Spin of random stationary light

José J. Gil,[1,*] Andreas Norrman,[2] Ari T. Friberg,[2] and Tero Setälä[2]
[1]Department of Applied Physics, University of Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain
[2]Center for Photonics Sciences, University of Eastern Finland, P. O. Box 111, FI-80101 Joensuu, Finland

(Received 1 March 2023; accepted 28 April 2023; published 22 May 2023)

We develop a theoretical foundation for the spin angular momentum (SAM) of random, statistically stationary polychromatic light fields within the framework of classical optical coherence theory. The formulation is valid for fields of arbitrary frequency bandwidth and dimensionality. Both temporal and spectral representations are given, and we further elucidate the relationship between the SAM and the polarization characteristics of such fields as compared to monochromatic light. The special cases of quasimonochromatic light and planar fields are analyzed separately. Generally, our paper offers deeper insights into the SAM and polarization structures as well as their interlinked connections in random stationary light, which could be beneficial in exploiting SAM in stochastic optical near fields and tightly focused beams exhibiting complex polarization character.

DOI: 10.1103/PhysRevA.107.053518

I. INTRODUCTION

Spin angular momentum (SAM) is a fundamental physical quantity of light [1–8], which is closely intertwined with the concept of polarization [9,10]. As suggested very early on by Poynting [11] and experimentally observed by Beth [12], the intrinsic angular momentum of a fully polarized light beam is determined by the amount of circular polarization associated with the polarization state. The spin and its transverse character in optical evanescent waves [4,5,13] and focal fields [4,5,14–16] have especially attracted a considerable amount of research recently. A vast majority of these investigations involve deterministic monochromatic light and, thus, two-dimensional (2D) polarization states for which the local electric field is bounded to a plane. However, every optical field in nature exhibits at least some degree of random fluctuations, which cannot be observed directly but inferred merely through statistical averages and correlations [17]. This fact may not render only a monochromatic treatment inadequate but also a 2D polarization description insufficient, since for random polychromatic light the local electric field can fluctuate in all three orthogonal spatial directions in any reference frame. Such general three-dimensional (3D) polarization states are known to entail some extraordinary spin characteristics [18–24] with no correspondence in the context of 2D polarization states, which motivates to develop a systematic theory for the SAM of randomly fluctuating, polychromatic optical fields of an arbitrary polarization state.

In this paper, by employing the classical theory of optical coherence, we establish a rigorous framework to characterize the SAM of random, statistically stationary light fields of any dimension and spectral distribution. Our formulation encompasses two central quantities: (1) the electric SAM density vector met in dual-symmetric electrodynamics [1,3,7,8,25] and (2) the electric spin vector encountered in polarization optics [18,19,22,26–28]. Although for monochromatic light these quantities are always directly proportional to each other, and hence practically interchangeable, we show that for random 3D polychromatic light this is not true. In particular, we elucidate the connection between the SAM density vector and the spin vector on a completely general level both in the time domain and in the frequency domain, providing important fundamental insights into the relationship between the SAM and polarization structures of arbitrary random stationary light. Reductions to 2D and quasimonochromatic fields are also considered, and it is shown that, in these cases, the SAM density vector is connected to the spin vector in a manner that shares similarity with the one for monochromatic light.

The succeeding sections of the present paper contain the following material. In Sec. II, we recall the spin of monochromatic light, which serves as a basis for later sections. In Sec. III, after providing background material about polarization of random stationary light, the spectral and temporal representations for the spin of general 3D polychromatic light are established. Section IV addresses the spin of quasimonochromatic fields whereas Sec. V deals with the spin of 2D polarization states. Finally, in Sec. VI we summarize the main results of this paper.

II. SPIN OF MONOCHROMATIC LIGHT

Monochromaticity is an idealized concept related to optical fields with zero spectral bandwidth, so their spectral profiles are determined by a single (angular) frequency ω and, consequently, their coherence time is infinite. Even though any real electromagnetic field has nonzero bandwidth, the assumption of monochromaticity allows for a simple and sufficient description in various instances. Here, for the sake of clarity and comprehensiveness and as a step preceding the description of more realistic physical situations, we consider the spin and polarization of monochromatic light. Since optical

*ppgil@unizar.es

interactions take place predominantly via the electric field, we henceforward focus solely on the electric component of the total electromagnetic field.

Let $\mathbf{E}(t) = \mathrm{Re}(\mathbf{e}e^{-i\omega t})$ be a monochromatic electric field at time t, where Re stands for the real part and $\mathbf{e} = (e_x, e_y, e_z)^{\mathrm{T}}$ is the complex 3D Jones vector [29–31], with T denoting transpose, which represents the local polarization state in Cartesian coordinates. The SAM density vector associated with the electric field is in vacuum given by [4,5]

$$\mathbf{S} = \frac{\varepsilon_0}{4\omega}\mathrm{Im}(\mathbf{e}^*\times\mathbf{e}), \tag{1}$$

where the superscript * indicates complex conjugate, $\times$ represents the vector product, ε_0 is the vacuum permittivity, and Im refers to the imaginary part. The result in Eq. (1) is obtained by time averaging the dual-symmetric form of the total electromagnetic SAM density vector [1,3,7,8] when the magnetic contribution is neglected, i.e.,

$$\mathbf{S}(t) = \frac{\varepsilon_0}{2}\mathbf{E}(t)\times\mathbf{A}(t). \tag{2}$$

Here $\mathbf{A}(t)$ is the (real) vector potential that obeys $\mathbf{E}(t) = -d\mathbf{A}(t)/dt$.

The SAM density vector $\mathbf{S}$ in Eq. (1) represents angular momentum per unit volume. However, we will henceforth drop the term "density" and for brevity simply refer to $\mathbf{S}$ as the SAM vector (its local character should be understood in this context). Moreover, in polarization optics, the focus is on the relative magnitude of the polarization descriptors rather than their dimensions. For example, measurable quantities, such as the Stokes parameters (see below), are defined with dimensions of intensity $I = \mathbf{e}^\dagger\mathbf{e}$ where the dagger indicates conjugate transpose. The spin vector $\mathbf{n}$ associated with the polarization state is then introduced by neglecting the factor $\varepsilon_0/4\omega$ in Eq. (1) as [18,19,22]

$$\mathbf{n} = \mathrm{Im}(\mathbf{e}^*\times\mathbf{e}) = \mathrm{Im}\begin{pmatrix} e_y^*e_z - e_z^*e_y \\ -e_x^*e_z + e_z^*e_x \\ e_x^*e_y - e_y^*e_x \end{pmatrix}. \tag{3}$$

Combining Eqs. (1) and (3) allows us to write the SAM vector of a general monochromatic field as

$$\mathbf{S} = \frac{\varepsilon_0}{4\omega}\mathbf{n}. \tag{4}$$

Importantly, for monochromatic light, the SAM and spin vectors point exactly in the same direction.

Before proceeding to the next section concerning spin of random polychromatic stationary light, we make some further remarks about the present idealization of monochromatic light. In this case, the electric field traces a stable polarization ellipse whose shape and size do not fluctuate. Hence, a monochromatic light field is fully polarized. For simplicity and without loss of generality, we take the polarization ellipse to lie in the xy plane, viz., $e_z = 0$. By then considering the Stokes parameters

$$s_0 = e_x^*e_x + e_y^*e_y, \quad s_1 = e_x^*e_x - e_y^*e_y, \quad s_2 = e_x^*e_y + e_y^*e_x,$$
$$s_3 = i(e_y^*e_x - e_x^*e_y), \tag{5}$$

the spin vector $\mathbf{n}$ in Eq. (3) can be written as

$$\mathbf{n} = s_3\mathbf{k}. \tag{6}$$

Here, $\mathbf{k}$ is the unit vector along the positive z axis and s_3 is called the helicity of the polarization state [2]. In particular, the degree of circular polarization of the field P_c is precisely $P_c = |\hat{\mathbf{n}}| = |\hat{s}_3|$ [26–28], where $\hat{\mathbf{n}} = \mathbf{n}/I$ is the intensity-normalized spin vector and $\hat{s}_3 = s_3/I$ is the intensity-normalized helicity.

III. SPECTRAL AND TEMPORAL SPIN OF POLYCHROMATIC LIGHT

Having discussed the spin of monochromatic light, we now turn to establish the frequency-domain and time-domain representations for the spin of random stationary light of arbitrary dimension and bandwidth. To this end, we first recall some basic notions regarding random polarization and optical coherence theory.

A. Polarization of random stationary light

In general, the following polarimetric situations may occur for random polychromatic fields:

2D polarization state. The local electric field evolves in a fixed plane, which allows one to take a reference frame in which the field is represented through two components only.

3D polarization state. The local electric field fluctuates such that the strengths of its three orthogonal components are nonzero in any reference frame [18,27,32,33].

Pure state. A fully polarized field that behaves polarimetrically like monochromatic light. Even though the intensity fluctuates (i.e., the size of the polarization ellipse fluctuates), the shape of the polarization ellipse is fixed [34,35]. Obviously, a pure state is necessarily a 2D state.

Mixed state. A partially polarized field that can be either 2D or 3D.

In addition, when considering the spectral profile of polychromatic light, the following cases can be distinguished:

Quasimonochromatic. The shape of the spectral profile is sufficiently narrow and there is a well-defined and representative mean (central) frequency $\bar{\omega}$. The electric field traces out a well-defined ellipse (instantaneous polarization ellipse) for times involving a sufficiently large number of optical cycles (with equivalent periods $2\pi/\bar{\omega}$). If the instantaneous polarization plane and the shape of the polarization ellipse vary (do not vary) for times shorter than the measurement time, the field is partially (totally) polarized. As we will see, quasimonochromaticity ensures the applicability of the concept of an instantaneous Jones vector. Polarization states of quasimonochromatic fields can be either 2D or 3D.

Broadband. In this case the concepts of an instantaneous polarization ellipse and an instantaneous Jones vector are not generally applicable. Nevertheless, for any given measurement time, the polarization state of a broadband field is well defined via the Stokes parameters (either in their 2D or 3D versions depending on the case [36–45]) or the associated polarization matrix (see below).

Regardless of the dimensionality and the spectral profile, all the information about the polarization state in the time domain is included in the 3×3 temporal polarization matrix

$$\mathbf{J} = \langle \mathbf{e}^*(t)\mathbf{e}^{\mathrm{T}}(t)\rangle = \begin{pmatrix} \langle e_x^*(t)e_x(t)\rangle & \langle e_x^*(t)e_y(t)\rangle & \langle e_x^*(t)e_z(t)\rangle \\ \langle e_y^*(t)e_x(t)\rangle & \langle e_y^*(t)e_y(t)\rangle & \langle e_y^*(t)e_z(t)\rangle \\ \langle e_z^*(t)e_x(t)\rangle & \langle e_z^*(t)e_y(t)\rangle & \langle e_z^*(t)e_z(t)\rangle \end{pmatrix}. \quad (7)$$

Here the brackets $\langle\,\rangle$ stand for time averaging and, as customary in optical coherence theory, the field is expressed as a complex-analytic signal $\mathbf{e}(t) = [e_x(t), e_y(t), e_z(t)]^{\mathrm{T}}$ defined by [17]

$$\mathbf{e}(t) = \int_0^\infty \mathbf{E}(\omega)e^{-i\omega t}d\omega, \quad (8)$$

which does not contain negative frequencies and where $\mathbf{E}(\omega)$ is the Fourier component of the field. In analogy to monochromatic fields, the physical (real) field $\mathbf{E}(t)$ is obtained as $\mathbf{E}(t) = \mathrm{Re}[\mathbf{e}(t)]$. The polarization matrix is Hermitian and positive semidefinite with the elements J_{ij} $(i, j = x, y, z)$ being the second-order moments of the three Cartesian components of $\mathbf{e}(t)$. In addition, for ergodic fields (considered here) the time average in Eq. (7) coincides with the ensemble average over sample realizations [17]. Note that, whereas the convention taken in Eq. (7) for the definition of the polarization matrix is common in optical coherence theory, the alternative convention $\mathbf{J} = \langle \mathbf{e}(t)\mathbf{e}^\dagger(t)\rangle$ is used frequently under the scope of polarization theory [10,18,19,27,38,46] in which case the signs of the imaginary parts of the off-diagonal elements of the polarization matrix should be inverted.

On the other hand, the complete information on the polarization state in the frequency domain is contained in the 3×3 spectral polarization matrix

$$\mathbf{\Phi}(\omega) = \langle \boldsymbol{\varepsilon}^*(\omega)\boldsymbol{\varepsilon}^{\mathrm{T}}(\omega)\rangle = \begin{pmatrix} \langle \varepsilon_x^*(\omega)\varepsilon_x(\omega)\rangle & \langle \varepsilon_x^*(\omega)\varepsilon_y(\omega)\rangle & \langle \varepsilon_x^*(\omega)\varepsilon_z(\omega)\rangle \\ \langle \varepsilon_y^*(\omega)\varepsilon_x(\omega)\rangle & \langle \varepsilon_y^*(\omega)\varepsilon_y(\omega)\rangle & \langle \varepsilon_y^*(\omega)\varepsilon_z(\omega)\rangle \\ \langle \varepsilon_z^*(\omega)\varepsilon_x(\omega)\rangle & \langle \varepsilon_z^*(\omega)\varepsilon_y(\omega)\rangle & \langle \varepsilon_z^*(\omega)\varepsilon_z(\omega)\rangle \end{pmatrix}, \quad (9)$$

where $\langle\,\rangle$ now stands for the ensemble average and $\boldsymbol{\varepsilon}(\omega)$ belongs to a set of monochromatic realizations all at frequency ω. We emphasize that $\boldsymbol{\varepsilon}(\omega)$ is *not* the Fourier component $\mathbf{E}(\omega)$ present in Eq. (8), although they both have the same units. Nevertheless, the ensemble $\{\boldsymbol{\varepsilon}(\omega)\}$ provides a rigorous representation of $\mathbf{\Phi}(\omega)$ of the field as a statistical average [17,47]. Note also that, even though each realization represents a pure state with a well-defined polarization ellipse, the ellipses and the polarization planes of the different realizations can be different, which thus in general necessitates a 3D polarization treatment of the field. As with the time domain, when the convention $\mathbf{\Phi}(\omega) = \langle \boldsymbol{\varepsilon}(\omega)\boldsymbol{\varepsilon}^\dagger(\omega)\rangle$ is taken, the signs of the imaginary parts of the off-diagonal elements of spectral polarization matrix should be inverted with respect to those derived from Eq. (9).

B. Spectral spin

The spin vector of a monochromatic realization appearing in Eq. (9) is $\mathrm{Im}[\boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)]$ [cf. Eq. (3)]. Ensemble averaging over these spin-vector realizations then yields the field's spectral spin vector

$$\boldsymbol{\nu}(\omega) = \mathrm{Im}\langle \boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)\rangle = 2\mathrm{Im}[\Phi_{yz}(\omega), -\Phi_{xz}(\omega), \Phi_{xy}(\omega)]^{\mathrm{T}}, \quad (10)$$

where $\Phi_{ij}(\omega)$ $(i, j = x, y, z)$ are the off-diagonal elements of the spectral polarization matrix $\mathbf{\Phi}(\omega)$. Likewise, averaging over the SAM vectors of the realizations, $(\varepsilon_0/4\omega)\mathrm{Im}[\boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)]$ [cf. Eq. (1)], leads to the spectral SAM vector

$$\mathbf{\Sigma}(\omega) = \frac{\varepsilon_0}{4\omega}\boldsymbol{\nu}(\omega). \quad (11)$$

In particular, Eq. (11) shows that the spectral spin and SAM vectors point exactly in the same direction, akin to the case of monochromatic light in Eq. (4).

Further insight into the spectral spin is obtained by invoking the Wiener-Khintchine theorem [17,48],

$$\mathbf{\Phi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{\Gamma}(\tau)e^{i\omega\tau}d\tau, \quad (12)$$

where $\mathbf{\Gamma}(\tau) = \langle \mathbf{e}^*(t)\mathbf{e}^{\mathrm{T}}(t+\tau)\rangle$ is the electric mutual coherence matrix that for stationary fields depends only on the time difference τ. Since Eq. (12) holds for each matrix element, we find that

$$\langle \boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty \boldsymbol{\eta}(\tau)e^{i\omega\tau}d\tau, \quad (13)$$

where we introduced the quantity $\boldsymbol{\eta}(\tau) = \langle \mathbf{e}^*(t)\times\mathbf{e}(t+\tau)\rangle$. According to Eqs. (10) and (13), the spectral spin vector then takes the form

$$\boldsymbol{\nu}(\omega) = \frac{1}{2\pi}\mathrm{Im}\left[\int_{-\infty}^\infty \boldsymbol{\eta}(\tau)e^{i\omega\tau}d\tau\right]. \quad (14)$$

The complex vector $\boldsymbol{\eta}(\tau)$ consists of the elements of $\mathbf{\Gamma}(\tau)$ and, hence, provides information how temporal coherence affects the spectral spin.

C. Temporal spin

The temporal spin vector of the random stationary field follows directly from Eq. (7) [19]:

$$\mathbf{n} = \mathrm{Im}\langle \mathbf{e}^*(t)\times\mathbf{e}(t)\rangle = 2\mathrm{Im}(J_{yz}, -J_{xz}, J_{xy})^{\mathrm{T}}. \quad (15)$$

This spin vector describes the time-averaged direction around which the electric field whirls [49]. We especially observe that the temporal spin vector coincides with $\boldsymbol{\eta}(\tau)$ in Eq. (13) for $\tau = 0$, viz., $\mathbf{n} = \mathrm{Im}[\boldsymbol{\eta}(0)]$. Moreover, the inverse of Eq. (13) implies

$$\boldsymbol{\eta}(\tau) = \int_0^\infty \langle \boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)\rangle e^{-i\omega\tau}d\omega, \quad (16)$$

with the integration extending over positive frequencies only owing to the complex analytic signal representation. We thus find that

$$\mathbf{n} = \mathrm{Im}[\boldsymbol{\eta}(0)] = \int_0^\infty \boldsymbol{\nu}(\omega)d\omega, \quad (17)$$

showing that, for stationary light, the temporal spin vector can be viewed as an incoherent sum of the spectral spin vectors.

We next derive the associated temporal SAM vector. We start from Eq. (2) and express the electric field and vector potential via complex analytic signals, i.e., $\mathbf{E}(t) = \mathrm{Re}[\mathbf{e}(t)]$ and $\mathbf{A}(t) = \mathrm{Re}[\mathbf{a}(t)]$, where $\mathbf{e}(t)$ is given in Eq. (8) and

$$\mathbf{a}(t) = \int_0^\infty \mathbf{A}(\omega)e^{-i\omega t}d\omega. \tag{18}$$

The corresponding Fourier components are related as $\mathbf{A}(\omega) = (-i/\omega)\mathbf{E}(\omega)$. Inserting the above expressions for $\mathbf{E}(t)$ and $\mathbf{A}(t)$ into Eq. (2) leads to

$$\mathbf{S}(t) = \frac{-i\varepsilon_0}{8}\bigg[\int_0^\infty\int_0^\infty \frac{1}{\omega}\mathbf{E}(\omega)\times\mathbf{E}(\omega')e^{-i(\omega+\omega')t}d\omega d\omega' + \int_0^\infty\int_0^\infty \frac{1}{\omega}\mathbf{E}^*(\omega)\times\mathbf{E}(\omega')e^{i(\omega-\omega')t}d\omega d\omega' - \text{c.c.}\bigg], \tag{19}$$

with c.c. denoting the complex conjugate. Taking the average of Eq. (19) subsequently yields

$$\mathbf{S} = \langle\mathbf{S}(t)\rangle = \frac{\varepsilon_0}{4}\int_0^\infty \frac{1}{\omega}\mathrm{Im}\langle\boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)\rangle d\omega, \tag{20}$$

where we employed the conditions

$$\langle\mathbf{E}^*(\omega)\times\mathbf{E}(\omega')\rangle = \langle\boldsymbol{\varepsilon}^*(\omega)\times\boldsymbol{\varepsilon}(\omega)\rangle\delta(\omega-\omega'), \quad \langle\mathbf{E}(\omega)\times\mathbf{E}(\omega')\rangle = 0. \tag{21}$$

The first relation, including the Dirac delta function $\delta(\omega-\omega')$, highlights the fact that for stationary light different frequency components are uncorrelated. It follows from an elementwise utilization of the Wiener-Khintchine theorem in Eq. (12), which can be expressed formally as $\langle\mathbf{E}^*(\omega)\mathbf{E}^{\mathrm{T}}(\omega')\rangle = \boldsymbol{\Phi}(\omega)\delta(\omega-\omega')$ [17], with the spectral polarization matrix $\boldsymbol{\Phi}(\omega)$ constructed as an ensemble average over the monochromatic realizations $\boldsymbol{\varepsilon}(\omega)$ as before in Eq. (9). The second relation in Eq. (21) is proportional to $\delta(\omega+\omega')$ (see Secs. 2.2 and 2.4.1 in Ref. [17]) and thus reduces to zero due to the complex analytic signal representation in Eq. (8). By making use of the spectral quantities $\boldsymbol{\nu}(\omega)$ and $\boldsymbol{\Sigma}(\omega)$ in Eqs. (10) and (11), we eventually find that the temporal SAM vector in Eq. (20) obeys

$$\mathbf{S} = \int_0^\infty \boldsymbol{\Sigma}(\omega)d\omega = \frac{\varepsilon_0}{4}\int_0^\infty \frac{1}{\omega}\boldsymbol{\nu}(\omega)d\omega. \tag{22}$$

Two main conclusions can be drawn from Eq. (22).

The first equality in Eq. (22) shows that the temporal SAM vector of a random stationary light field corresponds to an incoherent sum of the spectral SAM vectors. This connection is similar to the link between the temporal and spectral spin vectors in Eq. (17). The second equality in Eq. (22) dictates that, in general, the temporal SAM vector is *not* parallel to the temporal spin vector in Eq. (17) due to the spectral weighting $1/\omega$ within the integral. This result is strikingly different from the one in the spectral domain [Eq. (11)] and the one for monochromatic light [Eq. (4)] for which the SAM and spin vectors always point in the same direction.

We emphasize that the above analysis, which relies on the complex analytic signal representation (applicable to any electromagnetic field in nature), provides both a frequency-domain formulation [Eq. (11)] and a time-domain formulation [Eq. (22)] for the SAM vector of random stationary light of arbitrary spectral profile and dimensionality. We also remark that whereas the temporal spin vector appearing in Eqs. (15) and (17) does not, in general, describe the direction of the temporal SAM vector in Eq. (22) of the field, it is a central quantity that reflects the time-domain polarization properties of the field.

IV. SPIN OF QUASIMONOCHROMATIC LIGHT

There are many experimental situations where the field can be considered quasimonochromatic, that is, the spectral width $\Delta\omega$ is very narrow compared to the central natural frequency $\bar{\omega}$ of the spectrum. The quasimonochromaticity condition ensures that the coherence time involves a large number of equivalent natural cycles so that the instantaneous polarization ellipse is well defined for time intervals comparable to the polarization time [50–52], which supports the consistency of the concept of an instantaneous Jones vector. Note that such a concept is not applicable, in general, to highly polychromatic states where the instantaneous polarization ellipse may not be well defined.

As given in Ref. [48, p. 175], the mutual electric coherence matrix of a quasimonochromatic field can for τ values (much) less than the coherence time be approximated as

$$\boldsymbol{\Gamma}(\tau) = \mathbf{J}e^{-i\bar{\omega}\tau}. \tag{23}$$

Since in this case $\boldsymbol{\eta}(\tau) = \langle\mathbf{e}^*(t)\times\mathbf{e}(t)\rangle\exp(-i\bar{\omega}\tau)$, it follows from Eq. (14) that $\boldsymbol{\nu}(\omega) = \mathbf{n}\delta(\omega-\bar{\omega})$ [53]. Consequently, we effectively have

$$\mathbf{n} = \boldsymbol{\nu}(\bar{\omega}), \quad \mathbf{S} = \boldsymbol{\Sigma}(\bar{\omega}) = \frac{\varepsilon_0}{4\bar{\omega}}\mathbf{n}, \tag{24}$$

which express that the temporal spin and SAM vectors are determined by the corresponding spectral quantities at the center frequency $\bar{\omega}$. In addition, we see that for quasimonochromatic light $\mathbf{n}$ and $\mathbf{S}$ are parallel, and thus their connection is similar to that for monochromatic light in Eq. (4). Obviously, the narrower the spectral width (larger coherence time) is, the more accurate the above approximate expressions are. A more rigorous analysis would involve a finite sharp spectral function instead of the Dirac delta function.

Equation (24) constitutes the spin version of the result obtained in Ref. [54] and in Ref. [55] (Theorem 1): If a statistically stationary light field of arbitrary bandwidth is filtered to become narrowband, of mean frequency $\bar{\omega}$, the temporal spin vector (SAM vector) of the filtered beam is equal to the spectral spin vector (SAM vector) at the mean frequency $\bar{\omega}$ of the original field.

V. SPIN OF 2D LIGHT

As explained in Sec. III A, for a 2D polarization state the electric field is restricted to a plane and, hence, it can be represented by merely two orthogonal components. In this case, the temporal and spectral polarization matrices in Eqs. (7) and (9) are reducible to 2×2 forms as [17]

$$\mathbf{J} = \begin{pmatrix} \langle e_x^*(t)e_x(t)\rangle & \langle e_x^*(t)e_y(t)\rangle \\ \langle e_y^*(t)e_x(t)\rangle & \langle e_y^*(t)e_y(t)\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} s_0+s_1 & s_2+is_3 \\ s_2-is_3 & s_0-s_1 \end{pmatrix}, \tag{25}$$

$$\boldsymbol{\Phi}(\omega) = \begin{pmatrix} \langle \varepsilon_x^*(\omega)\varepsilon_x(\omega)\rangle & \langle \varepsilon_x^*(\omega)\varepsilon_y(\omega)\rangle \\ \langle \varepsilon_y^*(\omega)\varepsilon_x(\omega)\rangle & \langle \varepsilon_y^*(\omega)\varepsilon_y(\omega)\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} s_0(\omega)+s_1(\omega) & s_2(\omega)+is_3(\omega) \\ s_2(\omega)-is_3(\omega) & s_0(\omega)-s_1(\omega) \end{pmatrix}, \tag{26}$$

where s_0, s_1, s_2, s_3 and $s_0(\omega)$, $s_1(\omega)$, $s_2(\omega)$, $s_3(\omega)$ are the temporal and spectral Stokes parameters [cf. Eq. (5)]. The related temporal and spectral spin vectors are [cf. Eq. (6)]

$$\mathbf{n} = s_3\mathbf{k}, \qquad \boldsymbol{\nu}(\omega) = s_3(\omega)\mathbf{k}, \tag{27}$$

whose magnitudes $|s_3|$ and $|s_3(\omega)|$ are determined univocally by the imaginary parts of the polarization matrices $\mathbf{J}$ and $\boldsymbol{\Phi}(\omega)$. Therefore, real-valued polarization matrices (either 2×2 or 3×3 [18,22,27]) correspond to states that lack spin.

Equations (17) and (27) indicate that the temporal helicity s_3 is connected to the spectral helicity $s_3(\omega)$ according to

$$s_3 = \int_0^\infty s_3(\omega)d\omega. \tag{28}$$

Provided that s_3 is nonzero, regardless of the bandwidth, we can recast Eq. (22) for a 2D field into the form

$$\mathbf{S} = \frac{\varepsilon_0}{4\Omega}\mathbf{n}, \tag{29}$$

by introducing the *spin frequency*

$$\Omega \equiv \frac{s_3}{\int_0^\infty [s_3(\omega)/\omega]d\omega}. \tag{30}$$

We see that the temporal SAM vector and the spin frequency of a 2D field are exclusively specified by the helicity properties. In particular, the introduction of the spin frequency allows us to express the SAM vector of arbitrary polychromatic 2D light in a form analogous to that of monochromatic light in Eq. (4). However, since Ω can be positive or negative, the temporal spin and SAM vectors of polychromatic 2D light can be parallel or antiparallel, whereas for monochromatic light they are always parallel. In general, the magnitude of Ω can be interpreted as the angular frequency of the monochromatic field whose SAM vector can be rendered to coincide with that of the polychromatic 2D field considered. In addition, $1/\Omega$ can be viewed as the spectral-helicity weighted average of $1/\omega$. For a quasimonochromatic 2D field, the spin frequency is simply the (positive) mean frequency, $\Omega = \bar{\omega}$, and for a strictly monochromatic field Eq. (29) reduces to Eq. (10).

VI. CONCLUSIONS

To summarize, we have established a unified framework to characterize the spin of random stationary light fields of arbitrary bandwidth and dimensionality by employing the foundations of classical optical coherence theory. Both time-domain and frequency-domain formulations were provided. We especially focused on two main physical quantities and their mutual relations: (1) the electric SAM density vector appearing in the dual-symmetric representation of electrodynamics and (2) the electric spin vector that is fundamental in polarization optics. For monochromatic light, these two quantities differ only by a positive proportionality factor that renders them essentially interchangeable. It was shown that for random polychromatic 3D light a similar relationship holds in the spectral domain but not generally in the temporal domain. In particular, the time-domain SAM density and spin vectors generally can point in different directions. This highlights the fact that for a polychromatic 3D field extra care must be taken in assessing the effect of polarimetric properties on SAM. The special cases of quasimonochromatic and 2D fields were studied separately. By introducing the spin frequency, we demonstrated that in both cases the temporal SAM density vector can be expressed in a form analogous to that of monochromatic light. On the other hand, in contrast to monochromatic and quasimonochromatic light for which the SAM density and spin vectors are strictly parallel, for 2D broadband fields they can also be antiparallel. Furthermore, it was found that the temporal spin characteristics of quasimonochromatic fields are determined by the spectral spin qualities at the center frequency, whereas all spin properties of 2D light can be written directly in terms of helicities. On a general level, our paper forms a transparent bridge between the SAM and polarization properties of random stationary light, which is anticipated to be beneficial in nanophotonics, nonparaxial optics, and optomechanics where stochastic optical fields exhibiting complex structures are harnessed.

ACKNOWLEDGMENTS

This work was supported by the Academy of Finland (Grants No. 349396 and No. PREIN 320166) and the Joensuu University Foundation.

[1] S. M. Barnett, Rotation of electromagnetic fields and the nature of optical angular momentum, J. Mod. Opt. **57**, 1339 (2010).
[2] M. Masuripur, Spin and orbital angular momenta of electromagnetic waves in free space, Phys. Rev. A **84**, 033838 (2011).
[3] K. Y. Bliokh, J. Dressel, and F. Nori, Conservation of spin and orbital angular momenta in electromagnetism, New J. Phys. **16**, 093037 (2014).
[4] K. Y. Bliokh and F. Nori, Transverse and longitudinal angular momenta of light, Phys. Rep. **592**, 1 (2015).
[5] A. Aiello, P. Banzer, M. Neugebauer, and G. Leuch, From transverse angular momentum to photonic wheels, Nat. Photonics **9**, 789 (2015).
[6] S. M. Barnett, L. Allen, R. P. Cameron, C. R. Gilson, M. J. Padgett, F. C. Speirits, and A. M. Yao, On the natures of the

spin and orbital parts of optical angular momentum, J. Opt. **18**, 064004 (2016).
[7] A. I. Arbab, New derivation of the spin of the electromagnetic field, Optik **184**, 436 (2019).
[8] A. Aiello, Helicity, chirality, and spin of optical fields without vector potentials, Phys. Rev. A **106**, 043519 (2022).
[9] C. Brosseau, *Fundamentals of Polarized Light: A Statistical Approach* (Wiley, New York, 1998).
[10] J. J. Gil and R. Ossikovski, *Polarized Light and the Mueller Matrix Approach* (CRC, Boca Raton, FL, 2022).
[11] J. H. Poynting, The wave-motion of a revolving shaft, and a suggestion as to the angular momentum in a beam of circularly polarised light, Proc. R. Soc. London, Ser. A **82**, 560 (1909).
[12] R. A. Beth, Mechanical detection and measurement of the angular momentum of light, Phys. Rev. **50**, 115 (1936).
[13] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, Extraordinary momentum and spin in evanescent waves, Nat. Commun. **5**, 3300 (2014).
[14] M. Neugebauer, T. Bauer, A. Aiello, and P. Banzer, Measuring the Transverse Spin Density of Light, Phys. Rev. Lett. **114**, 063901 (2015).
[15] M. Neugebauer, J. S. Eismann, T. Bauer, and P. Banzer, Magnetic and Electric Transverse Spin Density of Spatially Confined Light, Phys. Rev. X **8**, 021042 (2018).
[16] J. S. Eismann, P. Banzer, and M. Neugebauer, Spin-orbit coupling affecting the evolution of transverse spin, Phys. Rev. Res. **1**, 033143 (2019).
[17] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
[18] M. Dennis, Geometric interpretation of the three-dimensional coherence matrix for nonparaxial polarization, J. Opt. A: Pure Appl. Opt. **6**, S26 (2004).
[19] J. J. Gil, A. T. Friberg, A. Norrman, and T. Setälä, Effect of polarimetric nonregularity on the spin of three-dimensional polarization states, New J. Phys. **23**, 063059 (2021).
[20] Y. Chen, F. Wang, Z. Dong, Y. Cai, A. Norrman, J. J. Gil, A. T. Friberg, and T. Setälä, Structure of transverse spin in focused random light, Phys. Rev. A **104**, 013516 (2021).
[21] J. S. Eismann, L. H. Nicholls, D. J. Roth, M. A. Alonso, P. Banzer, F. J. Rodríguez-Fortuno, A. V. Zayats, F. Nori, and K. Y. Bliokh, Transverse spinning of unpolarized light, Nat. Photonics **15**, 156 (2021).
[22] J. J. Gil, A. Norrman, A. T. Friberg, and T. Setälä, Intensity and spin anisotropy of three-dimensional polarization states, Opt. Lett. **44**, 3578 (2019).
[23] Y. Chen, A. Norrman, S. A. Ponomarenko, and A. T. Friberg, Spin density in partially coherent surface-plasmon-polariton vortex fields, Phys. Rev. A **103**, 063511 (2021).
[24] Z. Wang, C. Yan, F. Wang, Y. Chen, and Y. Cai, Effect of optical spatial coherence on localized spin angular momentum density in tightly focused light [Invited], J. Opt. Soc. Am. A **39**, C58 (2022).
[25] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, Dual electromagnetism: Helicity, spin, momentum and angular momentum, New J. Phys. **15**, 033026 (2013).
[26] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed. (Pergamon, New York, 1982).
[27] J. J. Gil, Interpretation of the coherency matrix for three-dimensional polarization states, Phys. Rev. A **90**, 043858 (2014).
[28] J. J. Gil, Components of purity of a three-dimensional polarization state, J. Opt. Soc. Am. A **33**, 40 (2016).
[29] R. M. A. Azzam, Three-dimensional polarization states of monochromatic light fields, J. Opt. Soc. Am. A **28**, 2279 (2011).
[30] C. J. R. Sheppard, Jones and Stokes parameters for polarization in three dimensions, Phys. Rev. A **90**, 023809 (2014).
[31] J. J. Gil, I. San José, A. Norrman, A. T. Friberg, and T. Setälä, Sets of orthogonal three-dimensional polarization states and their physical interpretation, Phys. Rev. A **100**, 033824 (2019).
[32] A. Norrman, A. T. Friberg, J. J. Gil, and T. Setälä, Dimensionality of random light fields, J. Eur. Opt. Soc.-Rapid Publ. **13**, 36 (2017).
[33] J. J. Gil, Geometric interpretation and general classification of three-dimensional polarization states through the intrinsic Stokes parameters, Photonics **8**, 315 (2021).
[34] L. Mandel, Intensity fluctuations of partially polarized light, Proc. Phys. Soc. **81**, 1104 (1963).
[35] T. Setälä, K. Lindfors, M. Kaivola, J. Tervo, and A. T. Friberg, Intensity fluctuations and degree of polarization in three-dimensional thermal light fields, Opt. Lett. **29**, 2587 (2004).
[36] G. G. Stokes, On the composition and resolution of streams of polarized light from different sources, Trans. Cambridge Philos. Soc. **9**, 399 (1852).
[37] P. Roman, Generalized Stokes parameters for waves with arbitrary form, Nuovo Cimento **13**, 974 (1959).
[38] U. Fano, Description of states in quantum mechanics by density matrix and operator techniques, Rev. Mod. Phys. **29**, 74 (1957).
[39] R. Barakat, Degree of polarization and the principal idempotents of the coherency matrix, Opt. Commun. **23**, 147 (1077).
[40] T. Carozzi, R. Karlsson, and J. Bergman, Parameters characterizing electromagnetic wave polarization, Phys. Rev. E **61**, 2024 (2000).
[41] T. Setälä, Degree of polarization for optical near fields, Phys. Rev. E **66**, 016615 (2002).
[42] J. J. Gil, J. M. Correas, P. Melero, and C. Ferreira, Generalized polarization algebra, Monog. Sem. Mat. G. Galdeano **31**, 161 (2004).
[43] N. I. Petrov, Vector and tensor polarizations of light beams, Laser Phys. **18**, 522 (2008).
[44] C. J. R. Sheppard, M. Castello, and A. Diaspro, Three-dimensional polarization algebra, J. Opt Soc Am. A **33**, 1938 (2016).
[45] J. J. Gil, Intrinsic Stokes parameters for 2D and 3D polarization states, J. Eur. Opt. Soc.-Rapid Publ. **10**, 15054 (2015).
[46] E. Wolf, Coherence properties of partially polarized electromagnetic radiation, Nuovo Cimento **13**, 1165 (1959).
[47] J. Tervo, T. Setälä, and A. T. Friberg, Theory of partially coherent electromagnetic fields in the space-frequency domain, J. Opt. Soc. Am. A **21**, 2205 (2004).
[48] E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge University Press, Cambridge, UK, 2007).
[49] M. A. Alonso, Geometric descriptions for the polarization of nonparaxial light: A tutorial, Adv. Opt. Photon. **15**, 176 (2023).
[50] T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, Polarization time and length for random optical beams, Phys. Rev. A **78**, 033817 (2008).
[51] T. Voipio, T. Setälä, A. Shevchenko, and A. T. Friberg, Polarization dynamics and polarization time of random

three-dimensional electromagnetic fields, Phys. Rev. A **82**, 063807 (2010).

[52] A. Shevchenko, M. Roussey, A. T. Friberg, and T. Setälä, Polarization time of unpolarized light, Optica **4**, 64 (2017).

[53] T. Setälä, F. Nunziata, and A. T. Friberg, Differences between partial polarizations in the space–time and space–frequency domains, Opt. Lett. **34**, 2924 (2009).

[54] H. Partanen, B. J. Hoenders, A. T. Friberg, and T. Setälä, Young's interference experiment with electromagnetic narrowband light, J. Opt. Soc. Am. A **35**, 1379 (2018).

[55] M. Lahiri and E. Wolf, Does a light beam of very narrow bandwidth always behave as a monochromatic beam? Phys. Lett. A **374**, 997 (2010).