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Universal embedding of a non-Hermitian reciprocal scattering optical system into a Hermitian time-reversal-invariant system

Momchil Minkov,* Heming Wang,* and Shanhui Fan†
Department of Electrical Engineering and Edward L. Ginzton Laboratory, Stanford University, Stanford, California 94305, USA

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For Hermitian optical systems, reciprocity and time-reversal symmetry are equivalent. In non-Hermitian systems that have gain and/or loss, however, they are not equivalent. Here, we point out a connection between reciprocity and time-reversal symmetry in general. For a non-Hermitian system, we show that reciprocity can be viewed as a manifestation of the time-reversal symmetry of an enclosing Hermitian system. Our work deepens the understanding of the general constraints on the scattering matrix of non-Hermitian optical systems.

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I. INTRODUCTION

The behaviors of optical structures are subject to a set of general constraints that arise from energy conservation, reciprocity, or time-reversal symmetry [1–5]. Understanding the implications of these constraints [6–10] is of fundamental importance in designing optical devices [11–15]. For energy-conserving systems, time-reversal symmetry and reciprocity are equivalent to each other [1]. For systems with gain or loss, however, time-reversal symmetry and reciprocity are no longer equivalent, and it is possible to find optical structures that satisfy one but not the other. For example, materials described by a scalar complex dielectric function can exhibit gain and loss, and hence do not possess time-reversal symmetry. These materials however are reciprocal [4]. Similarly, it was recently pointed out that one can construct material systems that have time-reversal symmetry but are not reciprocal [16].

In this paper, we point out a general connection between reciprocity and time-reversal symmetry. Based on the scattering matrix formalism, we prove that any reciprocal system (referred to as the "original system" below) can be embedded in a larger, Hermitian system ("enclosing system") that has time-reversal symmetry. We also provide an explicit construction of the enclosing system for the cases where the original system has gain and/or loss. Our results suggest that the reciprocity of a system can be viewed as a consequence of the time-reversal symmetry of the enclosing system.

II. REVIEW OF SCATTERING MATRIX

We start with a brief review of the scattering matrix formalism, which is a powerful tool in the description of open optical systems [Fig. 1(a)]. For the original system, we assume that, sufficiently far away from the scattering region where light scattering happens, the fields can be separated out into well-defined ports, such that the steady-state field in each port k can be written as

$$\mathbf{E}_k(\mathbf{r},t) = \mathbf{e}_k(x_k,y_k)[(a_k e^{-i\beta_k z_k} e^{-i\omega t} + a_k^* e^{i\beta_k z_k} e^{i\omega t}) + (b_k e^{i\beta_k z_k} e^{-i\omega t} + b_k^* e^{-i\beta_k z_k} e^{i\omega t})], \quad (1)$$

where the local coordinates (x_k, y_k, z_k) at each port are chosen such that the z_k axis points to the direction of light propagation in the kth port away from the scattering region, β_k is the propagation constant of the mode, ω is the angular frequency, a_k (b_k) is the complex incoming (outgoing) field amplitude, and $\mathbf{e}_k(x_k, y_k)$ is a transverse field profile normalized such that $|a_k|^2$ and $|b_k|^2$ are in units of photon number flux. Although we have illustrated each mode with a physically separate port in Fig. 1(a), this does not need to be the case, for example, in multimode waveguides. After grouping all the incoming amplitudes a_k into vector $\mathbf{a}$ and all outgoing amplitudes b_k into vector $\mathbf{b}$, the linear relationship between them is given by the scattering matrix S:

$$\mathbf{b} = S\mathbf{a}. \quad (2)$$

This formalism also applies to infinitely many modes (and even to a continuum of modes), but for concreteness, we will consider a finite number of ports n in our construction of a physically realizable universal device.

For a system in which energy is conserved, we must have $||\mathbf{b}||^2 \equiv \sum_k b_k^* b_k = ||\mathbf{a}||^2$ given the normalization of amplitudes, which implies that S is unitary:

$$S^{-1} = S^\dagger. \quad (3)$$

A separate condition on S is imposed by Lorentz reciprocity [4,5]. Briefly stated, Lorentz reciprocity is the invariance of a system under the exchange of sources and probes, which, in the scattering matrix formalism, requires that S is symmetric:

$$S = S^T. \quad (4)$$

Finally, the time-reversal symmetry implies that

$$S^{-1} = S^*. \quad (5)$$

*These authors contributed equally to this work.
†Corresponding author: shanhui@stanford.edu

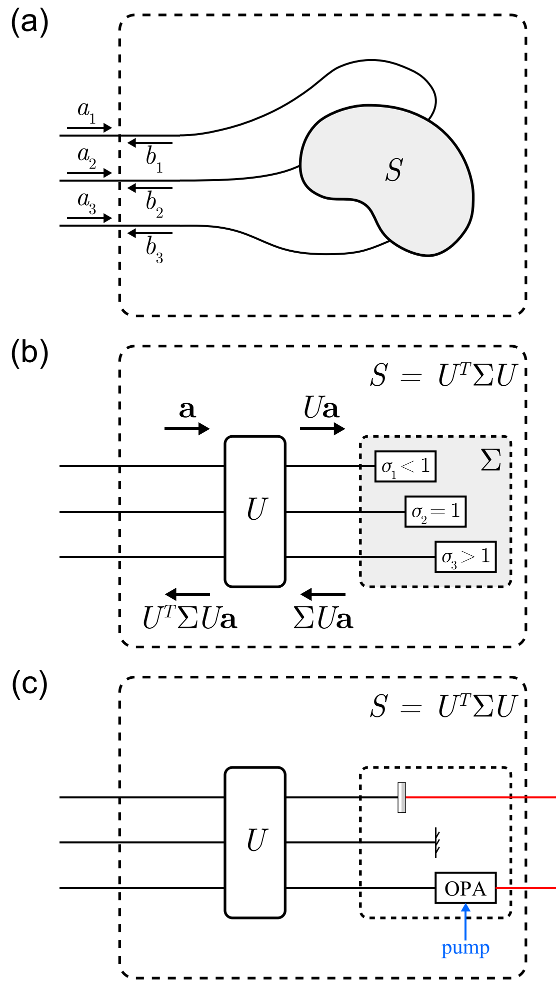

FIG. 1. (a) Schematic of a three-port system described by a scattering matrix. (b) Schematic of the circuit that implements the scattering matrix in the enclosing system, consisting of a mode converter and reflection components. (c) Realization of the reflection components using a partial mirror for $\sigma < 1$, an ideal mirror for $\sigma = 1$, and an optical parametric amplifier (OPA) with pump for $\sigma > 1$.

III. MAIN RESULTS

For an energy-conserving system, the concepts of reciprocity and time reversal are equivalent: If S is unitary, then Eqs. (4) and (5) are equivalent to each other. For non-Hermitian systems, i.e. systems with gain and loss, it is known that reciprocity and time-reversal symmetry are not equivalent. Here, however, we show a general connection between reciprocity and time-reversal symmetry, for systems with gain and/or loss.

As a starting point, we note that an arbitrary reciprocal S matrix, which is symmetric, admits a Takagi decomposition [17,18]

$$S = U^T \Sigma U, \tag{6}$$

with U unitary and Σ diagonal with real non-negative diagonal entries. The Takagi decomposition is the special form of the singular value decomposition applied to symmetric matrices. The diagonal entries of Σ (denoted as σ_k's) are the singular values.

The decomposition of a scattering matrix of Eq. (6) can be implemented with the circuit shown in Fig. 1(b). In the circuit, the unitary matrix U is implemented as a feedforward circuit element and converts between the physical ports and the singular modes (i.e., singular vectors of the scattering matrix). For light propagating along the forward direction, the element has a transmission matrix U. Such a feedforward circuit can be implemented using a Mach-Zehnder interferometer array, as shown in Refs. [19–21]. Here, we assume that the feedforward circuit element is energy conserving, reciprocal, and has no back reflection. Therefore, U is unitary, and moreover, its transmission matrix for light propagating in the backward direction is U^T. The diagonal matrix Σ is implemented as mirrors, with the amplitude reflectivity of the mirror corresponding to the singular values. Depending on the values of the singular values to be $\sigma_k < 1$, $\sigma_k = 1$, or $\sigma_k > 1$, the mirror can be lossy, lossless, or with gain, respectively. In the operation of the circuit, light passes through the feedforward circuit in the forward direction, reflects from the mirrors, and then passes through the feedforward circuit in the backward direction, resulting in the scattering matrix as described in Eq. (6).

We now show that the reciprocal S matrix, as considered in Eq. (6), can always be embedded in the scattering matrix of a larger Hermitian system that satisfies time-reversal symmetry. We consider the mirrors in Fig. 1(b), each of which is a one-port device. For our purposes, we first show that, for each mirror that has either gain or loss, we can embed it in a two-port device that is Hermitian and with time-reversal symmetry. (For a lossless mirror, there is no need to perform this embedding process, as the mirror is already Hermitian.) This process is schematically demonstrated in Fig. 1(c).

We consider the case of lossy mirrors first. To embed the lossy mirror, with an amplitude reflectivity $\sigma < 1$, in a Hermitian system, we consider a lossless partially reflecting mirror with a scattering matrix $S_<$ as defined by

$$\begin{pmatrix} b_s \\ b_t \end{pmatrix} = S_< \begin{pmatrix} a_s \\ a_t \end{pmatrix} = \begin{pmatrix} \sigma & i\sqrt{1-\sigma^2} \\ i\sqrt{1-\sigma^2} & \sigma \end{pmatrix} \begin{pmatrix} a_s \\ a_t \end{pmatrix}, \tag{7}$$

where subscript s (t) on the amplitude indicates the signal (through) port. We note that the scattering matrix of Eq. (7) is unitary and symmetric. Hence the partially reflecting mirror satisfies the constraints of energy conservation, reciprocity, and time-reversal symmetry.

We next consider the case of mirrors with gain, which could be realized using an optical parametric amplifier (OPA) based on difference frequency generation. Here, the OPA is chosen as a realization of the gain process since it admits a scattering matrix treatment as discussed below. Figure 2 shows a schematic implementation of the system. We assume the undepleted pump regime such that the pump amplitude is independent of the incident signal and idler fields, and the signal amplitude at ω_s is linearly coupled to the idler amplitude at $-\omega_d$ [the amplitude of the idler fields oscillating as $e^{i\omega_d t}$ in Eq. (1)] [22]. We further assume that the OPA is combined with a wavelength demultiplexer that separates the signal, idler, and pump waves, each directed toward their

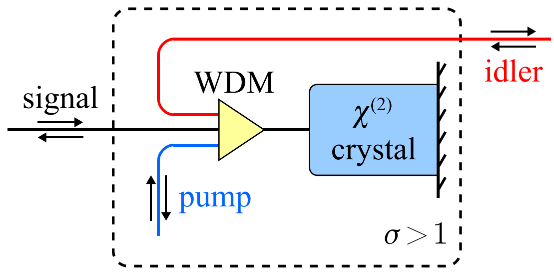

FIG. 2. A more in-depth view of the OPA system for the implementation of the gain mirror as shown in Fig. 1. WDM: wavelength demultiplexer.

respective ports. The scattering matrix $S_>$ for this gain mirror element, with $\sigma > 1$, can then be written as [23–25]

$$\begin{pmatrix} b_s \\ b_d^* \end{pmatrix} = S_> \begin{pmatrix} a_s \\ a_d^* \end{pmatrix} = \begin{pmatrix} \sigma & ie^{i\phi_p}\sqrt{\sigma^2-1} \\ -ie^{-i\phi_p}\sqrt{\sigma^2-1} & \sigma \end{pmatrix} \begin{pmatrix} a_s \\ a_d^* \end{pmatrix}, \tag{8}$$

where subscript s (d) indicates the signal (idler) port, and the pump phase difference appearing in the off-diagonal terms is defined as $\phi_p \equiv \varphi_p - \varphi_s - \varphi_d$, where φ_p, φ_s, and φ_d are the physical phases for the pump, signal, and idler fields. Note that the idler field amplitudes have been conjugated in order to form a linear relationship. The scattering matrix of Eq. (8) can be achieved with a Hamiltonian of the form $ga_sa_d + g^*a_s^\dagger a_d^\dagger$, where g is a coupling constant. This Hamiltonian, and thus the device, is Hermitian. However, the scattering matrix $S_>$ is not unitary. Here, we assumed the undepleted pump regime, hence the energy in the system is not conserved. The total photon number is also not conserved. Instead, since the signal and idler photons are created or destroyed in pairs, the difference between the signal and idler photon numbers is conserved. This is known as a "quasiunitary" condition and is consistent with the canonical commutation relations for the photon operators [23]. Specifically,

$$P_zS_>^{-1}P_z = S_>^\dagger, \quad P_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

and can be verified for Eq. (8) regardless of the value of ϕ_p.

We now discuss the issue of time-reversal symmetry for the OPA system. We start from frequency-domain Maxwell's equations for the signal and idler waves. In the absence of free charges and current sources, the equations read [22]

$$\nabla \times \nabla \times \mathbf{E}_s = \frac{\omega_s^2}{c^2}[\bar{\epsilon}_s\mathbf{E}_s + 2\bar{\bar{\chi}}\mathbf{E}_p\mathbf{E}_d^*], \tag{10}$$

$$\nabla \times \nabla \times \mathbf{E}_d = \frac{\omega_d^2}{c^2}[\bar{\epsilon}_d\mathbf{E}_d + 2\bar{\bar{\chi}}\mathbf{E}_p\mathbf{E}_s^*], \tag{11}$$

where $\bar{\epsilon}$ is the rank-2 relative permittivity tensor and $\bar{\bar{\chi}}$ is the rank-3 second-order nonlinear optical susceptibility tensor, both assumed to be real. For the time-reversal invariance, we require that $\mathbf{E}_s^*$ and $\mathbf{E}_d^*$ are also solutions to the above equations for the same $\mathbf{E}_p$, given that $\mathbf{E}_s$ and $\mathbf{E}_d$ are solutions. This is possible when $\mathbf{E}_p$ is real and can be seen by taking conjugates of Eqs. (10) and (11). This can be achieved when the pump wave forms a standing wave, as in the configuration shown in Fig. 2.

Based on the above discussions on Maxwell's equations, we now consider the time-reversal symmetry for the scattering matrix of the OPA system. Taking the conjugate of Eq. (8), we have

$$\begin{pmatrix} b_s^* \\ b_d \end{pmatrix} = S_>^* \begin{pmatrix} a_s^* \\ a_d \end{pmatrix}. \tag{12}$$

On the other hand, based on the discussion of Eqs. (10) and (11), we need to interpret the conjugate of the incoming (outgoing) wave amplitude as the outgoing (incoming) wave amplitudes. Moreover, assuming that the system has time-reversal symmetry, we have

$$\begin{pmatrix} a_s^* \\ a_d \end{pmatrix} = S_> \begin{pmatrix} b_s^* \\ b_d \end{pmatrix}. \tag{13}$$

Therefore, the time-reversal symmetry constraint on the OPA system reads

$$S_>^{-1} = S_>^*. \tag{14}$$

We see that the $S_>$ matrix in Eq. (8) indeed satisfies time-reversal symmetry when the pump phase is set as 0 or π, in consistency with the discussion above on Maxwell's equations. Therefore, with Eq. (8) we can embed a gain mirror into an OPA system that is Hermitian and satisfies time-reversal symmetry.

Now we return to consider the embedding of the entire S matrix as described in Eq. (6). To construct the enclosing system, we first rearrange the singular values of S (diagonal elements of Σ) in increasing order without loss of generality, and denote the number of singular values with $\sigma_k < 1$, $\sigma_k = 1$, and $\sigma_k > 1$ as $n_<$, $n_=$, and $n_>$, respectively. If all the singular values are smaller or equal to unity ($n_> = 0$), then gain is not required. We assemble all amplitudes for input and through ports as $\mathbf{a} = (a_1 \cdots a_n, a_{t1} \cdots a_{tn_<})^T$, and the $(n + n_<)$-dimensional scattering matrix defined on the enclosing system becomes

$$S_{\text{enc}} = \begin{pmatrix} U^T\Sigma U & iU^T\Lambda_<^T \\ i\Lambda_<U & \Sigma_< \end{pmatrix}, \tag{15}$$

where $\Sigma_< = \text{diag}(\sigma_1 \cdots \sigma_{n_<})$ is the lossy part of Σ, and $\Lambda_< = [(I_{n_<\times n_<} - \Sigma_<^2)^{1/2}, \mathbf{0}_{n_<\times(n-n_<)}]$. It is easy to check that Eq. (15) defines a unitary and symmetric matrix, given the unitarity of U and $\Sigma^2 + \Lambda_<^T\Lambda_< = I_{n\times n}$, and the upper left block matrix is the S matrix of the original system.

If there are some singular values greater than 1 ($n_> \geqslant 1$), the system then has gain. To construct an enclosing system, we need to consider an expanded input vector that contains both field amplitudes and conjugated amplitudes through [23,26]

$$\mathbf{a}_e = (a_1 \cdots a_n, a_{t1} \cdots a_{tn_<}, a_{d1}^* \cdots a_{dn_>}^*)^T. \tag{16}$$

The expanded $(n + n_< + n_>)$-dimensional scattering matrix of the entire system can thus be written as

$$S_{\text{enc}} = \begin{pmatrix} U^T\Sigma U & iU^T\Lambda_<^T & iU^T\Lambda_>^T \\ i\Lambda_<U & \Sigma_< & \mathbf{0} \\ -i\Lambda_>U & \mathbf{0} & \Sigma_> \end{pmatrix}, \tag{17}$$

where $\Sigma_> = \text{diag}(\sigma_{n_<+n_=+1} \cdots \sigma_n)$ is the gain part of Σ, and $\Lambda_> = [\mathbf{0}_{n_>\times(n-n_>)}, (\Sigma_>^2 - I_{n_>\times n_>})^{1/2}]$. As an example, the three-port original system with singular values $\sigma_1 < 1$, 1, and

$\sigma_3 > 1$, as shown in Fig. 1(b), can be embedded in a five-port enclosing system shown in Fig. 1(c), with one port added for each of the nonunity singular values. The full S_{enc}, in this case, can be expressed as

$$S_{\text{enc}} = \begin{pmatrix} & S & & iu_{11}\sqrt{1-\sigma_1^2} & iu_{31}\sqrt{\sigma_3^2-1} \\ & & & iu_{12}\sqrt{1-\sigma_1^2} & iu_{32}\sqrt{\sigma_3^2-1} \\ & & & iu_{13}\sqrt{1-\sigma_1^2} & iu_{33}\sqrt{\sigma_3^2-1} \\ iu_{11}\sqrt{1-\sigma_1^2} & iu_{12}\sqrt{1-\sigma_1^2} & iu_{13}\sqrt{1-\sigma_1^2} & \sigma_1 & 0 \\ -iu_{31}\sqrt{\sigma_3^2-1} & -iu_{32}\sqrt{\sigma_3^2-1} & -iu_{33}\sqrt{\sigma_3^2-1} & 0 & \sigma_3 \end{pmatrix}, \quad (18)$$

where u_{mn} is the matrix elements of U.

The entire S_{enc} matrix in Eq. (17) is "quasiunitary," expressed as

$$\overline{P}_z S_{\text{enc}}^{-1} \overline{P}_z = S_{\text{enc}}^{\dagger}, \quad \overline{P}_z = \begin{pmatrix} I_{n\times n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n_<\times n_<} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -I_{n_>\times n_>} \end{pmatrix}. \quad (19)$$

The matrix also satisfies time-reversal invariance $S_{\text{enc}}^{-1} = S_{\text{enc}}^*$. Finally, we can combine the time-reversal invariance with the quasiunitary condition and arrive at a generalized reciprocity relation for an S matrix:

$$\overline{P}_z S \overline{P}_z = S^T. \quad (20)$$

The S_{enc} in Eq. (17), by construction, naturally satisfies the generalized reciprocity relation. We provide a discussion of such reciprocity constraint from a direct analysis of Maxwell's equations in the Appendix.

IV. SUMMARY AND DISCUSSIONS

In summary, for any arbitrary non-Hermitian reciprocal system that has gain and/or loss as described by a symmetric scattering matrix S, we have shown that we can construct an enclosing system that is Hermitian and time-reversal invariant, with a scattering matrix that includes S as a submatrix. Our work points at a connection between reciprocity and time-reversal symmetry: For a non-Hermitian system, reciprocity can be viewed as a manifestation of the time-reversal symmetry of an enclosing Hermitian system.

We end the paper with a few clarifying remarks. Our paper concerns the three constraints of Hermiticity, reciprocity, and time-reversal symmetry in quantum electrodynamics. We emphasize that in discussing these constraints, one must consider an "open" system that interacts in some way with the surroundings. Here, by an "open" system, we refer to the fact that certain parameters of the system are considered external parameters. In contrast, a "closed" system is a system where there are no such external parameters. Any "open" system can always be embedded into a "closed" system by including the external parameters as dynamic variables. On the other hand, a "closed" system in quantum electrodynamics automatically satisfies the three constraints, as required by the fundamental Hamiltonian of quantum electrodynamics. Therefore, any practical discussion of Hermiticity, reciprocity, and time-reversal symmetry in quantum electrodynamics must by necessity refer to an "open" system.

The main result of our paper is that a reciprocal system can be embedded in an enclosing system that satisfies the three constraints. Here, the nontrivial aspect is that both systems are open in the sense discussed above. Obviously, a nonreciprocal system can be embedded in a closed system, but whether a nonreciprocal system can be embedded in an open system that satisfies these three constraints is an interesting question that we have not addressed. We note, however, that one cannot use our procedure to construct an open enclosing system satisfying these three constraints for a nonreciprocal system. Our results therefore suggest that reciprocal and nonreciprocal systems can be differentiated by studying the properties of their possible enclosing systems.

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APPENDIX: RECIPROCITY RELATIONS OF DIFFERENCE-FREQUENCY GENERATION

Here, we generalize the Lorentz reciprocity condition to nonlinear difference-frequency generation processes starting from Maxwell's equations. Assume that we have signal, idler, and pump fields denoted by subscripts s, d, and p, respectively, oscillating at angular frequencies such that $\omega_p = \omega_s + \omega_d$. The electric field can be defined through its frequency-domain components as

$$\mathbf{E}(\mathbf{r}, t) = \sum_{j=s,d,p} \mathbf{E}_j(\mathbf{r}) e^{-i\omega_j t} + \text{c.c.}, \quad (A1)$$

and the magnetic field $\mathbf{H}(\mathbf{r}, t)$ can be defined similarly. The complete frequency-domain Maxwell's equations, with current sources $\mathbf{J}_j(\mathbf{r})$ for each frequency component, read

$$\nabla \times \mathbf{E}_s = \frac{i\omega_s}{c^2\epsilon_0}\mathbf{H}_s, \quad (A2)$$

$$\nabla \times \mathbf{H}_s = -i\omega_s\epsilon_0(\bar{\epsilon}_s\mathbf{E}_s + 2\bar{\bar{\chi}}\mathbf{E}_p\mathbf{E}_d^*) + \mathbf{J}_s, \quad (A3)$$

$$\nabla \times \mathbf{E}_d = \frac{i\omega_d}{c^2\epsilon_0}\mathbf{H}_d, \quad (A4)$$

$$\nabla \times \mathbf{H}_d = -i\omega_d\epsilon_0(\bar{\epsilon}_d\mathbf{E}_d + 2\bar{\bar{\chi}}\mathbf{E}_p\mathbf{E}_s^*) + \mathbf{J}_d, \quad (A5)$$

$$\nabla \times \mathbf{E}_p = \frac{i\omega_p}{c^2\epsilon_0}\mathbf{H}_p, \quad (A6)$$

$$\nabla \times \mathbf{H}_p = -i\omega_p\epsilon_0(\bar{\epsilon}_p\mathbf{E}_p + 2\bar{\bar{\chi}}\mathbf{E}_s\mathbf{E}_d) + \mathbf{J}_p, \quad (A7)$$

where c is the vacuum speed of light, ϵ_0 is the vacuum permittivity, $\bar{\epsilon}$ is the rank-2 relative permittivity tensor, and $\bar{\bar{\chi}}$ is the rank-3 second-order nonlinear optical susceptibility tensor. All fields, $\bar{\epsilon}$, and $\bar{\bar{\chi}}$ may have spatial dependence. To process the equations, we first divide all equations by ω_j and introduce $\tilde{\mathbf{J}}_j = \mathbf{J}_j/\omega_j$. This is beneficial for nonlinear processes involving frequency mixing, where the conservation laws typically involve photon numbers rather than energy. Next, we take two copies of the system, where system A contains the fields $\mathbf{E}_j^A, \mathbf{H}_j^A$ and sources $\tilde{\mathbf{J}}_j^A$, and system B contains the fields $\mathbf{E}_j^B, \mathbf{H}_j^B$ and sources $\tilde{\mathbf{J}}_j^B$. For Eq. (A3) for system A, we left-multiply by $\mathbf{E}_s^B$ to get

$$\frac{1}{\omega_s}\mathbf{E}_s^B \cdot (\nabla \times \mathbf{H}_s^A) = -i\epsilon_0 \mathbf{E}_s^B \cdot [\bar{\epsilon}_s \mathbf{E}_s^A + 2\bar{\bar{\chi}}\mathbf{E}_p^A \mathbf{E}_d^{A*}] + \mathbf{E}_s^B \cdot \tilde{\mathbf{J}}_s^A. \tag{A8}$$

Performing integration over the system volume V, and using the identity $\mathbf{A} \cdot (\nabla \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B})$, leads to

$$\int_V d\mathbf{r} \frac{i\mathbf{H}_s^B \cdot \mathbf{H}_s^A}{c^2\epsilon_0} - \oint_{\partial V} d\mathbf{A} \cdot \frac{\mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} = \int_V d\mathbf{r}(-i\epsilon_0 \mathbf{E}_s^B) \cdot [\bar{\epsilon}_s \mathbf{E}_s^A + 2\bar{\bar{\chi}}\mathbf{E}_p^A \mathbf{E}_d^{A*}] + \int_V d\mathbf{r} \mathbf{E}_s^B \cdot \tilde{\mathbf{J}}_s^A. \tag{A9}$$

We subtract this equation from itself with A and B interchanged:

$$\begin{aligned}\int_V d\mathbf{r}(\tilde{\mathbf{J}}_s^A \cdot \mathbf{E}_s^B - \tilde{\mathbf{J}}_s^B \cdot \mathbf{E}_s^A) &- \oint_{\partial V} d\mathbf{A} \cdot \frac{\mathbf{E}_s^A \times \mathbf{H}_s^B - \mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} \\ &= i\epsilon_0 \int_V d\mathbf{r}(\mathbf{E}_s^B \cdot \bar{\epsilon}_s \mathbf{E}_s^A - \mathbf{E}_s^A \cdot \bar{\epsilon}_s \mathbf{E}_s^B) \\ &+ 2i\epsilon_0 \int_V d\mathbf{r}(\mathbf{E}_s^B \cdot \bar{\bar{\chi}}\mathbf{E}_p^A \mathbf{E}_d^{A*} - \mathbf{E}_s^A \cdot \bar{\bar{\chi}}\mathbf{E}_p^B \mathbf{E}_d^{B*}).\end{aligned} \tag{A10}$$

In the usual derivation of Lorentz reciprocity, the $\bar{\bar{\chi}}$ term is not present, and the argument is that if $\bar{\epsilon} = \bar{\epsilon}^T$, the $\bar{\epsilon}$ term also vanishes, and so the left-hand side must be zero. We now assume that the system is linearly reciprocal in this sense ($\bar{\epsilon}_j = \bar{\epsilon}_j^T$ for signal and idler), remove the $\bar{\epsilon}$ term on the right-hand side, and continue to consider the $\bar{\bar{\chi}}$ term. Now, for the signal field and the conjugated idler fields, the reciprocity relations read

$$\begin{aligned}\int_V d\mathbf{r}(\tilde{\mathbf{J}}_s^A \cdot \mathbf{E}_s^B - \tilde{\mathbf{J}}_s^B \cdot \mathbf{E}_s^A) &- \oint_{\partial V} d\mathbf{A} \cdot \frac{\mathbf{E}_s^A \times \mathbf{H}_s^B - \mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} \\ &= 2i\epsilon_0 \int_V d\mathbf{r}(\mathbf{E}_s^B \cdot \bar{\bar{\chi}}\mathbf{E}_p^A \mathbf{E}_d^{A*} - \mathbf{E}_s^A \cdot \bar{\bar{\chi}}\mathbf{E}_p^B \mathbf{E}_d^{B*}),\end{aligned} \tag{A11}$$

$$\begin{aligned}\int_V d\mathbf{r}(\tilde{\mathbf{J}}_d^A \cdot \mathbf{E}_d^B - \tilde{\mathbf{J}}_d^B \cdot \mathbf{E}_d^A)^* &- \oint_{\partial V} d\mathbf{A} \cdot \frac{(\mathbf{E}_d^A \times \mathbf{H}_d^B - \mathbf{E}_d^B \times \mathbf{H}_d^A)^*}{\omega_d} \\ &= -2i\epsilon_0 \int_V d\mathbf{r}(\mathbf{E}_d^{B*} \cdot \bar{\bar{\chi}}^* \mathbf{E}_p^{A*} \mathbf{E}_s^A - \mathbf{E}_d^{A*} \cdot \bar{\bar{\chi}}^* \mathbf{E}_p^{B*} \mathbf{E}_s^B).\end{aligned} \tag{A12}$$

Finally, subtracting the above two equations, we get

$$\begin{aligned}\int_V d\mathbf{r}[(\tilde{\mathbf{J}}_s^A \cdot \mathbf{E}_s^B - \tilde{\mathbf{J}}_d^{A*} \cdot \mathbf{E}_d^{B*}) - (\tilde{\mathbf{J}}_s^B \cdot \mathbf{E}_s^A - \tilde{\mathbf{J}}_d^{B*} \cdot \mathbf{E}_d^{A*})] &- \oint_{\partial V} d\mathbf{A} \cdot \left[\left(\frac{\mathbf{E}_s^A \times \mathbf{H}_s^B}{\omega_s} - \frac{\mathbf{E}_d^{A*} \times \mathbf{H}_d^{B*}}{\omega_d}\right) - \left(\frac{\mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} - \frac{\mathbf{E}_d^{B*} \times \mathbf{H}_d^{A*}}{\omega_d}\right)\right] \\ &= 2i\epsilon_0 \int_V d\mathbf{r}[\mathbf{E}_s^B \cdot (\bar{\bar{\chi}}\mathbf{E}_p^A - \mathbf{E}_p^{B*}\bar{\bar{\chi}}^\dagger)\mathbf{E}_d^{A*} \\ &- \mathbf{E}_s^A \cdot (\bar{\bar{\chi}}\mathbf{E}_p^B - \mathbf{E}_p^{A*}\bar{\bar{\chi}}^\dagger)\mathbf{E}_d^{B*}].\end{aligned} \tag{A13}$$

Based on the structure of this relation, we can extend the concept of Lorentz reciprocity to the nonlinear system by requiring that the right-hand side of Eq. (A13) vanishes. One general condition would be

$$\bar{\bar{\chi}}\mathbf{E}_p^A = \mathbf{E}_p^{B*}\bar{\bar{\chi}}^\dagger. \tag{A14}$$

In the special case of a real susceptibility $\bar{\bar{\chi}}$ that is symmetric over its signal and idler indices (i.e., when Kleinman's symmetry is satisfied [22]), the condition reduces to

$$\mathbf{E}_p^A(\mathbf{r}) = \mathbf{E}_p^{B*}(\mathbf{r}), \quad \text{for all } \mathbf{r} \text{ such that } \bar{\bar{\chi}}(\mathbf{r}) \neq \mathbf{0}. \tag{A15}$$

In the undepleted pump approximation, $\mathbf{E}_p$ is independent of $\mathbf{E}_s$ and $\mathbf{E}_d$, and can be solved from

$$\nabla \times \nabla \times \mathbf{E}_p - \frac{\omega_p^2}{c^2}\bar{\epsilon}_p \mathbf{E}_p = -i\mu_0\omega_p \mathbf{J}_p. \tag{A16}$$

Thus, when the solution to Eq. (A16) is real in the regions of the system where $\bar{\bar{\chi}} \neq \mathbf{0}$, the following reciprocity relation between signal and idler fields holds,

$$\begin{aligned}\int_V d\mathbf{r}\left[\left(\frac{\mathbf{J}_s^A \cdot \mathbf{E}_s^B}{\omega_s} - \frac{\mathbf{J}_d^{A*} \cdot \mathbf{E}_d^{B*}}{\omega_d}\right) - \left(\frac{\mathbf{J}_s^B \cdot \mathbf{E}_s^A}{\omega_s} - \frac{\mathbf{J}_d^{B*} \cdot \mathbf{E}_d^{A*}}{\omega_d}\right)\right] &= \oint_{\partial V} d\mathbf{A} \cdot \left[\left(\frac{\mathbf{E}_s^A \times \mathbf{H}_s^B}{\omega_s} - \frac{\mathbf{E}_d^{A*} \times \mathbf{H}_d^{B*}}{\omega_d}\right) \right. \\ &\left. - \left(\frac{\mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} - \frac{\mathbf{E}_d^{B*} \times \mathbf{H}_d^{A*}}{\omega_d}\right)\right],\end{aligned} \tag{A17}$$

or, when there are no current sources within the system boundary ∂V,

$$\oint_{\partial V} d\mathbf{A} \cdot \left(\frac{\mathbf{E}_s^A \times \mathbf{H}_s^B}{\omega_s} - \frac{\mathbf{E}_d^{A*} \times \mathbf{H}_d^{B*}}{\omega_d}\right) = \oint_{\partial V} d\mathbf{A} \cdot \left(\frac{\mathbf{E}_s^B \times \mathbf{H}_s^A}{\omega_s} - \frac{\mathbf{E}_d^{B*} \times \mathbf{H}_d^{A*}}{\omega_d}\right). \tag{A18}$$

Now we proceed to derive the reciprocity constraints on an S matrix based on Eq. (A18). For a scattering system with predefined ports, the enclosing surface will be chosen to perpendicularly cut through each port at $z = 0$. The complex electric and magnetic signal field orthogonal to the propagation axis can then be expressed as

$$\mathbf{E}_{s,k} = (a_{s,k} + b_{s,k})\mathbf{e}_{s,k}, \tag{A19}$$

$$\mathbf{H}_{s,k} = (a_{s,k} - b_{s,k})\mathbf{h}_{s,k}, \tag{A20}$$

where $\mathbf{e}_{s,k}$ and $\mathbf{h}_{s,k}$ are the field distributions for the signal ports. Here, we assume that the port consists of lossless and reciprocal materials, such that both the $\mathbf{e}_{s,k}$ and $\mathbf{h}_{s,k}$ components can be made real. This assumption is standard in the discussion of the S matrix [3,4] since it allows one to focus on the possibly unusual properties of the scattering matrix itself. Idler fields are defined similarly.

We now expand the reciprocity relation using the port amplitudes, which results in

$$\sum_k \left(a^A_{s,k}b^B_{s,k} - a^B_{s,k}b^A_{s,k}\right)\int_{A_k} d\mathbf{A}\cdot\frac{\mathbf{e}^A_{s,k}\times\mathbf{h}^B_{s,k}}{\omega_s} - \sum_k \left(a^{A*}_{d,k}b^{B*}_{d,k} - a^{B*}_{d,k}b^{A*}_{d,k}\right)\int_{A_k} d\mathbf{A}\cdot\frac{\mathbf{e}^A_{d,k}\times\mathbf{h}^B_{d,k}}{\omega_d} = 0. \tag{A21}$$

We will normalize the fields such that $|a|^2$ and $|b|^2$ represent the photon number flux. Specifically,

$$\int_{A_k} d\mathbf{A}\cdot\left(\mathbf{e}^A_{s,k}\times\mathbf{h}^B_{s,k}\right) = \hbar\omega_s, \tag{A22}$$

$$\int_{A_k} d\mathbf{A}\cdot\left(\mathbf{e}^A_{d,k}\times\mathbf{h}^B_{d,k}\right) = \hbar\omega_d. \tag{A23}$$

The integrals cancel out, and we are left with

$$\sum_k a^A_{s,k}b^B_{s,k} - \sum_k a^{A*}_{d,k}b^{B*}_{d,k} = \sum_k a^B_{s,k}b^A_{s,k} - \sum_k a^{B*}_{d,k}b^{A*}_{d,k}. \tag{A24}$$

Each side could be interpreted as an inner product between input and output amplitudes. Similar to the main text, we construct an input vector using both field amplitudes and conjugated amplitudes through $\mathbf{a} = (a_{s,1}\cdots a_{s,n_s}, a^*_{d,1}\cdots a^*_{d,n_d})^T$ and an auxiliary matrix,

$$\overline{P}_z = \begin{pmatrix} I_{n_s\times n_s} & \mathbf{0} \\ \mathbf{0} & -I_{n_d\times n_d}\end{pmatrix}. \tag{A25}$$

where n_s (n_d) is the number of signal (idler) ports. The reciprocity condition can then be rewritten as

$$\mathbf{a}^A\cdot\overline{P}_z\mathbf{b}^B = \mathbf{b}^A\cdot\overline{P}_z\mathbf{a}^B. \tag{A26}$$

As $\mathbf{b} = S\mathbf{a}$ and the reciprocity holds for all $\mathbf{a}^A$ and $\mathbf{a}^B$, we finally get

$$\overline{P}_z S = S^T\overline{P}_z. \tag{A27}$$

Moving the $\overline{P}_z$ on the right to the left side recovers Eq. (20).

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