Nuclear polarizability effects in $^3He^+$ hyperfine splitting

Vojtěch Patkóš,[1] Vladimir A. Yerokhin,[2] and Krzysztof Pachucki[3]
[1]*Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Prague 2, Czech Republic*
[2]*Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*
[3]*Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland*

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The nuclear polarizability effects in the hyperfine splitting of light atomic systems are not well known. The only system for which they were previously calculated is the hydrogen atom, where these effects were shown to contribute about 5% of the total nuclear correction. One generally expects the polarizability effects to become more pronounced for composite nuclei. In the present paper we determine the nuclear polarizability correction to the hyperfine splitting in He^+ by comparing the effective Zemach radius deduced from experimental hyperfine splitting with the Zemach radius obtained from electron scattering. We obtain a surprising result that the nuclear polarizability of the helion yields just 3% of the total nuclear correction, which is smaller than for the proton.

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I. INTRODUCTION

The hyperfine structure (HFS) in atoms and ions is determined not only by the value of the nuclear magnetic moment but also by the distribution of the charge and the magnetic moment over the nucleus and by the nuclear vector polarizability. These effects cannot be calculated accurately at present and are the main source of uncertainty in theoretical predictions.

The nuclear effects in HFS are typically divided into two parts: elastic and inelastic ones. The elastic effects are expressed in terms of charge and magnetic form factors, whereas the dominant inelastic effect is nuclear polarizability. It is well known that the dominant nuclear effect is of the elastic kind and is proportional to the so-called Zemach radius [1], which is the convolution of electric and magnetic form factors.

Little is known about the nuclear polarizability in HFS, mainly due to the complexity of its theoretical description. The effect is most pronounced for the muonic deuterium (μD) HFS, where it is supposed to be as large as the elastic nuclear contribution. The theoretical predictions for the μD HFS [2] are in conflict with the experimental results [3].

For electronic atoms the nuclear polarizability effects are much smaller than for muonic atoms but are not negligible. Even for hydrogen the inelastic effects were shown to be significant and yield about 5% of the elastic contribution [4]. For other atomic systems the inelastic HFS contributions are merely unknown. Low [5] has derived a simple formula for the leading nuclear-structure contribution, treating the nucleus as a composite system of protons and neutrons and avoiding the use of elastic nuclear form factors. Friar [6] estimated the inelastic contribution to HFS in deuterium beyond the Low formula and concluded that it is not significant. Khriplovich *et al.* [7,8] claimed to derive the leading logarithmic part of the inelastic contribution, but later it was demonstrated by one of us [9] that this contribution vanishes in a more complete treatment. In that work a formula for the inelastic contribution to atomic HFS was derived, but its complexity prevented any practical applications so far. So, the inelastic contribution to HFS in light atomic systems is merely unknown at present.

In the absence of theoretical calculations of the nuclear polarizability correction, in the present paper we perform its determination from experimental HFS splitting. We rely on the fact that all HFS corrections originating from the relativistic and quantum electrodynamics (QED) effects for the point nucleus can be calculated up to very high accuracy and that the elastic form factors of the nucleus can be extracted from analyzing the electron scattering data.

We introduce the effective Zemach radius $\widetilde{r}_Z$ which includes the inelastic nuclear contribution and can be accurately determined from high-precision experimental results for the HFS splitting. On the other hand, the standard elastic Zemach radius r_Z was determined from the electron-scattering data by Sick [10]. The difference $\widetilde{r}_Z - r_Z$ gives us the result for the nuclear polarizability correction.

The significance of the effective Zemach radius is that it can be used to obtain highly accurate theoretical predictions for the HFS of the neutral helium, which will be a subject of our forthcoming investigations.

II. HFS IN HYDROGENIC ATOMS

The leading hyperfine splitting of an $1S$ state is delivered by the so-called Fermi energy E_F, which is given by

$$E_F = \frac{8}{3}(Z\alpha)^4\frac{\mu^3}{m\,M}(1+k), \tag{1}$$

where Z and M are the nuclear charge number and the mass, respectively, μ is the reduced mass of the atom, and k is the nuclear magnetic moment anomaly $k = (g-2)/2$, with the natural nuclear g factor defined as

$$\vec{\mu} = \frac{Z\,e}{2\,M}\,g\,\vec{I}. \tag{2}$$

Here, $\vec{\mu}$ and $\vec{I}$ are the magnetic moment and the spin of the nucleus, respectively. The natural g factor is related to the standard nuclear g_I factor by

$$g = g_I \frac{M}{Z\, m_p}. \tag{3}$$

g_I can be obtained from the recent measurement of the shielded g_I in $^3He^+$ in Ref. [11] and the most accurate calculation of the shielding factor in Ref. [12], namely

$$g_I = -4.255\,250\,699\,9(34), \tag{4}$$

and therefore

$$g = -6.368\,307\,500\,5(51). \tag{5}$$

The complete hyperfine splitting is conveniently represented as

$$E_{\rm hfs} = E_F\,(1+\delta), \tag{6}$$

where δ represents the correction to the Fermi energy due to relativistic, QED, and nuclear effects. Within the approach of the nonrelativistic QED (NRQED), δ is represented as an expansion in terms of the fine-structure constant α,

$$\delta = \kappa + \delta^{(2)} + \delta^{(3)} + \delta^{(4)} + \delta^{(1)}_{\rm nuc} + \delta^{(1)}_{\rm rec} + \delta^{(2)}_{\rm nuc} + \delta^{(2)}_{\rm rec}, \tag{7}$$

where κ is the magnetic moment anomaly of the free electron, $\kappa = \alpha/(2\pi) + O(\alpha^2)$, and $\delta^{(i)}$, $\delta^{(i)}_{\rm nuc}$, and $\delta^{(i)}_{\rm rec}$ are the QED, nuclear, and recoil corrections of order α^i, respectively.

The QED corrections of order α^2, α^3, and α^4 are given by

$$\delta^{(2)} = \frac{3}{2}(Z\alpha)^2 + \alpha(Z\alpha)\left(\ln(2) - \frac{5}{2}\right), \tag{8}$$

$$\begin{aligned}\delta^{(3)} = {}& \frac{\alpha(Z\alpha)^2}{\pi}\left[-\frac{8}{3}\ln(Z\alpha)\left(\ln(Z\alpha) - \ln(4) + \frac{281}{480}\right)\right.\\ &\left. +17.122\,338\,751\,3 - \frac{8}{15}\ln(2) + \frac{34}{225}\right]\\ &+ \frac{\alpha^2(Z\alpha)}{\pi}\,0.770\,99(2),\end{aligned} \tag{9}$$

$$\begin{aligned}\delta^{(4)} = {}& \frac{17}{8}(Z\alpha)^4 + \alpha(Z\alpha)^3\left[\left(\frac{547}{48} - 5\ln(2)\right)\ln(Z\alpha)\right.\\ &\left. -4.493\,23(3) + \frac{13}{24}\ln 2 + \frac{539}{288}\right]\\ &- \frac{\alpha^2(Z\alpha)^2}{\pi^2}\left[\frac{4}{3}\ln^2(Z\alpha) + 1.278\ln(Z\alpha) + 10.0(2.5)\right]\\ &\pm \frac{\alpha^3(Z\alpha)}{\pi^2}.\end{aligned} \tag{10}$$

Most of the results summarized by Eqs. (8)–(10) can be found in Refs. [13,14]. We mention that the $\alpha(Z\alpha)^2$ part of $\delta^{(3)}$ contains the improved numerical value for the constant term from the Appendix, and the $\alpha(Z\alpha)^3$ part of $\delta^{(4)}$ includes higher orders in $Z\alpha$ for $Z = 2$ from Ref. [15]. The last term in $\delta^{(4)}$ represents the estimate of the unknown three-loop QED correction.

$\delta^{(1)}_{\rm nuc}$ is the leading $O(Z\alpha)$ nuclear structure correction. It is a sum of the elastic contribution proportional to the Zemach radius r_Z and the nuclear polarizability contribution. The Zemach radius is defined as

$$r_Z = \int d^3r_1 \int d^3r_2\, \rho_E(\vec r_1)\,\rho_M(\vec r_2)\,|\vec r_1 - \vec r_2|, \tag{11}$$

TABLE I. Contributions to HFS in the $^3He^+$ ion and determination of $\widetilde{r}_Z$. The nuclear charge radius $r_C = 1.973(14)$ fm [10].

Term	Value	$\times E_F$ (kHz)
1	1	−8 656 527.892(7)
κ	0.001 159 65	−10 038.6
$\delta^{(2)}$	0.000 127 07	−1100.0
$\delta^{(3)}$	−0.000 019 49	168.7
$\delta^{(4)}$	−0.000 000 75	6.5
$\delta^{(1)}_{\rm rec}$	−0.000 012 17(60)	105.4(5.3)
$\delta^{(2+)}_{\rm nuc}$	−0.000 002 89(3)	25.0
$\delta^{(2)}_{\rm rec}$	−0.000 001 16(18)	10.1(1.6)
Theory without $\delta^{(1)}_{\rm nuc}$	1.001 250 26(63)	−8 667 350.8(5.5)
Experiment [11]	1.001 053 77	−8 665 649.865 77(26)
$\delta^{(1)}_{\rm nuc}$	−0.000 196 49(63)	1701.0(5.5)
$\widetilde{r}_Z$ This work	2.600(8) fm	
r_Z [11]	2.608(24) fm	
r_Z [10] expt.	2.528(16) fm	
r_Z [16] nucl. theor.	2.539(3)(19) fm	
$\widetilde{r}_Z - r_Z$(expt.) =	0.072(18) fm	

where ρ_E and ρ_M are the Fourier transforms of the electric and magnetic form factors of the nucleus normalized to unity.

In this paper we introduce the *effective* Zemach radius $\widetilde{r}_Z$ that includes both the elastic and the inelastic contributions of order α. It is related to $\delta^{(1)}_{\rm nuc}$, by definition, as

$$\delta^{(1)}_{\rm nuc} = -2\,Z\,\alpha\,\mu\,\widetilde{r}_Z. \tag{12}$$

The difference $\widetilde{r}_Z - r_Z$ can then be interpreted as the inelastic nuclear-polarizability contribution. In the present paper, we determine $\delta^{(1)}_{\rm nuc}$ and therefore $\widetilde{r}_Z$ by taking the difference of the experimental HFS value and the theoretical prediction without $\delta^{(1)}_{\rm nuc}$.

The higher-order nuclear-structure corrections and the recoil corrections are much smaller than $\delta^{(1)}_{\rm nuc}$. They will be calculated assuming that the distributions of the nuclear charge and the magnetic moment are the same, $\rho_E(r) = \rho_M(r) \equiv \rho(r)$. Here, we will use the exponential and the Gaussian models for $\rho(r)$ (see Tables I and II). The parameters of the models will be fixed by matching the Zemach radius to the experimental value from Ref. [10]. In order to estimate the model dependence of our results, we take twice the difference of values obtained with the exponential and the Gaussian models.

The α^2 nuclear-structure correction is given by the sum of the relativistic and the radiative corrections,

$$\delta^{(2)}_{\rm nuc} = \delta^{(2)}_{\rm nuc,rel} + \delta^{(2)}_{\rm nuc,rad}. \tag{13}$$

The relativistic correction is given by [2]

$$\delta^{(2)}_{\rm nuc,rel} = \frac{4}{3}(m\,r_C\,Z\alpha)^2\left[-1 + \gamma + \ln(2\,m\,r_{CC}\,Z\alpha) + \frac{r_M^2}{4\,r_C^2}\right], \tag{14}$$

where r_M is the root-mean-square magnetic radius and $r_{CC}/r_C = 5.274\,565$ for the exponential charge distribution.

TABLE II. Various results for the exponential and Gaussian models of the nuclear charge and magnetization distributions. F_e is the charge distribution function, $V_C(r) = -{}^{Z\alpha}/_r\, F_e(r)$, whereas F_m is the magnetic moment distribution function, $H_\mu = {}^{|e|}/_{4\pi}\, \boldsymbol{\alpha}\cdot\boldsymbol{\mu}\times{}^{\vec r}/_{r^3}\, F_m(r)$.

	Exponential	Gaussian
$\rho(q^2)$	$\frac{\lambda^4}{(\lambda^2+q^2)^2}$	$e^{-aq^2/2}$
$\rho(r)$	$\frac{\lambda^3}{8\pi} e^{-\lambda r}$	$\frac{1}{(2\pi a)^{3/2}}\, e^{-r^2/(2a)}$
r_C	$2\sqrt{3}/\lambda$	$\sqrt{3a}$
r_Z	$35/(8\,\lambda)$	$4\sqrt{a/\pi}$
$F_e(r)$	$1 - e^{-\lambda r}(1 + \lambda r/2)$	$\mathrm{erf}\left(\frac{r}{\sqrt{2a}}\right)$
$F_m(r)$	$1 - e^{-\lambda r}(1 + \lambda r + (\lambda r)^2/2)$	$\mathrm{erf}\left(\frac{r}{\sqrt{2a}}\right) - \frac{\sqrt{2}\,r}{\sqrt{\pi a}}\, e^{-r^2/(2a)}$

The numerical contribution of this correction is quite small, $\delta^{(2)}_{\rm nuc,rel} = -5.4\times10^{-8}$ for He$^+$. Surprisingly, the next order in $Z\alpha$ correction yields a numerically larger contribution, because it is approximately proportional to $m\,r_Z$, instead of $(m\,r_C)^2$. For this reason we replace $\delta^{(2)}_{\rm nuc,rel}$ by $\delta^{(2+)}_{\rm nuc,rel}$, which is evaluated numerically in this work for $Z = 2$. The resulting contribution is

$$\delta^{(2+)}_{\rm nuc,rel} = -49.1(5)\times10^{-8}. \qquad (15)$$

The above value is obtained with the exponential model; its uncertainty represents the expected model dependence and is obtained as twice the difference from the results obtained with the Gaussian model.

The radiative finite nuclear size correction is given within the exponential distribution model by [17]

$$\delta^{(2)}_{\rm nuc,rad} = -2\,Z\,\alpha\,\mu\,r_Z\,\frac{\alpha}{\pi}\left(-\frac{5}{4} + \frac{2}{3}\ln\frac{\lambda^2}{m^2} - \frac{634}{315}\right), \qquad (16)$$

where the first term comes from the electron self-energy and next two from the vacuum polarization. Its numerical contribution for He$^+$ is

$$\delta^{(2)}_{\rm nuc,rad} = -\,240.1(2.4)\times10^{-8}, \qquad (17)$$

assuming a similar 1% uncertainty as for $\delta^{(2+)}_{\rm nuc,rel}$.

$\delta^{(1)}_{\rm rec}$ is the leading-order (in α) nuclear recoil correction, given by [18]

$$\delta^{(1)}_{\rm rec} = -\frac{Z\alpha}{\pi}\frac{m}{M}\frac{3}{8}\left\{g\left[\gamma - \frac{7}{4} + \ln(m\,r_{M^2})\right] -4\left[\gamma + \frac{9}{4} + \ln(m\,r_{EM})\right] - \frac{12}{g}\left[\gamma - \frac{17}{12} + \ln(m\,r_{E^2})\right]\right\}, \qquad (18)$$

where $\gamma \approx 0.577\,216$ is Euler's gamma constant,

$$\ln r_{EM} = \int d^3r_1 \int d^3r_2\, \rho_E(\vec r_1)\rho_M(\vec r_2)\ln|\vec r_1 - \vec r_2|, \qquad (19)$$

and $\ln r_{M^2}$ and $\ln r_{E^2}$ defined analogously. Within the exponential model,

$$\ln m\,r_{EM} = \ln m\,r_{E^2} = \ln m\,r_{M^2} = -\ln\frac{\lambda}{m} - \gamma + \frac{23}{12}, \qquad (20)$$

the numerical contribution for He$^+$ is

$$\delta^{(1)}_{\rm rec} = -1217(60)\times10^{-8}, \qquad (21)$$

where we ascribed a 5% uncertainty due to an approximate exponential parametrization of the helion form factors.

The higher-order recoil correction is the sum of the relativistic and the radiative-recoil contributions,

$$\delta^{(2)}_{\rm rec} = \delta^{(2)}_{\rm rec,rel} + \delta^{(2)}_{\rm rec,rad}. \qquad (22)$$

The relativistic recoil correction was derived in Ref. [19]. It has a finite point-nucleus limit and is given by

$$\delta^{(2)}_{\rm rec,rel} = (Z\alpha)^2\frac{\mu^2}{m\,M}\left\{-\left[1 + 7\,k + \frac{7}{1+k}\right]\frac{\ln(Z\alpha)}{4} - \left[9 + 11\,k + \frac{23}{1+k}\right]\frac{\ln 2}{4} + \frac{1}{36}\left[-20 + 31\,k + \frac{150}{1+k}\right]\right\}. \qquad (23)$$

The numerical contribution for He$^+$ is $\delta^{(2)}_{\rm rec,rel} = -116.3\times10^{-8}$.

The radiative recoil effect to HFS is in general unknown. Karshenboim [17] presented only a rough estimation for this correction in hydrogen. Instead of using this estimate, we assume the logarithmic enhancement for this correction and estimate the uncertainty due to its omission as

$$\delta^{(2)}_{\rm rec,rad} = \pm\delta^{(1)}_{\rm rec}\frac{\alpha}{\pi}\ln\frac{\lambda}{m} = \pm18.4\times10^{-8}. \qquad (24)$$

There are further higher-order corrections, such as the muonic and hadronic vacuum polarization, weak interactions, etc. These corrections are smaller than the uncertainty of $\delta^{(1)}_{\rm rec}$ and thus neglected (see the Supplemental Material of Ref. [11]).

III. RESULTS AND DISCUSSION

In Table I we collect all theoretical contributions to the ground-state hyperfine splitting of the ^{3}He$^+$ ion. It is interesting to note that the recoil correction $\delta^{(1)}_{\rm rec}$ yields about 6% of the total nuclear contribution, despite the smallness of the electron-nucleus mass ratio.

The difference of the theoretical prediction without $\delta^{(1)}_{\rm nuc}$ and the experimental value from Ref. [11] determines $\delta^{(1)}_{\rm nuc}$ and therefore the effective Zemach radius $\widetilde{r}_Z$. The difference of the effective Zemach radius and the elastic Zemach radius obtained in Ref. [10] from the electron-scattering data yields $\widetilde{r}_Z - r_Z = 0.072(18)$ fm. We interpret this difference as the contribution of the nuclear polarizability.

It is very remarkable that the numerical contribution of the nuclear polarizability is quite small, just about 3% of the elastic nuclear correction. This is smaller than for hydrogen, where the inelastic HFS contribution is about 5% [4]. We find this smallness very intriguing for the following reason.

The helion, being a composite nucleus, is a relatively weakly bound system as compared to the proton. This can be illustrated by comparing the proton mean excitation energy of 400 MeV with the helion proton-separation energy of 5 MeV. The nuclear polarizability is expected to be roughly proportional to the inverse of these energies. Indeed, for the Lamb shift the corresponding energy shifts are −0.109(12) kHz [4] and −55(5.5) kHz [20], for hydrogen and $^3He^+$, respectively. However, for the HFS the expected relation between hydrogen and He^+ fails spectacularly. At present we do not have any explanation for why the inelastic HFS nuclear contribution for a helion is smaller than for the proton. It should be noted that the authors of Ref. [11] claimed to evaluate the nuclear polarizability correction and obtained a very small result (vanishing within their uncertainty), so within their uncertainty $\widetilde{r}_Z = r_Z$.

Summarizing and bearing in mind the discrepancy for μD HFS [2] and for ^{6}Li HFS [21], a comprehensive theory for the inelastic (polarizability) correction to the atomic HFS with the composite nucleus is still lacking.

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APPENDIX: ONE-LOOP $\alpha(Z\alpha)^2$ SELF-ENERGY CORRECTION TO HFS

In a previous work [22] devoted to the $\alpha(Z\alpha)^2$ one-loop self-energy contribution the hyperfine splitting bugs crept into the formulas for intermediate contributions, while the final result was correct. Here, we remove all these bugs and present numerical integrals with a higher precision, which might be relevant in future studies of HFS in light atomic systems.

Using the notation from a former work [22] and thus setting for convenience $Z = 1$, the one-loop self-energy contribution to HFS in a hydrogenlike system is represented in terms of the dimensionless function F.

$$\Delta E = E_F \frac{\alpha}{\pi} \alpha^2 F, \tag{A1}$$

which is split into three parts,

$$F = F_L + F_M + F_H. \tag{A2}$$

The low-energy part,

$$\begin{aligned} F_L = {} & \frac{781}{18} + \frac{4\pi^2}{3} - \frac{166\ln(2)}{3} - \frac{2\ln(2)^2}{3} - 4\ln(\alpha) \\ & + 8\ln(2)\ln(\alpha) - \frac{8\ln(\alpha)^2}{3} + 2\ln(\epsilon) - 4\ln(2)\ln(\epsilon) \\ & + \frac{8\ln(\alpha)\ln(\epsilon)}{3} - \frac{2\ln(\epsilon)^2}{3} + n_1 + n_2, \end{aligned} \tag{A3}$$

is expressed in terms of two integrals, Eqs. (26) and (27) of Ref. [22],

$$n_1 = -0.085\,740\,323\,701\,4, \tag{A4}$$
$$n_2 = 0.067\,496\,936\,500\,3, \tag{A5}$$

which we present here with a much higher precision.

The middle-energy part,

$$F_M = F_{M1} + F_{M2} + F_{M3} + F_{M4}, \tag{A6}$$

consists of four subparts,

$$F_{M1} = \frac{1}{2}\left[1 - \frac{1}{2}\ln\left(\frac{2\alpha}{\rho}\right)\right], \tag{A7}$$

$$F_{M2} = -\frac{8}{3}\left[\frac{1}{2} - \ln\left(\frac{2\alpha}{\rho}\right)\right]\ln\left(\frac{m}{\mu}\right), \tag{A8}$$

$$F_{M3} = 4\ln\left(\frac{m}{\mu}\right) - \frac{1}{2}, \tag{A9}$$

$$F_{M4} = \ln\left(\frac{2\alpha}{\rho}\right). \tag{A10}$$

Their sum is

$$F_M = \frac{8}{3}\ln\left(\frac{m}{\mu}\right) + \frac{3}{4}\ln\left(\frac{2\alpha}{\rho}\right) + \frac{8}{3}\ln\left(\frac{m}{\mu}\right)\ln\left(\frac{2\alpha}{\rho}\right) \tag{A11}$$

$$= \frac{20}{9} - \frac{8}{3}\ln(2\epsilon) + \frac{107}{36}\ln\left(\frac{2\alpha}{\rho}\right) - \frac{8}{3}\ln(2\epsilon)\ln\left(\frac{2\alpha}{\rho}\right). \tag{A12}$$

The high-energy part is

$$\begin{aligned} F_H = {} & -\frac{335}{36} - \frac{11\pi^2}{18} + \frac{190\ln(2)}{9} + \frac{2\ln(2)^2}{3} + \frac{2\ln(\epsilon)}{3} \\ & + \frac{20\ln(2)\ln(\epsilon)}{3} + \frac{2\ln(\epsilon)^2}{3} + \frac{107\ln(\rho)}{36} \\ & - \frac{8\ln(2)\ln(\rho)}{3} - \frac{8\ln(\epsilon)\ln(\rho)}{3} - \frac{5\zeta(3)}{4}. \end{aligned} \tag{A13}$$

The final result is a sum of three parts as given in Eq. (A2),

$$\begin{aligned} F = {} & n_1 + n_2 + \frac{1307}{36} + \frac{13\pi^2}{18} - \frac{407\ln(2)}{12} - \frac{8\ln(2)^2}{3} \\ & - \frac{37\ln(\alpha)}{36} + \frac{16\ln(2)\ln(\alpha)}{3} - \frac{8\ln(\alpha)^2}{3} - \frac{5\zeta(3)}{4}, \end{aligned} \tag{A14}$$

and the constant term is equal to 17.122 338 751 3 . . . .

In the calculation of individual parts, we sometimes used the photon mass μ regularization and equivalently the photon cutoff ϵ. The conversion formulas from ϵ to μ are the following:

$$\ln(\epsilon) = \ln\left(\frac{\mu}{2}\right) + \frac{5}{6}, \tag{A15}$$

$$\ln^2(\epsilon) = \left[\ln\left(\frac{\mu}{2}\right) + \frac{5}{6}\right]^2 + \frac{31}{36} - \frac{\pi^2}{12}. \tag{A16}$$

[1] A. C. Zemach, Phys. Rev. **104**, 1771 (1956).
[2] M. Kalinowski, K. Pachucki, and V. A. Yerokhin, Phys. Rev. A **98**, 062513 (2018).
[3] R. Pohl, F. Nez, L. M. P. Fernandes, F. D. Amaro, F. Biraben, J. M. R. Cardoso, D. S. Covita, A. Dax, S. Dhawan, M. Diepold *et al.*, Science **353**, 669 (2016).
[4] O. Tomalak, Phys. Rev. D **99**, 056018 (2019).
[5] F. Low, Phys. Rev. **77**, 361 (1950).
[6] J. L. Friar and G. L. Payne, Phys. Lett. B **618**, 68 (2005).
[7] I. B. Khriplovich, A. I. Milstein, and S. S. Petrosyan, Zh. Eksp. Teor. Fiz. **109**, 1146 (1996) [Sov. Phys. JETP **82**, 616 (1996)]; Phys. Lett. B **366**, 13 (1996).
[8] I. B. Khriplovich and A. I. Milstein, Zh. Eksp. Teor. Fiz. **125**, 205 (2004) [J. Exp. Theor. Phys. **98**, 181 (2004)].
[9] K. Pachucki, Phys. Rev. A **76**, 022508 (2007).
[10] I. Sick, Phys. Rev. C **90**, 064002 (2014).
[11] A. Schneider, B. Sikora, S. Dickopf, M. Müller, N. S. Oreshkina, A. Rischka, I. A. Valuev, S. Ulmer, J. Walz, Z. Harman *et al.*, Nature (London) **606**, 878 (2022).
[12] D. Wehrli, A. Spyszkiewicz-Kaczmarek, M. Puchalski, and K. Pachucki, Phys. Rev. Lett. **127**, 263001 (2021).
[13] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rep. **342**, 63 (2001).
[14] E. Tiesinga, P. J. Mohr, D. B. Newell, and B. N. Taylor, Rev. Mod. Phys. **93**, 025010 (2021).
[15] V. A. Yerokhin and U. D. Jentschura, Phys. Rev. Lett. **100**, 163001 (2008).
[16] N. Nevo Dinur, O. J. Hernandez, S. Bacca, N. Barnea, C. Ji, S. Pastore, M. Piarulli, and R. B. Wiringa, Phys. Rev. C **99**, 034004 (2019).
[17] S. G. Karshenboim, Phys. Lett. A **225**, 97 (1997).
[18] K. Pachucki, Phys. Rev. A **106**, 022802 (2022).
[19] G. T. Bodwin and D. R. Yennie, Phys. Rev. D **37**, 498 (1988).
[20] K. Pachucki and A. M. Moro, Phys. Rev. A **75**, 032521 (2007).
[21] W. Sun, P.-P. Zhang, P.-P. Zhou, S.-L. Chen, Z.-Q. Zhou, Y. Huang, X.-Q. Qi, Z.-C. Yan, T.-Y. Shi, G. W. F. Drake *et al.*, arXiv:2303.07939.
[22] K. Pachucki, Phys. Rev. A **54**, 1994 (1996).