Editors' Suggestion

Nonradiating orbital motions

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Oscillations of extended source distributions that produce no radiation outside the domain of the source have been long known to be possible in all wave problems, including electromagnetism, acoustics, and even gravitational waves. In almost all demonstrations of these sources, called nonradiating sources in optics and nonpropagating excitations in acoustics, the oscillations are monochromatic or small displacement vibrations of a rigid distribution. Here, we demonstrate how it is possible to theoretically construct nonradiating sources of any size that possess orbital motion of any radius. Examples are given for two-dimensional scalar wave problems.

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I. INTRODUCTION

It is well known in classical electromagnetic theory that accelerating electric charges produce electromagnetic waves. It is much less known, however, that this is not always the case: One can produce extended distributions of charges that accelerate yet produce no radiation outside the source domain itself. Such *nonradiating sources*, as they have become known, have been investigated on and off for over a century. The first significant paper on the subject appeared in 1911, by Ehrenfest [1]; he provided several illustrative examples of radiationless motions as well as a mathematical formalism for constructing an arbitrary number of radiationless sources.

Ehrenfest was motivated at the time to explain how an electron can seemingly orbit an atomic nucleus without quickly losing all of its energy to radiation; with the advent of quantum physics, such an explanation became unnecessary. Nonradiating sources, however, would be rediscovered by many researchers over the next few decades. The most notable of these works was by Schott [2], who demonstrated that the center of a uniformly charged spherical shell can move in an almost arbitrary path, provided its period of motion T is related to the radius a of the sphere by $2a = mcT$, with m an integer and c the speed of light in vacuum. Other significant papers include that of Bohm and Weinstein [3], who showed that it is theoretically possible to create a charge distribution that oscillates without radiation and whose motion is driven by its own self-field, and that of Goedecke [4], who elaborated in great detail upon the properties of radiationless sources.

More recently, monochromatic nonradiating sources have been shown to be closely connected to the nonuniqueness of the inverse source problem [5–7]. In short, the existence of nonradiating objects implies the nonuniqueness of the inverse source problem because the properties of a nonradiating source can never be determined from radiation measurements. This led to a more general connection between uniqueness of inverse problems and the existence of "invisible" objects [8]. The mathematical properties of nonradiating sources have now been studied extensively [9,10]. It has been shown that nonradiating sources can exist in a variety of wave systems, including weak gravitational systems [11] as well as vibrating strings, where they are known as nonpropagating excitations [12,13]. Radiationless sources are therefore ubiquitous in wave problems.

But the motions in all the aforementioned papers are either monochromatic oscillations of a spatially fixed charge distribution or approximately so. In the solution presented by Schott, the condition $cT = 2a/m$ implies that the sphere must move a distance less than the diameter of the sphere in one period, making its motion more a "wobble" than an "orbit." This naturally raises the question of whether it is possible to construct radiationless sources which move in true orbits or even more complicated accelerated motions.

In this paper we demonstrate how one can generate nonpropagating excitations in a two-dimensional wave problem which are of arbitrary size and have an arbitrary orbital radius. We discuss the physics of such excitations and how such cases may be used to experimentally demonstrate the nonradiating effect, which has long been elusive. Such orbiting radiationless motions greatly broaden the class of nonradiating sources that have been considered.

II. THEORY OF NONRADIATING ORBITAL MOTIONS

We begin with the two-dimensional wave equation, given by

$$\nabla^2 U(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U(\mathbf{r}, t) = -4\pi Q(\mathbf{r}, t), \qquad (1)$$

where $U(\mathbf{r}, t)$ is a scalar field produced by a source $Q(\mathbf{r}, t)$, $\mathbf{r} = (x, y)$, ∇^2 is the two-dimensional Laplacian, and c represents the speed of the waves. We consider the two-dimensional problem for three reasons: First, it is mathematically simpler than the three-dimensional case, while possessing most of its important features. Second, two-dimensional nonradiating sources are relatively unexplored (with a few exceptions [14,15]), but even-dimensional radiation problems have notable differences from odd-dimensional ones (see

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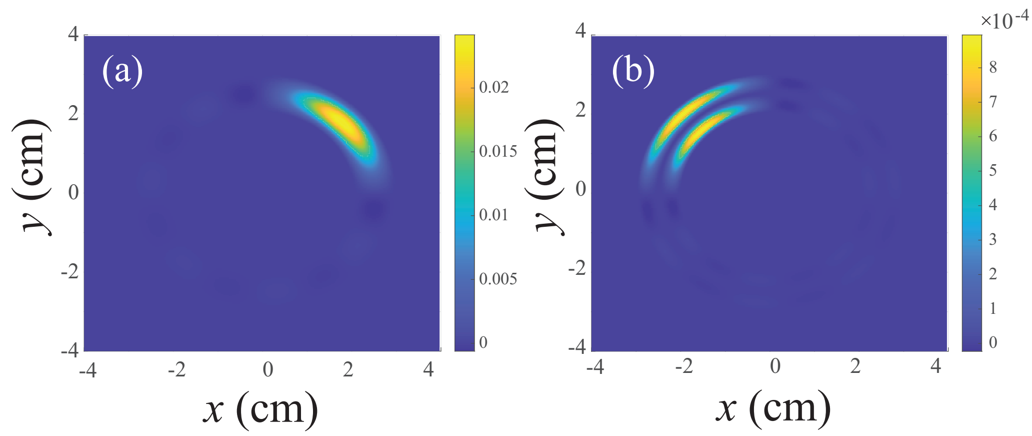

FIG. 1. Two snapshots of orbiting nonradiating fields, with (a) $m = 0$, $n = 0$ with $t = 0$, and (b) $m = 2$, $n = 0$, $t = T/4$. We have taken $\phi = 0$ and $\psi = \pi/2$ for both cases.

Chap. 19 in Ref. [16]), making their study of general interest. And finally, the two-dimensional geometry is perhaps the best for experimental realizations of nonradiating sources. We will use the electromagnetic terms "fields," "sources," and "charges" to describe the physics in this paper, though we stress that these results apply to any system that satisfies a wave equation.

We begin our construction with monochromatic fields and sources, with $U(\mathbf{r}, t) = u(\mathbf{r}, \omega)e^{-i\omega t}$, and $u(\mathbf{r}, \omega)$ satisfies the Helmholtz equation,

$$\nabla^2 u(\mathbf{r}, \omega) + k^2 u(\mathbf{r}, \omega) = -4\pi q(\mathbf{r}, \omega), \tag{2}$$

where $k = \omega/c$. It is known from Gamliel *et al.* [10] that the field of any nonradiating source must satisfy a pair of boundary conditions, namely that the field and its normal derivative must vanish on the boundary. With an eye towards developing an orbiting nonradiating source, we choose our source domain to be an annular region of inner radius a and outer radius b. In polar coordinates (r, θ), the field must therefore satisfy the following four conditions,

$$u(a, \theta) = u(b, \theta) = 0, \quad u_r(a, \theta) = u_r(b, \theta) = 0, \tag{3}$$

where the subscript r indicates the radial derivative.

There are endless possibilities for a function with continuous second derivatives to satisfy Eq. (2) and the nonradiating boundary conditions; we consider a separable function of the form $u(\mathbf{r}) = v(r)w(\theta)$, and for $a \leqslant r \leqslant b$ we set

$$v(r) = c_m(r - K)^m + c_{m+2}(r - K)^{m+2} + c_{m+4}(r - K)^{m+4}, \tag{4}$$

with $v(r) = 0$ for $r < a$ or $r > b$, where $K = (a + b)/2$ and m is an arbitrary non-negative integer. We readily find that we may satisfy the radial boundary conditions with

$$c_m = c_{m+4}\frac{(a - b)^4}{16}, \quad c_{m+2} = -c_{m+4}\frac{(a - b)^2}{2}. \tag{5}$$

Without loss of generality, we set $c_{m+4} = 1$; we will also take $m = 0$ and $m = 2$ as illustrative examples.

For the dependence on θ, we consider a source constrained to an arc from angle ϕ to angle ψ; to be nonradiating, this means that the function $w(\theta)$ must satisfy the conditions

$$w(\phi) = w(\psi) = 0, \quad w'(\phi) = w'(\psi) = 0, \tag{6}$$

with the prime representing the derivative with respect to θ. We may solve this with an expression analogous to Eq. (4), and inside the range $\phi \leqslant \theta \leqslant \psi$ we have

$$w(\theta) = d_n(\theta - H)^n + d_{n+2}(\theta - H)^{n+2} + d_{n+4}(\theta - H)^{n+4}, \tag{7}$$

where $w(\theta) = 0$ whenever $\theta < \phi$ or $\theta > \psi$, where $H = (\phi + \psi)/2$, and where n a non-negative integer. The expressions for the d_n are analogous to those of the c_m of Eq. (5).

We now have an expression for the field $u(\mathbf{r})$ of a nonradiating source that lies within the polar segment $a \leqslant r \leqslant b$ and $\phi \leqslant \theta \leqslant \psi$. It is to be noted that this solution $u(\mathbf{r})$ works for any frequency ω; only the radial shape of the source distribution, derived from Eq. (2), will change with frequency.

Because the function $w(\theta)$ is periodic, we may expand it in a Fourier series representation,

$$w(\theta) = \sum_{p=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\phi}^{\psi} w(\theta')e^{-ip\theta'}d\theta'\right)e^{ip\theta}. \tag{8}$$

We are now in a position to make a nonradiating source that orbits around the origin. This can done in a straightforward way by replacing θ by $\theta - \omega_0 t$ in Eq. (8), which produces a source whose azimuthal origin orbits with an angular frequency ω_0. The function $w(\theta)$ can then be written as

$$w(\theta, t) = \sum_{p=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\phi}^{\psi} w(\theta')e^{-ip\theta'}d\theta'\right)e^{ip(\theta-\omega_0 t)}. \tag{9}$$

The entire field, represented by $U(\mathbf{r}, t) = v(r)w(\theta, t)$, is therefore localized to a finite domain and orbits the origin with an inner radius a, outer radius b, and with a period $T = 2\pi/\omega_0$. The field in question is now a multifrequency field with frequencies $p\omega_0$, with p an integer, but every frequency component of the field satisfies the nonradiating boundary conditions given by Eqs. (3).

III. EXAMPLES AND OBSERVATIONS

We now consider several examples. For simplicity, we take $c = 1.5 \times 10^{10}$ cm/s, or half the vacuum speed of light, $a = 2$ cm, $b = 3$ cm, and we define the angular frequency by setting the maximum speed of the source to be $1/1000$ of c, i.e., the speed along the outer circumference C of the

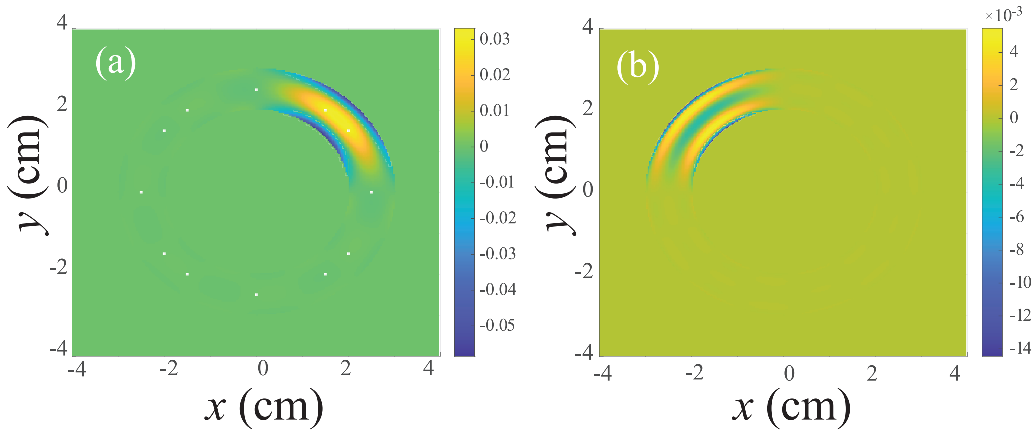

FIG. 2. Snapshots of orbiting nonradiating sources, with (a) $m = 0$, $n = 0$ with $t = 0$, and (b) $m = 2$, $n = 0$, $t = T/4$. We have taken $\phi = 0$ and $\psi = \pi/2$ for both cases.

source $v = c/1000$. With the outer circumference $C = 2\pi b$, this implies an angular frequency $\omega_0 = 5 \times 10^6$ rad/s.

The field resulting from Eqs. (4), (7), and (9) is shown in Fig. 1 for several choices of parameters. In Fig. 1(a), the field has been generated with $m = 0$, $n = 0$, producing a single spot. In Fig. 1(b), the field has been generated with $m = 2$, $n = 0$, producing two spots along the radial direction; the time was taken to be $T/4$, showing the propagation of the field around the circle.

The source distributions $Q(\mathbf{r}, t)$ for a field can be determined directly from Eq. (1); in cylindrical coordinates, the formula for $Q(\mathbf{r}, t)$ is given by

$$Q(\mathbf{r}, t) = -\frac{1}{4\pi}\left[wv_{rr} + \frac{1}{r}v_r w + \frac{1}{r^2}vw_{\theta\theta} - \frac{1}{c^2}vw_{tt}\right], \quad (10)$$

where the subscripts represent partial derivatives.

The source distributions corresponding to the examples of Fig. 1 are shown in Fig. 2. In the figure, the derivatives of Eq. (10) were done analytically. It is to be noted that the distributions consist of regions of alternating positive and negative charge along the radial direction. This is expected, as nonradiating sources are generally known to arise from an unusual form of complete destructive interference [9].

To derive our orbiting nonradiating source distributions, we first constructed fields that satisfy a set of boundary conditions and then determined the source distributions from the wave equation. To confirm that these source distributions are in fact nonradiating, we may numerically evaluate the field distribution from the source distribution frequency by frequency using a Green's function formula,

$$u(\mathbf{r}, \omega) = \int_D q(\mathbf{r}', \omega)G(|\mathbf{r} - \mathbf{r}'|, \omega)d^2\mathbf{r}', \quad (11)$$

where D is the domain of integration and $G(|\mathbf{r} - \mathbf{r}'|, \omega)$ is

$$G(|\mathbf{r} - \mathbf{r}'|, \omega) = i\pi H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|), \quad (12)$$

with $H_0^{(1)}$ the Hankel function of the first kind.

The zero-frequency ($p = 0$) case presented a unique challenge, as the Green's function for Poisson's equation in two dimensions is $G(|\mathbf{r} - \mathbf{r}'|, 0) = -2\ln(|\mathbf{r} - \mathbf{r}'|)$, which is divergent both for small and large values of its argument. To determine the zero-frequency component of the field, we instead evaluate the field $u(\mathbf{r}, 0)$ by a Fourier transform approach, using the relation

$$u(\mathbf{r}, 0) = 4\pi \int_D \frac{\tilde{q}(\mathbf{K}, 0)}{K^2 + i\epsilon} e^{i\mathbf{K}\cdot\mathbf{r}} d^2\mathbf{K}, \quad (13)$$

which comes from the direct Fourier transform of the Poisson equation. The quantity ϵ is a small regularization parameter used to keep the denominator of the integrand from diverging. The Fourier transforms of the source and its inverse can readily be done with fast Fourier transforms (FFTs).

The complete formula for the computation is of the form,

$$U(\mathbf{r}, t) = 4\pi \int_D \frac{\tilde{q}(\mathbf{K}, 0)}{K^2 + i\epsilon} e^{i\mathbf{K}\cdot\mathbf{r}} d^2\mathbf{K} + \sum_{p\neq 0} e^{-ip\omega_0 t} \int_D q(\mathbf{r}', p\omega_0)G(|\mathbf{r} - \mathbf{r}'|, p\omega_0)d^2\mathbf{r}'. \quad (14)$$

An example of one such calculation is shown in Fig. 3, using the same parameters as Fig. 1(b). Good agreement with the analytic result can be seen; we attribute the small nonzero parts of the field outside the source domain to numerical error and the sensitivity of the nonradiating condition to such errors. Within the circle of the orbit, the small nonzero field is attributed to a mismatch between the zero-frequency and nonzero-frequency parts of the calculation.

A few observations concerning these orbiting nonradiating distributions are worth noting. For the fields of Fig. 1, the fundamental frequency ω_0 corresponds to a fundamental wavelength $\lambda_0 = 18\,850$ cm, much larger than every other spatial scale in the system. Our orbiting sources are therefore subwavelength in size. Nevertheless, our results show that such a subwavelength size source can be designed in such a way as to be nonradiating.

The nonradiating condition depends only on the boundary conditions given by Eqs. (3) and (6) and not directly on the choice of spatial size of the localized field, which means we have great freedom in the choice of this size. Figure 4 shows two examples of this freedom. In Fig. 4(a), the source was taken with larger inner and outer radii; in Fig. 4(c), the inner radius was taken to be much smaller than the outer, resulting in a sweeping "searchlight" nonradiating field distribution.

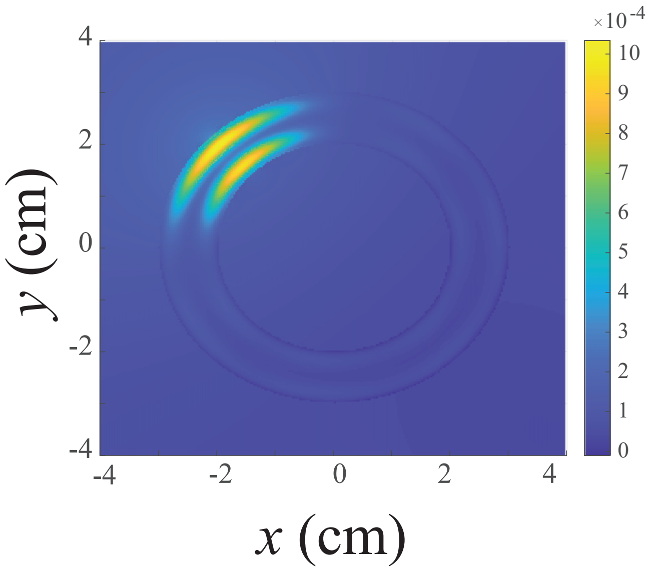

FIG. 3. Green's function calculation of the field $U(\mathbf{r}, t)$, with $m = 2$, $n = 0$, $t = T/4$, $\phi = 0$, and $\psi = \pi/2$, to be compared with Fig. 1(b).

The results presented in this paper can be readily extended to three-dimensional distributions by using a cylindrical coordinate system. The three-dimensional field distribution must satisfy boundary conditions comparable to those of Eq. (3), except the field and its normal derivative must vanish on every bounding surface of the source region. A function can be chosen to match these conditions in the ρ and z directions, and the behavior in the azimuthal direction can be expanded in a Fourier series to construct the orbiting solution.

IV. NONRADIATING ORBITAL MOTIONS AND SURFACE WAVES

It is worth exploring how these nonradiating orbital motions might be verified experimentally. Though nonradiating sources have been discussed for over a century, experimental demonstrations of the nonradiating effect have largely remained elusive. This is likely in part due to the difficulty of generating a three-dimensional radiation source of significant size, in any wave system, with tailored properties. Though it is intriguing to imagine a nonradiating synchrotron source, in practice such a system would be difficult, if not impossible, to realize. One class of nonradiating sources that have gotten much attention in recent years are nonradiating anapoles [17], in which the fields of an electric dipole and a toroidal dipole destructively interfere. Several experimental demonstrations of nonradiating anapoles have now been made, both in toroidal metamaterials [18] and in dielectric nanoparticles [19]. However, an anapole is a very specific case of the general nonradiating effect, and an approximate one in which the dipole moments cancel and not the higher-order multipoles; a demonstration of an exact nonradiating solution remains to be done.

Two-dimensional wave problems, such as those discussed in this paper, present a plausible approach for creating nonradiating sources. One possibility is to use surface waves such as surface plasmons [20] to demonstrate the effect. In the Otto or Kretschmann configuration [21], for example, an evanescent wave is used to locally excite a plasmon-supporting metal surface, producing propagating surface plasmons. A structured light beam could be used to generate an evanescent spot on the metal surface that matches a nonradiating source distribution. "Nonradiating" in this context refers to the surface plasmon field being confined to the area of excitation; part of the field can couple back off of the surface through the same mechanism that was used to excite it. A monochromatic nonradiating source could be generated in this fashion; one could also look to design an "orbiting" nonradiating spot by sweeping an exciting beam in a circle across the surface of the metal.

In such a case, the nonradiating orbits created will differ from the previous examples because the source is itself a beam of light with its own frequency, that we refer to as the carrier frequency. The field of a nonradiating orbital spot can then be written in the form

$$U(\mathbf{r}, t) = U_0(\mathbf{r}, t)\cos(\omega_c t), \tag{15}$$

where $U_0(\mathbf{r}, t)$ is an orbiting nonradiating source as described previously and ω_c is the frequency of the carrier wave. This wave will still satisfy our nonradiating boundary conditions, and should therefore generate a nonradiating field.

The introduction of the carrier wave, however, changes the form of the source distribution required to produce the excitation. On substituting from Eq. (15) into Eq. (10), we have

$$\begin{aligned} Q(\mathbf{r}, t) = Q_0(\mathbf{r}, t)\cos(\omega_c t) - \frac{1}{4\pi}\frac{\omega_c^2}{c^2}U_0(\mathbf{r}, t)\cos(\omega_c t) \\ + \frac{1}{4\pi}\frac{2\omega_c}{c^2}\partial_t U_0(\mathbf{r}, t)\sin(\omega_c t), \end{aligned} \tag{16}$$

where $Q_0(\mathbf{r}, t)$ is the source defined by the nonradiating orbital field $U_0(\mathbf{r}, t)$ alone.

In general, this equation suggests that the spatial distribution of the source will evolve in time as well as its orbital position. To evaluate the effects, we consider the behavior of a nonradiating distribution with $m = 2$, $n = 2$ at two different carrier frequencies.

We first consider a carrier frequency $\omega_c = 600 \times 10^6$ rad/s, which is comparable in magnitude to the orbital frequency, $\omega_0 = 5 \times 10^6$ rad/s. Figure 5 shows the squared field, $I(\mathbf{r}, t) = |U(\mathbf{r}, t)|^2$, and the source intensity $P(\mathbf{r}, t) = |Q(\mathbf{r}, t)|^2$, for $t = 0$. In this case, we find that the last two terms of Eq. (16), which depend on the carrier frequency, are generally negligible compared to the first term on the right-hand side, which represents the source term of the orbital motion alone. In this case, the source is approximately equal to the orbital term $Q_0(\mathbf{r}, t)$ multiplied by $\cos(\omega_c t)$. The source and the field both behave as an orbital nonradiating motion modulated by $\cos(\omega_c t)$, as can be seen in Supplemental Video 1 [22]. To create a nonradiating orbital motion with a carrier frequency comparable to the orbital frequency, one can simply use the design guidelines given in this paper for pure orbital motions.

We next consider a carrier frequency $\omega_c = 600 \times 10^{12}$ rad/s, which is much greater than the orbital frequency and comparable to an optical frequency. In this case, any

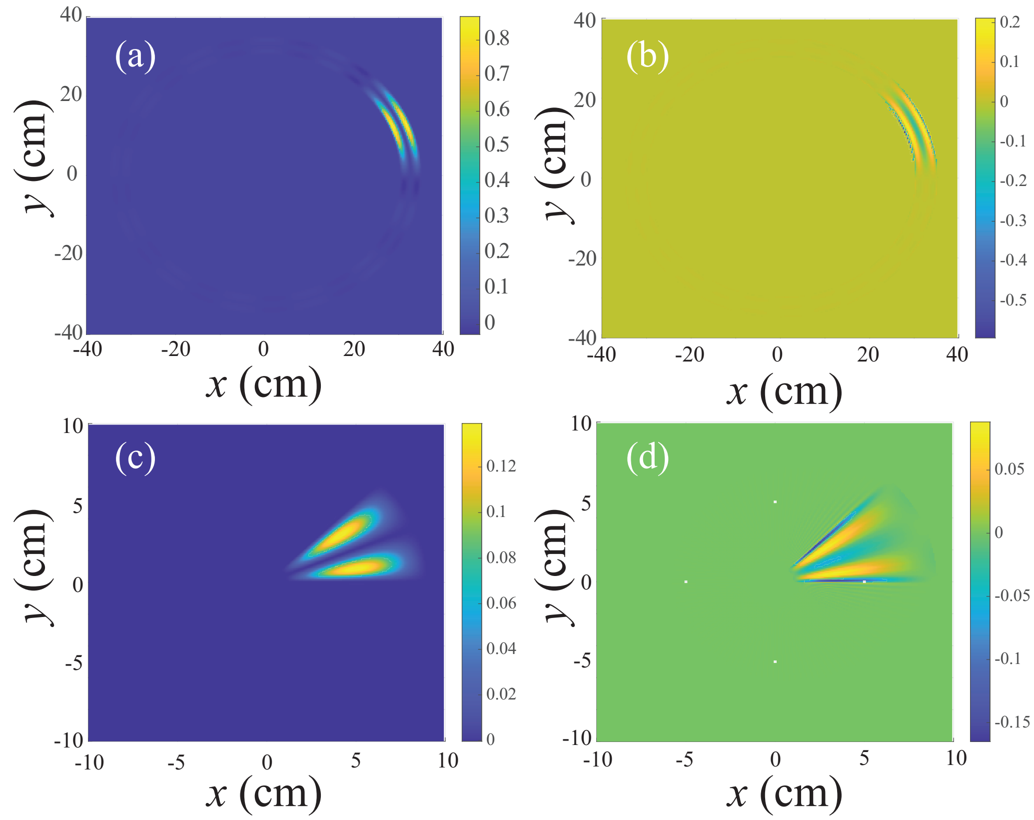

FIG. 4. Distributions of orbiting nonradiating sources. Large orbit: (a) Field of source with $m = 2$, $n = 0$, $a = 30$ cm, $b = 35$ cm, $\phi = 0$, $\psi = \pi/4$, $t = 0$, and (b) the corresponding source distribution. "Searchlight" source: (c) Field of source with $m = 0$, $n = 2$, $a = 1$ cm, $b = 9$ cm, $\phi = 0$, $\psi = \pi/4$, $t = 0$, and (d) the corresponding source distribution.

experimental measurement will not measure the instantaneous oscillations of the field, but only the average over many cycles of the carrier frequency. We define a cycle average in the usual manner,

$$\langle A(t)B(t)\rangle = \frac{1}{T}\int_0^T A(t)B(t)dt, \tag{17}$$

with $T \equiv 2\pi/\omega_c$. Over such a short timescale, only the carrier frequency components $\cos(\omega_c t)$ and $\sin(\omega_c t)$ will be averaged over, with other time-dependent terms effectively constant. We then introduce $\langle I(\mathbf{r}, t)\rangle = \langle |U(\mathbf{r}, t)|^2\rangle$, and the source intensity $\langle P(\mathbf{r}, t)\rangle = \langle |Q(\mathbf{r}, t)|^2\rangle$, and consider the behavior of these in Fig. 6.

In this limit, it can be seen that the source term $\langle P(\mathbf{r}, t)\rangle$ is approximately proportional to the field term $\langle I(\mathbf{r}, t)\rangle$. This can be traced to the ω_c^2 term of Eq. (16), which will be significantly larger than all other source terms. We may evidently create a nonradiating orbital motion with a high carrier frequency to

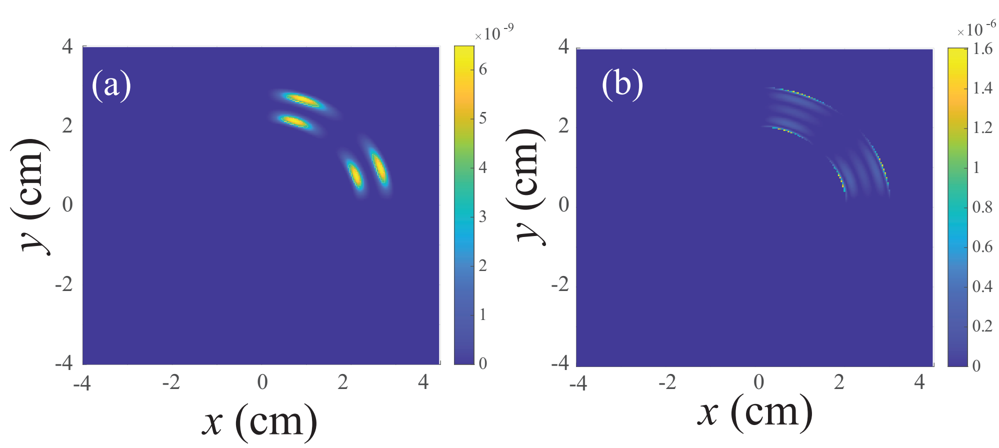

FIG. 5. The (a) intensity $I(\mathbf{r}, t)$ and (b) source intensity $P(\mathbf{r}, t)$ of a nonradiating orbital motion with carrier frequency $\omega_c = 600 \times 10^6$ rad/s and $m = 2$, $n = 2$, with all other parameters as given in Fig. 1.

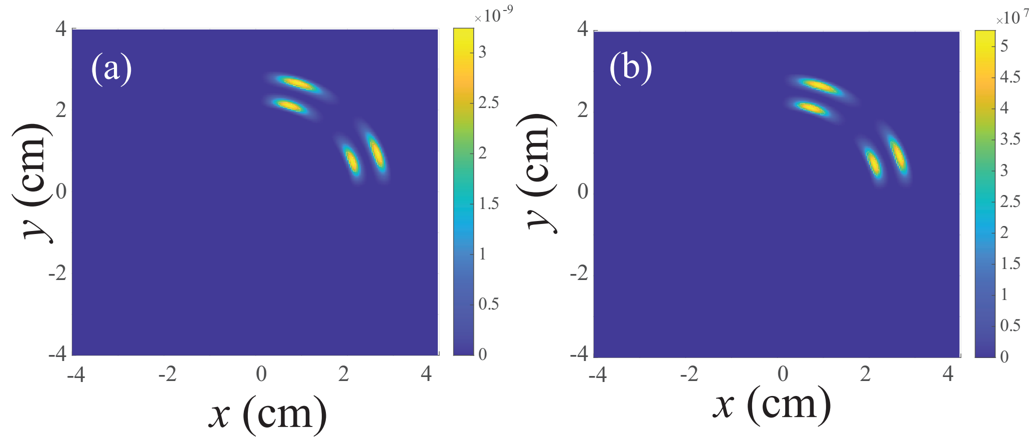

FIG. 6. The (a) intensity $\langle I(\mathbf{r},t)\rangle$ and (b) source intensity $\langle P(\mathbf{r},t)\rangle$ of a nonradiating orbital motion with carrier frequency $\omega_c = 600 \times 10^{12}$ rad/s and $m = 2$, $n = 2$, with all other parameters as given in Fig. 1.

a good approximation by using a source term that is proportional to the nonpropagating field.

V. CONCLUSIONS

In this paper, we have theoretically demonstrated the possibility of orbiting nonradiating source distributions. Though we used the simplest possible structure of these sources, in the choice of Eqs. (4) and (7), it is to be noted that we have great freedom in the choice of our local field functions, and can therefore generate a great diversity of field and source structures. These nonradiating results show that surprises can still be found even in some of the most basic wave propagation problems.

ACKNOWLEDGMENT

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[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.107.053517 for video of a nonradiating motion with a low carrier frequency.