Hierarchical activation of quantum nonlocality: Stronger than local indistinguishability

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The quantum state discrimination primitive becomes highly nontrivial in the limited measurement setting and leads to different classes of impossibility, viz., indistinguishability, unmarkability, irreducibility, etc. These phenomena, often referred to as another nonlocal aspect of quantum theory, have utmost importance in the domain of data hiding, secret sharing, etc. Motivated by this, recently a significant effort has been devoted to activate local indistinguishability from locally distinguishable nontrivial sets of quantum states. In the present work, we introduce other stronger notions of quantum nonlocality and depict a series of hierarchical nonlocality activation from nontrivial sets of locally distinguishable entangled states. Moreover, we appropriately moderate the strict condition of nontriviality, introduced in earlier literature, and come up with interesting examples in support of our claim.

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I. INTRODUCTION

The formalism of quantum theory assures perfect discrimination among any set of orthogonal preparations. However, identification of a multipartite quantum state, given from an orthogonal set, by some spatially separated players is a nontrivial task whenever they are restricted in their actions, i.e., allowed to perform only local operations along with classical communication (LOCC) [1–13]. Such an inadequacy of local operations in the task of state discrimination is often identified as a nonlocal feature of quantum theory according to Bennett *et al.* [14]. However, the unstructured configuration of LOCC protocols has motivated researchers to introduce a weaker version of the same as a preliminary step for state discrimination, namely, state elimination under orthogonality preserving local measurements (OPLM) [15]. The impossibility of such an elimination, termed local irreducibility of orthogonal quantum states, is depicted as a stronger variant of quantum nonlocality and has fueled a vast amount of studies [16–21]. For instance, while the two-qubit Bell basis is simultaneously indistinguishable as well as irreducible, the three-qubit Greenberger-Horne-Zeilinger (GHZ) basis can be reduced under OPLM, although it remains indistinguishable, in every bipartition [15]. On the other hand, recently another weak variant of state discrimination was introduced by Sen *et al.* [22] as an economic alternative for data hiding [23–25] and secret sharing [26,27]. Instead of identifying a multipartite quantum state given from an orthogonal set, the task is there to mark the complete set, and hence the impossibility to accomplish such a task is termed *local unmarkability* [22]. While an explicit example of locally indistinguishable four ququad states [8] has been shown to be *locally markable* with three-bit of residual entanglement [22], the possibility of local distinguishability always implies local markability.

Motivated by the immense cryptographic importance of such local indistinguishabilities [23–27], recently, Bandopadhyay and Halder introduced a notion of activating such nonlocal features from locally distinguishable sets [28]. They characterized several sets of locally distinguishable orthogonal entangled states, which can be converted to a locally indistinguishable one via OPLM. Notably, the presence of the same feature has recently been reported for product states [29], which might possibly be the strongest versions of such indistinguishabilities [30]. Such nonlocality activation phenomena help the spatially separated parties to modify their shared key according to their trustworthiness, should they change after the key has been shared [30]. However, the impossibility of self-testing and device-independent security proof for product states suggests the distribution of entangled quantum states (which can be self-tested) among the agents, as an initial secret key. Then according to their updated trustworthiness, they may want to obtain a key which is locally indistinguishable but can be decoded when all of them join together. This demands performing OPLM only, by the players on their individual subsystems. To activate an even stronger secrecy, they may wish to disallow all others from collaborating and even partially updating the information regarding the shared key. This simply implies obtaining a set of orthogonal states, none of which can be eliminated even if all but one of the parties join together. On the other hand, the distributor may supply a complete set of orthogonal entangled states among the players and dictate them to decode the ordering among the elements of that set, as the hidden secret. This task of local state marking consumes lesser resources than that of the local state discrimination, as mentioned earlier [22]. Consequently, the players, upon their updated trustworthiness, may be curious to activate another stronger nonlocality, from the state-marking primitive, at any later instant. In the present work, we have dealt with all these stronger notions of nonlocality activation from the set of locally distinguishable and hence locally markable set of entangled states. A notable issue in all these activation phenomena is to share *nontrivial* sets of orthogonal entangled states, which is free from local redundancy. This demands that the locally distinguishable secret should not be trivially updated to an indistinguishable one, just by tracing out any of its subsystems. A plausible restriction to avoid such redundant solutions is to consider only those sets as the initial secret, where discarding any of the constituents produces a nonorthogonal set and then the question of discrimination becomes irrelevant [28–30]. However, here we will revisit this criterion once again and relax it further to incorporate an interesting example for activating a stronger nonlocal feature.

II. ACTIVATION OF LOCAL UNMARKABILITY

In this section, we will delve into the activation of several variants of *local unmarkability*, in a hierarchical fashion, starting from a perfectly locally distinguishable set of entangled quantum states. As mentioned earlier, a set of orthogonal quantum states $\mathcal{S} := \{|\psi\rangle\}_{i=1}^{N}$ distributed among k spatially separated parties is said to be *locally unmarkable*, if it is impossible to mark their order of indices under LOCC [22].

To begin with, let us consider a set of orthogonal states, $\mathcal{S}_1 \equiv \{|\psi_i\rangle_{AB}\}_{i=1}^{4} \subset \mathbb{C}^2 \otimes \mathbb{C}^4$, where

$$|\psi_1\rangle_{AB} = |0\mathbf{0}\rangle + |0\mathbf{2}\rangle + |1\mathbf{1}\rangle - |1\mathbf{3}\rangle\,, \tag{1a}$$

$$|\psi_2\rangle_{AB} = |0\mathbf{0}\rangle - |0\mathbf{2}\rangle - |1\mathbf{1}\rangle - |1\mathbf{3}\rangle\,, \tag{1b}$$

$$|\psi_3\rangle_{AB} = |0\mathbf{1}\rangle - |1\mathbf{2}\rangle - |1\mathbf{0}\rangle - |0\mathbf{3}\rangle\,, \tag{1c}$$

$$|\psi_4\rangle_{AB} = |0\mathbf{1}\rangle - |1\mathbf{2}\rangle + |1\mathbf{0}\rangle + |0\mathbf{3}\rangle\,. \tag{1d}$$

The ququad system in Bob's possession can be thought of as a bipartite composition of two qubits, viz., $|\mathbf{0}\rangle_B := |00\rangle_{b_1b_2}\,, |\mathbf{1}\rangle_B := |01\rangle_{b_1b_2}\,, |\mathbf{2}\rangle_B := |10\rangle_{b_1b_2}\,, |\mathbf{3}\rangle_B := |11\rangle_{b_1b_2}\,.$

In the following, we will prove that the locally distinguishable set of states $\mathcal{S}_1$ is free from local redundancy. Before going to the details, we will first recall the definition of *local redundancy* as introduced in [28] and also used in a couple of successive results [29,30]. Note that it is always possible to construct examples of locally distinguishable states, which become indistinguishable (preserving their orthogonality) after reducing subsystem(s); hence one can claim this phenomenon as an activation of nonlocality. To get rid of such trivial examples, the authors in [28] have identified a set of orthogonal states as *locally redundant*, if discarding any of its subsystem(s) preserves their orthogonality.

Proposition 1. The set $\mathcal{S}_1$ is locally distinguishable and free from local redundancy.

Proof. To identify the shared ensemble, given from the set $\mathcal{S}_1$, Alice performs a measurement $\mathcal{M}_A \equiv \{M_1^A := P[|0\rangle_A], M_2^A := P[|1\rangle_A]\}$. Here, $P[(|i\rangle\,, |j\rangle)_\#] := (|i\rangle\,\langle i| + |j\rangle\,\langle j|)_\#$, and # denotes the party. Regardless of Alice's outcome, Bob measures in the basis $\{(|\mathbf{0}\rangle \pm |\mathbf{2}\rangle), (|\mathbf{1}\rangle \pm |\mathbf{3}\rangle)\}$. Then, by communicating their results, they can identify the state.

To check the condition of local redundancy, observe that discarding Bob's first qubit reduces both $|\psi_3\rangle_{AB}$ and $|\psi_4\rangle_{AB}$ to $\frac{1}{2}(|0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_{b_2} + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_{b_2})$, while discarding the second qubit produces $\frac{1}{2}(|0\rangle\langle 0|_A \otimes |-\rangle\langle -|_{b_1} + |1\rangle\langle 1|_A \otimes |+\rangle\langle +|_{b_1})$ from both $|\psi_2\rangle_{AB}$ and $|\psi_3\rangle_{AB}$. On the other hand, discarding Alice's qubit reduces both $|\psi_2\rangle_{AB}$ and $|\psi_4\rangle_{AB}$ to a uniform ensemble of $|+\rangle_{b_1}|1\rangle_{b_2}$ and $|-\rangle_{b_1}|0\rangle_{b_2}$. Hence, tracing out any of the subsystems leads these states to a nonorthogonal set and hence completes the proof. ■

Note that, being a sufficient condition, the demand of the nonorthogonality under subsystem(s) reduction for local redundancy excludes not only all possible trivial constructions, but also several possible classes of nontrivial structures. For instance, consider two Bell states $|\phi^\pm\rangle_{A_1B_1}$, respectively tagged with $|00\rangle_{A_2B_2}$ and $|01\rangle_{A_2B_2}$. Evidently, these states become identical if the subsystem A_1B_2 is discarded, while discarding any of the A_1B_1, A_2B_2, or A_2B_1 subsystems leaves them orthogonal indeed. However, the remaining subsystem(s) for the latter case ($\{|00\rangle\,, |01\rangle\}_{A_2B_2}$, $\{|\phi^+\rangle\,, |\phi^-\rangle\}_{A_1B_1}$ or $\{\frac{\mathbb{I}}{2} \otimes |0\rangle\langle 0|, \frac{\mathbb{I}}{2} \otimes |1\rangle\langle 1|\}_{A_1B_2}$, respectively) are perfectly distinguishable under LOCC and hence no activation of nonlocality is possible simply by discarding the subsystem(s) for this case. Now, if one can come up with an OPLM construction, which can convert the set $\{|\phi^+\rangle_{A_1B_1} \otimes |00\rangle_{A_2B_2}\,, |\phi^-\rangle_{A_1B_1} \otimes |01\rangle_{A_2B_2}\}$ to the locally indistinguishable (mixed) states, then the activation should be admitted as a nontrivial one. However, according to the existing definition for *local redundancy* [28], the set is not a potential candidate for nontrivial nonlocality activation. Therefore, we have further relaxed the strict criterion for local redundancy and define the following:

Definition 1. A set of orthogonal quantum states is said to be free from relaxed local redundancy if the set gets converted to either (i) a nonorthogonal set or (ii) a locally distinguishable orthogonal set, after discarding any of its subsystem(s).

Needless to say, if a set of states is free from local redundancy, then it must be free from the relaxed version since the earlier considers situation (i) only.

To justify that Definition 1 conceives a broader class of potential nonlocality activating states, let us now consider the set $\mathcal{S}_2 \equiv \{|\xi_i\rangle_{AB}\}_{i=1}^{4} \subset \mathbb{C}^4 \otimes \mathbb{C}^8$, where

$$|\xi_1\rangle_{AB} = |\psi_1\rangle_{A_1B_1} \otimes |\phi^+\rangle_{A_2B_2}\,, \tag{2a}$$

$$|\xi_2\rangle_{AB} = |\psi_2\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}\,, \tag{2b}$$

$$|\xi_3\rangle_{AB} = |\psi_3\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}\,, \tag{2c}$$

$$|\xi_4\rangle_{AB} = |\psi_4\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}\,. \tag{2d}$$

Here, $|\psi_i\rangle$'s are same as in the set $\mathcal{S}_1$ and $|\phi^\pm\rangle$ denote the Bell states $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. While the set of states is not free from the earlier definition of local redundancy [28–30], here we will show the following:

Proposition 2. The set of states $\mathcal{S}_2$ is locally distinguishable and free from relaxed local redundancy.

Proof. Note that the set $\mathcal{S}_2$ can be obtained just by tagging $|\phi^\pm\rangle$ with the set of states in $\mathcal{S}_1$. Therefore, Alice and Bob can follow the same scheme elucidated in Proposition 1 for the bipartite subsystems A_1B_1, which, in turn, perfectly distinguishes the set $\mathcal{S}_2$ via LOCC.

To prove that $\mathcal{S}_2$ is free from relaxed local redundancy, we will consider every possible subsystem(s) reductions case by case.

Case I. First consider that both Alice and Bob discard their subsystems A_1B_1. Consequently, the four resulting states $\{|\zeta_1'\rangle := |\phi^+\rangle_{A_2B_2}, |\zeta_2'\rangle := |\phi^-\rangle_{A_2B_2}, |\zeta_3'\rangle := |\phi^-\rangle_{A_2B_2}, |\zeta_4'\rangle := |\phi^-\rangle_{A_2B_2}\}$ are not mutually orthogonal.

Further, it follows from the proof of Proposition 1 that discarding any subsystem(s) from A_1B_1, the remaining reduced sets in $\mathbb{C}^2 \otimes \mathbb{C}^8$ or $\mathbb{C}^4 \otimes \mathbb{C}^2$ are also nonorthogonal.

Case II. Let us now consider that Alice and Bob discard the subsystems A_2B_2. In this case, it is evident that the reduced states $\{|\psi_i\rangle_{A_1B_1}\}_{i=1}^4$ are orthogonal and locally distinguishable (see Proposition 1). So, obviously, this operation does not make the set nonlocal.

Similarly, it is also evident that whenever any of the players discards A_2 or B_2, the set of reduced states remains locally distinguishable in A_1B_1. This completes our proof. ■

Evidently, it follows from Propositions 1 and 2 that both sets of orthogonal states $\mathcal{S}_1$ and $\mathcal{S}_2$ are locally markable [22]. Also it is trivial to argue that any set of orthogonal entangled states $\mathcal{S}_3 \subset \mathcal{S}_1$, with cardinality *three*, is locally distinguishable and hence locally markable. Moreover, it follows from [28] that the set $\mathcal{S}_3$ is free from local redundancy.

Now, we are at a position to present a series of hierarchical nonlocality activations, which are stronger than local indistinguishability. The initial quantum secret in each of these cases is chosen from the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. As a weakest variant, in the following, we will first present the result for simply local indistinguishability activation.

Theorem 1. The set $\mathcal{S}_2$ can be deterministically converted to a locally indistinguishable, but markable set of states.

Proof. Suppose Bob performs an OPLM, $\mathcal{N}_{B_1} \equiv \{N_1^{B_1} := P[(|\mathbf{0}\rangle, |\mathbf{1}\rangle)_{B_1}], N_2^{B_1} := P[(|\mathbf{2}\rangle, |\mathbf{3}\rangle)_{B_1}]\}$ on the subsystem B_1. Here, $P[(|\psi\rangle, |\phi\rangle)_k]$ denotes the projector onto the subspace spanned by $\{|\psi\rangle, |\phi\rangle\}$, on the k party's subsystem. For each of the clicks, the players are left with the following four orthogonal states in $\mathbb{C}^{2\otimes 2} \otimes \mathbb{C}^{2\otimes 2}$:

$$(|0\mathbf{p}\rangle + |1\mathbf{q}\rangle)_{A_1B_1} \otimes |\phi^+\rangle_{A_2B_2}, \tag{3a}$$

$$(|0\mathbf{p}\rangle - |1\mathbf{q}\rangle)_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \tag{3b}$$

$$(|0\mathbf{q}\rangle - |1\mathbf{p}\rangle)_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \tag{3c}$$

$$(|0\mathbf{q}\rangle + |1\mathbf{p}\rangle)_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}, \tag{3d}$$

where $\mathbf{p} = \mathbf{0}$ and $\mathbf{q} = \mathbf{1}$ whenever $N_1^{B_1}$ clicks, while for $N_2^{B_1}$, $\mathbf{p} = \mathbf{2}$ and $\mathbf{q} = \mathbf{3}$ (up to a local phase of $e^{i\pi}$). In both cases, the orthogonal set in Eq. (3) is known to be locally indistinguishable [8], but perfectly markable under LOCC with three-bits of remaining entanglement [22]. ■

Now we consider the next level of stronger nonlocality activation, i.e., to obtain an orthogonal set of entangled states unmarkable under one-way LOCC only. Notably, one-way LOCC deserves importance over the both-way scenario whenever one of the players, sharing the quantum secret, is assumed to completely trustworthy.

Theorem 2. The set $\mathcal{S}_3$ can be deterministically converted to a locally indistinguishable set which is also unmarkable under one-way LOCC.

Proof. Note that the local measurement $\mathcal{N}_{B_1}$ performed at Bob's possession (on the B_1 subsystem), in Theorem 1, updates the A_1B_1 subsystem, i.e., the set $\mathcal{S}_1$, to *four* orthogonal Bell states. Now, since the set $\mathcal{S}_3$ is a subset of $\mathcal{S}_1$, the same OPLM converts $\mathcal{S}_3$ to the corresponding *three* Bell states. The local indistinguishability of any three Bell states follows from [1,2], while the unmarkability under one-way LOCC was recently proved in [22]: This proves the claim of the present theorem. ■

Notably, the activation of local indistinguishability from the set $\mathcal{S}_3$ has also been independently depicted in [28].

Importantly, while the local markability for three Bell states under one-way LOCC is impossible, the status is the same, considering that both-way LOCC has been an open problem [22]. As a natural consequence, one may ask whether there exists any orthogonal set of entangled states from which the strongest form (until now) of quantum nonlocality, i.e., unmarkability under both-way LOCC, can be activated. In the following, we will answer this affirmatively.

Theorem 3. The set $\mathcal{S}_1$ can be deterministically converted to a locally unmarkable set of states by performing OPLM.

Proof. As mentioned in Theorem 2, performing the OPLM, $\mathcal{N}_B \equiv \{N_1^B := P[(|\mathbf{0}\rangle, |\mathbf{1}\rangle)_B], N_2^B := P[(|\mathbf{2}\rangle, |\mathbf{3}\rangle)_B]\}$, on his possession, Bob can transform the set $\mathcal{S}_1$ to four orthogonal entangled states, which are equivalent to the two-qubit Bell basis. In particular, the updated states will be $\{|0\mathbf{0}\rangle \pm |1\mathbf{1}\rangle, |0\mathbf{1}\rangle \pm |1\mathbf{0}\rangle\}$ and $\{|0\mathbf{2}\rangle \pm |1\mathbf{3}\rangle, |1\mathbf{2}\rangle \pm |0\mathbf{3}\rangle\}$ for the clicks N_1^B and N_2^B, respectively. Finally, the local unmarkability of the Bell basis has been shown in [22], which further implies their local indistinguishability [1,2]. ■

With all three above theorems, we can conclude that the activation of all possible nonlocal features in question of the state markability, via one-sided OPLM, is a generic feature for quantum theory. However, one may be curious to ask whether the OPLM performed by a single party suffices for all these activation phenomena. In the following, we will come up with another set $\mathcal{S}_4$ of orthogonal entangled states, the activation of local unmarkability from which involves both of the parties to perform OPLM and communicate their outcomes to each other. In particular, we choose $\mathcal{S}_4 := \{|\xi_i\rangle\}_{i=1}^8$ of eight bipartite states in $\mathbb{C}^4 \otimes \mathbb{C}^4$, where

$$|\xi_{1(2)}\rangle_{AB} = |\mathbf{00}\rangle \pm |\mathbf{02}\rangle \pm |\mathbf{31}\rangle - |\mathbf{33}\rangle, \tag{4a}$$

$$|\xi_{3(4)}\rangle_{AB} = |\mathbf{01}\rangle \mp |\mathbf{32}\rangle \mp |\mathbf{30}\rangle - |\mathbf{03}\rangle, \tag{4b}$$

$$|\xi_{5(6)}\rangle_{AB} = |\mathbf{10}\rangle + |\mathbf{12}\rangle \pm |\mathbf{21}\rangle \mp |\mathbf{23}\rangle, \tag{4c}$$

$$|\xi_{7(8)}\rangle_{AB} = |\mathbf{11}\rangle - |\mathbf{22}\rangle \pm |\mathbf{20}\rangle \pm |\mathbf{13}\rangle. \tag{4d}$$

Here, the bases of each subsystem can be assumed as a two-qubit composite system, i.e., $|\mathbf{0}\rangle := |00\rangle, |\mathbf{1}\rangle := |01\rangle, |\mathbf{2}\rangle := |10\rangle, |\mathbf{3}\rangle := |11\rangle$.

Proposition 3. The set $\mathcal{S}_4 := \{|\xi_i\rangle\}_{i=1}^8$ is locally distinguishable and free from local redundancy.

Proof. We start by describing the local distinguishability protocol. First, Alice performs a measurement $\mathcal{R}_A \equiv \{R_1^A := P[(|\mathbf{0}\rangle, |\mathbf{3}\rangle)_A], R_2^A := P[(|\mathbf{1}\rangle, |\mathbf{2}\rangle)_A]\}$. The postmeasurement states are $\{|\xi_i\rangle\}_{i=1}^4$ or $\{|\xi_i\rangle\}_{i=5}^8$ for the first and second outcome, respectively. However, it is evident that those two sets of qubit(equivalently)-ququad states are in one-to-one correspondence to the set $\mathcal{S}_1$. The rest of the distinguishability protocol is thus a straightforward extension of the protocol provided in Proposition 1.

It is also straightforward to show that the set does not have local redundancy. If both of the parties discard one of their subsystems, they will remain with eight states in $\mathbb{C}^2 \otimes \mathbb{C}^2$, which cannot be orthogonal in any way. Further, discarding Bob's first (second) qubit, the states $|\xi_3\rangle$ and $|\xi_4\rangle$ will produce a uniform ensemble of $|001\rangle$ and $|110\rangle$ ($|00-\rangle$ and $|11+\rangle$). Similarly, discarding any of Alice's qubit will produce $\frac{1}{2}(|0\rangle\langle 0| \otimes |-\rangle\langle -| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |+\rangle\langle +| \otimes |0\rangle\langle 0|)$ both from $|\xi_3\rangle$ and $|\xi_4\rangle$. ■

Now, we will show that the strongest form of nonlocality, i.e., the unmarkability under both-way LOCC, can be activated from the set $\mathcal{S}_4$ whenever both of the parties perform OPLM on their respective subsystems.

Theorem 4. The set $\mathcal{S}_4$ can be deterministically converted to a locally unmarkable set of states via local OPLMs.

Proof. In order to activate the nonlocality of the orthogonal set $\mathcal{S}_4$, Bob performs an OPLM, $\mathcal{N}_B \equiv \{N_1^B := P[(|\mathbf{0}\rangle, |\mathbf{1}\rangle)_B], N_2^B := P[(|\mathbf{2}\rangle, |\mathbf{3}\rangle)_B]\}$. On the other side, Alice measures $\mathcal{R}_A \equiv \{R_1^A := P[(|\mathbf{0}\rangle, |\mathbf{3}\rangle)_A], R_2^A := P[(|\mathbf{1}\rangle, |\mathbf{2}\rangle)_A]\}$. For all possible outcomes of these two measurements, it is quite evident that the players are left with any of the four two-qubit Bell states (equivalently) which is known to be locally unmarkable under general LOCC [22]. This completes our proof. ■

Thus we can conclude that quantum theory admits the complete hierarchy for the activation of local unmarkability, a stronger notion than local indistinguishability. Notably, the final states obtained in Theorems 2 and 3 (hence, in Theorem 4) are locally irreducible indeed. In particular, the states obtained in Theorem 2 are *three* two-qubit Bell states, for which the local irreducibility under OPLM follows from the fact that any two orthogonal bipartite states are always locally distinguishable [1]. On the other hand, the irreducibility of the complete two-qubit Bell basis (i.e., the final states obtained in Theorems 3 and 4) can be shown via explicit OPLM construction [15]. However, a pertinent question in this direction would be regarding the activation of a stronger nonlocal feature of local irreducibility in the multipartite scenario, which we will answer affirmatively in the following.

III. ACTIVATION OF MULTIPARTITE LOCAL IRREDUCIBILITY

For the sake of completeness, let us begin with defining the feature *local irreducibility* [15] for a set of orthogonal quantum states. A set of multipartite orthogonal quantum states is said to be locally irreducible when spatially separated parties, each holding one part of a state chosen arbitrarily from the given set, cannot rule out even one possibility by performing orthogonality preserving local measurements.

Now, consider a set of N-partite states $\mathcal{S}_5^{(N)} := \{|\eta_k(\pm)\rangle\}_{k=0}^{\alpha_N} \in \mathbb{C}^4 \otimes \mathbb{C}^{2^{\otimes(N-1)}}$, such that

$$|\eta_k(\pm)\rangle = |\mathbf{0}, k\rangle \pm |\mathbf{1}, (\alpha_N - k)\rangle \pm [|\mathbf{2}, k\rangle \mp |\mathbf{3}, (\alpha_N - k)\rangle], \tag{5}$$

where $\alpha_p := 2^{(p-1)} - 1$, k is the decimal equivalent of the corresponding $(N-1)$-bit string, and the states $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ can be assumed as a composition of two qubits $\{00, 01, 10, 11\}$, respectively. In the following, we will first comment on the LOCC distinguishability of this set.

Proposition 4. The set $\mathcal{S}_5^{(N)}$ is distinguishable under LOCC and free from local redundancy.

Proof. To discriminate this ensemble, all but the first party will measure their respective qubits in $\mathcal{M}_i \equiv \{M_1^i := P[|0\rangle_i], M_2^i := P[|1\rangle_i]\}$ and, via classical communication, they can identify the decimal index k and correspondingly the state will be either $|\eta_k(\pm)\rangle$ or $|\eta_{\alpha_N-k}(\pm)\rangle$. Now, performing a four-outcome measurement $\mathcal{N}_A \equiv \{N_1^A := P[|\mathbf{0}+\mathbf{2}\rangle_A], N_2^A := P[|\mathbf{0}-\mathbf{2}\rangle_A], N_3^A := P[|\mathbf{1}+\mathbf{3}\rangle_A], N_4^A := P[|\mathbf{1}-\mathbf{3}\rangle_A]\}$ on the first party's possession, they can discriminate the state perfectly. Precisely, the outcomes of the first party's measurement correspond to the states $|\eta_k(\pm)\rangle$ and $|\eta_{\alpha_N-k}(\mp)\rangle$, respectively, for every value of k.

Now, we will argue that the set of states is free from local redundancy. Note that if the first (second) qubit of the first party (personified as Alice) is discarded, then the reduced density matrix for each pair of states $|\eta_k(\pm)\rangle$ ($\{|\eta_k(\pm)\rangle, |\eta_{\alpha_N-k}(\mp)\rangle\}$) will be identical. Further, if the second party discards their qubit, then the reduced system for each of the pairs $\{|\eta_k(\pm)\rangle, |\eta_{k+\alpha_{N-1}+1}(\pm)\rangle\}$ will be identical and a similar argument runs for all the $(N-1)$ parties due to the party-symmetric nature of these states. ■

Our next theorem will depict the possibility for a stronger form of nonlocality activation from the set $\mathcal{S}_5^{(N)}$.

Theorem 5. The set $\mathcal{S}_5^{(N)}$ can be deterministically converted to a class of locally irreducible N-partite genuinely entangled states, which are even indistinguishable when all but one party come together.

Proof. To activate local irreducibility, Alice will perform a measurement $\mathcal{N}_A \equiv \{N_1^A := P[(|\mathbf{0}\rangle, |\mathbf{1}\rangle)_A], N_2^A := P[(|\mathbf{2}\rangle, |\mathbf{3}\rangle)_A]\}$ on her possession. For each of her clicks, the set $\mathcal{S}_5^{(N)}$ will be converted to a set of genuinely entangled states in $\mathbb{C}^{2^{\otimes N}}$, which can be represented as

$$|\phi_k(\pm)\rangle = |\mathbf{p}, k\rangle \pm |\mathbf{q}, (\alpha_N - k)\rangle, \tag{6}$$

where $|\mathbf{p}\rangle = |\mathbf{0}\rangle$ ($|\mathbf{2}\rangle$) and $|\mathbf{q}\rangle = |\mathbf{1}\rangle$ ($-|\mathbf{3}\rangle$) whenever the projector N_1^A (N_2^A) clicks.

Clearly, these states are N-partite genuinely entangled GHZ states, up to a local unitary ($|\mathbf{2}\rangle \to |\mathbf{0}\rangle$; $-|\mathbf{3}\rangle \to |\mathbf{1}\rangle$) for the click N_2^A. These states are known to be locally irreducible [15]. Moreover, in the following, we will show that they are locally indistinguishable, even if all but one of the parties collaborate.

It is easy to see that any N-qubit GHZ state, in any $1:(N-1)$ bipartition, can be written as

$$|\mathcal{G}_k(\pm)\rangle = |0, k\rangle \pm |1, (\alpha_N - k)\rangle, \tag{7}$$

where $k \in \{0, \ldots, \alpha_N\}$ represents the decimal equivalent of $(N-1)$-bits.

Now, it is evident that when all the $(N-1)$ parties come together, they can construct the unitary $\mathbb{U}_k^{\pm}$, which takes $|0\rangle^{\otimes(N-1)} \to |k\rangle$ and $|1\rangle^{\otimes(N-1)} \to \pm|(\alpha_N - k)\rangle$. Therefore, one can write

$$|\mathcal{G}_k(\pm)\rangle = (\mathbb{I} \otimes \mathbb{U}_k^{\pm})\,|\mathcal{G}_0(+)\rangle. \tag{8}$$

Further, by noting that all the 2^N states in Eq. (7) are entangled in $\mathbb{C}^2 \otimes \mathbb{C}^{2^{\otimes(N-1)}}$ and using Eq. (8), we can assure that these states can only be distinguished under LOCC with a probability of not more than $\frac{1}{2}$ [31]. This proves that the states are indistinguishable even in one versus $(N-1)$ bipartitions. ∎

Lastly, we will consider even a stronger nonlocality activation for which the final states are locally irreducible whenever all but one of the parties collaborate, which readily implies the local indistinguishability in the same bipartition. Remember that the practical scenario for all these nonlocality activations from local sets involves a trustworthy agent (say, Alice) who has complete freedom to activate different levels of local state discrimination complexities, depending upon her updated belief regarding the others. This motivates us to consider the activation of strong irreducibility, in the Alice versus others bipartition only.

Let us consider the set $\mathcal{S}_6^{(N)} \subset \mathcal{S}_5^{(N)}$ containing *four* N-partite states $\{|\eta_k(\pm)\rangle$, such that $k = 0, \alpha_N\}$. Following Definition 1 for relaxed local redundancy, we can argue the following:

Proposition 5. The set $\mathcal{S}_6^{(N)}$ is distinguishable under LOCC and free from relaxed local redundancy.

Proof. Note that the set $\mathcal{S}_6^{(N)} \subset \mathcal{S}_5^{(N)}$. Now, the local distinguishability of $\mathcal{S}_5^{(N)}$ (as in Proposition 4) simply implies that the set $\mathcal{S}_6^{(N)}$ is distinguishable under LOCC.

In a similar spirit, it follows from Proposition 4 that discarding the first or second qubit for the first party makes the set $\mathcal{S}_6^{(N)}$ nonorthogonal, as it contains both the pair of states $\{|\eta_k(\pm)\rangle\}$ and $\{|\eta_{\alpha_N-k}(\pm)\rangle\}$ for $k \in \{0, \alpha_N\}$.

However, unlike Proposition 4, when any of the $(N-1)$ parties (except the first party) discards the corresponding subsystem, the final set remains orthogonal to each other. The set of the four final states then reads

$$\begin{aligned}\rho_{\eta_0(\pm)} = \; & \tfrac{1}{4}(|\mathbf{0}\pm\mathbf{2}\rangle\langle\mathbf{0}\pm\mathbf{2}| \otimes |0\rangle\langle 0|^{\otimes(N-2)} \\ & +|\mathbf{1}\mp\mathbf{3}\rangle\langle\mathbf{1}\mp\mathbf{3}| \otimes |1\rangle\langle 1|^{\otimes(N-2)}), \\ \rho_{\eta_{\alpha_N}(\pm)} = \; & \tfrac{1}{4}(|\mathbf{0}\pm\mathbf{2}\rangle\langle\mathbf{0}\pm\mathbf{2}| \otimes |1\rangle\langle 1|^{\otimes(N-2)} \\ & +|\mathbf{1}\mp\mathbf{3}\rangle\langle\mathbf{1}\mp\mathbf{3}| \otimes |0\rangle\langle 0|^{\otimes(N-2)}).\end{aligned}$$

Interestingly, all four states can discriminated under LOCC only: the first party (say, Alice) performs a measurement $\mathcal{N}_A \equiv \{N_1^A{:=}P[|\mathbf{0}+\mathbf{2}\rangle_A], N_2^A{:=}P[|\mathbf{0}-\mathbf{2}\rangle_A], N_3^A{:=}P[|\mathbf{1}+\mathbf{3}\rangle_A], N_4^A := P[|\mathbf{1}-\mathbf{3}\rangle_A]\}$, while any one among the other $(N-2)$ parties performs the computational basis measurement on the respective quantum system. Evidently, the mutual communication between them can single out the final state, and hence this completes the proof. ∎

In the following, we will demonstrate the possible notion of strong irreducibility activation from the set $\mathcal{S}_6^{(N)}$ under OPLM.

Theorem 6. The set $\mathcal{S}_6^{(N)}$ can be deterministically converted to a class of strongly irreducible N-partite genuinely entangled states.

Proof. Note that if Alice performs the same measurement as depicted in Theorem 5, the set $\mathcal{S}_6^{(N)}$ will be converted to the states in Eq. (6), with $k \in \{0, \alpha_N\}$. Now, the action of the local unitary $(|2\rangle \to |0\rangle\, ; -|3\rangle \to |1\rangle)$ for the click N_2^A on Alice's system makes the set identical to $\{|00\rangle \pm |1\alpha_N\rangle\,, |0\alpha_N\rangle \pm |\mathbf{10}\rangle\}$. Observing that these states are equivalent to two-qubit Bell states between Alice versus all $(N-1)$ others, we can conclude regarding their strong irreducibility in the same bipartition [15]. ∎

IV. DISCUSSION

In summary, we have shown that quantum theory allows one to activate several stronger variants of local indistinguishability from a set of locally distinguishable entangled states, under local operations only. Dealing with two different complexity levels of local quantum state discrimination, namely, *local unmarkability* and *local irreducibility*, we have concluded that both of them can be activated in a hierarchical fashion. In particular, as the weaker variants, we have come up with two different examples in $\mathbb{C}^4 \times \mathbb{C}^8$ which can be activated to locally indistinguishable sets of states, respectively markable and unmarkable under one-way LOCC only. Further, the stronger notion, i.e., the local unmarkability under both-way LOCC, has been introduced for another set of states in $\mathbb{C}^2 \times \mathbb{C}^4$ and $\mathbb{C}^{4^{\otimes 2}}$. The latter among them is important to conclude that the activation of such nonlocal aspects is not a generic feature for a single-sided local measurement; instead, it may involve all the parties to perform OPLM on their individual subsystems. It is also possible to decrease the cardinality of the activated set more than that of the initial one, which is again supported by the last example. On the other hand, we have also presented two generic examples in $\mathbb{C}^4 \times \mathbb{C}^{2^{\otimes(N-1)}}$, which can be activated to the sets of states that are indistinguishable in any bipartition. Moreover, while the final set of states obtained from the first one is locally irreducible, the activated irreducibility from the latter one is stronger in the sense that none of them can be eliminated even if all but the trustworthy agent collaborate.

Besides exploring a vivid range of quantum nonlocality from the perspective of state discrimination, our work opens up a number of different directions for future research. Although the activation of indistinguishability and irreducibility for the product states has been reported very recently [29,30], the idea of unmarkability activation has not been widely studied. Further, it will be interesting to study the possibilities of all such activations for the nonmaximally entangled states, which, in turn, may provide an answer towards the activation of the strongest possible state indistinguishability with two copies under adaptive LOCC [11]. While the present paper considers the notions of nonlocality in terms of the complexities associated to the local decodability of the quantum secrets, an alternative quantification can be made in terms of the resource requirement for their perfect decoding, i.e., under separable superoperator [8,32–34], positive partial transpose (PPT)-preserving operation [35], etc. In this direction, it is noteworthy that the activated nonlocality in Theorem 1 is secured even under a PPT-preserving operation [8]. However, the complete characterization of this particular aspect of nonlocality activation can be an interesting open direction.

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