Hybrid completely positive Markovian quantum-classical dynamics

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A concise and self-contained derivation of hybrid quantum-classical dynamics is given in terms of Markovian master equations. Many previously known results are rederived and revised and some of them are completed or corrected. Using a method as simple as possible, our goal is a brief introduction to the state of the art of hybrid dynamics, with a limited discussion of the implications for foundations and without discussion of further relevance in the measurement problem, quantum gravity, chemistry, numeric methods, etc. Hybrid dynamics is defined as a special case of composite quantum dynamics where the observables of one of the two subsystems are restricted to the commuting set of diagonal operators in a fixed basis. With this restriction, the derivation of hybrid dynamical equations is clear conceptually and simple technically. Jump and diffusive dynamics follow in the form of hybrid master equations. Their stochastic interpretation (called unravelings) is derived. We discuss gauge-type ambiguities, problems of uniqueness, and covariance of the diffusive master equation. Also conditions of minimum noise and of monitoring the quantum trajectory are derived. We conclude that the hybrid formalism is equivalent to the standard Markovian theory of time-continuous quantum measurement (monitoring) on the one hand and is a motivating alternative formalism on the other.

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I. INTRODUCTION

In the real world, quantum and classical phenomena coexist and evolve according to their own rules. They do interact, of course, and we know very well the mathematical models of some particular interactions. The action of a classical system on the quantum one is modeled if we make the Hamiltonian $\hat{H}$ depend on the classical system's variables x. The backaction, i.e., the quantum system's impact on a classical one, is also known well from the quantum measurement: The quantum system rules the pointer position x of a classical meter. This backaction is extremely specific. More general backaction is the central problem to understand and to describe mathematically when quantum and classical systems are interacting.

The central object of composite quantum-classical systems is the hybrid state, represented by the hybrid density [1]

$$\hat{\rho}(x) = \hat{\rho}_x \rho(x), \tag{1}$$

where $\rho(x)$ is the normalized probability density of the classical variables x and $\hat{\rho}_x$ is the density operator of the quantum system conditioned on the value x of the classical variable.

There has been much effort put forth and many results concerning the possible evolution equations for $d\hat{\rho}(x)/dt$. The bottleneck is the backaction, although its elementary pattern is known from all quantum theory textbooks. The von Neumann measurement of the complete orthogonal set $\{\hat{P}_x\}$ of Hermitian projectors imposes the following change of the hybrid state (1):

$$\left\{\hat{\rho} \rightarrow \frac{\hat{P}_x\hat{\rho}\hat{P}_x}{\mathrm{tr}(\hat{P}_x\hat{\rho})} \text{ with probability } \rho(x) = \mathrm{tr}(\hat{P}_x\hat{\rho})\right\}$$
$$\Longleftrightarrow \{\hat{\rho}(x) \longrightarrow \hat{P}_x\hat{\rho}_x\hat{P}_x\}. \tag{2}$$

As we see, the textbook stochastic jumps are equivalent to a single deterministic map of the hybrid state. If we construct a stochastic dynamics underlying the process on the left-hand side, we have an equivalent deterministic hybrid dynamics for $\hat{\rho}(x)$. von Neumann's statistical interpretation of quantum states (also called the Born rule) follows from the statistical interpretation of the hybrid state.

For a long time, the dynamics of the von Neumann measurement, the left-hand side of (2), was missing from the textbooks, as it was considered irrelevant. Recently, for very different motivations, it was constructed in the continuous limit of discrete von Neumann measurements. The prevailing formalism has been Markovian stochastic equations, not the hybrid formalism. References [2–4] were milestones; reviews are in [5–9].

Will the hybrid dynamics, developed on their own, yield more than the time-continuous measurements in the hybrid formalism, i.e., time-continuous extension of the right-hand side of (2)? Before answering, we introduce the calculus of hybrid dynamics.

Many investigations of quite different concepts, motivations, methods, level of mathematical rigor, etc., have emerged in 40 years, from the earliest attempts to couple quantum and canonical classical systems [1,10,11] to the present author's work [12–19] and other's results [20–28] directly related to the present work, as well as many other contributions, e.g., in Refs. [29–41]. Here we use a unique approach of elementary derivations and economic presentation, in order to give a complete but concise account of Markovian hybrid quantum-classical dynamics modeled by hybrid master equations (HMEs). In Sec. II the canonical jump HME and its stochastic interpretation, called unraveling, are derived. From this HME, the limit of a continuous, diffusive HME is derived

(Sec. III), which, in Sec. IV, is put into the general covariant form. Conditions of minimum irreversibility, covariant and noncovariant unravelings, and monitoring are discussed as well. Comparison with and comments about previous results are made in Sec. V. Section VI summarizes lessons about hybrid dynamics. Appendix A provides an elementary example and Appendix B contains an application to canonical classical subsystems, with a simple example.

II. HYBRID MASTER EQUATION

Quantum theory is universal in our approach and classical systems are a special case of quantum ones: Operators are restricted for a commutative subset, i.e., they are diagonal in a fixed basis. Accordingly, consider the Hilbert space $\mathcal{H}_{\rm QC}=\mathcal{H}_{\rm Q}\otimes\mathcal{H}_{\rm C}$, where $\mathcal{H}_{\rm C}$ will host our classical system in a fixed basis $\{|x\rangle\}$. The composite quantum state $\widehat{\rho}$ and composite Hamiltonian $\widehat{H}$ (as well as the composite observables) are diagonal in the fixed basis:

$$\widehat{\rho}=\sum_x \hat{\rho}(x)\otimes|x\rangle\langle x|, \tag{3}$$

$$\widehat{H}=\sum_x \hat{H}(x)\otimes|x\rangle\langle x|. \tag{4}$$

The block diagonality of the objects $\hat{\rho}(x)$ and $\hat{H}(x)$ ensures classicality of the subsystem in $\mathcal{H}_{\rm C}$, i.e., that its state is and remains diagonal in the fixed basis. The block-diagonal objects $\hat{\rho}(x)$ and $\hat{H}(x)$ will be called the hybrid state and hybrid Hamiltonian, respectively. Note the notational difference: Operators on $\mathcal{H}_{\rm QC}$ have wider circumflexes than operators on $\mathcal{H}_{\rm Q}$. We are looking for a Markovian evolution equation for $\widehat{\rho}$, which is completely positive (CP) and preserves the block-diagonal form of $\widehat{\rho}$.

Since CP maps represent the most general quantum dynamics, we start from the CP map Λ of the composite state $\widehat{\rho}$ and request that it preserve block diagonality, i.e., classicality of the subsystem in $\mathcal{H}_{\rm C}$. Conveniently, we can write the general form of the map for the hybrid representation $\hat{\rho}(x)$ of $\widehat{\rho}$. With the Einstein convention of summation, it reads

$$\hat{\rho}_\Lambda(x)=\sum_y D_{\beta\alpha}(x,y)\hat{L}_\alpha\hat{\rho}(y)\hat{L}_\beta^\dagger, \tag{5}$$

where $\{\hat{L}_\alpha\}$ is an operator basis in $\mathcal{H}_{\rm Q}$. The Hermitian matrix D satisfies

$$\sum_x D_{\beta\alpha}(x,y)\hat{L}_\beta^\dagger\hat{L}_\alpha=\hat{I} \tag{6}$$

for all y. If we diagonalize D at each point (x,y), the form of the map becomes

$$\hat{\rho}_\Lambda(x)=\sum_{y,\alpha}\lambda_\alpha(x,y)\hat{L}_\alpha(x,y)\hat{\rho}(y)\hat{L}_\alpha^\dagger(x,y), \tag{7}$$

where $\{\hat{L}_\alpha(x,y)\}$ is an operator basis depending on (x,y). Hence all $\lambda_\alpha(x,y)\geqslant 0$ for all α and (x,y); otherwise the map Λ cannot be CP. Now we can absorb each factor λ_α into $\hat{L}_\alpha$ and return to the full operator formalism. We can write the CP map, preserving the block-diagonal form of $\widehat{\rho}$ in the standard quantum-mechanical form

$$\widehat{\rho}_\Lambda=\widehat{L}_\alpha\widehat{\rho}\widehat{L}_\alpha^\dagger, \tag{8}$$

with the linearly independent Kraus operators defined by

$$\widehat{L}_\alpha=\sum_y \hat{L}_\alpha(x,y)\otimes|x\rangle\langle y|, \tag{9}$$

where $\hat{L}_1(x,y),\hat{L}_2(x,y),\ldots$ are linearly independent. If Λ is a semigroup and generates time evolution of $\widehat{\rho}$ then, according to the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) theorem [42,43], the evolution is governed by the quantum master equation of the form

$$\frac{d\widehat{\rho}}{dt}=-i[\widehat{H},\widehat{\rho}]+\widehat{L}_\alpha\widehat{\rho}\widehat{L}_\alpha^\dagger-\mathbb{H}\widehat{L}_\alpha^\dagger\widehat{L}_\alpha\widehat{\rho}, \tag{10}$$

where the $\widehat{L}_\alpha$'s are called the Lindblad generators. They can (and will) be chosen such that $\widehat{I},\widehat{L}_1,\widehat{L}_2,\ldots$ are linearly independent. As expected, this quantum master equation preserves block diagonality by construction; hence it obtains a closed equation in the hybrid formalism [19–22,25,27,28]

$$\frac{d\hat{\rho}(x)}{dt}=-i[\hat{H}(x),\hat{\rho}(x)]+\sum_y[\hat{L}_\alpha(x,y)\hat{\rho}(y)\hat{L}_\alpha^\dagger(x,y)-\mathbb{H}\hat{L}_\alpha^\dagger(y,x)\hat{L}_\alpha(y,x)\hat{\rho}(x)], \tag{11}$$

where $\mathbb{H}$ means the Hermitian part of the subsequent expression. This is the canonical form of the CP Markovian HME where the classical system is discrete. The hybrid generators are only restricted by their linear independence and their linear independence of $\hat{I}\delta(x,y)$ as well. Their dependence on (x,y) can otherwise be arbitrary. Throughout our work $\delta(x,y)$ will denote the usual discrete delta function. The backaction is encapsulated by the generators $\hat{L}_\alpha(x,y)$. Note the basic lesson: When the quantum-classical interaction is mutual influence between the subsystems the hybrid system can never be reversible; the evolution is governed by hybrid master (kinetic) equations. Quantitative lower bounds on their irreversibility (noise) will be derived in Sec. III.

The choice of the hybrid Hamiltonian and the hybrid generators is unique up to arbitrary complex functions $\ell_\alpha(x)$ since the HME (11) is invariant for the following shifts:

$$\hat{L}_\alpha(x,y)\to\hat{L}_\alpha(x,y)+\ell_\alpha(x)\delta(x,y),$$
$$\hat{H}(x)\to\hat{H}(x)-\frac{i}{2}[\ell_\alpha^\star(x)\hat{L}_\alpha(x,x)-\text{H.c.}]. \tag{12}$$

Special cases may allow much larger groups of such gauge freedom when the dependence of the generators on (x,y) is degenerate, like in the continuous HMEs of Secs. III and IV.

A. Stochastic unraveling

Master equations are deterministic. Like any quantum or classical master equation, also our HME (11) possesses statistical interpretation. The solution $\rho(x)$ of a classical master (kinetic) equation can be decomposed into a unique stochastic process x_t of random trajectories. Quantum master equations can be decomposed into stochastic processes ψ_t of random quantum trajectories; the decomposition is called unraveling. Quantum unravelings are not unique. Here we define the hybrid of the classical and quantum stochastic decompositions (unravelings).

Let us formulate the mathematical condition of hybrid unraveling. If $\hat{\rho}(z,t)$ is the solution of the HME (11) then it is the stochastic mean $\mathbb{M}$ over the contribution of hybrid trajectories (x_t, ψ_t),

$$\hat{\rho}(z, t) = \mathbb{M}\psi_t\psi_t^\dagger\delta(z, x_t), \tag{13}$$

where the x_t and ψ_t are correlated jump stochastic processes, one in the space of classical coordinates and the other in the Hilbert space of state vectors. Consider first the unique unraveling of the distribution $\rho(x) = \mathrm{tr}\hat{\rho}(x)$ of the classical subsystem. The trace of the HME (11) yields the classical master (kinetic) equation

$$\frac{d\rho(x)}{dt} = \sum_\alpha\sum_y[T_\alpha(x, y)\rho(y) - T_\alpha(y, x)\rho(x)], \tag{14}$$

with the ψ-dependent transition (jump) rate from x to y for each α:

$$T_\alpha(y, x) = \mathrm{tr}[\hat{L}_\alpha(y, x)\hat{\rho}_x\hat{L}_\alpha^\dagger(y, x)] = \langle\hat{L}_\alpha^\dagger(y, x)\hat{L}_\alpha(y, x)\rangle_x. \tag{15}$$

We introduce the total transition rate from x:

$$T(x) = \sum_\alpha\sum_y T_\alpha(y, x). \tag{16}$$

To unravel the quantum subsystem, we need the anti-Hermitian (frictional) hybrid Hamiltonian defined by

$$-i\hat{H}_{\mathrm{fr}}(x) = -\frac{1}{2}\sum_y\hat{L}_\alpha^\dagger(y, x)\hat{L}_\alpha(y, x). \tag{17}$$

The corresponding frictional Schrödinger equation is

$$\frac{d\psi}{dt} = \left(-i\hat{H}(x) - i\hat{H}_{\mathrm{fr}}(x) + \frac{1}{2}T(x)\right)\psi, \tag{18}$$

where $\frac{1}{2}T(x)$ restores the norm $\psi^\dagger\psi$ since $\frac{1}{2}T(x) = -\mathrm{Im}\langle\hat{H}_{\mathrm{fr}}\rangle$ [cf. Eqs. (15)–(17)], with Im denoting the imaginary part.

The hybrid unraveling consists of the following two correlated jump stochastic (piecewise deterministic) processes, one for x_t and the other for ψ_t (cf. [28]),

$$x = \mathrm{const}, \tag{19}$$

$$\frac{d\psi}{dt} = \left(-i\hat{H}(x) - i\hat{H}_{\mathrm{fr}}(x)\psi + \frac{1}{2}T(x)\right)\psi, \tag{20}$$

and for jumps,

$$x \to x', \psi \to \frac{\hat{L}_\alpha(x, x')\psi}{\sqrt{T_\alpha(x', x)}} \tag{21}$$

at rate $T_\alpha(x', x)$. The proof that the unraveling satisfies the condition (13) is the following. With the notation $\hat{P} = \psi\psi^\dagger$, the condition reads

$$d\hat{\rho}(z, t) = d\mathbb{M}\hat{P}_t\delta(z, x_t), \tag{22}$$

where $d\hat{\rho}(z, t)$ is determined by the HME (11). The unraveling yields two terms for the change of $\hat{P}\delta(z, x)$ in time dt:

$$\begin{aligned}\hat{P}\delta(z, x) \to\ & [1 - T(x)dt)]\{\hat{P} - i[\hat{H}(x), \hat{P}]dt \\ & - i[\hat{H}_{\mathrm{fr}}(x), \hat{P}]_+dt + T(x)\hat{P}dt\}\delta(z, x) \\ & + \sum_\alpha T_\alpha(x', x)dt\frac{\hat{L}_\alpha(x, x')\hat{P}\hat{L}_\alpha^\dagger(x, x')}{T_\alpha(x', x)}\delta(z, x').\end{aligned} \tag{23}$$

The first term comes from the deterministic equations (19) and (20) and the second term represents the average of jumps (21). The ψ-dependent transition rates $T_\alpha(x', x)$ and $T(x)$ cancel and we are are left with an expression linear in $\hat{P}\delta(z, x)$. Taking its mean results in

$$\begin{aligned}\frac{d}{dt}\mathbb{M}\hat{P}\delta(z, x) =\ & \mathbb{M}\{-i[\hat{H}(x), \hat{P}] - i[\hat{H}_{\mathrm{fr}}(x), \hat{P}]_+\}\delta(z, x) \\ & + \mathbb{M}\hat{L}_\alpha(x, x')\hat{P}\hat{L}_\alpha^\dagger(x, x')\delta(z, x').\end{aligned} \tag{24}$$

If we recall the expression (17) of $\hat{H}_{\mathrm{fr}}$ then we can recognize that $\mathbb{M}\hat{P}\delta(z, x)$ satisfies the HME (11).

The unraveling (19)–(21) is ambiguous. The shifts (12) leave the HME (11) invariant, but we get different unravelings. Nevertheless, the unraveling becomes invariant and unique with the replacement (cf. [44])

$$\hat{L}_\alpha(x, y) \to \hat{L}_\alpha(x, y) - \langle\hat{L}_\alpha(x, y)\rangle\delta(x, y). \tag{25}$$

One would think that in Eq. (21) the states ψ_t and x_t are jumping together. This is not true if $\hat{L}_\alpha(x, x) \neq 0$ since $\hat{L}_\alpha(x, x)$ generates a nontrivial jump of ψ_t and no jump for x_t. So to synchronize the jumps of ψ and x either we consider the invariant modification (25) of the unraveling or we just request that $\hat{L}_\alpha(x, x)$ vanishes for all α and x.

B. Monitoring the quantum trajectory

The quantum trajectory ψ_t is not observable in general since its detection inevitably perturbs it. That is the problem of monitoring the quantum system. Can we, just by monitoring the classical subsystem's x_t, monitor the evolution of ψ_t? Generally we cannot. If the number of generators $\hat{L}_\alpha(x, y)$ is more than one, detecting a jump of x_t does not tell us which generator made ψ_t jump. The jump of x_t leaves the jump of ψ_t in Eq. (21) undetermined. This ambiguity can be fixed in a natural class of special hybrid systems.

Let us add a vectorial structure $\{x^\alpha\}$ to the classical discrete space and assume the specific form of the hybrid generators $\hat{L}_\alpha(x, y)$,

$$\hat{L}_\alpha(x^\alpha, y^\alpha)\prod_{\beta\neq\alpha}\delta(x^\beta, y^\beta), \tag{26}$$

also with $\hat{L}_\alpha(x^\alpha, x^\alpha) = 0$. The jump rates $T_\alpha(x^\alpha, y^\beta)$ vanish if $\alpha \neq \beta$ [cf. Eq. (15)]. Now if we observe a jump of x_t^α we can uniquely determine the jump of ψ_t. With the vectorized classical variables x^α, the unraveling (19)–(21) of the HME (11) becomes unique and the quantum trajectory can be monitored.

III. FROM THE DISCRETE TO THE DIFFUSIVE HYBRID MASTER EQUATION

Although discrete classical systems are important, most classical systems of interest, like the Hamiltonian ones, are continuous. Therefore, we are going to construct the continuous limit of the obtained discrete HME (11). As is known, the only continuous classical Markovian process is diffusion with additional deterministic drift. We start with the generic HME (11) on the discrete subset $\{x^n\}$ of the multidimensional continuum and generate a diffusion process in the continuous limit $\epsilon \to 0$.

Note that some hybrid generators can be of the form $\hat{L}_\alpha(y,x)=\hat{I}L_\alpha(x,y)\equiv L_\alpha(x,y)$, where $\hat{I}$ is the unit operator (which we have omitted in the notation). For some such classical generators we introduce a new label n and the classical generators $L_n(x,y)$. So we have hybrid generators $\hat{L}_\alpha(x,y)$ and the classical ones $L_n(x,y)$. The classical terms are intended to generate diffusion terms for the classical subsystem in the continuous limit. The two classes of generators will be chosen as follows:

$$\hat{L}_\alpha(x,y)=\hat{L}_\alpha(y,y)\prod_n\frac{\delta(x-y,\epsilon)+\delta(y-x,\epsilon)}{\sqrt{2}}, \quad (27)$$

$$L_n(x,y)=\frac{\delta(x^n-y^n,\epsilon)-\delta(y^n-x^n,\epsilon)}{\sqrt{2}\epsilon}\times\prod_{m\neq n}\frac{\delta(x^m-y^m,\epsilon)+\delta(y^m-x^m,\epsilon)}{\sqrt{2}}. \quad (28)$$

We introduce the positive-semidefinite complex decoherence matrix $D_\mathrm{Q}^{\alpha\beta}$, the positive-semidefinite real diffusion matrix D_C^{nm}, and the arbitrary complex matrix $G_\mathrm{CQ}^{n\alpha}$ of backaction. Consider the following Hermitian block matrix and let it be positive semidefinite:

$$\mathcal{D}=\begin{bmatrix}D_\mathrm{Q} & G_\mathrm{CQ}^\dagger\\ G_\mathrm{CQ} & D_\mathrm{C}\end{bmatrix}\geqslant 0. \quad (29)$$

We can see that nonzero backaction will require both nonzero decoherence and diffusion. In addition to the constraints $D_\mathrm{Q}\geqslant 0$ and $D_\mathrm{C}=\overline{D}_\mathrm{C}\geqslant 0$ that we always take for granted, there are further constraints on matrices D_Q, D_C, and G_CQ, equivalent to (29), to be shown in Sec. IV A.

The following HME generates CP maps [since it reduces to the HME (11) if we diagonalize $\mathcal{D}$]:

$$\frac{d\hat{\rho}(x)}{dt}=-i[\hat{H}(x),\hat{\rho}(x)]+D_\mathrm{Q}^{\beta\alpha}\sum_y[\hat{L}_\alpha(x,y)\hat{\rho}(y)\hat{L}_\beta^\dagger(x,y)-\mathbb{H}\hat{L}_\beta^\dagger(x,y)\hat{L}_\alpha(y,x)\hat{\rho}(x)]+D_\mathrm{C}^{nm}\sum_y[L_n(x,y)L_m(x,y)\hat{\rho}(y)-L_n(y,x)L_m(y,x)\hat{\rho}(x)]+\overline{G}_\mathrm{CQ}^{n\alpha}\sum_y[L_n(x,y)\hat{L}_\alpha(x,y)\hat{\rho}(y)-L_n(y,x)\mathbb{H}\hat{L}_\alpha(x,y)\hat{\rho}(x)]. \quad (30)$$

In the continuous limit $\epsilon\to 0$, the terms with $\hat{L}_\alpha$ and $\hat{L}_\beta$ contribute to standard GKLS structures

$$\hat{L}_\alpha(x)\hat{\rho}(x)\hat{L}_\beta^\dagger(x)-\mathbb{H}\hat{L}_\beta^\dagger(x)\hat{L}_\alpha(x)\hat{\rho}(x), \quad (31)$$

with $\hat{L}_\alpha(x)\equiv\hat{L}_\alpha(x,x)$. The terms with L_n and L_m yield diffusion of the classical variable x. With ϵ_n denoting a vector whose only nonzero component is the nth one (which is ϵ), the yield at $n\neq m$ reads

$$\frac{\hat{\rho}(x+\epsilon_n+\epsilon_m)+\hat{\rho}(x-\epsilon_n-\epsilon_m)-\hat{\rho}(x-\epsilon_n+\epsilon_m)-\hat{\rho}(x+\epsilon_n-\epsilon_m)}{2\epsilon^2}\underset{\epsilon\to 0}{\longrightarrow}\frac{1}{2}\partial_n\partial_m\hat{\rho}(x), \quad (32)$$

where $\partial_n\equiv\partial/\partial x^n$. We get a similar yield for $n=m$:

$$\tfrac{1}{2\epsilon^2}[\hat{\rho}(x+\epsilon_n)+\hat{\rho}(x-\epsilon_n)-2\hat{\rho}(x)]\underset{\epsilon\to 0}{\longrightarrow}\tfrac{1}{2}\partial_n\partial_n\hat{\rho}(x). \quad (33)$$

The cross terms with $\hat{L}_\alpha$ and L_n generate the nontrivial backaction of the quantum system on the classical part:

$$\tfrac{1}{2\epsilon}[\hat{L}_\alpha(x+\epsilon_n)\hat{\rho}(x+\epsilon_n)-\hat{L}_\alpha(x-\epsilon_n)\hat{\rho}(x-\epsilon_n)]+\mathrm{H.c.}\underset{\epsilon\to 0}{\longrightarrow}\partial_n[\hat{L}_\alpha(x)\hat{\rho}(x)]+\mathrm{H.c.} \quad (34)$$

Using these limits in Eq. (30) and adding a deliberate classical drift of velocity $V(x)$, we obtain the continuous limit of the discrete HME (11):

$$\frac{d\hat{\rho}(x)}{dt}=-i[\hat{H}(x),\hat{\rho}(x)]+D_\mathrm{Q}^{\beta\alpha}[\hat{L}_\alpha(x)\hat{\rho}(x)\hat{L}_\beta^\dagger(x)-\mathbb{H}\hat{L}_\beta^\dagger(x)\hat{L}_\alpha(x)\hat{\rho}(x)]+\frac{1}{2}D_\mathrm{C}^{nm}\partial_n\partial_m\hat{\rho}(x)+\{\overline{G}_\mathrm{CQ}^{n\alpha}\partial_n[\hat{L}_\alpha(x)\hat{\rho}(x)]+\mathrm{H.c.}\}-\partial_n[V^n(x)\hat{\rho}(x)]. \quad (35)$$

The three constant parameter matrices D_Q, D_C, and G_CQ are constrained by the semidefiniteness $\mathcal{D}\geqslant 0$ of the block matrix (29) formed by them.

The above HME (35) is not yet the general diffusive one. Obviously, we can add an extra x-dependent decoherence $\Delta D_\mathrm{Q}(x)$ as well as an extra diffusion $\Delta D_\mathrm{C}(x)$ as long as $\mathcal{D}\geqslant$ holds true after the replacements $D_\mathrm{Q}\to D_\mathrm{Q}+\Delta D_\mathrm{Q}(x)$ and $D_\mathrm{C}\to D_\mathrm{C}+\Delta D_\mathrm{C}(x)$, that is, the validity of the diffusive HME (35) extends for x-dependent parameters $D_\mathrm{Q}(x)$ and $D_\mathrm{C}(x)$ provided $\mathcal{D}(x)\geqslant 0$. In the next section we show that also the backaction matrix G_CQ can depend on x.

IV. COVARIANT HYBRID MASTER EQUATION

The form (35) of the HME is explicitly covariant under the global linear transformations of the operator basis $\hat{L}_\alpha(x)$ and the classical variables x. We are interested in the explicitly covariant form under local, i.e., x-dependent complex linear transformation of the operator basis and under general coordinate transformations of x. The form of such a HME (35) should be (cf. [25])

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + D_{\mathrm{Q}}^{\beta\alpha}(\hat{L}_\alpha\hat{\rho}\hat{L}_\beta^\dagger - \mathbb{H}\hat{L}_\beta^\dagger\hat{L}_\alpha\hat{\rho}) + \frac{1}{2}\partial_n\partial_m(D_{\mathrm{C}}^{nm}\hat{\rho}) + \partial_n(\overline{G}_{\mathrm{CQ}}^{n\alpha}\hat{L}_\alpha\hat{\rho} + \mathrm{H.c.}) - \partial_n(V^n\hat{\rho}), \tag{36}$$

where every object is a function of x, a fact that our above notation hides for the sake of compactness. The coefficients $D_{\mathrm{Q}}(x)$, $D_{\mathrm{C}}(x)$, and $G_{\mathrm{CQ}}(x)$ satisfy the same constraint (29) $\mathcal{D}(x) \geqslant 0$ as before, now understood for all x:

$$\mathcal{D}(x) = \left[\begin{array}{c|c} D_{\mathrm{Q}}^{\alpha\beta}(x) & \overline{G}_{\mathrm{CQ}}^{m\beta}(x) \\ \hline G_{\mathrm{CQ}}^{n\alpha}(x) & D_{\mathrm{C}}^{nm}(x)\end{array}\right] \geqslant 0. \tag{37}$$

We prove the equivalence of the covariant HME (36) to (35). By a suitable choice of the operator basis $\hat{L}_\alpha(x)$ and the classical coordinates x^n, one can always transform $G_{\mathrm{CQ}}(x)$ into a constant matrix. This results in the HME (35), which, as we argued there, is valid for x-dependent D_{Q} and D_{C}.

The covariant diffusive HME (36) is the most general continuous HME to generate CP dynamics. Every object in it is x dependent. The condition (37) is necessary and sufficient when the generators $\hat{L}_\alpha(x)$ are linearly independent and also linearly independent of the unit operator $\hat{I}$. The hybrid Hamiltonian $\hat{H}$ and the classical drift V are arbitrary. The Hermitian matrix $D_{\mathrm{Q}} \geqslant 0$ of decoherence and the real matrix $D_{\mathrm{C}} \geqslant 0$ of diffusion must form the positive-semidefinite block matrix (37) in corners with the matrix G_{CQ} (and $G_{\mathrm{CQ}}^\dagger$) of backaction. We list three useful alternatives that can always be achieved by transformation of the reference frames: a fixed operator basis $\{\hat{L}_\alpha\}$, simultaneously diagonal D_{Q} and D_{C} with zeros and ones, or G_{CQ} with zeros and ones in the main diagonal and zeros elsewhere. (These coordinate transformations may request the embedding of x in higher dimensions than they are of.) In addition to the explicit covariance, there is a further gauge freedom, the descendant of the shifts (12) in the discrete HME,

$$\begin{aligned} \hat{L}_\alpha &\to \hat{L}_\alpha + \ell_\alpha, \\ \hat{H} &\to \hat{H} - \frac{i}{2}(\ell_\alpha^\star\hat{L}_\alpha - \mathrm{H.c.}), \\ V^n &\to V^\alpha - (\overline{G}_{\mathrm{CQ}}^{n\alpha}\ell_\alpha + \mathrm{H.c.}), \end{aligned} \tag{38}$$

where $\ell_\alpha(x)$ is an arbitrary complex function.

A. Minimum-noise threshold

We are going to break down the condition $\mathcal{D} \geqslant 0$ (37) into constraints between $\mathcal{D}$'s building blocks. The necessary conditions are

$$\mathrm{range}D_{\mathrm{Q}} \geqslant \mathrm{range}G_{\mathrm{CQ}}^\dagger G_{\mathrm{CQ}}, \tag{39}$$

$$\mathrm{range}D_{\mathrm{C}} \geqslant \mathrm{range}G_{\mathrm{CQ}}G_{\mathrm{CQ}}^\dagger. \tag{40}$$

They express that the range of decoherence D_{Q} cannot be narrower than the range of $\hat{L}_\alpha$ that are coupled to the classical x^n by backaction G_{CQ}. Similarly, the range of classical diffusion D_{C} cannot be smaller than the range of x^n that are coupled to the $\hat{L}_\alpha$. Nonzero backaction means mandatory noise: both decoherence and diffusion.

Suppose that we have a given nonzero matrix G_{CQ} of backaction and we are interested in a certain minimum of the total irreversibility, i.e., a certain minimum of the block matrix $\mathcal{D}$, implying a certain minimum of the decoherence D_{Q} and diffusion D_{C}, as we will see below. Obviously, the strict positivity $\mathcal{D} > 0$ means more noise than the minimum. We are interested in the maximum degenerate $\mathcal{D}$ meaning the lowest rank$\mathcal{D}$ which is limited by the rank $r_{\mathrm{CQ}} = \mathrm{rank}G_{\mathrm{CQ}}$; otherwise $\mathcal{D} \geqslant 0$ cannot be true. Therefore, we define the minimum-noise threshold by

$$\mathrm{rank}\mathcal{D} = \mathrm{rank}G_{\mathrm{CQ}}. \tag{41}$$

Then the inequalities (39) saturate, $\mathrm{range}D_{\mathrm{Q}} = \mathrm{range}G_{\mathrm{CQ}}^\dagger G_{\mathrm{CQ}}$ and $\mathrm{range}D_{\mathrm{C}} = \mathrm{range}G_{\mathrm{CQ}}G_{\mathrm{CQ}}^\dagger$, also meaning that $\mathrm{rank}D_{\mathrm{Q}} = \mathrm{rank}D_{\mathrm{C}} = r_{\mathrm{CQ}}$. Both the number of coupled independent generators $\hat{L}_\alpha$ and coordinates x^n coincide with r_{CQ}.

For a given x, transform G_{CQ} into the frame where its elements are zero except for an $r_{\mathrm{CQ}} \times r_{\mathrm{CQ}}$ unit matrix in the top left corner. Then, because of (39), both D_{Q} and D_{C} must have an $r_{\mathrm{CQ}} \times r_{\mathrm{CQ}}$ strictly positive matrix in their top left corners and zeros elsewhere. Recall the general condition $\mathcal{D} \geqslant 0$. If we drop rows and columns of zeros then the said nonzero $r_{\mathrm{CQ}} \times r_{\mathrm{CQ}}$ submatrices, denoted invariably, must be each other's inverses [12]:

$$D_{\mathrm{C}}(x)D_{\mathrm{Q}}(x) = I. \tag{42}$$

This is the condition of the minimum-noise threshold (41) in the special frame fitted to the backaction matrix.

We can now identify quantitatively and directly the lower bound of irreversibility that hybrid systems must undergo even if the quantum and classical subsystems were reversible in themselves. At a given backaction strength G_{CQ} (scaled now to be unity) and at the minimum-noise threshold, the quantum decoherence strength D_{Q} and classical diffusion strength D_{C} are inverses of each other; lower decoherence requests higher diffusion and vice versa.

It is possible to decompose the constraint $\mathcal{D} \geqslant 0$ as well as the minimum-noise condition (41) into covariant relationships (i.e., valid in any reference frame) for the three parameter matrices. Two equivalent forms of $\mathcal{D} \geqslant 0$ are [25]

$$G_{\mathrm{CQ}}\frac{1}{D_{\mathrm{Q}}}G_{\mathrm{CQ}}^\dagger \leqslant D_{\mathrm{C}}, \tag{43}$$

$$G_{\mathrm{CQ}}\frac{1}{D_{\mathrm{Q}}}D_{\mathrm{Q}}G_{\mathrm{CQ}}^\dagger = G_{\mathrm{CQ}}G_{\mathrm{CQ}}^\dagger \tag{44}$$

and

$$G_{\mathrm{CQ}}^\dagger\frac{1}{D_{\mathrm{C}}}G_{\mathrm{CQ}} \leqslant D_{\mathrm{Q}}, \tag{45}$$

$$G_{\mathrm{CQ}}^\dagger\frac{1}{D_{\mathrm{C}}}D_{\mathrm{C}}G_{\mathrm{CQ}} = G_{\mathrm{CQ}}^\dagger G_{\mathrm{CQ}}, \tag{46}$$

where $1/D_{\mathrm{Q}}$ and $1/D_{\mathrm{C}}$ are generalized inverses. The threshold condition $\mathrm{rank}\mathcal{D} = \mathrm{rank}G_{\mathrm{CQ}}$ of minimum noise corresponds to the saturation of inequalities into equalities. Note that $(1/D_{\mathrm{Q}})D_{\mathrm{Q}} = \mathrm{range}D_{\mathrm{Q}}$ and $(1/D_{\mathrm{C}})D_{\mathrm{C}} = \mathrm{range}D_{\mathrm{C}}$; hence the second lines in Eqs. (43) and (45) correspond to the mentioned identities $\mathrm{range}D_{\mathrm{Q}} = \mathrm{range}G_{\mathrm{CQ}}^\dagger G_{\mathrm{CQ}}$ and

$\mathrm{range}D_{\mathrm{C}}=\mathrm{range}G_{\mathrm{CQ}}G_{\mathrm{CQ}}^{\dagger}$, respectively. Under the above covariant parametric constraints, the covariant HME (36) and its unravelings (49) and (50) include all possible dynamics that contain noise on and above the threshold of consistency. (Appendix A shows the sharpness of the above conditions on a simplest HME.)

B. Stochastic unravelings

When we construct unravelings of the HME (36), we follow the steps of Sec. II A and construct the two correlated stochastic processes for x_t and ψ_t satisfying the condition of unraveling (13). The two processes will be diffusive this time. First, take the trace of the diffusive HME (35) and obtain the classical Fokker-Planck equation of the classical subsystem

$$\frac{d\rho(x)}{dt}=\frac{1}{2}D_{\mathrm{C}}^{nm}\partial_n\partial_m\rho(x)-\partial_n\big[V^n(x)\rho(x)-2\,\mathbb{R}\overline{G}_{\mathrm{CQ}}^{n\alpha}(x)\langle\hat{L}_\alpha(x)\rangle\rho(x)\big],\tag{47}$$

where $\mathbb{R}$ denotes the real part of the subsequent expression. The unraveling of this equation will be a unique Brownian motion with a unique ψ-dependent drift. To unravel the quantum subsystem, we need the anti-Hermitian (frictional) hybrid Hamiltonian defined by

$$-i\hat{H}_{\mathrm{fr}}(x)=-\tfrac{1}{2}D_{\mathrm{Q}}^{\beta\alpha}\{[\hat{L}_\alpha^\dagger(x)-\langle\hat{L}_\alpha^\dagger(x)\rangle][\hat{L}_\beta(x)-\langle\hat{L}_\beta(x)\rangle]+[\langle\hat{L}_\alpha^\dagger(x)\rangle\hat{L}_\beta(x)-\mathrm{H.c.}]\}.\tag{48}$$

The hybrid unraveling consists of two correlated diffusive stochastic processes, one for x_t and the other for ψ_t,

$$dx^n=V^n(x)dt-2\,\mathbb{R}\overline{G}_{\mathrm{CQ}}^{n\alpha}(x)\langle\hat{L}_\alpha(x)\rangle dt+dW^n(x),\tag{49}$$

$$d\psi=-i[\hat{H}(x)+\hat{H}_{\mathrm{fr}}(x)]\psi dt+(\hat{L}_\alpha(x)-\langle\hat{L}_\alpha(x)\rangle)\psi d\overline{\xi}^\alpha(x),\tag{50}$$

where $dW^n(x)$ is real and $d\xi^\alpha(x)$ is complex zero-mean Itô differential of auxiliary stochastic processes, correlated as follows:

$$dW^n dW^m=D_{\mathrm{C}}^{nm}dt,\qquad d\xi^\alpha d\overline{\xi}^\beta=D_{\mathrm{Q}}^{\alpha\beta}dt,\qquad dW^n d\xi^\alpha=G_{\mathrm{CQ}}^{n\alpha}dt.\tag{51}$$

Let us use the vector symbols dW and $d\xi$, then we get the equivalent compact form of correlations:

$$\begin{bmatrix} d\xi d\xi^\dagger & d\xi dW^T\\ dW d\xi^\dagger & dW dW^T\end{bmatrix}=\mathcal{D}dt.\tag{52}$$

We prove that the unraveling (49) and (50) satisfies the condition (13). Like in Sec. II A, we use the denotation $\hat{P}=\psi\psi^\dagger$ and the same form (22) of the condition to be proved:

$$d\hat{\rho}(z,t)=d\mathbb{M}\hat{P}_t\delta(z-x_t).\tag{53}$$

The Itô differential on the right-hand side contains three terms which we are going to express by the equations of $d\psi$ and dx of the unraveling (see also [27]). First, to calculate $d\hat{P}=d\psi\psi^\dagger+\psi d\psi^\dagger+d\psi d\psi^\dagger$ we use the stochastic equation (50) of $d\psi$:

$$d\hat{P}=-i[\hat{H}(z),\hat{P}]dt+D_{\mathrm{Q}}^{\beta\alpha}[\hat{L}_\alpha(x)\hat{P}\hat{L}_\beta^\dagger(x)-\mathbb{H}\hat{L}_\beta^\dagger(x)\hat{L}_\alpha(x)\hat{P}]dt+\{[\hat{L}_\alpha(x)-\langle\hat{L}_\alpha(x)\rangle]\hat{P}d\overline{\xi}^\alpha(x)+\mathrm{H.c.}\}.\tag{54}$$

Second, we calculate $d\delta(z-x)$ using the stochastic equation (49) of dx,

$$d\delta(z-x)=\tfrac{1}{2}D_{\mathrm{C}}^{nm}\partial_n\partial_m\delta(z-x)dt+\big[V^n(x)-2\,\mathbb{R}\overline{G}_{\mathrm{CQ}}^{n\alpha}\langle\hat{L}_\alpha(x)\rangle\big]\partial_n\delta(z-x)dt+[\partial_n\delta(z-x)]dW^n(x),\tag{55}$$

where the partial derivations refer to x obviously. From here we get the three terms of $d\mathbb{M}\hat{P}\delta(z-x)$ after using $\partial\delta(z-x)/\partial x^n=-\partial\delta(z-x)/\partial z^n$ and then taking the stochastic mean over dW, $d\xi$, and x:

$$\begin{aligned}\mathbb{M}d\hat{P}\delta(z-x)&=-i[\hat{H}(z),\hat{\rho}(z)]dt+D_{\mathrm{Q}}^{\beta\alpha}[\hat{L}_\alpha(z)\hat{\rho}(z)\hat{L}_\beta^\dagger(z)-\mathbb{H}\hat{L}_\beta^\dagger(z)\hat{L}_\alpha(z)\hat{\rho}(z)]dt,\\ \mathbb{M}\hat{P}d\delta(z-x)&=\tfrac{1}{2}D_{\mathrm{C}}^{nm}\partial_n\partial_m\hat{\rho}(z)-\partial_n\big[V^n(z)-2\,\mathbb{R}\overline{G}_{\mathrm{CQ}}^{n\alpha}\langle\hat{L}_\alpha(z)\rangle\hat{\rho}(z)\big]dt,\\ \mathbb{M}d\hat{P}d\delta(z-x)&=-\overline{G}_{\mathrm{CQ}}^{n\alpha}\partial_n[(\hat{L}_\alpha-\langle\hat{L}_\alpha\rangle)\hat{\rho}(z)]dt+\mathrm{H.c.}\end{aligned}\tag{56}$$

By taking the sum of these three equations we recognize that $\mathbb{M}\hat{P}\delta(z-x)$ satisfies the HME (35).

The hybrid unraveling corresponds to the time-continuous measurement of the observables $\overline{G}_{\mathrm{CQ}}^{n\alpha}\hat{L}_\alpha(x)+\mathrm{H.c.}$ The hybrid unraveling contains an autonomous drift of the classical variables and general feedback: The Hamiltonian, the decoherence matrix, the measured observable, and the measurement noise can depend on the measured signal x. These dependences could be part of the time-continuous measurement, but usually they are not (except for typical feedback Hamiltonians, linear in dx/dt).

As we can easily inspect, the unraveling (49) and (50) is invariant for the shifts (38); however, in general, it is not covariant under the linear transformations of the operator basis $\{\hat{L}_\alpha(x)\}$ of the HME. Therefore, the unraveling of the HME is not unique; it inherits the ambiguity of unravelings [5] of the quantum-mechanical GKLS dynamics. Observe that we left the non-Hermitian correlation $d\xi^\alpha(x)d\xi^\beta(x)$ unspecified, whereas the stochastic processes (x_t,ψ_t) depend on it. For deliberate choices of $d\xi^\alpha d\xi^\beta$ we get different unravelings of the same HME (36).

Covariance of the unraveling can be achieved if the complex noise $d\xi^\alpha$ is covariant. The simplest way is if we set

$$d\xi^\alpha d\xi^\beta=0\tag{57}$$

(see [45,46], as well as [44]). Another option of covariant (and real) $d\xi^\alpha$ will be shown in Sec. IV C.

Just like unravelings of standard GKLS master equations, the hybrid unravelings are not necessarily in terms of pure states ψ_t. We will discuss generic mixed-state unravelings later in this section. Before that, a specific case is considered which is a closest generalization of the pure-state unravelings

(49) and (50). Extension from pure states ψ_t for mixed states $\hat\sigma_t$ is straightforward since the formalism is very similar. The pure-state density operator $\hat P$ gives way to $\hat\sigma$ and the equation of $d\psi$ is replaced by an equation of $d\hat\sigma$. Accordingly, the definition of unraveling reads

$$\hat\rho(z,t)=\mathbb{M}\hat\sigma_t\delta(z-x_t). \tag{58}$$

Equation (49) for dx^n is the same as before; Eq. (50) gives way to the equation for $d\hat\sigma$:

$$\begin{aligned} d\hat\sigma =& -i[\hat H(x),\hat\sigma]dt\\ &+D_{\rm Q}^{\beta\alpha}[\hat L_\alpha(x)\hat\sigma\hat L_\beta^\dagger(x)-\mathbb{H}\hat L_\beta^\dagger(x)\hat L_\alpha(x)\hat\sigma]dt\\ &+\{[\hat L_\alpha(x)-\langle\hat L_\alpha(x)\rangle]\hat\sigma d\overline{\xi}^\alpha(x)+\text{H.c.}\}. \end{aligned} \tag{59}$$

With the replacement $\hat P\to\hat\sigma$, Eqs. (47)–(56) of the previous proof apply exactly in the same form and we conclude that the process $(x_t,\hat\sigma_t)$ unravels the HME (36). Notice the purification feature of (59) known from standard unravelings of GKLS master equations. If the state is not pure, i.e., $\text{tr}\hat\sigma^2<1$, then

$$\begin{aligned} \frac{d}{dt}\mathbb{M}\text{tr}\hat\sigma^2 &= \frac{2}{dt}\mathbb{M}\text{tr}(\hat\sigma d\hat\sigma)+\frac{1}{dt}\text{tr}(d\hat\sigma)^2\\ &=2D_{\rm Q}^{\beta\alpha}\text{tr}(\hat\sigma\hat L_\alpha\hat\sigma\hat L_\beta^\dagger-\mathbb{H}\hat\sigma\hat L_\beta^\dagger\hat L_\alpha\hat\sigma)\\ &\quad+\text{tr}[(\hat L_\alpha-\langle\hat L_\alpha\rangle)\hat\sigma^2(\hat L_\beta^\dagger-\langle\hat L_\beta^\dagger\rangle)]\\ &=2D_{\rm Q}^{\beta\alpha}\text{tr}\{[\sqrt{\hat\sigma}(\hat L_\beta-\langle\hat L_\beta\rangle)\sqrt{\hat\sigma}]^\dagger\\ &\quad\times[\sqrt{\hat\sigma}(\hat L_\alpha-\langle\hat L_\alpha\rangle)\sqrt{\hat\sigma}]\}>0. \end{aligned} \tag{60}$$

An arbitrary initial mixed state $\hat\sigma_t$ will be purified asymptotically until $\text{tr}\hat\sigma^2=1$ and then the mixed state (59) becomes equivalent with the pure state (20).

Given a covariant HME (36), the family of unravelings is larger than the above family of perfectly purifying ones. There are partially purifying unravelings if the HME is above the threshold of minimum noise [cf. (45)]

$$D_{\rm Q}=G_{\rm CQ}^\dagger\frac{1}{D_{\rm C}}G_{\rm CQ}+\Delta D_{\rm Q}\equiv D_{\rm Q min}+\Delta D_{\rm Q}, \tag{61}$$

where $\Delta D_{\rm Q}\geqslant 0$. Then Eq. (59) for the mixed-state unraveling remains the same but the correlation $d\xi d\xi^\dagger=D_{\rm Q}dt$ will be reduced to the minimum-noise threshold

$$d\xi d\xi^\dagger=D_{\rm Q\,min}dt=G_{\rm CQ}^\dagger\frac{1}{D_{\rm C}}G_{\rm CQ}dt. \tag{62}$$

The other correlations $dWd\xi=G_{\rm CQ}dt$ and $dWdW^T=D_{\rm C}dt$ are unchanged. The price we pay for the reduced noise is illustrated if we group the terms of (59) as follows:

$$\begin{aligned} d\hat\sigma =& -i[\hat H(x),\hat\sigma]dt\\ &+\tfrac12 D_{\rm Q\,min}^{\beta\alpha}[\hat L_\alpha(x)\hat\sigma\hat L_\beta^\dagger(x)-\mathbb{H}\hat L_\beta^\dagger(x)\hat L_\alpha(x)\hat\sigma]dt\\ &+\{[\hat L_\alpha(x)-\langle\hat L_\alpha(x)\rangle]\hat\sigma d\overline{\xi}^\alpha(x)+\text{H.c.}\}\\ &+\tfrac12\Delta D_{\rm Q}^{\beta\alpha}[\hat L_\alpha(x)\hat\sigma\hat L_\beta^\dagger(x)-\mathbb{H}\hat L_\beta^\dagger(x)\hat L_\alpha(x)\hat\sigma]dt. \end{aligned} \tag{63}$$

The first line corresponds to the perfectly purifying unraveling, at the minimum noise (62), whereas the decoherence term in the second line counters the purification. A highly mixed state becomes purer and a low level of mixture becomes higher. Mixedness may have a stationary value for certain HMEs and certain unravelings. We notice that the family of mixed-state unravelings is even larger since we can always set

$$d\xi d\xi^\dagger=\eta D_{\rm Q\,min}\quad(0<\eta\leqslant 1). \tag{64}$$

C. Monitoring the diffusive quantum trajectory

We are going to show that, similarly to the jump trajectories in Sec. II B, also diffusive quantum trajectories ψ_t can be monitored if the classical trajectories x_t are observed. As we will see, this option of monitoring constrains the parameters of the HME (36) and singles out a unique unraveling among the infinite many.

Monitoring the quantum trajectory ψ_t via monitoring the classical x_t is possible if and only if $d\psi$ (50) uniquely depends on dx (49). Hence, the vector $d\xi$ must be a linear function of the vector dW:

$$d\xi=F_{\rm QC}dW. \tag{65}$$

This deterministic relationship removes the ambiguity of the unraveling because it also specifies $d\xi d\xi$ that was free correlation [see Eq. (52)] in general unravelings. The above relationship should be consistent with the correlations (52). They imply two equations $D_{\rm Q}=F_{\rm QC}D_{\rm C}F_{\rm QC}^\dagger$ and $G_{\rm CQ}^\dagger=F_{\rm QC}D_{\rm C}$. These equations possesses the solution

$$F_{\rm QC}=G_{\rm CQ}^\dagger\frac{1}{D_{\rm C}} \tag{66}$$

and a constraint on the HME's parameters:

$$D_{\rm Q}=G_{\rm CQ}^\dagger\frac{1}{D_{\rm C}}G_{\rm CQ}. \tag{67}$$

This constraint is the condition that monitoring ψ_t be possible, namely, we insert the solution (66) into (65) to obtain the desired map of dW into $d\xi$:

$$d\xi=G_{\rm CQ}^\dagger\frac{1}{D_{\rm C}}dW. \tag{68}$$

It is remarkable that this equation defines covariant $d\xi$ and removes the ambiguity of $d\xi d\xi$.

The condition (67) of monitoring coincides with the saturated condition (45) but, importantly, the other condition (46) of minimum noise is not necessary for monitoring. Accordingly, the option of monitoring is still guaranteed above the noise threshold: range$D_{\rm C}$ can be larger than range$G_{\rm CQ}G_{\rm CQ}^\dagger$. At some x, there can be some components of x that are not coupled to the quantum subsystem, that represent above-threshold noise, and that are redundant for and do not prevent the monitoring of ψ_t.

Using (68), we can eliminate $d\xi$ from the equations of unraveling. Then both dx and $d\psi$ (or $d\hat\sigma$) are driven by the same real noise dW. Accordingly, Eq. (50) becomes [27]

$$\begin{aligned} d\psi =& -i[\hat H(x)+\hat H_{\rm fr}(x)]\psi dt+dW^n(x)\big[D_{\rm C}^{-1}(x)\big]_{nm}\\ &\times G_{\rm CQ}^{m\alpha}(x)[\hat L_\alpha(x)-\langle\hat L_\alpha(x)\rangle]\psi. \end{aligned} \tag{69}$$

In the special reference frame where $G_{\rm CQ}^{n\alpha}=\delta^{n\alpha}$ and we consider the minimum-noise threshold where the nondegenerate $D_{\rm Q}$ and $D_{\rm C}$ are real and inverses of each other, we have $d\xi=D_{\rm C}^{-1}dW=D_{\rm Q}dW$ and Eqs. (49) and (50) of unraveling reduce to

$$dx^\alpha = V^\alpha(x)dt - 2\,\mathbb{R}\langle\hat{L}_\alpha(x)\rangle dt + dW_\alpha(x),$$
$$d\psi = -i[\hat{H}(x) - i\hat{H}_{\rm fr}(x)]\psi dt + [\hat{L}_\alpha(x) - \langle\hat{L}_\alpha(x)\rangle]\psi D_{\rm Q}^{\alpha\beta} dW_\beta(x). \quad (70)$$

Notice the new denotation $dW_\alpha \equiv dW^n|_{n=\alpha}$. We recognize the equations of correlated time-continuous measurements (monitoring) of the observables $\hat{L}_\alpha + \text{H.c.}$ where x_t is the measured signal. Here the model is a bit more general because the signal can have an autonomous drift, the Hamiltonian, and the monitored observables can depend on the measured signal.

V. DISCUSSION

In this work we revisited, clarified, and completed earlier results on the Markovian master and stochastic equations of hybrid quantum-classical dynamics, paying attention to simplicity and brevity.

The starting concept and the derivation of the HME in Sec. II are most similar to the rigorous formulations of Blanchard and Jadczyk [20,21]. Here we notice the shift invariance (12) of the HME and the related nonuniqueness of the unraveling. Using vectorized classical variables is a useful alternative to the sophisticated conditions of monitoring the jump quantum trajectory in [28].

Derivation in Sec. III of the diffusive HME (35) from the discrete (11) is an attempt to replace the complicated though perhaps more rigorous procedure of Oppenheim *et al.* [25]. Our derivation is based on the discrete forerunners of $\delta(x-y)$ and $\partial/\partial x^n$. These are nontrivial while allowing for an elementary derivation. Importantly, the naive choice of $\hat{L}_\alpha(x, y) = \hat{L}_\alpha(x)\delta(x-y)$ and $L_n(x, y) = |x\rangle\partial_n\langle x|$ instead of (27) and (28) turns out to be incorrect because off-block-diagonal terms play a role [47]. The naive choice gave the correct structure of the HME but below the correct threshold of minimum noise by a factor of $\frac{1}{2}$.

In Sec. IV we rederived the HME (36), which was derived already by Oppenheim *et al.* [25] and Layton *et al.* [27]. These works did not mention the covariance of their result, nor did they put it in the usual form of co- and contravariant indices. Our work emphasizes and exploits that the HME is explicitly covariant for local linear (i.e., not necessarily unitary) transformations of the Lindblad generators and for general transformations of the classical coordinates.

Section IV A presented a fairly compact condition (41) of the threshold for minimum noise. The phenomenon and equation (42) of trade-off between decoherence and diffusion were recognized in [12] and were extended recently for the general diffusive HME in [25] [see Eqs. (43)–(46)] using general matrix inverses to cover degenerate matrices of decoherence, diffusion, and backaction; their degeneracies are not exceptions but typical.

Our important contribution in Sec. IV B is that the pure-state diffusive unravelings of a diffusive HME are always possible and are exactly as ambiguous as the standard unravelings of the GKLS master equations. Since the ambiguity coincides with that of the unravelings of pure quantum GKLS master equations, we can fix them in the same way. The choice $d\xi d\xi = 0$ is well known in theory of quantum state diffusion and has been used for covariance in [44] and developed by Gisin and Percival [45,46] in the GKLS and Itô formalisms. The full multitude of unravelings, applicable to the hybrid dynamics as well, was discussed by Wiseman and the present author in [5].

Section IV C postulates the covariant condition (65) of quantum trajectory monitoring. The resulting equations of monitoring coincide with those in [27]. The claim therein that these equations are in one-to-one correspondence with the HME is confirmed by covariance when the HME is at the threshold of minimum noise. The option of monitoring is not restricted for the HME of minimum noise; diffusion (not decoherence) can be higher than the threshold.

VI. CONCLUSION

Although rarely stated explicitly, the interaction between quantum and classical systems has no other consistent mathematical model than time-continuous quantum measurement and feedback, where measurement outcomes form the variables of the classical system. This echoes von Neumann's visionary postulate. To obtain a classical variable correlated with an unknown quantum state, the only consistent mathematical model is the von Neumann quantum measurement. Not too surprising, the equations of hybrid stochastic unravelings, both discrete and continuous, coincide with the respective equations of time-continuous measurement, provided the classical system is identified with the measurement outcomes, as in the elementary case (2).

The unravelings (statistical interpretation) of hybrid master equations are mathematical equivalents of time-continuous quantum measurements as mentioned, e.g., in [19]. The advantage of hybrid master equations and unravelings over time-continuous quantum measurement is not yet conceptual. The hybrid formalism may be fairly convenient in many applications, e.g., foundations or improved semiclassical gravity. No doubt, it may develop its own metaphysics as well.

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APPENDIX A: DIFFUSIVE HME OF TWO-LEVEL QUANTUM SYSTEM

Consider a two-level quantum system coupled to a classical system of a single variable x. In Pauli's formalism, we can write the hybrid density (1) in the general form

$$\hat{\rho}(x)=\tfrac{1}{2}[1+\hat{s}(x)]\rho(x), \tag{A1}$$

where $\hat{s}(x)=s_1(x)\hat{\sigma}_1+s_2\hat{\sigma}_2+s_3\hat{\sigma}_3$ and the length $s=|\mathbf{s}|$ of the Bloch vector $\mathbf{s}=(s_1,s_2,s_3)$ must satisfy $s\leqslant 1$. Let the diffusive HME be the simple one

$$\frac{d\hat{\rho}(x)}{dt}=\hat{\sigma}_3\hat{\rho}(x)\hat{\sigma}_3-\hat{\rho}(x)+G[\hat{\sigma}_3,\hat{\rho}\,'(x)]_+ +\frac{1}{2}\hat{\rho}\,''(x). \tag{A2}$$

This corresponds to $D_\mathrm{Q}=D_\mathrm{C}=1$ and G is real. We prove that $|G|$ cannot be larger than 1.

Substitute Eq. (A1) and multiply both sides by 2, yielding

$$\frac{d}{dt}[(1+\hat{s})\rho]=[\hat{\sigma}_3(1+\hat{s})\hat{\sigma}_3-(1+\hat{s})]\rho + G[\hat{\sigma}_3,[(1+\hat{s})\rho]']_+ +\frac{1}{2}[(1+\hat{s})\rho]''. \tag{A3}$$

An equivalent form reads

$$\frac{d\hat{s}}{dt}\rho+\hat{s}\frac{d\rho}{dt}=-2\hat{s}_\perp\rho+2G\hat{\sigma}_3\rho'+\frac{1}{2}\hat{s}\rho''+\hat{s}'\rho'+\frac{1}{2}\hat{s}''\rho. \tag{A4}$$

We take the trace of both sides, yielding the equation $d\rho/dt=+2G(s_3\rho)'+\frac{1}{2}\rho''$. If we substitute it back, we get

$$\frac{d\hat{s}}{dt}\rho=-2\hat{s}_\perp\rho+2G\hat{\sigma}_3\rho'+\hat{s}'\rho'+\frac{1}{2}\hat{s}''\rho-2G\hat{s}(s_3\rho)', \tag{A5}$$

where $\hat{s}_\perp=s_1\hat{\sigma}_1+s_2\hat{\sigma}_2$. We multiply both sides by $\hat{s}$ and take $\frac{1}{2}$ times their trace again:

$$\frac{1}{2}\frac{ds^2}{dt}\rho=-2s_\perp^2\rho+2Gs_3(1-s^2)\rho'+(s^2)'\rho' +\frac{1}{2}\mathbf{s}\mathbf{s}''\rho-2Gs^2s_3'\rho. \tag{A6}$$

If we suppose $s^2=1$, then ds^2/dt cannot be positive and $(s^2)'$ must vanish. Thus we have the following inequality:

$$0\geqslant -2s_\perp^2+\tfrac{1}{2}\mathbf{s}\mathbf{s}''-2Gs^2s_3'. \tag{A7}$$

Now we insert the ansatz $\mathbf{s}(x)=(\cos(x),0,\sin(x))$ into the inequality, leading to

$$0\geqslant\frac{1}{2}\frac{ds^2}{dt}=-2\cos^2(x)-2G\cos(x)-\frac{1}{2}, \tag{A8}$$

which must be satisfied for all x. This is equivalent to the upper bound on the backaction coupling:

$$G^2\leqslant 1. \tag{A9}$$

APPENDIX B: HAMILTONIAN HME

Let $\hat{H}(x)$ be a hybrid Hamiltonian where the classical subsystem is canonical. The first N canonical variables $\{x^n\mid n=1,\dots,N\}$ are the coordinates and the second N ones $\{x^n\mid n=N+1,\dots,2N\}$ are the momenta. The HME takes the form

$$\frac{d\hat{\rho}}{dt}=-i[\hat{H},\hat{\rho}]+\mathbb{H}\{\hat{H},\hat{\rho}\}+\cdots, \tag{B1}$$

where the ellipsis stands for mandatory decoherence and diffusion terms. The bracket $\{\hat{H},\hat{\rho}\}$ is the Poisson bracket of classical canonical theory

$$\{\hat{H},\hat{\rho}\}=\varepsilon^{nm}(\partial_n\hat{H})(\partial_m\hat{\rho}), \tag{B2}$$

where ε^{nm} is the $2N\times 2N$ symplectic matrix. To determine the decoherence and diffusion terms, we note that the hybrid Hamiltonian can always have the form

$$\hat{H}(x)=\hat{H}_\mathrm{Q}+H_\mathrm{C}(x)+\hat{H}_\mathrm{CQ}(x), \tag{B3}$$

with

$$\hat{H}_\mathrm{CQ}(x)=h^\alpha(x)\hat{L}_\alpha, \tag{B4}$$

where $h^\alpha(x)$ is a real function and the $\hat{L}_\alpha$ are linearly independent Hermitian operators, also linearly independent of $\hat{I}$. Accordingly, the HME contains a purely Hamiltonian term $-i[\hat{H}_\mathrm{Q}+\hat{H}_\mathrm{CQ},\hat{\rho}]$, a purely classical term $\{H_\mathrm{C},\hat{\rho}\}$, a backaction term $\mathbb{H}\{\hat{H}_\mathrm{CQ},\hat{\rho}\}$, and the mandatory decoherence and diffusive terms. It is easy to see that $\{H_\mathrm{C},\hat{\rho}\}$ corresponds to classical drift velocity $V^n=\varepsilon^{nm}\partial_m H_\mathrm{C}$. By identifying the backaction term

$$\mathbb{H}\{\hat{H}_\mathrm{CQ},\hat{\rho}\}=\mathbb{H}\{h^\alpha\hat{L}_\alpha,\hat{\rho}\}=\varepsilon^{nm}\mathbb{H}(\partial_n h^\alpha)\hat{L}_\alpha(\partial_m\hat{\rho}) \tag{B5}$$

in Eq. (B1) with the covariant HME's backaction term $2\mathbb{H}\partial_n(\overline{G}_\mathrm{CQ}^{n\alpha}\hat{L}_\alpha\hat{\rho})$, we read out the matrix of backaction:

$$G_\mathrm{CQ}^{n\alpha}=\overline{G}_\mathrm{CQ}^{n\alpha}=-\tfrac{1}{2}\varepsilon^{nm}\partial_m h^\alpha. \tag{B6}$$

Hence, the Hamiltonian HME takes the general form

$$\frac{d\hat{\rho}}{dt}=-i[\hat{H}_\mathrm{Q}+\hat{H}_\mathrm{CQ},\hat{\rho}]+\{H_\mathrm{C},\hat{\rho}\}+\mathbb{H}\{\hat{H}_\mathrm{CQ},\hat{\rho}\} + D_\mathrm{Q}^{\beta\alpha}(\hat{L}_\alpha\hat{\rho}\hat{L}_\beta^\dagger-\mathbb{H}\hat{L}_\beta^\dagger\hat{L}_\alpha\hat{\rho})+\frac{1}{2}\partial_n\partial_m\big(D_\mathrm{C}^{nm}\hat{\rho}\big) \tag{B7}$$

and the matrices D_Q, D_C, and G_CQ should satisfy the non-negativity constraints discussed in Secs. III and IV. Note that the decoherence and diffusion terms do not retain the covariance for canonical transformations of the classical coordinates $\{x^n\}$.

Let us apply this Hamiltonian HME to the simple hybrid system of two identical harmonic oscillators, one quantized and the other classical. The classical canonical variables are $(x^1,x^2)\equiv(q,p)$ and the quantized ones are $(\hat{Q},\hat{P})$. Let the hybrid Hamiltonian (B3) consist of

$$\hat{H}_\mathrm{Q}=\tfrac{1}{2}(\hat{Q}^2+\hat{P}^2), \tag{B8}$$

$$H_\mathrm{C}(q,p)=\tfrac{1}{2}(q^2+p^2), \tag{B9}$$

$$\hat{H}_\mathrm{CQ}(q,p)=g(q\hat{Q}+p\hat{P}). \tag{B10}$$

In the expansion (B4) of $\hat{H}_\mathrm{CQ}$ we choose $\hat{L}_1=\hat{Q}$ and $\hat{L}_2=\hat{P}$ and then $h^1=gq=gx^1$ and $h^2=gp=gx^2$; hence $G^{12}=-G^{21}=g/2$ while $G^{11}=G^{22}=0$. The minimum-noise condition reads, e.g., $GD_\mathrm{Q}^{-1}G^T=D_\mathrm{C}$, which means that both D_Q and D_C are 2×2 diagonal matrices and satisfy

$$D_\mathrm{Q}^{11}D_\mathrm{C}^{22}=D_\mathrm{Q}^{22}D_\mathrm{C}^{11}=\frac{g^2}{4}. \tag{B11}$$

The resulting HME of $\hat\rho(q,p)$ takes the form

$$\frac{d\hat\rho}{dt} = -i\left[\frac{1}{2}(\hat Q^2+\hat P^2)+g(q\hat Q+p\hat P),\hat\rho\right]+[q\partial_p\hat\rho - p\partial_q\hat\rho + g\mathbb{H}(\hat Q\partial_p\hat\rho - \hat P\partial_q\hat\rho)]$$
$$+\frac{1}{2}D_{\rm Q}^{11}[\hat Q,[\hat Q,\hat\rho]]+\frac{1}{2}D_{\rm Q}^{22}[\hat P,[\hat P,\hat\rho]]+\frac{1}{2}D_{\rm C}^{11}\partial_q^2\hat\rho+\frac{1}{2}D_{\rm C}^{22}\partial_q^2\hat\rho. \tag{B12}$$

We see that both $\hat Q$ and $\hat P$ contribute to nonvanishing decoherence terms and both q and p must diffuse. As we noticed, the mandatory decoherence and diffusion terms violate the canonical invariance of the classical subsystem. Of course, they remain nonunique even after taking the constraints (B11) into the account.

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