single term only.

dimension than the first one; the ellipsis stands for other operators of yet lower length dimension. Despite the infinite number of interaction terms in (6), only a finite number of Feynman diagrams [constructed using only a finite subset of the interaction terms of (6)]—only those dictated by the so-called power counting rules [2–17]—contribute to a term proportional to a given power[3] of (kR) in the computed quantity. The couplings $C_0, C_2, C_2', \ldots$ of the effective-field theory are fixed by matching the amplitude of the elastic fermion-fermion scattering computed using the effective-field theory onto the general form of such an amplitude expanded in powers of the relative particle momenta and parametrized by the scattering lengths a_ℓ, the effective radii r_ℓ, etc., which with sufficient accuracy characterize the fundamental binary interaction. Of course, since the effective interaction is local, divergences appear and must be regularized (in [7] a cutoff Λ on the wave vectors circulating in loops was imposed); the relations between computed physical quantities like $a_\ell, r_\ell, E_\Omega, \ldots$ and the effective-field-theory parameters $C_0, C_2, \ldots$ depend then on the regularization, but this dependence disappears when the computed energy is expressed in terms of the scattering lengths and radii. To the order which is needed to compute the third-order corrections to the ground-state energy of the gas of fermions, the relations between the couplings C_0, C_2, and C_2' and the scattering lengths a_0, a_1, and the radius r_0 are known [5] and (when the cutoff Λ is used as the regulator) read

$$C_0(\Lambda) = \frac{4\pi\hbar^2}{m_f} a_0 \left(1 + \frac{2}{\pi} a_0 \Lambda + \frac{4}{\pi^2} a_0^2 \Lambda^2 + \cdots\right), \quad (7)$$

$$C_2(\Lambda) = \frac{4\pi\hbar^2}{m_f} \frac{1}{2} a_0^2 r_0 + \cdots, \quad C_2'(\Lambda) = \frac{4\pi\hbar^2}{m_f} a_1^3 + \cdots. \quad (8)$$

The power counting rules state that the order $k_F R$ and $(k_F R)^2$ corrections to the ground-state energy of the gas of fermions, that is, the order $k_F a_0$ and $(k_F a_0)^2$ terms in the formula (2), are generated solely by the term in (6) proportional to C_0. The correction of order $k_F a_0$ is therefore given by the sum over all possible choices of the (different) pairs of spin projections σ' and σ circulating in the two loops of a simple two-loop diagram that can be formed by joining the corresponding pairs of lines representing the interaction vertex proportional to C_0, while that of order $(k_F a_0)^2$ is given by a similar sum over spin projections circulating in the three-loop diagram shown in Fig. 1 on the left. This diagram can be composed in three equivalent ways (corresponding to different assignments of the momenta to its internal lines)

$$\begin{aligned} \frac{1}{i\hbar}\frac{E_\Omega^{(2)}}{V} &= \frac{1}{2!}\left(\frac{C_0}{i\hbar}\right)^2 \sum_{\sigma'<\sigma} \int \frac{d^4q}{(2\pi)^4} [A_{\sigma',\sigma}(q)]^2 \\ &= \frac{1}{2!}\left(\frac{C_0}{i\hbar}\right)^2 \sum_{\sigma'<\sigma} \int \frac{d^4q}{(2\pi)^4} [B_{\sigma',\sigma}(q)]^2 \\ &= \frac{1}{2!}\left(\frac{C_0}{i\hbar}\right)^2 \sum_{\sigma'<\sigma} \int \frac{d^4q}{(2\pi)^4} B_{\sigma,\sigma}(q) B_{\sigma',\sigma'}(q), \end{aligned} \quad (9)$$

[3]Terms proportional to higher powers of (kR) are also modified by logarithms [2].

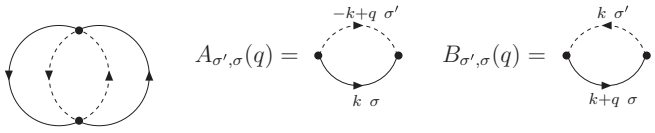

FIG. 1. The only type of diagrams contributing to the order $(k_F R)^2$ correction to the ground-state energy of the gas of spin-s fermions and two elementary one-loop diagrams out of which the second-order and third-order corrections with the C_0 couplings can be constructed. Solid and dashed lines denote propagators of fermions with different spin projections.

out of the two elementary blocks (corresponding to the two elementary loops shown in Fig. 1)

$$\begin{aligned} A_{\sigma',\sigma}(q) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} i \Bigg(& \frac{\theta(|\mathbf{k}-\mathbf{q}| - p_{F\sigma'})\theta(|\mathbf{k}| - p_{F\sigma})}{q^0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}-\mathbf{q}} + i0} \\ & - \frac{\theta(p_{F\sigma'} - |\mathbf{k}-\mathbf{q}|)\theta(p_{F\sigma} - |\mathbf{k}|)}{q^0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}-\mathbf{q}} - i0} \Bigg), \end{aligned} \quad (10)$$

$$\begin{aligned} B_{\sigma',\sigma}(q) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} i \Bigg(& - \frac{\theta(p_{F\sigma'} - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - p_{F\sigma})}{q^0 + \omega_{\mathbf{k}} - \omega_{\mathbf{k}+\mathbf{q}} + i0} \\ & + \frac{\theta(|\mathbf{k}| - p_{F\sigma'})\theta(p_{F\sigma} - |\mathbf{k}+\mathbf{q}|)}{q^0 + \omega_{\mathbf{k}} - \omega_{\mathbf{k}+\mathbf{q}} - i0} \Bigg), \end{aligned} \quad (11)$$

in which $\omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m_f$, obtained by the standard integration over k^0 using the residue method. Each of these blocks is a sum of two parts: those in the elementary block $A_{\sigma',\sigma}(q)$, which will be needed in the discussion of the approach of [14], can be interpreted as corresponding to the propagation of two particles (the first part) and the propagation of two holes (the other part). In either form of (9), after the integration over dq^0 only two (out of the possible four) terms are nonzero (and equal to one another), because in the two other terms all poles are on the same side of the real q^0 axis.

The final form of the order $(k_F a_0)^2$ contribution to the energy density is obtained (see [7]) by making the appropriate change of the remaining integration variables. In this way one arrives at

$$\frac{E_\Omega^{(2)}}{V} = \frac{64 m_f C_0^2}{\hbar^2 (2\pi)^6} \sum_{\sigma'<\sigma} J(p_{F\sigma'}, p_{F\sigma}), \quad (12)$$

with the function $J(p_{F\sigma}, p_{F\sigma'}) = J(p_{F\sigma'}, p_{F\sigma})$ given by the integral

$$\begin{aligned} J(p_{F\sigma'}, p_{F\sigma}) = \int_0^{s_{\max}} ds\, s^2 \frac{1}{4\pi} \int d^3\mathbf{t}\, \theta(p_{F\sigma'} \\ - |\mathbf{u}+\mathbf{s}|)\theta(p_{F\sigma} - |\mathbf{t}-\mathbf{s}|) g_{\sigma',\sigma}(t,s), \end{aligned} \quad (13)$$

in which $s_{\max} = \frac{1}{2}(p_{F\sigma'} + p_{F\sigma})$. The function $g_{\sigma',\sigma}(t,s) \equiv g_{\sigma',\sigma}(|\mathbf{t}|, |\mathbf{s}|) = g_{\sigma,\sigma'}(t,s)$, given by the integral (symmetric in the labels σ and σ')

$$\begin{aligned} g_{\sigma',\sigma}(t,s) &\equiv \frac{1}{4\pi} \int d^3\mathbf{u} \frac{\theta(|\mathbf{u}+\mathbf{s}| - p_{F\sigma'})\theta(|\mathbf{u}-\mathbf{s}| - p_{F\sigma})}{\mathbf{t}^2 - \mathbf{u}^2 + i0} \\ &= -\Lambda + g_{\sigma',\sigma}^{\text{fin}}(t,s) + \frac{t^2}{\Lambda} + \cdots, \end{aligned} \quad (14)$$

has been obtained in [7] in an analytic form which assumes (without losing generality) that $p_{\mathrm{F}\sigma'} \leqslant p_{\mathrm{F}\sigma}$. It reads

$$g^{\mathrm{fin}}_{\sigma',\sigma}(t,s) = \frac{1}{2}p_{\mathrm{F}\sigma} + \frac{t}{4}\ln\frac{(p_{\mathrm{F}\sigma}-t)^2-s^2}{(p_{\mathrm{F}\sigma}+t)^2-s^2} + \frac{p^2_{\mathrm{F}\sigma}-s^2-t^2}{8s}\ln\frac{(p_{\mathrm{F}\sigma}+s)^2-t^2}{(p_{\mathrm{F}\sigma}-s)^2-t^2}, \quad (15)$$

when $0 < s \leqslant \frac{1}{2}(p_{\mathrm{F}\sigma} - p_{\mathrm{F}\sigma'})$, and

$$g^{\mathrm{fin}}_{\sigma',\sigma}(t,s) = \frac{1}{4}(p_{\mathrm{F}\sigma'} + p_{\mathrm{F}\sigma} + 2s) + \frac{t}{4}\ln\frac{p_{\mathrm{F}\sigma'}+s-t}{p_{\mathrm{F}\sigma'}+s+t} + \frac{t}{4}\ln\frac{p_{\mathrm{F}\sigma}+s-t}{p_{\mathrm{F}\sigma}+s+t} + \frac{p^2_{\mathrm{F}\sigma'}-t^2-s^2}{8s}\ln\frac{(p_{\mathrm{F}\sigma'}+s)^2-t^2}{u_0^2-t^2} + \frac{p^2_{\mathrm{F}\sigma}-t^2-s^2}{8s}\ln\frac{(p_{\mathrm{F}\sigma}+s)^2-t^2}{u_0^2-t^2}, \quad (16)$$

where

$$u_0^2 = \tfrac{1}{2}\left(p^2_{\mathrm{F}\sigma'} + p^2_{\mathrm{F}\sigma}\right) - s^2, \quad (17)$$

when $\frac{1}{2}(p_{\mathrm{F}\sigma} - p_{\mathrm{F}\sigma'}) < s \leqslant s_{\max}$.

It is convenient to write the function $J(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ as the sum

$$J = J_{\mathrm{div}} + J_{\mathrm{fin}} + J_{1/\Lambda} + \cdots,$$

where the successive terms directly correspond to the terms in (14). Therefore, J_{fin} is independent of the cutoff Λ, while $J_{1/\Lambda}$ and the ellipsis stand for terms vanishing in the limit $\Lambda \to \infty$. As

$$\int_0^{s_{\max}} ds\, s^2 \frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(p_{\mathrm{F}\sigma'} - |\mathbf{u}+\mathbf{s}|)\theta(p_{\mathrm{F}\sigma} - |\mathbf{t}-\mathbf{s}|) = \frac{p^3_{\mathrm{F}\sigma'}p^3_{\mathrm{F}\sigma}}{72} \quad (18)$$

[7], $J_{\mathrm{div}} = -(\Lambda/72)p^3_{\mathrm{F}\sigma'}p^3_{\mathrm{F}\sigma}$ and it is straightforward to check that the divergences arising from J_{div} cancel against similar terms proportional to Λ arising when C_0 in the correction

$$\frac{E^{(1)}_\Omega}{V} = \frac{C_0}{36\pi^4}\sum_{\sigma'<\sigma} p^3_{\mathrm{F}\sigma'}p^3_{\mathrm{F}\sigma}, \quad (19)$$

arising from the mentioned two-loop diagrams, is expressed in terms of the s-wave scattering length a_0 as in (7). The cutoff independent terms of (19) give then the term proportional to $k_{\mathrm{F}}a_0$ in (2) and the terms proportional to inverse powers of Λ (which would be absent if the dimensional regularization were used instead of the cutoff Λ) can be, if the computation is restricted to the second order, simply discarded. In the computation including higher-order corrections they cancel against similar spurious contributions arising in higher orders.

If the densities of fermions having different spin projections are all equal ($N_\sigma = N/g_s$ and $p_{\mathrm{F}\sigma} = k_{\mathrm{F}}$ for all $\sigma = 1,\ldots,g_s$), one numerically finds that

$$J_{\mathrm{fin}}(k_{\mathrm{F}}, k_{\mathrm{F}}) = \frac{k_{\mathrm{F}}^7}{840}(11 - 2\ln 2).$$

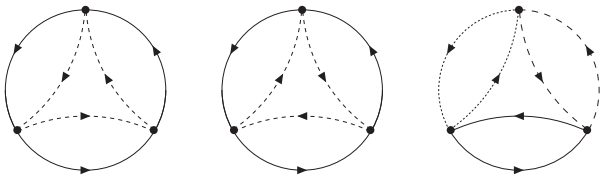

FIG. 2. Single particle-particle and two different particle-hole vacuum diagrams contributing in the order C_0^3 to E_Ω. Different types of lines represent propagators of fermions having different spin projections. The second particle-hole diagram exists only if there are more than two different spin projections ($s > \frac{1}{2}$).

The sum over pairs of different spin projections gives in this case the factor $\frac{1}{2}g_s(g_s-1)$ and one recovers the well-known result rederived in [2] using the effective-field theory. If the densities of fermions with different spin projections are not equal, numerical evaluation of the integrals defining the function $J(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ shows it is proportional to the function J_K (4) introduced by Kanno, with the precise relation being[4]

$$160 J_{\mathrm{fin}}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = k_{\mathrm{F}}^7 J_K(x_{\sigma'}, x_\sigma).$$

This leads to the result (2) obtained recently in a somewhat different way in [8].

III. THIRD-ORDER CORRECTIONS PROPORTIONAL TO C_0^3

According to the power counting rules, the order $(k_{\mathrm{F}}R)^3$ corrections to the ground-state energy density arise from the four-loop diagrams with three interaction vertices proportional to C_0 and from the two-loop diagrams with a single C_2 or C_2' interaction vertex [the terms $V_{\mathrm{int}}^{(C_2)}$ and $V_{\mathrm{int}}^{(C_2')}$ in (6)]. In this section we work out the contribution of the first class of diagrams; the contribution of diagrams with the vertices generated by the interaction terms proportional to C_2 and C_2' will be computed in Sec. IV.

If $s > \frac{1}{2}$, three kinds of nonvanishing four-loop diagrams with three interaction vertices proportional to the C_0 coupling can be formed. They are shown in Fig. 2. The first one is the so-called particle-particle diagram, while the remaining two are called particle-hole diagrams.

The contribution of the order C_0^3 particle-particle diagram shown in Fig. 2 on the left is (2 is the combinatoric factor)

$$\frac{1}{i\hbar}\frac{E^{(3)\mathrm{pp}}_\Omega}{V} = \frac{1}{3!}\left(\frac{C_0}{i\hbar}\right)^3 2\sum_{\sigma'<\sigma}\int\frac{d^4q}{(2\pi)^4}[A_{\sigma',\sigma}(q)]^3. \quad (20)$$

As it is clear from the spin structure of the interaction term proportional to C_0 and the flow of the spin projections in the diagram, the sum runs only over pairs of different spin projection labels $\sigma' \neq \sigma$. There are $g_s(g_s-1)/2$ such pairs and, if densities of fermions with different spin projections are all equal, the summation reproduces the usual spin factor associated with this contribution [2]. For the sake of further discussion, we note that the product of three A blocks gives

[4]From the formulas (13) and (14) it is clear that $J(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) \equiv J(x_{\sigma'}k_{\mathrm{F}}, x_\sigma k_{\mathrm{F}}) = k_{\mathrm{F}}^7 J(x_{\sigma'}, x_\sigma)$.

rise to $2^3 = 8$ terms out of which two are eliminated by the integration over dq^0 (they have all poles on the same side of the real q^0 axis), while the remaining six split into three identical terms which arise from the products of two terms corresponding in (10) (appealing to the interpretation of the two terms of the A block) to the propagation of particles and one corresponding to the propagation of holes, and another three identical ones in which this composition is reversed. As a result, this contribution to the energy density can be written in terms of two functions $G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ and $G^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ (corresponding respectively to these two kinds of compositions):

$$\frac{E_\Omega^{(3)\mathrm{pp}}}{V} = \frac{128 m_f^2 C_0^3}{(2\pi)^8 \hbar^4} \sum_{\sigma'<\sigma} [G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) + G^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})]. \tag{21}$$

The functions $G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ and $G^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ are given by the integrals

$$G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^{s_{\max}} ds\, s^2 \frac{1}{4\pi} \int d^3\mathbf{t}\, \theta(p_{\mathrm{F}\sigma'} - |\mathbf{t}+\mathbf{s}|)\theta(p_{\mathrm{F}\sigma} - |\mathbf{t}-\mathbf{s}|)[g_{\sigma',\sigma}(t,s)]^2, \tag{22}$$

$$G^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^{s_{\max}} ds\, s^2 \frac{1}{4\pi} \int d^3\mathbf{t}\, \theta(|\mathbf{t}+\mathbf{s}| - p_{\mathrm{F}\sigma'})\theta(|\mathbf{t}-\mathbf{s}| - p_{\mathrm{F}\sigma})[h_{\sigma',\sigma}(t,s)]^2, \tag{23}$$

in which $g_{\sigma',\sigma}(t,s)$ is the function defined by (14) and $h_{\sigma',\sigma}(t,s) \equiv h_{\sigma',\sigma}(|\mathbf{t}|, |\mathbf{s}|) = h_{\sigma,\sigma'}(t,s)$ is given by the finite integral

$$h_{\sigma',\sigma}(t,s) = \frac{1}{4\pi} \int d^3\mathbf{u} \frac{\theta(p_{\mathrm{F}\sigma'} - |\mathbf{u}+\mathbf{s}|)\theta(p_{\mathrm{F}\sigma} - |\mathbf{u}-\mathbf{s}|)}{\mathbf{t}^2 - \mathbf{u}^2 - i0}. \tag{24}$$

The explicit form of the function $h_{\sigma',\sigma}(t,s)$ has been obtained[5] using the same method (based on the trick introduced in Appendix C of [19]) by which the function $g_{\sigma',\sigma}(t,s)$ was computed in [7]. Assuming without loss of generality that $p_{\mathrm{F}\sigma'} \leqslant p_{\mathrm{F}\sigma}$, it reads

$$h_{\sigma',\sigma}(t,s) = -\frac{1}{2}p_{\mathrm{F}\sigma'} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}\sigma'}-s)}{t+(p_{\mathrm{F}\sigma'}-s)} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}\sigma'}+s)}{t+(p_{\mathrm{F}\sigma'}+s)} + \frac{t^2-\left(p_{\mathrm{F}\sigma'}^2-s^2\right)}{8s}\ln\frac{t^2-(p_{\mathrm{F}\sigma'}+s)^2}{t^2-(p_{\mathrm{F}\sigma'}-s)^2}, \tag{25}$$

if $0 \leqslant s \leqslant \frac{1}{2}(p_{\mathrm{F}\sigma} - p_{\mathrm{F}\sigma'})$, and [$u_0^2$ is again given by (17)]

$$\begin{aligned} h_{\sigma',\sigma}(t,s) = {} & \frac{1}{2}(2s - p_{\mathrm{F}\sigma'} - p_{\mathrm{F}\sigma}) - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}\sigma'}-s)}{t+(p_{\mathrm{F}\sigma'}-s)} - \frac{t}{4}\ln\frac{t-(p_{\mathrm{F}\sigma}-s)}{t+(p_{\mathrm{F}\sigma}-s)} - \frac{1}{8s}\left[(p_{\mathrm{F}\sigma}-s)^2 + (p_{\mathrm{F}\sigma'}-s)^2 - 2u_0^2\right] \\ & - \frac{t^2 - p_{\mathrm{F}\sigma}^2 + s^2}{8s}\ln\frac{t^2-(p_{\mathrm{F}\sigma}-s)^2}{t^2-u_0^2} - \frac{t^2 - p_{\mathrm{F}\sigma'}^2 + s^2}{8s}\ln\frac{t^2-(p_{\mathrm{F}\sigma'}-s)^2}{t^2-u_0^2}, \end{aligned} \tag{26}$$

if $\frac{1}{2}(p_{\mathrm{F}\sigma} - p_{\mathrm{F}\sigma'}) \leqslant s \leqslant s_{\max} = \frac{1}{2}(p_{\mathrm{F}\sigma} + p_{\mathrm{F}\sigma'})$. Expanding both these expression, one finds that the function $h_{\sigma',\sigma}(t,s)$ behaves as $1/t^2$ when $t = |\mathbf{t}| \to \infty$. Therefore, although the integral over $d^3\mathbf{t}$ in (23) covers the infinite exterior of the two Fermi spheres, it is convergent; unlike the function $G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$, which depends on the divergent function $g_{\sigma',\sigma}(t,s)$, the function $G^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ is finite.

Since the remaining order-C_0^3 contributions to E_Ω/V will turn out to be finite, already at this point one can check the cancellation of ultraviolet divergences in the expression

$$\frac{\Delta E_\Omega}{V} = \sum_{\sigma'<\sigma}\left(\frac{p_{\mathrm{F}\sigma'}^3 p_{\mathrm{F}\sigma}^3}{36\pi^4}C_0 + \frac{64 m_f C_0^2}{(2\pi)^6\hbar^2}J(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) + \frac{128 m_f^2 C_0^3}{(2\pi)^8\hbar^4}G^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})\right) + \cdots.$$

Inserting here the expansions (7) of C_0 (truncated to the appropriate order), writing

$$[g_{\sigma',\sigma}(t,s)]^2 = \Lambda^2 - 2\Lambda g_{\sigma',\sigma}^{\mathrm{fin}} + \left(g_{\sigma',\sigma}^{\mathrm{fin}}\right)^2 - 2t^2 + \cdots, \tag{27}$$

and using the result (18), it is a matter of a simple algebra to check that all terms diverging with $\Lambda \to \infty$ as well as the spurious terms specific for the adopted regularization [that is, the finite term arising from the product of the term proportional to $a_0\Lambda$ and the term t^2/Λ from $g_{\sigma',\sigma}(t,s)$ in the contribution proportional to C_0^2 and the term $-2t^2$ in (27)] completely cancel out, as they should. As a result, to obtain the relevant contribution to E_Ω/V one can simply drop all Λ-dependent terms in the function $g_{\sigma',\sigma}(t,s)$ and set $C_0 = (4\pi\hbar^2/m_f)a_0$ (the remaining part of the function $G^{(1)}$ will be denoted by $G_{\mathrm{fin}}^{(1)}$).

The first type of particle-hole diagrams (the middle one in Fig. 2) was computed in [16] in the case of spin $s = \frac{1}{2}$ fermions. The generalization of the formulas obtained there is straightforward and reduces to taking again the sum over pairs of different spin projections:

$$\frac{1}{i\hbar}\frac{E_\Omega^{(3)\mathrm{ph1}}}{V} = \frac{1}{3!}\left(\frac{C_0}{i\hbar}\right)^3 2\sum_{\sigma'<\sigma}\int\frac{d^4q}{(2\pi)^4}[B_{\sigma',\sigma}(q)]^3. \tag{28}$$

[5] Analysis of the integration domains of (24) shows that the pole at $\mathbf{t}^2 = \mathbf{u}^2$ is never reached.

(Again 2 is the combinatoric factor.) This gives

$$\frac{E_\Omega^{(3)\mathrm{ph1}}}{V} = -\frac{32m_f^2C_0^3}{(2\pi)^8\hbar^4}\sum_{\sigma'<\sigma}[K^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) + K^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})], \tag{29}$$

where the functions $K^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ and $K^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ are given by

$$K^{(1)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^\infty ds\, s^2 \frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(|\mathbf{t}+\mathbf{s}| - p_{\mathrm{F}\sigma'}) \times \theta(p_{\mathrm{F}\sigma} - |\mathbf{t}-\mathbf{s}|)\left[f^{(1)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s)\right]^2, \tag{30}$$

$$K^{(2)}(p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^\infty ds\, s^2 \frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(p_{\mathrm{F}\sigma'} - |\mathbf{t}+\mathbf{s}|) \times \theta(|\mathbf{t}-\mathbf{s}| - p_{\mathrm{F}\sigma})\left[f^{(2)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s)\right]^2 \tag{31}$$

and the functions $f^{(1)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s)$ and $f^{(2)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s)$ are defined by the integrals

$$f^{(1)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s) = \frac{1}{4\pi}\int d^3\mathbf{u}\frac{\theta(p_{\mathrm{F}\sigma'} - |\mathbf{u}+\mathbf{s}|)\theta(|\mathbf{u}-\mathbf{s}| - p_{\mathrm{F}\sigma})}{(\mathbf{u}-\mathbf{t})\cdot\mathbf{s} + i0} \tag{32}$$

and

$$f^{(2)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s) = \frac{1}{4\pi}\int d^3\mathbf{u}\frac{\theta(|\mathbf{u}+\mathbf{s}| - p_{\mathrm{F}\sigma'})\theta(p_{\mathrm{F}\sigma} - |\mathbf{u}-\mathbf{s}|)}{(\mathbf{u}-\mathbf{t})\cdot\mathbf{s} - i0}. \tag{33}$$

Finally, the contribution of the last type of diagrams (the rightmost one in Fig. 2) is given by

$$\frac{1}{i\hbar}\frac{E_\Omega^{(3)\mathrm{ph2}}}{V} = \frac{1}{3!}\left(\frac{C_0}{i\hbar}\right)^3(-3!)\sum_{\sigma'',\sigma',\sigma}\int\frac{d^4q}{(2\pi)^4} \times B_{\sigma'',\sigma''}(q)B_{\sigma',\sigma'}(q)B_{\sigma,\sigma}(q), \tag{34}$$

where $-3!$ is the sign-combinatoric factor and the sum is over all triplets of different spin projections ($\sigma \neq \sigma' \neq \sigma''$ and $\sigma \neq \sigma''$). After the standard steps this leads to

$$\frac{E_\Omega^{(3)\mathrm{ph2}}}{V} = \frac{32m_f^2C_0^3}{(2\pi)^8\hbar^4}\sum_{\sigma'',\sigma',\sigma}[\tilde{K}^{(1)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) + \tilde{K}^{(2)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) + \tilde{K}^{(1)}(p_{\mathrm{F}\sigma'}; p_{\mathrm{F}\sigma}, p_{\mathrm{F}\sigma''}) + \tilde{K}^{(2)}(p_{\mathrm{F}\sigma'}; p_{\mathrm{F}\sigma}, p_{\mathrm{F}\sigma''}) + \tilde{K}^{(1)}(p_{\mathrm{F}\sigma}; p_{\mathrm{F}\sigma''}, p_{\mathrm{F}\sigma'}) + \tilde{K}^{(2)}(p_{\mathrm{F}\sigma}; p_{\mathrm{F}\sigma''}, p_{\mathrm{F}\sigma'})], \tag{35}$$

where the functions $\tilde{K}^{(1)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$ and $\tilde{K}^{(2)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma})$, symmetric in their second and third Fermi momentum labels, are given by the nested integrals

$$\tilde{K}^{(1)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^\infty ds\, s^2\frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(|\mathbf{t}+\mathbf{s}| - p_{\mathrm{F}\sigma''})\theta(p_{\mathrm{F}\sigma''} - |\mathbf{t}-\mathbf{s}|)f^{(1)}_{\sigma',\sigma'}(\mathbf{t}\cdot\mathbf{s}, s)f^{(1)}_{\sigma,\sigma}(\mathbf{t}\cdot\mathbf{s}, s), \tag{36}$$

$$\tilde{K}^{(2)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \int_0^\infty ds\, s^2\frac{1}{4\pi}\int d^3\mathbf{t}\,\theta(p_{\mathrm{F}\sigma''} - |\mathbf{t}+\mathbf{s}|)\theta(|\mathbf{t}-\mathbf{s}| - p_{\mathrm{F}\sigma''})f^{(2)}_{\sigma',\sigma'}(\mathbf{t}\cdot\mathbf{s}, s)f^{(2)}_{\sigma,\sigma}(\mathbf{t}\cdot\mathbf{s}, s). \tag{37}$$

However, the change $\mathbf{s} \to -\mathbf{s}$ in the integrals defining the functions $\tilde{K}^{(1)}$ and $\tilde{K}^{(2)}$ makes it obvious that

$$\tilde{K}^{(2)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}) = \tilde{K}^{(1)}(p_{\mathrm{F}\sigma''}; p_{\mathrm{F}\sigma'}, p_{\mathrm{F}\sigma}). \tag{38}$$

This allows us to simplify the right-hand side of (35) by retaining in it (and doubling) only, say, the functions $\tilde{K}^{(2)}$.

If the densities N_σ/V of fermions having different spin projection are all equal ($p_{\mathrm{F}\sigma} = p_{\mathrm{F}\sigma'} = p_{\mathrm{F}\sigma''} = k_\mathrm{F}$), then $\tilde{K}^{(2)}(k_\mathrm{F}; k_\mathrm{F}, k_\mathrm{F}) = K^{(1)}(k_\mathrm{F}, k_\mathrm{F}) = K^{(2)}(k_\mathrm{F}, k_\mathrm{F})$. The spin factors of the sum of the contributions (29) and (35) combine in this case into

$$-\frac{g_s(g_s-1)}{2} + 3\frac{g_s(g_s-1)(g_s-2)}{6} = \frac{1}{2}g_s(g_s-1)(g_s-3),$$

which is the spin factor associated with the particle-hole diagram in this case [2].

Analysis of the integration domains in the formulas defining the K functions reveals that the poles of the integrands at $\mathbf{u}\cdot\mathbf{s} = \mathbf{t}\cdot\mathbf{s}$ are never reached, so the prescriptions $\pm i0$ can be dropped. Still, obtaining the analytic forms of the functions $f^{(1,2)}_{\sigma',\sigma}(\mathbf{t}\cdot\mathbf{s}, s)$ is rather cumbersome; hence we do not give the technical details of the computation. The final formulas (different in the regimes of small, intermediate, and large s) for these functions can be found in [16]; the formulas for the function $f^{(2)}_{\sigma,\sigma}(\mathbf{s}\cdot\mathbf{t}, s)$ (not given in [16]) can in principle be obtained as a $p_{\mathrm{F}\sigma'} = p_{\mathrm{F}\sigma}$ limit of the function $f^{(2)}_{\sigma',\sigma}(\mathbf{s}\cdot\mathbf{t}, s)$, but for this particular case a much simpler formula can be obtained.[6] It

[6]We have of course checked numerically that $f^{(2)}_{\sigma',\sigma}(\mathbf{s}\cdot\mathbf{t}, s)$ agrees with $f^{(2)}_{\sigma,\sigma}(\mathbf{s}\cdot\mathbf{t}, s)$ in the limit $p_{\mathrm{F}\sigma'} = p_{\mathrm{F}\sigma}$

reads

$$f^{(2)}_{\sigma\sigma}(\mathbf{t}\cdot\mathbf{s},s)=s(p_{\mathrm{F}\sigma}-t)+\frac{1}{4}\left(p^2_{\mathrm{F}\sigma}-s^2+2st\right)\ln\left(\frac{\left(t^2-p^2_{\mathrm{F}\sigma}+s^2-2st\right)\left(t\,p_{\mathrm{F}\sigma}-p^2_{\mathrm{F}\sigma}+s^2-st\right)}{t^2\left(t\,p_{\mathrm{F}\sigma}+k^2-s^2+st\right)}\right)$$
$$-\frac{1}{4}\left(p^2_{\mathrm{F}\sigma}-s^2-2st\right)\ln\left(\frac{\left(t^2-p^2_{\mathrm{F}\sigma}+s^2+2st\right)\left(t\,p_{\mathrm{F}\sigma}-p^2_{\mathrm{F}\sigma}+s^2+st\right)}{t^2\left(t\,p_{\mathrm{F}\sigma}+p^2_{\mathrm{F}\sigma}-s^2-st\right)}\right)$$
$$+\frac{1}{2}\left(p^2_{\mathrm{F}\sigma}-s^2\right)\ln\frac{p_{\mathrm{F}\sigma}+s}{p_{\mathrm{F}\sigma}-s}-t^2\ln\frac{t-s-p_{\mathrm{F}\sigma}}{t+s-p_{\mathrm{F}\sigma}},$$

with $f^{(2)}_{\sigma\sigma}(\mathbf{t}\cdot\mathbf{s},s)=-f^{(1)}_{\sigma\sigma}(\mathbf{t}\cdot\mathbf{s},s)$.

With the analytical formulas for these functions, the remaining integrals in (30), (31), and (37) can be evaluated numerically using the standard *Mathematica* built-in routine for multidimensional integration over a prescribed domain. The resulting order $(k_{\mathrm{F}}a_0)^3$ correction that adds up to the terms in large parentheses in (2) is[7]

$$\frac{160}{\pi^3}(k_{\mathrm{F}}a_0)^3\Bigg(\sum_{\sigma'<\sigma}\left[4G^{(1)}_{\mathrm{fin}}(x_{\sigma'},x_\sigma)+4G^{(2)}(x_{\sigma'},x_\sigma)-K^{(1)}(x_{\sigma'},x_\sigma)-K^{(2)}(x_{\sigma'},x_\sigma)\right]$$
$$+\sum_{\sigma'',\sigma',\sigma}\left[2\tilde{K}^{(2)}(x_{\sigma''};x_{\sigma'},x_\sigma)+2\tilde{K}^{(2)}(x_{\sigma'};x_\sigma,x_{\sigma''})+2\tilde{K}^{(2)}(x_\sigma;x_{\sigma''},x_{\sigma'})\right]\Bigg),\tag{39}$$

where the second sum is over all triplets of different indices σ'', σ', and σ.

With these results, one can try to verify the reliability of the approach developed in [14]. Its authors considered only the case of spin-$\frac{1}{2}$ fermions and only the contributions to the ground-state energy density which, like the order-C_0^2 contribution (9) and the order-C_0^3 contribution (20), can be composed of the A blocks. Of the corresponding nth-order contribution [$(n-1)!$ is the combinatoric factor]

$$\frac{1}{i\hbar}\frac{E^{(n)\mathrm{pp}}_\Omega}{V}=\frac{1}{n!}\left(\frac{C_0}{i\hbar}\right)^n(n-1)!\sum_{\sigma'<\sigma}\int\frac{d^4q}{(2\pi)^4}[A_{\sigma',\sigma}(q)]^n,\tag{40}$$

which gives rise to 2^n-2 nonvanishing terms, they retain only n identical ones obtained by taking $n-1$ parts of the A block corresponding to the propagation of particles and one part corresponding to the propagation of holes (again appealing to the interpretation introduced earlier). Thus, out of the complete order-C_0^3 contribution obtained above, they take only the function $G^{(1)}_{\mathrm{fin}}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$, discarding the contributions of the function $G^{(2)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$ and of the sum $\frac{1}{4}[K^{(1)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})+K^{(2)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})]$. As Fig. 3 shows, the first of these two functions is indeed one order of magnitude smaller than $G^{(1)}_{\mathrm{fin}}$ in the entire range of the ratio $r=p_{\mathrm{F}\sigma'}/p_{\mathrm{F}\sigma}$ (both these functions, similarly to the functions $K^{(1,2)}$ and $\tilde{K}^{(2)}$, vanish at $r=0$ as a result of the absence of s-wave interaction of fermions obeying the Pauli exclusion principle), but the same is not true of the second function, which is always comparable to $G^{(1)}_{\mathrm{fin}}$ and enters (39) with the opposite sign strongly reducing the contribution of the function $G^{(1)}_{\mathrm{fin}}$. Moreover, in higher orders the number of discarded terms arising from (40) rapidly grows and if each of the discarded $(2^n-2)-n$ terms of the contribution (40) is only one order of magnitude smaller than each of the n terms taken into account, the domination of the latter can be invalidated (as the transition to the ordered state is expected to occur at $k_{\mathrm{F}}a_0\lesssim 1$, the higher-order contributions are not much suppressed by the powers of the expansion parameter $k_{\mathrm{F}}a_0$). Moreover, in higher orders the number of other types of diagrams (with topologies different from those shown in Fig. 2) grows and their contribution, like the one of $K^{(1)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})+K^{(2)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$, may dominate the one of the retained terms. Thus the reliability of the approximations made in [14] may be questioned even though the numerical results obtained there agree quite well with those obtained by the quantum Monte Carlo methods.

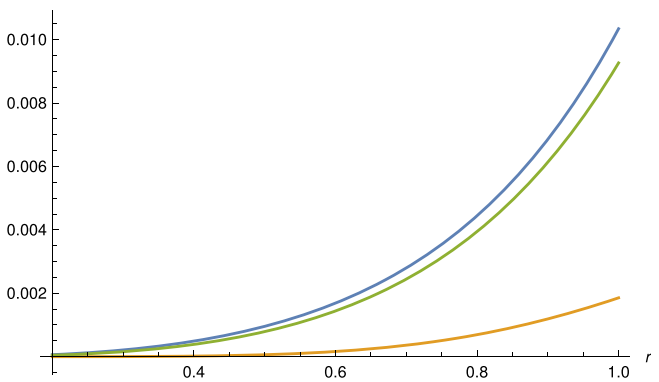

FIG. 3. Plot of the functions $G^{(1)}_{\mathrm{fin}}(r,1)$ (the highest, blue, curve), $G^{(2)}(r,1)$ (the lowest, orange, curve), and $\frac{1}{4}[K^{(1)}(r,1)+K^{(2)}(r,1)]$ (the middle, green, curve).

[7]From the defining formulas it is clear that all the functions $G^{(1,2)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$, $K^{(1,2)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$, and $\tilde{K}^{2)}(p_{\mathrm{F}\sigma''};p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})$ scale like the eighth power of the wave vector k_{F} which effectively reduces them to functions of the single variable $r=x_{\sigma'}/x_\sigma$: $G^{(1)}(p_{\mathrm{F}\sigma'},p_{\mathrm{F}\sigma})=p^8_{\mathrm{F}\sigma}G^{(1)}(x_{\sigma'}/x_\sigma,1)$, etc.

IV. CONTRIBUTION OF THE INTERACTION TERMS PROPORTIONAL TO C_2 AND C_2'

According to the power counting rule in the order $k_{\rm F}^3(\hbar^2k_{\rm F}^2/2m_f)(k_{\rm F}R)^3$, one has to take into account also the operators

$$V_{\rm int}^{(C_2)} = -\frac{C_2}{8}\int d^3\mathbf{x}\sum_{\sigma<\sigma'}[\psi_\sigma^\dagger\psi_{\sigma'}^\dagger(\psi_{\sigma'}\nabla^2\psi_\sigma - 2\nabla\psi_{\sigma'}\cdot\nabla\psi_\sigma + \nabla^2\psi_{\sigma'}\psi_\sigma) + (\nabla^2\psi_\sigma^\dagger\psi_{\sigma'}^\dagger - 2\nabla\psi_\sigma^\dagger\cdot\nabla\psi_{\sigma'}^\dagger + \psi_\sigma^\dagger\nabla^2\psi_{\sigma'}^\dagger)\psi_{\sigma'}\psi_\sigma] \tag{41}$$

and

$$V_{\rm int}^{(C_2')} = -\frac{C_2'}{8}\int d^3\mathbf{x}\sum_{\sigma,\sigma'}(\nabla\psi_\sigma^\dagger\psi_{\sigma'}^\dagger - \psi_\sigma^\dagger\nabla\psi_{\sigma'}^\dagger)\cdot(\nabla\psi_{\sigma'}\psi_\sigma - \psi_{\sigma'}\nabla\psi_\sigma). \tag{42}$$

(Notice that in the second operator the sum involves also terms with $\sigma=\sigma'$.) Their contribution to the ground-state energy density in the case of nonequal densities of spin-$\frac{1}{2}$ fermions having different spin projections has been explicitly written down in [16]. Generalization of these formulas to the case of spin-s fermions with different densities of their g_s spin projections is straightforward. They read

$$\frac{E_\Omega^{(C_2)}}{V} = \frac{C_2}{240\pi^4}\sum_{\sigma'<\sigma}p_{{\rm F}\sigma'}^3p_{{\rm F}\sigma}^3\left(p_{{\rm F}\sigma'}^2+p_{{\rm F}\sigma}^2\right)$$
$$= \frac{k_{\rm F}^3}{6\pi^2}\frac{3}{5}\frac{\hbar^2k_{\rm F}^2}{2m_f}\frac{1}{6\pi}k_{\rm F}^3a_0^2r_0\sum_{\sigma'<\sigma}x_{\sigma'}^3x_\sigma^3\left(x_{\sigma'}^2+x_\sigma^2\right), \tag{43}$$

$$\frac{E_\Omega^{(C_2')}}{V} = \frac{C_2'}{120\pi^4}\left(\sum_{\sigma=1}^{g_s}p_{{\rm F}\sigma}^8 + \frac{1}{2}\sum_{\sigma'<\sigma}p_{{\rm F}\sigma'}^3p_{{\rm F}\sigma}^3(p_{{\rm F}\sigma'}^2+p_{{\rm F}\sigma}^2)\right)$$
$$= \frac{k_{\rm F}^3}{6\pi^2}\frac{3}{5}\frac{\hbar^2k_{\rm F}^2}{2m_f}\frac{2}{3\pi}(k_{\rm F}a_1)^3$$
$$\times\left(\sum_{\sigma=1}^{g_s}x_\sigma^8 + \frac{1}{2}\sum_{\sigma'<\sigma}x_{\sigma'}^3x_\sigma^3\left(x_{\sigma'}^2+x_\sigma^2\right)\right). \tag{44}$$

For equal densities of all spin projections this reduces to the known results given, e.g., in [2]. The contribution $E_\Omega^{(C_2')}$ is special in that, unlike the remaining ones (to this order), it does not vanish if all fermions have the same spin projection. This has important consequences for the transition to the ordered state.

V. PHASE TRANSITION TO THE ORDERED STATE

A short-range repulsive interaction, cooperating with the Pauli exclusion rule, can, if its strength is high enough and the gas density is not too low, cause the transition to the ordered state. The qualitative explanation of this phenomenon is that because of the Pauli exclusion, the dominant s-wave interaction energy of two fermions with the same spin projection must vanish (it is positive if the spin projections are different); hence assuming by the majority of fermions the same spin projection and decreasing thereby the interaction energy may become energetically more favorable than minimizing the kinetic energy by having all spin projections equally populated. Since the effect is due to the competition between the energy of the free system and the interaction energy, it can occur only if the interaction is sufficiently strong, which implies that it hardly can be treated perturbatively. In the mean-field approximation, [equivalent to taking into account in (2) only the order $k_{\rm F}a_0$ correction], and for spin $s=\frac{1}{2}$ fermions this happens when $k_{\rm F}a_0 > \pi/2$ (see, e.g., [9,10,20]).

In this simplest case the role the order parameter naturally plays is the polarization $P=(N_+-N_-)/N$ and seeking the minimum of the ground-state energy $E_\Omega(P)$ as a function of P is straightforward. In the case of spin-s fermions there are in principle many possible configurations with nonequal densities of different spin projections, but from the heuristic argument given above it follows that in this case the minimum of the ground-state energy should be in the configuration in which only one spin projection has density higher than the remaining ones, which all have equal densities.[8] Of course, in the ordered phase there are then $g_s=2s+1$ degenerate ground states related to one another by the SU(g_s) symmetry because in the absence of an external field any of the g_s spin projections can be that one which is populated by the majority of fermions. Therefore, the natural order parameter P defined by setting (without loss of generality we assume that it is the first spin component which is populated differently from the remaining ones)

$$x_1=(1+P)^{1/3},\quad x_\sigma=\left(1-\frac{P}{g_s-1}\right)^{1/3},\quad \sigma=2,\ldots,g_s, \tag{45}$$

$$x_1^3+x_2^3+\cdots+x_{g_s}^3=g_s,\quad -1\leqslant P\leqslant g_s-1, \tag{46}$$

which corresponds to

$$P=\frac{g_sN_1-N}{N}=(g_s-1)\frac{N_1-N_2}{N}, \tag{47}$$

ceases to have the simple interpretation of the system's polarization.[9] In terms of the factors $x_1(P)$ and $x_2(P,g_s)$ given by (45), the term in large parentheses in the formula (2) takes

[8] The authors of [8] claim to have proved this mathematically, but a cursory look at their argument, which relies on the obvious symmetry of the equations determining the densities of the spin projections other than the distinguished one, shows it is not valid.

[9] Obviously, if the spontaneous ordering is approached in the usual thermodynamic limit by switching off an external magnetic field coupled to the fermion spin, it will be the largest spin projection in the direction of the field that will be singled out. In this case the magnetization will be given by $N\mu P/(g_s-1)$, where μ is the magnetic moment associated with a single spin.

the form

$$f_s(P) = x_1^5 + (g_s-1)x_2^5 + \frac{20}{9\pi}k_Fa_0(g_s-1)x_2^3 \times\left(x_1^3+\frac{1}{2}(g_s-2)x_2^3\right) + \frac{2}{\pi^2}(k_Fa_0)^2(g_s-1) \times\left(J_K(x_1,x_2)+\frac{1}{2}(g_s-2)\frac{44-8\ln 2}{21}\right). \quad (48)$$

In the case of spin-$\frac{1}{2}$ fermions the phase transition to the ordered state at $k_Fa_0=\pi/2$ in the mean-field approximation, equivalent to retaining in (2) only the first two terms in the large parentheses, is known to be continuous. This results however from a numerical coincidence which makes the second derivative of $E_\Omega^{\rm MF}(P)$ with respect to P to vanish at $P=0$ for $k_Fa_0=\pi/2$, precisely when the nontrivial minimum first appears. This does not happen if $s>\frac{1}{2}$ and in the mean-field approximation the phase transitions to the ordered states are in all these cases of first order.[10] It can also be easily checked that the critical value of k_Fa_0 at which it occurs decreases with the spin s but rather slowly: $(k_Fa_0)_{\rm cr}^{\rm MF}\approx 1.43$ for $s=\frac{3}{2}$, $(k_Fa_0)_{\rm cr}^{\rm MF}\approx 1.21$ for $s=\frac{7}{2}$, and $(k_Fa_0)_{\rm cr}^{\rm MF}\approx 1.14$ for $s=\frac{9}{2}$. In view of the qualitative explanation of the reasons for the transition, such a behavior is easy to understand: The gas kinetic energy is linear in the number g_s of possible spin projections, while the interaction depends on it (in the mean-field approximation) quadratically. One can also observe that only for $s=\frac{3}{2}$ (and of course $s=\frac{1}{2}$) is the spontaneous polarization P not maximal at $(k_Fa_0)_{\rm cr}^{\rm MF}$; for all higher spins it becomes maximal ($P=g_s-1$) already at the transition point.

If the corrections of order $(k_Fa_0)^2$ are included, i.e., the full formula (48) is used, the transition to the ordered state (at zero temperature) becomes of first order also for $s=\frac{1}{2}$ (and retains, of course, this character for $s>\frac{1}{2}$). The relevant plots for $s=\frac{1}{2}$ can be found in [12] as well as in [8] and for $s>\frac{1}{2}$ in [12]. The transition occurs now at $(k_Fa_0)_{\rm cr}^{\rm 2nd}\approx 1.054$ for $s=\frac{1}{2}$, $(k_Fa_0)_{\rm cr}^{\rm 2nd}\approx 0.954$ for $s=\frac{3}{2}$, $(k_Fa_0)_{\rm cr}^{\rm 2nd}\approx 0.840$ for $s=\frac{7}{2}$, and $(k_Fa_0)_{\rm cr}^{\rm 2nd}\approx 0.804$ for $s=\frac{9}{2}$. Moreover, in this approximation only for $s=\frac{1}{2}$ does the polarization gradually approach the maximal value $P=1$ [$P\approx 0.58$ exactly at the critical coupling $k_Fa_0=(k_Fa_0)_{\rm cr}^{\rm 2nd}$]; in the remaining cases it jumps immediately to the maximal possible value $P_{\max}=g_s-1$.

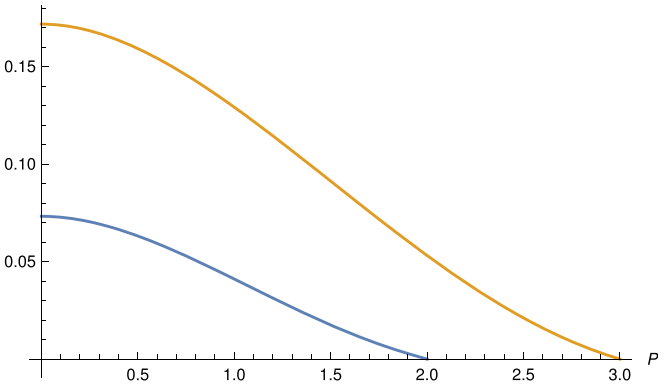

FIG. 4. Plot of the term in large square brackets in the expression (49) as a function of the order parameter P defined by (45) for $g_s=3$ (the lower, blue, curve) and $g_s=4$ (the upper, orange, curve). Its vanishing for the maximal value of P ($P_{\max}=g_s-1$) follows from the vanishing of the s-wave interaction of fermions in the same spin state.

Inclusion of the order-$(k_Fa_0)^3$ corrections to the energy density treated as a function of the order parameter P amounts to adding to the function $f_s(P)$ given by (48) the term

$$\frac{160}{\pi^3}(k_Fa_0)^3(g_s-1)\big\{4G_{\rm fin}^{(1)}(x_2,x_1)+4G^{(2)}(x_2,x_1) - K^{(1)}(x_2,x_1)-K^{(2)}(x_2,x_1)+\tfrac{1}{2}(g_s-2)\big[4G_{\rm fin}^{(1)}(x_2,x_2) +4G^{(2)}(x_2,x_2)-K^{(1)}(x_2,x_2)-K^{(2)}(x_2,x_2)\big] +(g_s-2)\big[\tilde K^{(2)}(x_1;x_2,x_2)+2\tilde K^{(2)}(x_2;x_2,x_1) +(g_s-3)\tilde K^{(2)}(x_2;x_2,x_2)\big]\big\}. \quad (49)$$

(The content of the curly brackets is plotted as a function of the order parameter P if Fig. 4 for $g_s=3$ and 4.) This has the following consequences. First, the critical value of the expansion parameter k_Fa_0 is further reduced: The transition occurs now at $(k_Fa_0)_{\rm cr}^{\rm 3rd}=0.990\,96$ for $s=\frac{1}{2}$ [and at $(k_Fa_0)_{\rm cr}^{\rm 3rd}=0.741\,90$ and $0.494\,45$ for $s=\frac{3}{2}$ and $s=\frac{9}{2}$, respectively]. Second, the first-order character of the transition, which is quite clear without this correction, becomes now much less pronounced: while for $s=\frac{1}{2}$ the height of the hill (separating the minimum at $P=0$ from the one at $P\neq 0$) remains almost unchanged, greatly reduced is the nonzero value of the order parameter right at the transition. This is clear from the comparison of Figs. 5(a) and 5(b) showing the shape of the function $f_s(P)$ for three values of k_Fa_0 close to the critical one (corresponding to the approximation used): In Fig. 5(a) the order-$(k_Fa_0)^3$ corrections are not included while in Fig. 5(b) they are. This aspect, the reduction of the the nonzero value of the order parameter right at the transition, is even more dramatic in the case of higher spins: Without the third-order corrections this value is maximal $P=g_s-1$; with these corrections it is reduced to approximately 0.12 (practically independent of the value of s). Moreover, for $s>\frac{1}{2}$ also the height of the hill separating the symmetry breaking minimum from the symmetry preserving one at $P=0$ is re-

[10] Taking the effects of the interaction in the first nontrivial order and restricting oneself to the configurations specified by (45), one can repeat the steps taken in [9] and obtain the full low-temperature profile of the transition to the ordered phase for any spin s. At zero magnetic field H, the equation

$$g_s\mu H+\frac{4g_s}{3\pi}k_Fa_0P = (g_s-1)\left\{(1+P)^{2/3}-\left(1-\frac{P}{g_s-1}\right)^{2/3} -\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F^{(s)}}\right)^2\left[(1+P)^{-2/3}-\left(1-\frac{P}{g_s-1}\right)^{-2/3}\right]+\cdots\right\},$$

which determines the parameter P starts, for $s>\frac{1}{2}$ to have a nontrivial solution before the slopes at $P=0$ of its both sides equalize; this shows that the transition is of first order.

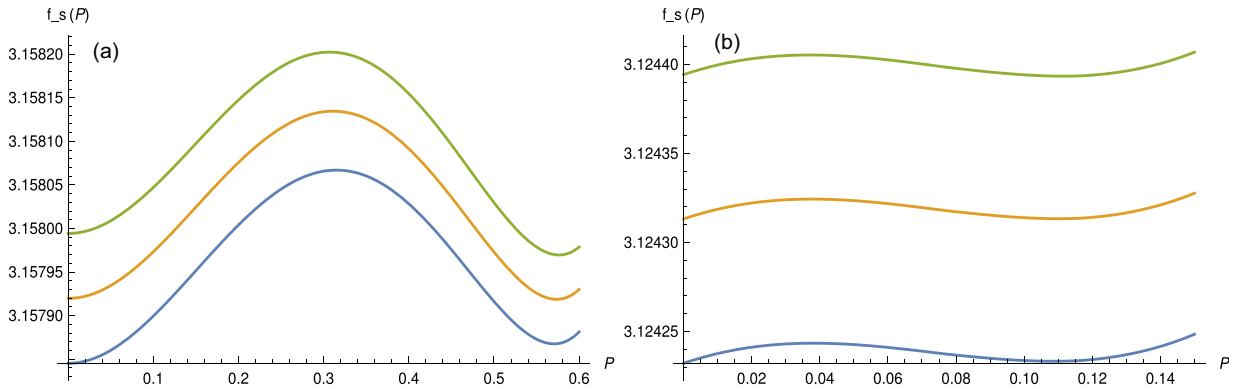

FIG. 5. Plots of the ground-state energy of the system of spin-$\frac{1}{2}$ fermions as a function of the order parameter P in units of $(k_F^3/6\pi^2)(\frac{3}{5})(\hbar^2k_F^2/2m_f)$, i.e., of the function $f_s(P)$, (a) without the order $(k_Fa_0)^3$ corrections, with the lines corresponding to $k_Fa_0 = 1.054\,04$ (the lowest, blue, line), 1.054 09 (the middle, orange, line), and 1.105 414 (the highest, green, line), and (b) with the order $(k_Fa_0)^3$ corrections, with the lines corresponding to $k_Fa_0 = 0.990\,91$ (the lowest, blue, line), 0.990 96 (the middle, orange, line), and 0.991 01 (the highest, green, line). In both panels the middle line corresponds to the respective critical value of k_Fa_0.

duced by roughly three orders of magnitude for $s = \frac{3}{2}$ and up to four for $s = \frac{9}{2}$. In the case of spin $\frac{3}{2}$ this is shown in Fig. 6. The results for different values of the spin s are presented in Table I, in which we have included also integer spin values; in the case of atomic gases different hyperfine levels can play the role of the effective spin and therefore its integer values are also compatible with the Fermi statistics. The dependence of the ground-state energy on the polarization P in the case of three states (spin 1), most studied in experiments with cold atoms, is shown in Fig. 7.

These purely perturbative results strongly point towards the possibility that the transition to the ordered state is, when all effects associated with the interactions other than that proportional to the coupling C_0 in (6) are ignored (which is equivalent to setting to zero all the higher partial wave scattering lengths a_ℓ and all the effective radii r_ℓ), indeed of the continuous type. Most probably with the inclusion of higher and higher corrections its character becomes less and less first order and becomes truly continuous when a resummation of the sort performed in [14] is made but becomes practically indistinguishable from such a transition already at a finite (not very high, as our computation shows) order of the ordinary perturbative expansion.

Finally, we can briefly discuss effects induced by the potential presence of a non-negligibly small s-wave effective radius r_0 and/or p-wave scattering length a_1. They are introduced through the two lower length dimension operators in (6) which are given explicitly by the formulas (41) and (42). In the configuration of densities assumed in this section these interactions add to the function $f_s(P)$ the terms

$$\frac{1}{6\pi}k_F^3a_0^2r_0(g_s-1)\left[x_1^3x_2^3\left(x_1^2+x_2^2\right)+(g_s-2)x_2^8\right] + \frac{2}{3\pi}(k_Fa_1)^3\left[x_1^8+\frac{1}{2}g_s(g_s-1)x_2^8 + \frac{1}{2}(g_s-1)x_1^3x_2^3\left(x_1^2+x_2^2\right)\right]. \tag{50}$$

If the effects of these operators are subleading compared to the effects caused by the order-k_F^3 corrections of the operator proportional to C_0 (that is, when $|r_0|, |a_1| \ll a_0$), they modify the general picture described above only slightly. For instance,

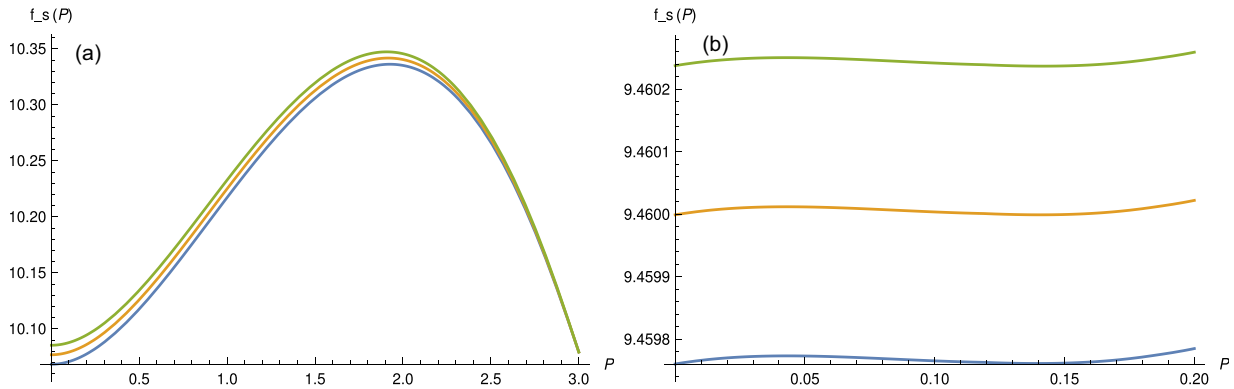

FIG. 6. Plots of the ground-state energy of the system of spin-$\frac{3}{2}$ fermions as a function of the order parameter P in units of $(k_F^3/6\pi^2)(\frac{3}{5})(\hbar^2k_F^2/2m_f)$, i.e., of the function $f_s(P)$, (a) without the order $(k_Fa_0)^3$ corrections, with the lines corresponding to $k_Fa_0 = 0.9532$ (the lowest, blue, line), 0.9542 (the middle, orange, line), and 0.9552 (the highest, green, line), and (b) with the order $(k_Fa_0)^3$ corrections, with the lines corresponding to $k_Fa_0 = 0.741\,88$ (the lowest, blue, line), 0.741 90 (the middle, orange, line), and 0.741 92 (the highest, green, line). In both panels the middle line corresponds to the respective critical value of k_Fa_0.

TABLE I. Characteristics of the transitions to the ordered state at $T = 0$ for different values of the spin s of fermions.

s	$(k_F a_0)_{\rm cr}^{\rm MF}$	$P_{\rm cr}^{\rm MF}$	$(k_F a_0)_{\rm cr}^{\rm 2nd}$	$P_{\rm cr}^{\rm 2nd}$	$(k_F a_0)_{\rm cr}^{\rm 3rd}$	$P_{\rm cr}^{\rm 3rd}$
$\frac{1}{2}$	$\pi/2$	0	1.05408	0.58	0.99097	0.11
1	1.51075	1.45	1.001	$P_{\rm max}$	0.83396	0.13
$\frac{3}{2}$	1.4298	2.78	0.9543	$P_{\rm max}$	0.74190	0.14
2	1.3598	3.95	0.9176	$P_{\rm max}$	0.67636	0.14
$\frac{5}{2}$	1.3016	$P_{\rm max}$	0.8877	$P_{\rm max}$	0.62560	0.125
3	1.2531	$P_{\rm max}$	0.8627	$P_{\rm max}$	0.58432	0.12
$\frac{7}{2}$	1.2118	$P_{\rm max}$	0.8405	$P_{\rm max}$	0.54976	0.115
4	1.1756	$P_{\rm max}$	0.8212	$P_{\rm max}$	0.52018	0.11
$\frac{9}{2}$	1.1439	$P_{\rm max}$	0.8040	$P_{\rm max}$	0.49445	0.105
5	1.0901	$P_{\rm max}$	0.7886	$P_{\rm max}$	0.47178	0.10
$\frac{11}{2}$	1.0901	$P_{\rm max}$	0.7753	$P_{\rm max}$	0.45164	0.10

in the case of $s = \frac{1}{2}$ the presence of $r_0 \approx a_0/10$ decreases by less than 1% the critical value of the parameter $k_{\rm F}a_0$, while $r_0 \approx -a_0/10$ increases it by the same amount, in agreement with the observation made in [14]. In both cases the character of the phase transition is not appreciably changed. Somewhat surprising is however the observation that for larger spin values this pattern is reversed: Negative r_0 decreases slightly the critical value of $k_{\rm F}a_0$, while positive r_0 increases it. The reason for this is that while for $s = \frac{1}{2}$ the first line of (50) is a function of P monotonically decreasing to zero (the contribution to the energy density of the interaction proportional to C_2 is by the Pauli exclusion principle also bound to vanish at $P = 1$), for higher spins it is a slightly increasing function, precisely in the range in which the new minimum of the energy density forms.

In general, the correction induced by the operator (42) is larger than the one induced by (41). Moreover, in the perturbation series for E_Ω/V it is the first correction which is not bound to vanish at $P = 1$ by the Pauli exclusion principle. Nevertheless, if one assumes that $|a_1|$ is of the same order as $|r_0|$ and much smaller than a_0, its contribution is not larger than that of (41) simply because it is proportional to a_1^3. In the configuration of the densities considered here, the expression in the second line of (50) is for all values of the spin s a monotonically increasing function. For this reason the contribution increases the value of $(k_{\rm F}a_0)_{\rm cr}$ for $a_1 > 0$ and lowers if for $a_1 < 0$ by roughly 1% so long as $|a_1| \lesssim a_0/10$ without affecting significantly the character of the transition. Thus, it seems likely that if all the scattering lengths a_ℓ, $\ell = 1, 2, \ldots$, all the radii r_ℓ, etc., are subleading with respect to a_0, the continuous character of the transition will emerge after all corrections are taken into account (being, from the practical point of view, indistinguishable from continuous already starting from some fixed order of the perturbative expansion).

VI. CONCLUSION

Computing the ground-state energy of a finite density system of fermions interacting through a binary spin-independent repulsive interaction is a classic problem of many-body quantum mechanics. The modern effective-field-theory approach greatly simplifies this task. In particular, it allowed the completion of the computation of the fourth-order (i.e., proportional to $k_{\rm F}^4$) corrections in the case of spin-s fermions and equal densities of different spin projections [5] and the extension to the third order the computation of energy of spin-$\frac{1}{2}$ fermions for an arbitrary value of the system's polarization [16]. Here we have extended the latter result to the

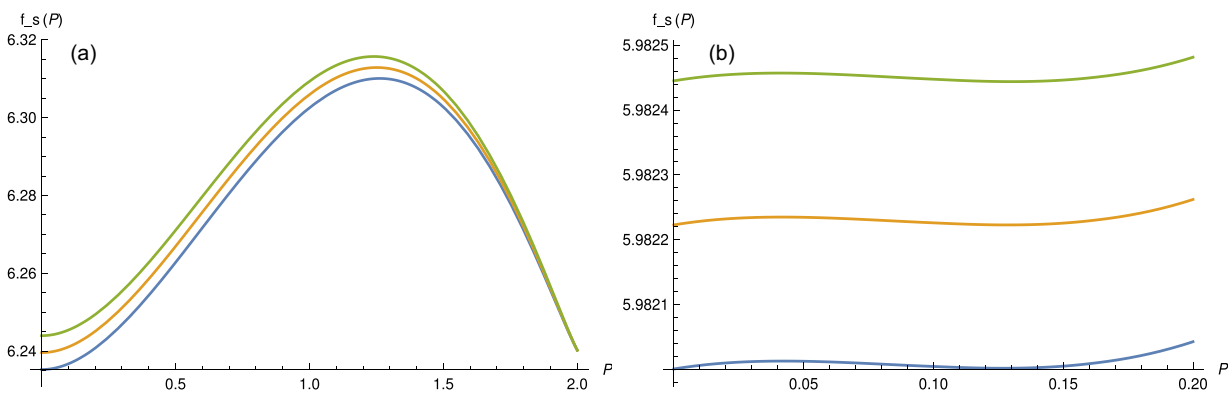

FIG. 7. Plots of the ground-state energy of the system of spin-1 ($g_s = 3$) fermions as a function of the order parameter P in units of $(k_{\rm F}^3/6\pi^2)(\frac{3}{5})(\hbar^2 k_{\rm F}^2/2m_f)$, i.e., of the function $f_s(P)$, (a) without the order $(k_{\rm F}a_0)^3$ corrections, with the lines corresponding to $k_{\rm F}a_0 = 1.1000$ (the lowest, blue, line), 1.1001 (the middle, orange, line), and 0.1002 (the highest, green, line), and (b) with the order $(k_{\rm F}a_0)^3$ corrections, with the lines corresponding to $k_{\rm F}a_0 = 0.833\,92$ (the lowest, blue, line), 0.833 96 (the middle, orange, line), and 0.834 00 (the highest, green, line). In both panels the middle line corresponds to the respective critical value of $k_{\rm F}a_0$.

case of spin-s fermions and arbitrary densities of different spin projections. The derived formulas allowed us to compute the ground-state energy semianalytically exploiting only the built-in routines of the *Mathematica* package.

We have used this result to discuss two issues. First, we checked numerically how the magnitude of the third-order term included in the resummation of an infinite subset of corrections done in [14] compares with the magnitudes of the rejected terms and found that it is not obviously dominant. Second, we discussed the impact the third-order corrections have on the characteristics of the system's transition to the ordered phase at zero temperature. We found that already the third-order corrections tend to erase the first-order character of this transition independently of the value s of the spin of fermions. Although our observations were made on the basis of the perturbative expansion used in the regime, which is probably beyond the domain of its applicability (the comparison of the results of the quantum Monte Carlo computations of the ground-state energy of the unpolarized gas of spin-$\frac{1}{2}$ fermions with the perturbative computation revealed [15] that the expansion is reliable up to $k_{\mathrm{F}}a_0 \lesssim 0.6$; in the case of $s > \frac{1}{2}$ the value of the expansion parameter $k_{\mathrm{F}}a_0$ at which the transition to the ordered state occurs is smaller, but since at the same time the magnitude of the successive terms of the perturbation series is increased by the growing powers of the factor g_s, the limit of the reliability of the expansion probably also decreases with s), they nevertheless seem to lend some support to the claim made in [14] that the transition is continuous rather than first order. It seems that the picture which emerges is quite sensible: Since a continuous transition is associated with fluctuations at all length scales, at any finite order of the perturbative expansion such a transition should look like a first-order one (the continuous transition in the system of spin-$\frac{1}{2}$ fermions obtained in the mean-field approximation is from this perspective a mere numerical coincidence), though its first-order character may rapidly disappear with an increased order of the approximation (so that it quickly may become indistinguishable from a continuous one from the practical point of view); the truly continuous character of a transition can really be revealed only by a (partial) resummation of all order contributions, of the sort proposed in [14].

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