Coherence generation, symmetry algebras, and Hilbert space fragmentation

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Hilbert space fragmentation is a novel type of ergodicity breaking in closed quantum systems. Recently, an algebraic approach was utilized to provide a definition of Hilbert space fragmentation characterizing families of Hamiltonian systems based on their (generalized) symmetries. In this paper we reveal a simple connection between the aforementioned classification of physical systems and their coherence generation properties, quantified by the coherence generating power (CGP). The maximum CGP (in the basis associated with the algebra of each family of Hamiltonians) is exactly related to the number of independent Krylov subspaces K, which is precisely the characteristic used in the classification of the system. In order to gain further insight, we numerically simulate paradigmatic models with both ordinary symmetries and Hilbert space fragmentation, comparing the behavior of the CGP in each case with the system dimension. More generally, allowing the time evolution to be any unitary channel in a specified algebra, we show analytically that the scaling of the Haar averaged value of the CGP depends only on K. These results illustrate the intuitive relationship between coherence generation and symmetry algebras.

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I. INTRODUCTION

Nonequilibrium quantum dynamics of closed systems has been an extensively studied issue in recent years. Generic isolated quantum systems are expected to thermalize in the thermodynamic limit, in the sense that for any small subsystem the rest of the system acts as a bath so that the expectation value of local observables coincides with that derived by statistical-mechanical ensembles [1,2]. A central role in the understanding of this phenomenon is played by the so-called eigenstate thermalization hypothesis (ETH) [3–5], according to which, under certain conditions, the energy eigenstates are conjectured to be thermal, i.e., the expectation values of observables with local support over the energy eigenstates coincide with those of a thermal ensemble with temperature related to the energy [6].

While the ETH is expected to hold for generic systems [5,7–15], several ETH-violating classes of models are known. Integrable systems and many-body localized (MBL) systems are the two prime examples, where an extensive number of conserved quantities prevent the system from thermalizing. The conserved quantities are a result of symmetries readily encoded in the Hamiltonian (in the integrable systems) or from emergent symmetries created by strong disorder (in the MBL case). More novel types of disorder-free ergodicity-breaking models were recently observed and studied, dubbed quantum many-body scars (QMBS) [16–40] and Hilbert space fragmentation [29,41–53]. The QMBS correspond to a weak violation of the ETH, in which only a small set of eigenstates in the bulk of the spectrum are nonthermal. The emergence of scar dynamics has been linked with the existence of dynamical symmetries [38,54–56], which allow the formation of towers of scar eigenstates inside an invariant subspace. Hilbert space fragmentation corresponds to the broader phenomenon of "shattering" of the Hilbert space in exponentially many dynamically disconnected subspaces (referred to as Krylov subspaces) with no obvious associated conserved quantities. In addition to weak violation, such systems can exhibit a strong violation of the ETH, where a nonzero measure subset of all the eigenstates is nonthermal.

Recently, an algebraic framework was utilized to analyze the phenomenon of Hilbert space fragmentation for families of Hamiltonians [53], establishing the role of nonconventional symmetries. Such an approach reveals the central role of the symmetry algebra in the classification of the system and is applicable to conventional models as well [57]. Another inherent advantage is that the properties exhibited by the families of Hamiltonians are free from fine-tuning. In general, systems exhibiting novel ergodic behavior (such as QMBS and Hilbert space fragmentation) belong to special classes of Hamiltonians [53] that are of great interest in attempts to fully describe the behavior of nonequilibrium closed quantum systems [20,22,42,58–60].

The ergodic properties of a system are correlated with information-theoretic signatures, such as eigenstate entanglement [61–63] and information scrambling [18,64,65]. In this paper we establish a connection between the classification framework introduced in [53] and coherence, quantified by the coherence generating power (CGP) [66–69]. The CGP bound established in this paper becomes particularly relevant for

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the aforementioned special classes of Hamiltonians exhibiting Hilbert space fragmentation.

This paper is structured as follows. In Sec. II B we present an analytical upper bound of the CGP of families of Hamiltonian evolutions, with respect to a basis induced by the algebra of each family; this bound is fully described by the aforementioned classification of the families of Hamiltonians. We also provide numerical results of systems with both ordinary symmetries and Hilbert space fragmentation and compare the behavior of the CGP as we increase the system size. In Sec. III we allow the evolution to be any unitary channel in a prescribed algebra and show that the Haar average value of the CGP is typical and its scaling with the system size is fully described by the classification of unitary evolutions in the algebra. In Sec. IV we conclude with a brief discussion of the results. The detailed proofs of the analytical results are included in the Appendix.

II. COHERENCE GENERATING POWER FOR CLASSES OF MODELS

A. Preliminaries

Let $\mathcal{H} \cong \mathbb{C}^d$ be a finite d-dimensional Hilbert space and denote the space of linear operators acting on it by $\mathcal{L}(\mathcal{H})$. The key mathematical structures of interest are $*$-closed unital algebras of observables $\mathcal{A}$, alongside their commutants,

$$\mathcal{A}' = \{Y \in \mathcal{L}(\mathcal{H}) | [X, Y] = 0 \quad \forall X \in \mathcal{A}\}.$$

Due to the double commutant theorem, these algebras always come in pairs, so as $(\mathcal{A}')' = \mathcal{A}$ [70].

Letting $\mathcal{Z}(\mathcal{A}) := \mathcal{A} \cap \mathcal{A}'$ denote the center of $\mathcal{A}$ with $d_{\mathcal{Z}} := \dim \mathcal{Z}(\mathcal{A})$, a fundamental structure theorem for C^* algebras implies a decomposition of the Hilbert space of the form [70]

$$\mathcal{H} \cong \bigoplus_{J=1}^{d_{\mathcal{Z}}} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J},$$

$$\mathcal{A} \cong \bigoplus_{J=1}^{d_{\mathcal{Z}}} \mathbb{1}_{n_J} \otimes \mathcal{L}(\mathbb{C}^{d_J}),$$

$$\mathcal{A}' \cong \bigoplus_{J=1}^{d_{\mathcal{Z}}} \mathcal{L}(\mathbb{C}^{n_J}) \otimes \mathbb{1}_{d_J}. \tag{1}$$

Due to the above decomposition,

$$\dim \mathcal{H} \equiv d = \sum_{J=1}^{d_{\mathcal{Z}}} n_J d_J,$$

$$\dim \mathcal{A} = \sum_{J=1}^{d_{\mathcal{Z}}} d_J^2 =: d(\mathcal{A}),$$

$$\dim \mathcal{A}' = \sum_{J=1}^{d_{\mathcal{Z}}} n_J^2 =: d(\mathcal{A}').$$

In general, the algebra $\mathcal{A}'$ is non-Abelian and we will denote by $\mathcal{M} \subset \mathcal{A}'$ the (possibly not unique) maximal Abelian subalgebra of the commutant with dimension $K := \dim \mathcal{M} = \sum_{J=1}^{d_{\mathcal{Z}}} n_J$. Note that K is exactly the number of linearly independent *common* invariant subspaces of all operators in $\mathcal{A}$.

TABLE I. Classification of families of Hamiltonians of the form (3) based on the scaling of the dimension K of the maximally Abelian subalgebra of the commutant $\mathcal{A}'$ with system size L in one dimension [53].

$\log K$	Class representative
$\sim O(1)$	Discrete global symmetry
$\sim \log L$	Continuous global symmetry
$\sim L$	Fragmentation

Given a basis $B = \{|i\rangle\}_{i=1}^d$ of $\mathcal{H}$, one can always express a pure state as a linear superposition of the states $|i\rangle$. This fact is experimentally materialized as what is called quantum coherence [71]. In general, one defines B-incoherent states ρ ($\rho \geqslant 0$ and $\mathrm{Tr}\rho = 1$) as states diagonal in B (coherent states are all states that are not incoherent). Given a unitary operator $U \in \mathcal{L}(\mathcal{H})$, a measure of its CGP with respect to the basis B is given by [67]

$$C_B(U) = 1 - \frac{1}{d} \sum_{i,j=1}^{d} |\langle i|U|j\rangle|^4. \tag{2}$$

Note that the above quantity coincides with a measure of information scrambling of B-diagonal (incoherent) degrees of freedom [72,73], which aligns with its ability to quantify the deviation of incoherent states from the incoherent space under evolution with U.

Reference [53] introduced an algebra-based classification of multisite Hamiltonian models. Given an algebra $\mathcal{A}$ that can be generated by local operators h_j, one considers the family of Hamiltonians

$$H = \sum_j J_j h_j, \tag{3}$$

where J_j are arbitrary coupling constants. Then the symmetries of the system are characterized by the scaling of K with the system size. For example, for one-dimensional systems of size L, if $\log K \sim L$, then the family of Hamiltonians in Eq. (3) possesses an exponentially large number of dynamically disconnected subspaces, which serves as the definition of Hilbert space fragmentation (see Table I). The models exhibiting Hilbert space fragmentation were shown to exhibit nonlocal conserved quantities, dubbed statistically localized integrals of motion [44,53].

B. Coherence generating power bound

We consider an algebra $\mathcal{A} = \langle h_j \rangle$ and families of Hamiltonians of the form (3). Clearly, $H \in \mathcal{A}$ and due to Eq. (1),

$$U := \exp(-itH) = \bigoplus_{J=1}^{d_{\mathcal{Z}}} \mathbb{1}_{n_J} \otimes U_J. \tag{4}$$

We also consider a basis $B_{\mathcal{A}} = \{|\phi_J\rangle \otimes |\psi_J\rangle | J = 1, \ldots, d_{\mathcal{Z}}; \phi_J = 1, \ldots, n_J; \psi_J = 1, \ldots, d_J\}$, where $|\phi_J\rangle$ and $|\psi_J\rangle$ span $\mathbb{C}^{n_J}$ and $\mathbb{C}^{d_J}$ of the decomposition (1). We can then show the following proposition.

Proposition 1. (a) The CGP of $U = \exp(-itH) \in \mathcal{A}$ in a basis $B_{\mathcal{A}}$ induced by the algebra decomposition is

$$C_{B_{\mathcal{A}}}(U) = 1 - \frac{1}{d} \sum_{J,\psi_J,\psi'_J} n_J \left|\langle\psi_J|U_J|\psi'_J\rangle\right|^4 =: 1 - \frac{1}{d} f_{B_{\mathcal{A}}}(U). \tag{5}$$

(b) The maximum value of the CGP is

$$\max_{U\in\mathcal{A}} C_{B_{\mathcal{A}}}(U) = 1 - \frac{1}{d}K \tag{6}$$

and is achieved when $|\langle\psi_J|U_J|\psi'_J\rangle| = d_J^{-1/2}$ for all J.

We note that if U is any unitary operator in $\mathcal{L}(\mathcal{H})$, then the maximum CGP is $C_{\max} = 1 - \frac{1}{d}$ [67,72,73]. The difference in the bound of Eq. (6) is an extra factor of K, which is exactly the number of independent common invariant subspaces for all evolutions generated by the family of Hamiltonians in Eq. (3). The scaling of K with the system dimension provides a classification of this family (see, e.g., Table I); thus from Eq. (6) the scaling of the maximum CGP of the entire family of Hamiltonians is also dependent solely on this classification as well. This observation constitutes an exact analytical implementation of the intuitive connection between (generalized) system symmetries and coherence generation. Note that the Hilbert space decomposition (1) is a basic element for the protection of quantum coherence in information processes, e.g., by decoherence-free subspaces and subsystems [74–77]. There the algebra $\mathcal{A}$ is generated by the system operators of the interaction Hamiltonian and K is simply the total dimension of decoherence-free subsystems $\mathbb{C}^{n_J}$.

C. Quantum spin-chain models

Note that the result in Proposition 1 only depends on the fact that $U \in \mathcal{A}$ and not on the fact that U is generated by a Hamiltonian of the form (3). As so, if we choose randomly a Hamiltonian from Eq. (3), the resulting evolution need not be typical. In order to evaluate the behavior of the CGP in such a case, we numerically simulate the evolution of spin-chain models in one dimension with L sites:

(i) the spin-$\frac{1}{2}$ XXZ model with an on-site magnetic field,

$$H_{\rm XXZ} = \sum_j \left[J_j^{\perp}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right) + J_j^z\sigma_j^z\sigma_{j+1}^z + h_j\sigma_j^z\right],$$

(ii) the spin-1 Temperley-Lieb (TL) model, and

$$H_{\rm TL} = \sum_j J_j e_{j,j+1},$$

where $e_{j,k} = 3(|\psi_{\rm sing}\rangle\langle\psi_{\rm sing}|)_{j,k}$ and $|\psi_{\rm sing}\rangle_{j,k} = \frac{1}{\sqrt{3}}\sum_{\alpha\in\{0,1,2\}}|\alpha\alpha\rangle_{j,k}$,

(iii) the fermionic t-J_z model

$$H_{t-J_z} = \sum_j \left(-t_{j,j+1}\sum_{\sigma\in\{\uparrow,\downarrow\}}\left(\tilde{c}_{j,\sigma}\tilde{c}^{\dagger}_{j+1,\sigma} + \text{H.c.}\right) + J^z_{j,j+1}S^z_jS^z_{j+1} + h_jS^z_j + g_j\left(S^z_j\right)^2\right), \tag{7}$$

where $S_j^z := \tilde{c}^{\dagger}_{j,\uparrow}\tilde{c}_{j,\uparrow} - \tilde{c}^{\dagger}_{j,\downarrow}\tilde{c}_{j,\downarrow}$ and $\tilde{c}_{j,\sigma} := c_{j,\sigma}(1 - c^{\dagger}_{j,-\sigma}c_{j,-\sigma})$. Here $\sigma_j^{\alpha}, \alpha \in x, y, z$ denote the Pauli matrices. The XXZ model has shown to be integrable by the Bethe ansatz [78] and possesses a conventional U(1) symmetry with the associated conserved quantity $\sigma^z_{\rm tot} = \sum_j \sigma_j^z$. The sectors of Eq. (1) are labeled by this conserved quantity ($J = L/2 - \sigma^z_{\rm tot} = 0, \ldots, L$) and the dimensions of the irreducible representations of $\mathcal{A}$ and $\mathcal{A}'$ are $d_J = \binom{L}{J}$ and $n_J = 1$, respectively [53]. For a given J there is one common Krylov subspace (since $n_J = 1$) which is spanned by the z-basis product states with $\sigma^z_{\rm tot} = L/2 - J$. Note that $K_{XXZ} = L + 1$, which scales linearly with the system size.

The TL model is an example of Hilbert space fragmentation. The dynamically disconnected Krylov subspaces are understood by the use of a basis of dots and noncrossing dimers [53,79,80]. Each basis state consists of a pattern of dimers that connect two sites and represent the state $|\psi_{\rm sing}\rangle_{j,k}$. The rest of the sites (that are not connected to either end of a dimer) make up the dots and the state $|\psi_{\rm dots}\rangle$ is chosen such that it is annihilated by all projectors $|\psi_{\rm dim}\rangle_{m,m+1}\langle\psi_{\rm dim}|_{m,m+1}$ (where m labels all dots while excluding the dimers). The algebra formed by the $L-1$ generators $e_{j,j+1}$ is the Temperley-Lieb algebra $\text{TL}_L(q)$, where $q + q^{-1} = 3$ (which is the number of local degrees of freedom in a spin-1 model) [53]. It has been shown that, for even L, the sectors of the decomposition (1) are labeled by $J = 0, \ldots, L/2$ with corresponding dimensions

$$n_J = [2J+1]_q, \quad d_J = \binom{L}{L/2+J} - \binom{L}{L/2+J+1},$$

where $[n]_q := (q^n - q^{-n})/(q - q^{-1})$ is a q-deformed integer [79]. It is then straightforward to show that $K = \frac{q^{-L}(q^{L+2}-1)^2}{(q^2-1)^2}$, which scales exponentially with the system size.

The t-J_z model also falls into the category of Hilbert space fragmentation. The Hamiltonian acts effectively in the Hilbert space with no double occupancy and it has been shown that all operators in the commutant algebra are diagonal in the product state basis [53]. The sectors are characterized by the pattern of spins up and down, which remains invariant under the action of the Hamiltonian [44]. For open boundary conditions there are $2^{L+1} - 1$ such sectors (all with $n_J = 1$), so $K = 2^{L+1} - 1$, which scales exponentially with L [53].

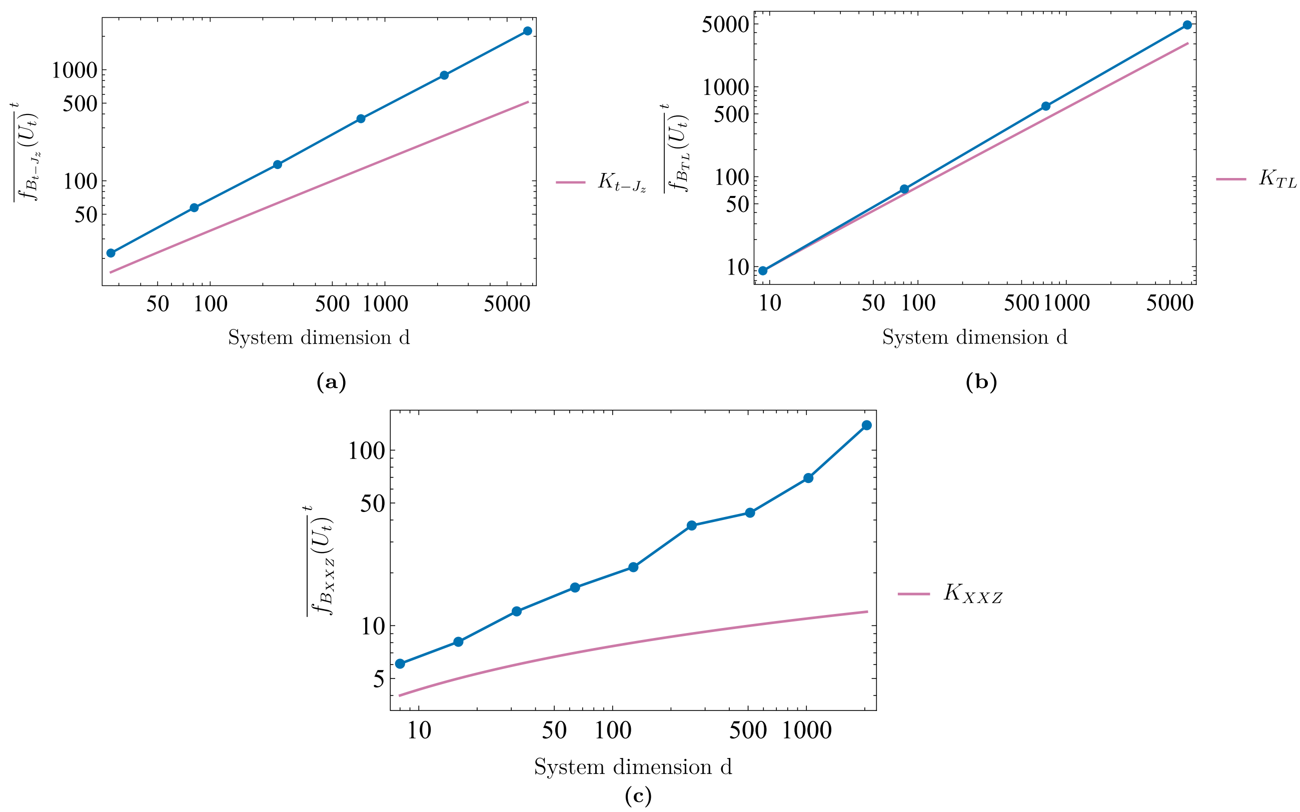

FIG. 1. Log-log plots of the long-time average of $f_{B_\mathcal{A}}(U) = 1 - C_{B_\mathcal{A}}(U)$ (connected dotted lines) in comparison to the bound K (solid lines) for (a) the t-J_z model, where $\overline{f_{B_{t-J_z}}(U_t)}^t \sim d^{0.83}$ scales exponentially with the system size and the bound set by the number of independent Krylov subspaces scales as $d^{\log_3 2}$; (b) the TL model, where $\overline{f_{B_{\rm TL}}(U_t)}^t \sim d^{0.95}$ scales exponentially with the system size and the scaling is very close to that set by the bound $d^{\log_3 q}$ (see Sec. 4 of the Appendix), $\log_3 q \simeq 0.88$, which can be understood in terms of the existence of Krylov-restricted thermalization in some of the large fragments of the TL model [47]; and (c) the XXZ model, where the behavior of $\overline{f_{B_{XXZ}}(U_t)}^t$ greatly deviates from the bound with the system size. Integrable systems (even with a random choice of couplings) are not expected to showcase generic properties, even after resolving for symmetries.

We simulate the exact dynamics for the above systems and for different system sizes with the coupling constants randomly drawn from [0,1] (each time we increase the system size we add one more set of randomly chosen couplings to the previously drawn ones). The choice of the coupling constants sets the energy scale of the Hamiltonians and in turn the timescale of the dynamics. For each model we construct the relevant bases described above and compute the long-time average of the CGP of the unitary evolution. In Figs. 1(a) and 1(b) we observe that for the fragmented models the long-time average of the quantity $f_{B_\mathcal{A}}(U)$ [related to the CGP via Eq. (5)] has a similar behavior with the number of Krylov subspaces K. Specifically, we find that (see also Sec. 4 of the Appendix)

$$\overline{f_{B_{t\text{-}J_z}}(U_t)}^t \sim d^{0.83} \quad \left(\text{compared to } K_{t\text{-}J_z} \sim d^{\log_3 2}\right),$$
$$\overline{f_{B_{\rm TL}}(U_t)}^t \sim d^{0.95} \quad \big[\text{compared to } K_{\rm TL} \sim d^{\log_3 q},$$
$$q = (3+\sqrt{5})/2 \simeq 2.62\big].$$

Fragmented systems can be further characterized into strongly or weakly fragmented depending on the size $\max_J d_J$ of the biggest Krylov subspace; specifically for strongly (weakly) fragmented systems, $\max_J d_J/d \to 0$ ($\max_J d_J/d \to 1$) as $d \to \infty$ [42,53]. When the size of the fragments is large enough (in the thermodynamic limit) and the Hamiltonian is nonintegrable in these fragments, signatures of thermalization within the fragment can be observed, a phenomenon referred to as Krylov-restricted thermalization [47]. In fact, the TL model has been shown to exhibit thermalization in some of its exponentially large Krylov subspaces [47], which implies that the behavior of the CGP is expected to be closer to that of a generic model (see also Sec. III). In contrast to the fragmented models, the behavior of the CGP of the integrable XXZ model with the system dimension is far away from the bound (6) [Fig. 1(c)]. This aligns with the expectation that integrable models do not showcase features of generic evolutions, even after resolving for the symmetries. Figure 2 emphasizes the vastly different CGP behaviors of the different classes of models simulated; for finite dimensions the bound (6) forces the CGP of the fragmented models to remain much lower than that of the integrable model. This shows that despite the fact that fragmented models are in general nonintegrable, emergent generalized symmetries from kinetic constraints lead to a milder mixing of different parts of the Hilbert space, with possible physical importance for information processing tasks, e.g., protection of information via decoherence-free subspaces

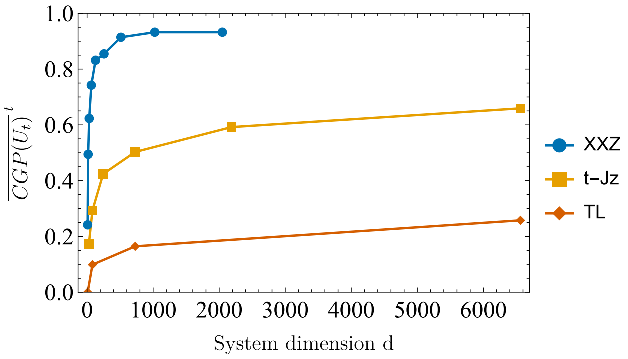

FIG. 2. Comparison of the scaling of the CGP for the XXZ, t-J_z, and TL models. We observe that for finite dimensions, the CGP of fragmented models is heavily constrained by the bound set in Eq. (6), leading to less mixing of different parts of the Hilbert space under evolution.

[74,76,81] or utilization of (dynamical) localization for quantum memory implementations [46,82].

III. HAAR AVERAGED COHERENCE GENERATING POWER AND SYSTEM-SIZE SCALING

We will now consider the case where $U \in \mathcal{A}$ can be any unitary in the algebra. It is straightforward to notice that Proposition 1 continues to hold, since Eq. (4) holds for all unitaries in the algebra. A natural question we wish to investigate is, given an algebra $\mathcal{A}$, what the typical value of the CGP is and how it is related to the number of independent Krylov subspaces K. To do so, we average $C_{B_\mathcal{A}}(U)$ over the Haar measure of the subgroup of unitaries $U \in \mathcal{A}$.

Proposition 2. Given an algebra $\mathcal{A}$, the Haar average of the CGP over the unitaries in the algebra is

$$\overline{C_{B_\mathcal{A}}(U)}^{U\in\mathcal{A}} := \mathbb{E}_{U\in\mathcal{A}}[C_{B_\mathcal{A}}(U)] = 1 - \frac{1}{d}\sum_J \frac{2d_J}{d_J+1}n_J. \tag{8}$$

Equation (8) provides an analytical expression for the Haar average value of the CGP in terms of the dimensions n_J and d_J related to the decomposition (1). Since $1 \leqslant d_J \leqslant d \,\forall J$ we can (loosely) bound the typical value as

$$1 - \frac{2}{d+1}K \leqslant \overline{C_{B_\mathcal{A}}(U)}^{U\in\mathcal{A}} \leqslant 1 - \frac{1}{d}K. \tag{9}$$

This shows that the scaling of the average value with the system dimension depends exactly on the classification of the model in terms of the scaling of K with the system size (Sec. II A). As seen in Proposition 4 of [67], Levy's lemma implies that the Haar average is expected to be typical inside sufficiently large Krylov subspaces (see Sec. 3 of the Appendix). This aligns with the observation that the weakly fragmented TL model seems to have close to maximal CGP, as in that case there is a dominant Krylov subspace such that $\max_J \frac{d_J}{d} \to 1$ in the thermodynamic limit.

Note that, due to the double commutant theorem, $\mathcal{A}$ is completely determined by the commutant $\mathcal{A}'$, which represents the set of symmetries imposed on the evolution. As so, the selection of a random unitary in $\mathcal{A}$ corresponds to a random unitary channel constrained by physically motivated symmetries.

IV. CONCLUSION

In this paper we revealed a connection between a classification of families of Hamiltonian evolutions and their coherence generating power with respect to a basis induced by the family. Specifically, the families of Hamiltonians were previously classified based on the scaling of the number K of independent dynamically disconnected subspaces (called Krylov subspaces) with the system size [53]. Each family of Hamiltonians is defined with respect to an algebra $\mathcal{A}$ generated by local terms that are used to compose the Hamiltonians. The generalized symmetries of the system are captured by the commutant algebra $\mathcal{A}'$ and K coincides with the dimension of the maximally Abelian subalgebra of $\mathcal{A}'$. The Krylov subspaces are described by an algebra-induced Hilbert space decomposition, which also specifies a relevant basis. Our main result was then the demonstration that the maximum CGP (with respect to this relevant basis) of such a family of Hamiltonians is exactly related to K; hence its scaling with the system dimension is precisely dependent on the classification of the system. This gives an exact quantitative implementation of the intuitive connection between (generalized) symmetries and coherence generation. A principal example is the situation of Hilbert space fragmentation, in which case K scales exponentially with the system size, leading to a substantially lower upper bound for the CGP in finite dimensions.

In order to further investigate the above observations, we numerically simulated different families of one-dimensional spin-chain models and computed long-time averages of their CGP with respect to the basis induced by the algebra of each family. We observed that for the fragmented systems and the fermionic t-J_z and spin-1 Temperley-Lieb models, the CGP follows closely the exponential behavior of the bound induced by the exponential number of Krylov subspaces. The particularly significant agreement in the TL case was connected with the previously observed Krylov restricted thermalization in some of its large Krylov subspaces in the thermodynamic limit. In contrast, in the case of the integrable spin-$\frac{1}{2}$ XXZ model, the CGP greatly deviates from the bound set by the linear number of Krylov subspaces. Naturally, the above picture ties into the fact that the maximum (or close to the maximum) CGP is expected from systems that exhibit generic features inside sufficiently large Krylov subspaces.

The above observation was made precise by allowing the unitary evolution to be generated by any unitary in $\mathcal{A}$ and performing the Haar average of the CGP over all unitaries in $\mathcal{A}$. We showed that the scaling of this average value with the system size depends exactly on the classification of the model in terms of the system size scaling of K. In addition, Levy's lemma ensures that the Haar average will be typical for sufficiently large Krylov subspaces.

A natural question one may ask is if there are more information-theoretic quantities that can be linked (in an exact fashion) with the classification of models induced by their symmetry algebra. Moreover, it would be interesting to further investigate the conditions under which families of models exhibit CGP close to the bound, as well as derive explicit connections with their ergodic and integrability properties inside the various Krylov subspaces.

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APPENDIX

1. Proof of Proposition 1

(a) Using Eq. (4) in Eq. (2) for the basis $B_{\mathcal{A}} = \{|\phi_J\rangle \otimes |\psi_J\rangle | J = 1, \ldots, d_{\mathcal{Z}}; \phi_J = 1, \ldots, n_J; \psi_J = 1, \ldots, d_J\}$ we get

$$\begin{aligned} C_{B_{\mathcal{A}}}(U) &= 1 - \frac{1}{d} \sum_{\phi_J, \phi'_{J'}} \sum_{J, J'} \sum_{\psi_J, \psi'_{J'}} |\langle\phi_J| \otimes \langle\psi_J| \oplus_{\tilde{J}} \mathbb{1}_{n_{\tilde{J}}} \otimes U_{\tilde{J}} |\phi'_{J'}\rangle \otimes |\psi'_{J'}\rangle|^4 \\ &= 1 - \frac{1}{d} \sum_{\phi_J, \phi'_{J'}} \sum_{J, J'} \sum_{\psi_J, \psi'_{J'}} \delta_{J\tilde{J}} \delta_{J'\tilde{J}} \delta_{\phi_J \phi'_{J'}} |\langle\psi_J| U_{\tilde{J}} |\psi'_{J'}\rangle|^4 \\ &= 1 - \frac{1}{d} \sum_{J, \phi_J, \psi_J, \psi'_J} |\langle\psi_J| U_J |\psi'_J\rangle|^4 \\ &= 1 - \frac{1}{d} \sum_{J, \psi_J, \psi'_J} n_J |\langle\psi_J| U_J |\psi'_J\rangle|^4. \end{aligned} \quad \text{(A1)}$$

(b) Let $X \in \mathcal{L}(\mathbb{C}^{d_J})$. Then, since $\|X - \frac{\mathrm{Tr}X}{d_J} \mathbb{1}_{d_J}\|_2^2 \geqslant 0$, we have

$$\|X\|_2^2 \geqslant \frac{|\mathrm{Tr}X|^2}{d_J} \quad \forall X \in \mathcal{L}(\mathbb{C}^{d_J}). \quad \text{(A2)}$$

Using the above identity for $X = \sum_{\psi_J} |\langle\psi_J|U_J|\psi'_J\rangle|^2 |\psi_J\rangle\langle\psi_J|$, we get

$$\sum_{\psi_J} |\langle\psi_J|U_J|\psi'_J\rangle|^4 \geqslant \frac{\sum_{\psi_J} |\langle\psi_J|U_J|\psi'_J\rangle|^2}{d_J} = \frac{1}{d_J}. \quad \text{(A3)}$$

Using Eq. (A3) in Eq. (A1), we get

$$C_{B_{\mathcal{A}}}(U) \leqslant 1 - \frac{1}{d} \sum_{J, \psi'_J} n_J \frac{1}{d_J} = 1 - \frac{1}{d} \sum_J d_J \frac{n_J}{d_J} = 1 - \frac{1}{d} \sum_J n_J = 1 - \frac{1}{d} K \quad \text{(A4)}$$

and clearly the maximum is achieved when $|\langle\psi_J|U_J|\psi'_J\rangle| = d_J^{-1/2}\, \forall J$, i.e., when U_J is mutually unbiased with respect to the basis $\{|\psi_J\rangle\}$ of $\mathbb{C}^{d_J}$.

2. Proof of Proposition 2

Due to Eq. (4), taking the Haar average over all $U \in \mathcal{A}$ corresponds to taking the Haar average over the unitaries in the subsystems $\mathbb{C}^{d_J}$. Defining $\Pi_{\psi_J} = |\psi_J\rangle\langle\psi_J|$, we can rewrite Eq. (5) as

$$\begin{aligned} C_{B_{\mathcal{A}}}(U) &= 1 - \frac{1}{d} \sum_{J, \psi_J, \psi'_J} n_J \big[\mathrm{Tr}\big(\Pi_{\psi_J} U_J \Pi_{\psi'_J} U_J^\dagger\big)\big]^2 \\ &= 1 - \frac{1}{d} \sum_{J, \psi_J, \psi'_J} n_J \mathrm{Tr}\big[\big(\Pi_{\psi_J} U_J \Pi_{\psi'_J} U_J^\dagger \otimes \Pi_{\psi_J} U_J \Pi_{\psi'_J} U_J^\dagger\big)\big] \\ &= 1 - \frac{1}{d} \sum_{J, \psi_J, \psi'_J} n_J \mathrm{Tr}\big(\Pi_{\psi_J}^{\otimes 2} U_J^{\otimes 2} \Pi_{\psi'_J}^{\otimes 2} U_J^{\dagger \otimes 2}\big). \end{aligned} \quad \text{(A5)}$$

By Schur-Weyl duality the commutant of the algebra $\mathcal{M}_J$ generated by $\{M_J^{\otimes 2} | M_J \in \mathcal{L}(\mathbb{C}^{d_J})\}$ is $\mathcal{M}'_J = \mathbf{C}S_2$, where $S_2 = \{\mathbb{1}, S\}$ is the symmetric group over the copies in $\mathbb{C}^{d_J} \otimes \mathbb{C}^{d_J}$ [83]. Since we can always find a unitary basis of $\mathcal{L}(\mathbb{C}^{d_J})$, it follows that

$\mathcal{M}_J$ is equivalently generated by $\{U_J^{\otimes 2} | U_J \in \mathcal{L}(\mathbb{C}^{d_J}), U_J U_J^\dagger = \mathbb{1}_{d_J}\}$. Also, note that $\mathbb{P}_{\mathcal{M}_J'}[\bullet] := \overline{U_J^{\otimes 2}[\bullet] U_J^{\dagger \otimes 2}}^{U_J}$ is an orthogonal projector on $\mathcal{M}_J'$ [84]. So we can express $\mathbb{P}_{\mathcal{M}_J'}$ in terms of the orthonormal basis $\{\frac{\mathbb{1}+S}{\sqrt{2d_J(d_J+1)}}, \frac{\mathbb{1}-S}{\sqrt{2d_J(d_J-1)}}\}$ of $\mathbf{C}S_2$:

$$\mathbb{P}_{\mathcal{M}_J'}[\bullet] = \overline{U_J^{\otimes 2}[\bullet] U_J^{\dagger \otimes 2}}^{U_J} = \sum_{\eta=\pm 1} \frac{\mathbb{1} + \eta S}{2d_J(d_J + \eta)} \langle \mathbb{1} + \eta S, \bullet \rangle. \tag{A6}$$

Taking the Haar average over the unitaries in $\mathbb{C}^{d_J}$ in Eq. (A5) and using Eq. (A6), we get

$$\begin{aligned}
\overline{C_{B_\mathcal{A}}(U)}^{U\in\mathcal{A}} &= 1 - \frac{1}{d} \sum_{J,\psi_J,\psi_J'} n_J \mathrm{Tr}\left(\Pi_{\psi_J}^{\otimes 2} \sum_{\eta=\pm 1} \frac{\mathbb{1} + \eta S}{2d_J(d_J+\eta)} \langle \mathbb{1} + \eta S, \Pi_{\psi_J'} \rangle \right) \\
&= 1 - \frac{1}{d} \sum_{J,\psi_J,\psi_J'} n_J \mathrm{Tr}\left(\Pi_{\psi_J}^{\otimes 2} \frac{\mathbb{1} + S}{d_J(d_J+1)} \right) \\
&= 1 - \frac{1}{d} \sum_{J,\psi_J,\psi_J'} n_J \frac{2}{d_J(d_J+1)} \\
&= 1 - \frac{1}{d} \sum_J \frac{2d_J}{d_J+1} n_J.
\end{aligned} \tag{A7}$$

3. Typicality inside sufficiently large Krylov subspaces

This follows from Proposition 4 of [67]. The key ingredient is Levy's lemma for Haar distributed unitaries in $\mathbb{C}^{d_J}$. The operator norms used are the Schatten k-norms [85] defined as $\|X\|_k := (\sum_i s_i^k)^{1/k}$, where $\{s_i\}_i$ are the singular values of X. For $k \to \infty$, $\|X\|_\infty = \max_i \{s_i\}$ is the usual operator norm.

*Lemma 1 (*Levy's lemma). If $X : U(d^J) \mapsto \mathbb{R}$ is a Lipschitz continuous function of constant λ, i.e., $|X(U_J) - X(V_J)| \leqslant \lambda \|X(U_J) - X(V_J)\|_2$, then

$$\Pr\left[\left| X(U_J) - \overline{X(U_J)}^{U_J \in \mathbb{C}^{d_J}} \right| \geqslant \epsilon \right] \leqslant e^{-d_J/\epsilon^2 4\lambda^2}. \tag{A8}$$

We will also need to known norm inequalities such as

$$|\mathrm{Tr}(AB)| \leqslant \|A\|_1 \|B\|_\infty \tag{A9}$$

and, if $\mathcal{T}$ is a positive trace-preserving map,

$$\|\mathcal{T}(X)\|_1 \leqslant \|X\|_1, \tag{A10}$$

$$\tfrac{1}{2} \| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 \leqslant \| |\psi\rangle - |\phi\rangle \| \leqslant 2. \tag{A11}$$

Choose $U = \oplus_J \mathbb{1}_{n_J} \otimes U_J$ and $V = \oplus_J \mathbb{1}_{n_J} \otimes V_J \in \mathcal{A}$. Note that using the swap trick $\mathrm{Tr}(AB) = \mathrm{Tr}(SA \otimes B)$, we can rewrite

$$C_{B_\mathcal{A}}(U) = 1 - \frac{1}{d} \sum_J n_J \sum_{\psi_J,\psi_J'} |\langle \psi_J | U_J | \psi_J' \rangle|^4 = 1 - \frac{1}{d} \sum_J n_J \sum_{\psi_J'} \mathrm{Tr}\{ S_J [\mathcal{D}_J \mathcal{U}_J (\Pi_{\psi_J'})]^{\otimes 2} \}, \tag{A12}$$

where $D_J(\bullet) = \sum_{\psi_J} \Pi_{\psi_J} \bullet \Pi_{\psi_J}$, $\mathcal{U}_J(\bullet) = U_J \bullet U_J^\dagger$, and S_J is the swap in $\mathbb{C}^{d_J} \otimes \mathbb{C}^{d_J}$. Now let $X(U_J) = \frac{1}{d_J} \sum_{\psi_J'} \mathrm{Tr}[S_J \mathcal{D}_J^{\otimes 2} \mathcal{U}_J^{\otimes 2} (\Pi_{\psi_J'}^{\otimes 2})]$. We then have

$$\begin{aligned}
|X(U_J) - X(V_J)| &= \frac{1}{d_J} \left| \sum_{\psi_J'} \mathrm{Tr}\left[S_J \mathcal{D}_J^{\otimes 2} \left(\mathcal{U}_J^{\otimes 2} - \mathcal{V}_J^{\otimes 2} \right) \left(\Pi_{\psi_J'}^{\otimes 2} \right) \right] \right| \leqslant \frac{1}{d_J} \sum_{\psi_J'} \left| \mathrm{Tr}\left[S_J \mathcal{D}_J^{\otimes 2} \left(\mathcal{U}_J^{\otimes 2} - \mathcal{V}_J^{\otimes 2} \right) \left(\Pi_{\psi_J'}^{\otimes 2} \right) \right] \right| \\
&\leqslant \frac{1}{d_J} \sum_{\psi_J'} \|S\|_\infty \left\| \mathcal{D}_J^{\otimes 2} \left(\mathcal{U}_J^{\otimes 2} - \mathcal{V}_J^{\otimes 2} \right) \left(\Pi_{\psi_J'}^{\otimes 2} \right) \right\|_1 \leqslant \frac{1}{d_J} \sum_{\psi_J'} \left\| \mathcal{U}_J^{\otimes 2} \left(\Pi_{\psi_J'}^{\otimes 2} \right) - \mathcal{V}_J^{\otimes 2} \left(\Pi_{\psi_J'}^{\otimes 2} \right) \right\| \\
&\leqslant \frac{1}{d_J} \sum_{\psi_J'} 2 \left\| U_J^{\otimes 2} |\psi_J'\rangle^{\otimes 2} - V_J^{\otimes 2} |\psi_J'\rangle^{\otimes 2} \right\| \leqslant \frac{1}{d_J} \sum_{\psi_J'} 2 \left\| U_J^{\otimes 2} - V_J^{\otimes 2} \right\|_\infty = 2 \left\| U_J^{\otimes 2} - V_J^{\otimes 2} \right\|_\infty \\
&= 2 \left\| \mathbb{1} - U_J^{\dagger \otimes 2} V_J^{\otimes 2} \right\|_\infty,
\end{aligned} \tag{A13}$$

where to go from line 1 to line 2 we used triangle inequality, from line 2 to line 3 Eq. (A9) with $A = \mathcal{D}_J^{\otimes 2}(\mathcal{U}_J^{\otimes 2} - \mathcal{V}_J^{\otimes 2})(\Pi_{\psi_J'}^{\otimes 2})$, $B = S$, from line 3 to line 4 the fact that $\|S\|_\infty = 1$ and Eq. (A10) with $\mathcal{T} = \mathcal{D}_J$, from line 4 to line 5 Eq. (A11) with $|\psi\rangle = U_J^{\otimes 2}|\psi_J'\rangle^{\otimes 2}$ and $|\phi\rangle = V_J^{\otimes 2}|\psi_J'\rangle^{\otimes 2}$, and in line 5 the definition of the operator norm. Let us define $M := \mathbb{1}_{d_J} - U_J^\dagger V_J$. Then

$$\begin{aligned} |X(U_J) - X(V_J)| &\leqslant 2\|\mathbb{1} - (\mathbb{1} + M \otimes M - \mathbb{1}_{d_J} \otimes M - M \otimes \mathbb{1}_{d_J})\|_\infty \\ &\leqslant 2\big(2\|M\|_\infty + \|M\|_\infty^2\big) \leqslant 8\|M\|_\infty = 8\|U_J - V_J\|_\infty \leqslant 8\|U_J - V_J\|_2, \end{aligned} \tag{A14}$$

where we used that $\|M\|_\infty \leqslant 2$. So $X : U(\mathbb{C}^{d_J}) \mapsto \mathbb{R}$ is Lipschitz continuous with constant $\lambda = 8$ and the result follows from Lemma 1.

4. System dimension scalings

The scaling of $\overline{f_{B_{t\text{-}J_z}}(U_t)}^t$ and $\overline{f_{B_{\rm TL}}(U_t)}^t$ in Sec. II C is found by finding the best exponential-law $[f(x) = Ax^B]$ fit of the numerical data. Specifically, we find $\overline{f_{B_{t\text{-}J_z}}(U_t)}^t \sim d^\lambda$ with $\lambda = 0.8340 \pm 0.0023$ and a root mean square error is 2.989 and $\overline{f_{B_{\rm TL}}(U_t)}^t \sim d^\lambda$ with $\lambda = 0.9468 \pm 0.0016$ and a root mean square error of 2.192.

The scalings of $K_{t\text{-}J_z}$ and $K_{\rm TL}$ follow directly from the expressions $K_{t\text{-}J_z} = 2^{L+1} - 1$ and $K_{\rm TL} = \frac{q^{-L}(q^{L+2}-1)^2}{(q^2-1)^2}$, where in both cases $L = \log_3 d$. Explicitly,

$$\begin{aligned} K_{t\text{-}J_z} &= 2^{L+1} - 1 = 2^{\log_3 d + 1} - 1 \sim 2^{\log_3 d} = d^{\log_3 2}, \\ K_{\rm TL} &= \frac{q^{-L}(q^{L+2} - 1)^2}{(q^2 - 1)^2} \sim q^L = q^{\log_3 d} = d^{\log_3 q}. \end{aligned}$$

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