Distributing quantum states with finite lifetime

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This paper considers a quantum node tasked with the teleportation of multiple information-carrying qubit (ICQ) streams, each to a different receiver, by means of entanglements, local operations, and classical communication. Our vision is that the node establishes entangled qubit pairs (EQPs) with the receivers before the arrival of ICQs, rather than waiting for their arrival. In this vein, the paper focuses on the class of protocols referred to as *class* $\mathcal{I}$ that instantaneously teleport arriving ICQs using preestablished EQPs, preventing arriving ICQs from decohering. The excess ratio $\varepsilon_{\rm r}$ is introduced as a quantifier of the system resources per arriving ICQ, and $\varepsilon_{\rm r} = 1$ is shown to be a critical threshold. With $\varepsilon_{\rm r} > 1$: for arrival streams characterized by interarrivals stochastically larger than exponential random variables, any member of class $\mathcal{I}$ teleports all arriving ICQs after a finite transient. With $\varepsilon_{\rm r} < 1$: for stationary ergodic arrival streams, there exists no protocol that teleports all arriving ICQs after a finite transient. This work thus establishes the ultimate limit for distributing quantum states with finite lifetime. Within class $\mathcal{I}$, a protocol referred to as fresh information delivery (FID) is introduced and its optimality is proven. The operational characteristic of FID is provided in terms of the tradeoff between the waiting time of the EQP before it is utilized for teleportation and the excess ratio. Numerical experiments, comparing the proposed FID protocol with alternatives, corroborate the theoretical results. The results in this paper can be used for designing quantum nodes, paving the way for the implementation of the future quantum internet.

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I. INTRODUCTION

The next technological revolution is expected to be driven by quantum information science [1–3], which will transform many fields including sensing, communication, control, computing, positioning, navigation, timing, and more [4–14]. A crucial step towards this revolution involves storing quantum states and distributing them to spatially separated locations, paving the way for the quantum internet [15–17]. The future quantum internet is among the foremost anticipated technological breakthrough [18].

One key physical phenomenon enabling the design of future quantum networks is entanglement, a purely quantum phenomenon with no classical counterpart [19–21]. Entanglement is the enabling mechanism for quantum teleportation and plays a key role in quantum information science [22–26]. The simplest form of a quantum state suitable for teleportation is the qubit, which is a representation of a two-level quantum system such as the polarization of a single photon or the spin of an electron [27,28].

Approaches, guidelines, and technological solutions that drove the development of the internet in the last few decades are not directly applicable to the design of the quantum internet [15–17]. While prototypical real-word applications of quantum networks composed of a few nodes are foreseeable [18], there are several challenges in realizing large-scale quantum networks.

Storing, processing, and transmitting quantum states are difficult tasks due to the decoherence phenomenon caused by the uncontrolled interactions between the quantum states and the environment. The environment behaves as a source of quantum noise which degrades quantum states [29,30]. In addition, quantum states cannot be cloned [24,31–33], making it impossible to replicate information as in classical repeaters. To efficiently transmit information over long distances, quantum repeaters [34,35], entanglement distribution operations [29,34], and distillation operations [36–38] have been proposed.[1]

A crucial building block of future quantum networks is a quantum node that can perform two basic operations: (i) stores local qubits of entangled qubit pairs (EQPs) shared between the source node and different destination nodes; and (ii) teleports arriving information-carrying qubits (ICQs) to the intended destinations utilizing the stored EQPs. Classical fundamental limits for the transmission of qubits that decohere while waiting in a queue have been derived in [41]. The design

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[1] Quantum error correction techniques [9,39,40] can also be used for direct transmission over long distances. However, these techniques require a large overhead, and their implementation remains largely problematic.

of the operative modality of a quantum node is addressed in [42], based on the assumption that quantum states do not decohere in time, i.e., they have an infinite lifetime. However, lifetime of quantum states is limited by the decoherence phenomenon, and it may significantly affect the performance of practical quantum networks. Thus, finite lifetime of quantum states needs to be accounted for in the design of quantum network.

A central question related to the distribution of quantum states with finite lifetime[2] is: "How can protocols be designed to prevent decoherence of the arriving ICQs while efficiently utilizing EQPs?" The answer will make a substantial stride towards enabling quantum networks using noisy intermediate-scale quantum (NISQ) technology. The goal of this paper is to design robust, scalable, and reliable protocols for distributing quantum states. Towards this end, protocols that immediately teleport the arriving ICQs utilizing *preestablished* EQPs with an appropriate establishment rate are conceived. The concept is motivated by observing that information carried by the arriving qubits cannot be replaced whereas established EQPs can be replenished. In designing the protocols the arrival time of ICQs and the establishment time of EQPs are modeled as point processes. The arriving ICQs and the established EQPs have finite lifetimes in the sense that their states decohere in time.

This paper develops a framework in which distributing quantum states with finite lifetime is formulated as a problem of matching points that belong to two point processes. The key contributions are as follows:

(a) identification of the excess ratio ε_{r} as a fundamental parameter providing a threshold for statistical consistency of the protocols;

(b) establishment of the ultimate limit for distributing quantum states with finite lifetime by means of statistically consistent protocols;

(c) proof of optimality for the fresh information delivery (FID) protocol among all the instantaneous protocols; and

(d) derivation of a simple and accurate closed-form formula for the operational characteristic of the FID protocol.

The remaining sections are organized as follows. Section II introduces the basic elements of the quantum node and describes the problem setting. Section III presents the quantum protocols studied in the paper. The optimality of the FID protocol is proven in Sec. IV. Characterization of the protocols is discussed in Sec. V. A brief recapitulation of the main analytical results is provided in Sec. VI. Section VII contains numerical experiments that corroborate the analysis. Finally, Sec. VIII summarizes the findings of the paper.

Notations: The set of integers, nonnegative integers, positive integers, real numbers, nonnegative real numbers, and positive real numbers are respectively denoted by $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_0^+$, and $\mathbb{R}^+$. The set of integers $\{1, 2, \dots, N\}$ is denoted by $\mathbb{I}_N$. Quantum states and quantum density operators are denoted by bold lowercase using Dirac notation (e.g., $|\boldsymbol{\phi}\rangle$) and bold uppercase (e.g., $\boldsymbol{\Xi}$) letters, respectively. Random variables (RVs) are displayed in sans serif, upright fonts; their instantiations in serif, italic fonts. For example, a RV and its instantiation are denoted by x and x, respectively. The functions $f_{\mathsf{x}}(x)$ and $F_{\mathsf{x}}(x)$ denote the probability distribution function (PDF) and the cumulative distribution function (CDF) of the RV x, respectively. The notation $\mathsf{x} \sim \mathcal{E}(L, \lambda)$ denotes that the RV x follows the indicated distribution where, as an example, the Erlang distribution with parameters $L \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ is used. Statistical expectation and probability operators are represented by $\mathbb{E}\{\cdot\}$ and $\mathbb{P}\{\cdot\}$, respectively. For two RVs x and y, x is stochastically larger than y, written $\mathsf{x} \succcurlyeq \mathsf{y}$, if $\mathbb{P}\{\mathsf{x} > t\} \geqslant \mathbb{P}\{\mathsf{y} > t\}$ for all $t \in \mathbb{R}$. The indicator function of the set $\mathcal{S}$ is denoted by $\mathbb{1}_{\mathcal{S}}(\cdot)$, i.e., $\mathbb{1}_{\mathcal{S}}(s) = 1$ if $s \in \mathcal{S}$ and $\mathbb{1}_{\mathcal{S}}(s) = 0$ if $s \notin \mathcal{S}$. The symbol $\mathbb{K}\{\mathcal{S}\}$ denotes the number of elements in the set $\mathcal{S}$, with $\mathbb{K}\{\emptyset\} = 0$ where $\emptyset$ is the empty set. For $a \in \mathbb{R}$, $(a, a] = \emptyset$. Finally, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ denotes the Gamma function and $\mathrm{u}(\cdot)$ denotes the unit step function, i.e., $\mathrm{u}(x) = 1$ for $x \geqslant 0$, and $\mathrm{u}(x) = 0$ otherwise.

II. MODEL AND ASSUMPTIONS

A quantum node is composed of: (i) $L \geqslant 1$ queues, each equipped with a dedicated quantum memory for storing ICQs intended for receiver $\ell \in \mathbb{I}_L$; (ii) a platform for establishing entanglements with the L receivers; and (iii) a shared quantum memory for storing local qubits of EQPs shared with each of the L receivers. For simplicity, this paper considers a single-hop connection between the quantum node and each of the L receivers. The problem being addressed here is complementary to that considered in [42]; there the lifetime of the quantum states is infinite and the memory size is limited, while in this work the lifetime of the quantum states is finite and the memory size is treated as unlimited.[3]

A. Memories

Two kinds of memories are involved, namely information qubit memory (IQM) and entangled qubit memory (EQM). IQM is not needed in the special case of immediate ICQ teleportation. This paper considers on-demand access to the memory, i.e., quantum states can be stored and retrieved instantaneously with no fidelity penalty associated with such accesses [30]. The considerations in this paper hold for general memory models. For concreteness, reference can be made to a hybrid implementation in which traveling light-based quantum states are linked to matter-based quantum memories.

Light-matter coupling is a promising area of research for the development of devices capable of storing, processing, and transmitting quantum states [15,27,43–45]. Using light-matter coupling technology, a decoherence-protected memory has

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[2]For brevity, a quantum state that decoheres in time is referred to as a "quantum state with finite lifetime."

[3]Note, however, that the finite lifetime of the quantum states prevents the accumulation of a large number of "alive" qubits in the system. Indeed, in practical implementations, qubits stored in the memory for a long time can be eliminated with negligible impact on the system performance.

been realized for a single-photon qubit with coherence times on the order of 100 ms, average fidelity of about 0.8, and storage-and-retrieval efficiency of 22% [45].

B. Arriving ICQs

The state of arriving ICQs is unknown to the quantum node and these ICQs arrive with a "label" indicating their destination.[4] Upon arrival, they are either teleported immediately or stored in the pertinent queue. For concreteness, ICQs as well as EQPs can be thought of as polarized photons which are excellent carriers and one of the most effective forms of "flying qubits" [28]. However, the approach developed in this paper is applicable to other kinds of physical qubits. Special attention will be paid to the case where arriving ICQs are immediately teleported,[5] which prevents them from decohering and relaxes the requirement of the L memories for storing them in the queues.

The theoretical results in Sec. IV are valid for ICQ arrival processes described by general point processes. Specific statistical models for the arrival times of the ICQs are used in Sec. V. In particular, Theorem 4 is valid for any renewal process with interarrivals stochastically larger than exponential RVs. Theorem 5 is valid for any stationary ergodic interarrival process. Both theorems cover the homogeneous Poisson point process (PPP) model [47] as a special case. For brevity, the term PPP will be used to refer to the natural and popular homogeneous PPP [48–51].

C. ICQ decoherence models

Let the two-dimensional Hilbert space $\mathcal{H}_2$, having $\{|0\rangle, |1\rangle\}$ as computational basis, be the reference space for ICQs. Let $\boldsymbol{\Xi}_0$ be the density operator representing the quantum state[6] of the arriving ICQ. As soon as an arriving ICQ is stored in the IQM, it starts to decohere. Let $\tau \in \mathbb{R}_0^+$ denote the waiting time (in seconds) of the ICQ, namely, the duration that the ICQ is stored in the IQM before it is teleported. The following three decoherence models deserve special attention.

(a) *Depolarizing model*: After a time interval of duration τ, the original quantum state $\boldsymbol{\Xi}_0$ becomes

$$\boldsymbol{\Xi}_\tau = p_\tau\, \boldsymbol{\Xi}_0 + (1-p_\tau)\boldsymbol{I}_2/2\,, \tag{1}$$

where $\boldsymbol{I}_2/2 = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ is the one-qubit maximally mixed state [24,25].

(b) *Erasure model*: After a time interval of duration τ, the original quantum state $\boldsymbol{\Xi}_0$ becomes

$$\boldsymbol{\Xi}_\tau = p_\tau\, \boldsymbol{\Xi}_0 + (1-p_\tau)|\boldsymbol{v}\rangle\langle\boldsymbol{v}|\,, \tag{2}$$

where $|\boldsymbol{v}\rangle$ is some pure state in a Hilbert space that is different from $\mathcal{H}_2$ [25,41].

(c) *Bistate model*: The original quantum state $\boldsymbol{\Xi}_0$ remains unaffected for a fixed time interval of duration $1/r_q$, $r_q \in \mathbb{R}^+$, and then becomes $|\boldsymbol{v}\rangle\langle\boldsymbol{v}|$, namely,

$$\boldsymbol{\Xi}_\tau = \mathbb{1}_{\mathcal{G}_{r_q}}(\tau)\, \boldsymbol{\Xi}_0 + \mathbb{1}_{\mathcal{B}_{r_q}}(\tau)\,|\boldsymbol{v}\rangle\langle\boldsymbol{v}|\,, \tag{3}$$

where $\mathcal{G}_{r_q} = [0, 1/r_q]$ and $\mathcal{B}_{r_q} = (1/r_q, \infty)$.

A generally accepted model for the probability p_τ in (1) and (2) is exponential [24], i.e.,

$$p_\tau = e^{-r_q \tau}\,, \tag{4}$$

where $r_q \in \mathbb{R}^+$ is the decoherence rate (in Hz) of ICQs.[7] The bistate model (3) can be obtained from the erasure model (2) with $p_\tau = 1$ for $\tau \in \mathcal{G}_{r_q}$ and $p_\tau = 0$ for $\tau \in \mathcal{B}_{r_q}$.

D. Establishment of EQPs

The platform for establishing entanglements in the quantum node supplies EQPs, entangled with each receiver, for teleporting the arriving ICQs. This paper considers a fully heralded entanglement establishment mechanism, similar[8] to that in [29,42], which makes multiple attempts until an EQP is successfully established with a given receiver. Let

$$\lambda_e \triangleq \text{EQP establishment rate (Hz)}\,, \tag{5}$$

$$\lambda_a \triangleq \text{entanglement attempt rate (Hz)}\,, \tag{6}$$

$$p_s \triangleq \text{probability of successful attempt}\,, \tag{7}$$

with $\lambda_e \in \mathbb{R}^+$, $\lambda_a \in \mathbb{R}^+$, and $p_s \in (0, 1]$. The number of attempts to obtain the first occurrence of a success is a discrete geometric RV [52,53]. Clearly, $\lambda_e = \lambda_a\, p_s$. Assuming $\lambda_a \gg \lambda_e$, namely, $p_s \ll 1$ [29], the geometric RV can be approximated by an exponential RV having expected value $1/\lambda_e$. Thus, the random establishment delay $\mathsf{z} \geqslant 0$ (in seconds) needed to successfully establish an EQP with a given receiver satisfies

$$\mathbb{P}\{\mathsf{z} > z\} = e^{-\lambda_e z}, \quad z \geqslant 0\,. \tag{8}$$

As soon as a success is achieved, the established EQP is stored in EQM[9] and a new series of attempts starts again, and so forth indefinitely. Successive establishment delays are independent and identically distributed (IID) [42]. These settings will be used in Sec. V A to derive the statistical model for the establishing times of the EQPs.

When the classical communication delay dominates platform-dependent delays, the time needed to complete an attempt is limited by the propagation distance. Denoting by c the speed of light, this time is $3 \times 10^3/c \approx 10$ µs for a receiver

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[4]For example, the insertion of labels can be obtained by "piggybacking" classical information on a stream of qubits [46].

[5]Here "immediately" means that the local operations and classical communication (LOCC) required to teleport the ICQ begin as soon as it arrives, in contrast to the case in which the ICQ is stored in a memory and retrieved at a later time to perform LOCC for teleportation.

[6]For brevity, the term "quantum state" will be used for "density operator describing the quantum state" whenever there is no ambiguity.

[7]The environment, e.g., a quantum memory, causing a quantum state to decohere exponentially in time is called "Ohmic." In this model, the state is coupled with a bath of harmonic oscillators.

[8]In [29] the system is composed of nitrogen vacancy diamond spin qubit nodes separated by only 2 m.

[9]In many scenarios of practical interest, the EQM consists of two local EQMs, each of which stores the local qubit of the EQP.

located at a distance of 3 km from the quantum node (100 μs at a distance of 30 km), which corresponds to the attempt rate $\lambda_a \lesssim 10^5$ Hz ($\lambda_a \lesssim 10^4$ Hz). These values will be used to compute λ_e in (16) of Sec. II E.

E. EQP decoherence models

Let the four-dimensional Hilbert space $\mathcal{H}_4$, having $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as computational basis, be the reference space for EQPs. Suppose, for concreteness, that the established EQP is a depolarized version of the maximally entangled Bell state $|\boldsymbol{\phi}^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. In this case,

$$\boldsymbol{\Xi}_0 = w_0\, |\boldsymbol{\phi}^+\rangle\langle\boldsymbol{\phi}^+| + (1-w_0)\boldsymbol{I}_4/4\,, \tag{9}$$

where $0 \leqslant w_0 \leqslant 1$ and $\boldsymbol{I}_4/4 = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)/4$ is the two-qubit maximally mixed state [24,25]. As soon as the established EQP is stored in the EQM, it starts to decohere. Let $\tau \in \mathbb{R}_0^+$ denote the waiting time (in seconds) of the EQP, namely, the duration that the local qubit of the EQP spent in the EQM before it is utilized for teleportation. The following three decoherence models, analogous to those in (1), (2), and (3) for a single qubit, concern a pair of qubits.

(a) *Depolarizing model*: After a time interval of duration τ, the original quantum state $\boldsymbol{\Xi}_0$ becomes

$$\boldsymbol{\Xi}_\tau = p_\tau\, \boldsymbol{\Xi}_0 + (1-p_\tau)\boldsymbol{I}_4/4\,. \tag{10}$$

(b) *Erasure model*: After a time interval of duration τ, the original quantum state $\boldsymbol{\Xi}_0$ becomes

$$\boldsymbol{\Xi}_\tau = p_\tau\, \boldsymbol{\Xi}_0 + (1-p_\tau)\,|\boldsymbol{v}\rangle\langle\boldsymbol{v}|\,, \tag{11}$$

where $|\boldsymbol{v}\rangle$ is some pure state in a Hilbert space that is different from $\mathcal{H}_4$.

(c) *Bistate* model: The original quantum state $\boldsymbol{\Xi}_0$ remains unaffected for a fixed time interval of duration $1/r_e$, $r_e \in \mathbb{R}^+$, and then becomes $|\boldsymbol{v}\rangle\langle\boldsymbol{v}|$, namely,

$$\boldsymbol{\Xi}_\tau = \mathbb{1}_{\mathcal{G}_{r_e}}(\tau)\,\boldsymbol{\Xi}_0 + \mathbb{1}_{\mathcal{B}_{r_e}}(\tau)\,|\boldsymbol{v}\rangle\langle\boldsymbol{v}|\,, \tag{12}$$

where $\mathcal{G}_{r_e} = [0, 1/r_e]$ and $\mathcal{B}_{r_e} = (1/r_e, \infty)$.

The probability p_τ in (10) and (11) is given by

$$p_\tau = e^{-r_e \tau}\,, \tag{13}$$

where $r_e \in \mathbb{R}^+$ is the decoherence rate (in Hz) of the EQP which depends on the implementation technology of the local EQMs [30]. The bistate model (12) can be obtained from the erasure model (11) with $p_\tau = 1$ for $\tau \in \mathcal{G}_{r_e}$ and $p_\tau = 0$ for $\tau \in \mathcal{B}_{r_e}$.

When arriving ICQs are immediately teleported, their degradation is mainly determined by the decoherence of the EQPs that are utilized to teleport them. Such degradation can be characterized by the fidelity

$$\varrho(\boldsymbol{\Sigma}, \boldsymbol{\Psi}) = \left(\mathrm{tr}\left\{\sqrt{\sqrt{\boldsymbol{\Psi}}\,\boldsymbol{\Sigma}\,\sqrt{\boldsymbol{\Psi}}}\right\}\right)^2 \in [0,1]\,, \tag{14}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Psi}$, respectively, denote the decohered and the original state of the EQP that is utilized for teleportation [24,25]. When the original state is a Bell state, e.g., $\boldsymbol{\Psi} = |\boldsymbol{\phi}^+\rangle\langle\boldsymbol{\phi}^+|$, the fidelity is called Bell-state fidelity. For the depolarizing model in (10), it can be shown that

$$\varrho(\boldsymbol{\Xi}_\tau, |\boldsymbol{\phi}^+\rangle\langle\boldsymbol{\phi}^+|) = \tfrac{1}{4} + \tfrac{3}{4}\, w_0\, e^{-r_e \tau}\,. \tag{15}$$

Let $\varrho_0 \triangleq \varrho(\boldsymbol{\Xi}_0, |\boldsymbol{\phi}^+\rangle\langle\boldsymbol{\phi}^+|)$ denote the initial fidelity of an established EQP before it is stored in the EQM. The success probability of a single attempt to establish an EQP is given by $p_s = 10^{-3}(1-\varrho_0)$ [42]. Thus, the corresponding EQP establishment rate is

$$\lambda_e = \lambda_a\, p_s = \lambda_a\, 10^{-3}\,(1-\varrho_0)\,. \tag{16}$$

Using the values of λ_a provided in Sec. II D gives $\lambda_e \leqslant 10$ Hz at a distance of 3 km ($\lambda_e \leqslant 1$ Hz at a distance of 30 km) for $\varrho_0 = 0.9$. Note from (15) that $\varrho_0 = 0.9$ corresponds to $w_0 \approx 0.87$.

III. QUANTUM NODE PROTOCOLS

This section first defines the point processes and the corresponding counting processes describing the arrival times of the ICQs and the establishment times of the EQPs. Then, the teleportation protocols are introduced.

A. Point processes

For brevity, "ICQ point" and "EQP point" will be used to refer to arrival time of the ICQ and establishment time of the EQP, respectively. ICQ points and EQP points are measured in seconds. In the following, reference is made to a generic receiver $\ell \in \mathbb{I}_L$.

Definition 1 (Point Processes). The ICQ point process over the (time) real half-axis $\mathbb{R}_0^+$ is a collection of nonnegative RVs $\{\mathsf{q}_i^{(\ell)},\ i \in \mathbb{N}_0\}$ with the properties that $\mathsf{q}_{k-1}^{(\ell)} < \mathsf{q}_k^{(\ell)}$ almost surely (a.s.), $k \in \mathbb{N}$, and $\lim_{k\to\infty} \mathsf{q}_k^{(\ell)} = \infty$ a.s. By convention, $\mathsf{q}_0^{(\ell)} = 0$. The corresponding counting process is $\{\mathsf{n}_q^{(\ell)}(t),\ t \in \mathbb{R}_0^+\}$ where $\mathsf{n}_q^{(\ell)}(t) \triangleq \mathbb{K}\{i \in \mathbb{N}_0 : 0 < \mathsf{q}_i^{(\ell)} \leqslant t\}$ satisfies $\mathsf{n}_q^{(\ell)}(t) < \infty$ and $\lim_{t\to\infty} \mathsf{n}_q^{(\ell)}(t) = \infty$. Similar definitions hold for the EQP point process $\{\mathsf{e}_j^{(\ell)},\ j \in \mathbb{N}_0\}$ and the corresponding counting process $\{\mathsf{n}_e^{(\ell)}(t),\ t \in \mathbb{R}_0^+\}$. □

It is assumed that $\mathsf{q}_i^{(\ell)} \neq \mathsf{e}_j^{(\ell)}$ a.s., for all $i, j \in \mathbb{N}$. ICQ point processes $\{\mathsf{q}_i^{(\ell)},\ i \in \mathbb{N}_0\}$ for different $\ell \in \mathbb{I}_L$ are IID point processes. Furthermore, ICQ point processes $\{\mathsf{q}_i^{(\ell)},\ i \in \mathbb{N}_0\}$, $\ell \in \mathbb{I}_L$, are independent of EQP point processes $\{\mathsf{e}_j^{(\ell)},\ j \in \mathbb{N}_0\}$, $\ell \in \mathbb{I}_L$.

In the remainder of Sec. III and in Sec. IV the superscript (ℓ) is omitted for notational simplicity. The random quantities $\mathsf{n}_q(t)$ and $\mathsf{n}_e(t)$ describe the number of arriving ICQs and established EQPs in the interval $(0, t]$, respectively, where $t \in \mathbb{R}_0^+$ denotes time. The interarrivals of the ICQ and the EQP point process are, respectively, given by

$$\mathsf{x}_k \triangleq \mathsf{q}_k - \mathsf{q}_{k-1}, \quad k \in \mathbb{N}\,, \tag{17a}$$

$$\mathsf{y}_k \triangleq \mathsf{e}_k - \mathsf{e}_{k-1}, \quad k \in \mathbb{N}\,. \tag{17b}$$

The point process is said to be generated by its interarrivals. Note that the definition of the point process does not require a specific statistical model for these interarrivals; in particular, the interarrivals are not necessarily independent.

Definition 2 (Superposed Process). The superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ is defined as the collection of elements from ICQ and EQP point processes, arranged in increasing order a.s.

$$\{\mathsf{s}_k,\ k \in \mathbb{N}\} = \{\mathsf{q}_i,\ i \in \mathbb{N}\} \cup \{\mathsf{e}_j,\ j \in \mathbb{N}\},\quad \mathsf{s}_1 < \mathsf{s}_2 < \dots . \tag{18}$$

By convention, $\mathsf{s}_0 = 0$. The corresponding counting process $\{\mathsf{n}_\mathrm{s}(t),\ t \in \mathbb{R}_0^+\}$ is given by $\mathsf{n}_\mathrm{s}(t) = \mathsf{n}_\mathrm{q}(t) + \mathsf{n}_\mathrm{e}(t)$. The superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ is *marked* so that a point of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ can be distinguished as belonging to $\{\mathsf{q}_i,\ i \in \mathbb{N}\}$ or $\{\mathsf{e}_j,\ j \in \mathbb{N}\}$. □

B. Protocols

The key aspect to efficiently distribute quantum states with finite lifetimes is the protocol that matches arriving ICQs to established EQPs. To introduce the protocols, let $\{\mathsf{q}_{i_m},\ m \in \mathbb{I}_M\}$ and $\{\mathsf{e}_{j_m},\ m \in \mathbb{I}_M\}$ respectively denote the subsequence of ICQs that are teleported and the subsequence of the EQPs that are utilized for teleportation, where M denotes the number of ICQs that are teleported, with $M = \infty$ not excluded. The subsequences of teleported ICQs and utilized EQPs are, respectively, extracted from the point processes $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ and $\{\mathsf{e}_j,\ j \in \mathbb{N}_0\}$, with q_0 and e_0 excluded, i.e., $i_m \neq 0$ and $j_m \neq 0$ for all $m \in \mathbb{I}_M$. For $m \in \mathbb{I}_M$, the mappings $m \mapsto i_m$ and $m \mapsto j_m$ indicate that the mth teleported ICQ arrived at q_{i_m} utilized the EQP established at e_{j_m}. The mappings $m \mapsto i_m$ and $m \mapsto j_m$ are one-to-one, since any teleported ICQ is matched to one and only one utilized EQP and any utilized EQP is matched to one and only one teleported ICQ.

The teleportation protocol is a rule for matching elements of $\{\mathsf{q}_i,\ i \in \mathbb{N}\}$ to elements of $\{\mathsf{e}_j,\ j \in \mathbb{N}\}$, i.e., a rule for determining the subsequences $\{\mathsf{q}_{i_m},\ m \in \mathbb{I}_M\}$ and $\{\mathsf{e}_{j_m},\ m \in \mathbb{I}_M\}$. The vector $[\mathsf{e}_{j_m}\ \mathsf{q}_{i_m}]$ is referred to as the mth matched vector and the sequence $\{[\mathsf{e}_{j_m}\ \mathsf{q}_{i_m}],\ m \in \mathbb{I}_M\}$ is referred to as the matched sequence. Any point of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ is called matched if it belongs to some vector in the matched sequence, otherwise it is called unmatched. The teleportation protocol can be executed at the points of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ sequentially; if previously stored in a memory, matched points are removed from the memory because either they (ICQs) are teleported or they (EQPs) are utilized for teleportation.

This paper focuses on the class $\mathcal{I}$ of *instantaneous* protocols, which consists of the following operations executed at the points of $\{\mathsf{q}_i,\ i \in \mathbb{N}\}$ sequentially.

> As soon as an ICQ arrives at the quantum node, it is immediately teleported utilizing an EQP that has been stored in the EQM, if any. The utilized EQP is effectively removed from the EQM. If the EQM is empty, the ICQ is deleted, i.e., it is irremediably lost.

With instantaneous protocols, the arriving ICQs are never stored in the IQM. If multiple EQPs are available in the EQM, one of them is selected for teleportation according to some rule. One member of the class $\mathcal{I}$ deserving special attention is the *FID protocol* h_F, which consists of the following operations executed at the points of $\{\mathsf{q}_i,\ i \in \mathbb{N}\}$ sequentially.

> As soon as an ICQ arrives at the quantum node, it is immediately teleported utilizing the *most recent* EQP that has been stored in the EQM, if any. The utilized EQP is effectively removed from the EQM. If the EQM is empty, the ICQ is deleted, i.e., it is irremediably lost.

Algorithm 1. Fresh information delivery.

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Input: q = [q_1 q_2 ... q_n], ICQ arrival times;
       e = [e_1 e_2 ... e_k], EQP establishment times,
       with e_k ⩾ q_n.
Output: δ = [δ_1 δ_2 ... δ_n], delays (∞ if not teleported).
Initialize: δ ← n-vector of ∞; e2q ← k-vector of 0.
for i ← 1 to n
    m ← find{e ⩽ q_i & e2q == 0, largest}
    if m is not empty then
        e2q_m ← i
        δ_i ← q_i − e_m
    end
end
return δ
```

The pseudocode[10] illustrating the FID protocol is shown in Algorithm 1. For each ICQ, the algorithm outputs a nonnegative number referred to as "the delay." The delay is equal to the waiting time of the EQP that is matched to the ICQ before it is utilized for teleportation. For the depolarizing model in (10), the sequence of delays can be used to compute the sequence of fidelity values by (15).

With the FID protocol, an ICQ is irremediably lost if the EQM is empty when the ICQ arrives. This is still the case even if an EQP is established shortly after the ICQ arrives. To avoid this, protocols not belonging to the class $\mathcal{I}$ are also conceived and one such protocol is now introduced. Define two time intervals of duration W_b and W_f (in seconds). The bounded delay delivery (BDD) protocol[11] h_B consists of the following operations executed at the points of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ sequentially.

(a) If the point corresponds to an ICQ, then it is teleported utilizing the most recent EQP that has been stored in EQM for no more than W_b seconds ago. The utilized EQP is effectively removed from the EQM. If such an EQP does not exist, the ICQ is stored in the IQM.

(b) If the point corresponds to an EQP, then it is utilized to teleport the most recent ICQ that has been stored in the IQM for no more than W_f seconds ago. The teleported ICQ is effectively removed from the IQM. If such an ICQ does not exist, the EQP is stored in the EQM.

The pseudocode illustrating the BDD protocol is shown in Algorithm 2. For each ICQ, the algorithm outputs a real

[10] Algorithms 1 and 2 utilize a MATLAB-like pseudocode; $\boldsymbol{e2q}_m$ denotes the mth element of vector $\boldsymbol{e2q}$; $\boldsymbol{e2q}_m = i$ indicates that the ith ICQ is teleported utilizing the mth EQP; $\boldsymbol{q2e}_m = i$ indicates that the mth ICQ is teleported utilizing the ith EQP; inequalities between vectors and scalars are applied element-wise and result in vectors of logicals; **find**$\{\boldsymbol{a}, \text{largest}\}$ returns the largest index of the "true" elements of the logical vector $\boldsymbol{a}$; and **cumsum**$\{\boldsymbol{a}\}$ returns the vector of the cumulative sums of vector $\boldsymbol{a}$.

[11] The BDD protocol is similar to the BGM algorithm used in [54,55].

Algorithm 2. Bounded delay delivery.

Input: $\boldsymbol{q} = [q_1\, q_2\, \ldots\, q_n]$, ICQ arrival times;
$\boldsymbol{e} = [e_1\, e_2\, \ldots\, e_k]$, EQP establishment times,
with $e_k \geqslant q_n$; W_b, W_f, window lengths.
Output: $\boldsymbol{\delta} = [\delta_1\, \delta_2\, \ldots\, \delta_n]$, delays ($\infty$ if not teleported).
Initialize: $\boldsymbol{e2q} \leftarrow k$-vector of 0;
$\boldsymbol{q2e} \leftarrow n$-vector of 0;
$\boldsymbol{\delta} \leftarrow n$-vector of ∞;
$\boldsymbol{isq} \leftarrow (n+k)$-vector of 0.
$\boldsymbol{s} \leftarrow$ superposition of $\boldsymbol{q}$ and $\boldsymbol{e}$ [see (18)]
Set to 1 all entries of $\boldsymbol{isq}$ corresponding to ICQs of $\boldsymbol{s}$
$\boldsymbol{indq} \leftarrow$ **cumsum**$\{\boldsymbol{isq} == 1\}$
$\boldsymbol{inde} \leftarrow$ **cumsum**$\{\boldsymbol{isq} == 0\}$
for $i \leftarrow 1$ **to** $n+k$ **do**
  if $\boldsymbol{isq}_i == 1$ **then**
    $m \leftarrow$ **find**$\{\boldsymbol{e} \leqslant \boldsymbol{s}_i \;\&\; \boldsymbol{e} \geqslant \boldsymbol{s}_i - W_\text{b}$
      $\&\; \boldsymbol{e2q} == 0$, largest$\}$
    if m is not empty **then**
      $\boldsymbol{q2e}_{\boldsymbol{indq}_i} \leftarrow m$
      $\boldsymbol{e2q}_m \leftarrow \boldsymbol{indq}_i$
      $\boldsymbol{\delta}_{\boldsymbol{indq}_i} \leftarrow \boldsymbol{s}_i - \boldsymbol{e}_m$
    end
  else
    $m \leftarrow$ **find**$\{\boldsymbol{q} \leqslant \boldsymbol{s}_i \;\&\; \boldsymbol{q} \geqslant \boldsymbol{s}_i - W_\text{f}$
      $\&\; \boldsymbol{q2e} == 0$, largest$\}$
    if m is not empty **then**
      $\boldsymbol{e2q}_{\boldsymbol{inde}_i} \leftarrow m$
      $\boldsymbol{q2e}_m \leftarrow \boldsymbol{inde}_i$
      $\boldsymbol{\delta}_m \leftarrow \boldsymbol{q}_m - \boldsymbol{s}_i$
    end
  end
end
return $\boldsymbol{\delta}$

---

number referred to as "the delay." By convention, a nonnegative delay occurs when the EQP precedes the ICQ matched to it, in which case the delay is equal to the waiting time of the EQP in the EQM. Conversely, a negative delay occurs when the EQP follows the ICQ matched to it, in which case the negative of the delay is equal to the waiting time of the ICQ in the IQM.

The EQP utilized for teleporting the arriving ICQ is contained in a window beginning W_b seconds prior to the arrival and ending W_f seconds after the arrival. Quantum states stored in EQM and IQM for more than W_b seconds and W_f seconds, respectively, can be removed from the memory. With $W_\text{b} = \infty$ and $W_\text{f} = 0$, the BDD protocol reduces to FID. When $W_\text{b} = \infty$ and $W_\text{f} > 0$, BDD retains most advantages of FID at the cost of an IQM for storing the ICQs. In the following, the quantities W_b and W_f are considered to be finite.

IV. OPTIMALITY ANALYSIS

This section first introduces the performance indicators for characterizing the teleportation protocols and then proves the path optimality of the FID protocol.

A. Performance indicators

Let $h \in \mathcal{P}$ denote a teleportation protocol, where $\mathcal{P}$ is a given class of protocols. Let $\{\mathsf{\delta}_i(h),\ i \in \mathbb{N}\}$ be the sequence of delays (in seconds) associated with the sequence of ICQ points $\{\mathsf{q}_i,\ i \in \mathbb{N}\}$ and let $\{\mathsf{\delta}_{i_m}(h),\ m \in \mathbb{I}_M\}$ be the subsequence of delays associated with the teleported ICQs arriving at times described by the subsequence $\{\mathsf{q}_{i_m},\ m \in \mathbb{I}_M\}$, i.e.,

$$\mathsf{\delta}_{i_m}(h) \triangleq \mathsf{q}_{i_m} - \mathsf{e}_{j_m}\,. \tag{19}$$

The delays associated to the arriving ICQs that are not teleported take on a dummy value.

The arithmetic mean of the absolute value of the delays (in seconds) using protocol $h \in \mathcal{P}$ is defined as

$$\mathsf{d}(h) \triangleq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\mathsf{A}(h)}(i)\,|\mathsf{\delta}_i(h)|}{\sum_{i=1}^{n} \mathbb{1}_{\mathsf{A}(h)}(i)}\,, \tag{20}$$

where $\mathsf{A}(h) = \{\mathsf{i}_1, \mathsf{i}_2, \ldots\}$, $\mathsf{i}_n \in \mathbb{N}$, denotes the set of indices describing arrival times of the teleported ICQs using protocol h. For $h \in \mathcal{I}$, the absolute value in (20) is immaterial and the delay $\mathsf{\delta}_{i_m}(h)$ in (19) coincides with the waiting time in the EQM of the EQP established at time e_{j_m}. The fraction of teleported ICQs using protocol $h \in \mathcal{P}$ is defined as

$$\mathsf{\eta}_\text{t}(h) \triangleq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\mathsf{A}(h)}(i)}{n}\,. \tag{21}$$

The two limits in (20) and (21) are assumed to exist in the a.s. sense, and they exist as deterministic values in most cases of practical interest.[12] The quantities in (20) and (21) are key parameters for determining the performance of the teleportation protocols. The former quantity, $\mathsf{d}(h)$, characterizes the quality of the teleported ICQs. Indeed, the delay in (19) regulates the quantum state degradation for the decoherence models (1)–(3) and (10)–(12). The latter quantity, $\mathsf{\eta}_\text{t}(h)$, characterizes the probability of teleportation.

Next, the indicators for characterizing the performance of teleportation protocols are introduced; these indicators will be used in Sec. VII to illustrate the performance of FID and BDD protocols.

Definition 3 (Path-Optimality). A teleportation protocol $h^* \in \mathcal{P}$ is path-optimal in the class $\mathcal{P}$ if, for any instantiation of the point processes $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ and $\{\mathsf{e}_j,\ j \in \mathbb{N}_0\}$, it simultaneously solves the following two optimization problems:

$$\inf_{h\in\mathcal{P}} d(h)\,, \tag{22a}$$

$$\sup_{h\in\mathcal{P}} \eta_\text{t}(h)\,, \tag{22b}$$

where $d(h)$ and $\eta_\text{t}(h)$, respectively, denote the instantiations of $\mathsf{d}(h)$ in (20) and $\mathsf{\eta}_\text{t}(h)$ in (21). □

[12] For a stationary sequence $\{|\mathsf{\delta}_{i_m}(h)|,\ m \in \mathbb{N}\}$, rewrite (20) as $\mathsf{d}(h) = \lim_{\nu\to\infty} \frac{1}{\nu}\sum_{m=1}^{\nu} |\mathsf{\delta}_{i_m}(h)|$. A sufficient condition for the existence of this limit as a RV is $\mathbb{E}\{|\mathsf{\delta}_{i_m}(h)|\} < \infty$, $m \in \mathbb{N}$; if in addition the sequence is ergodic, then the limit is deterministic and given by $\mathbb{E}\{|\mathsf{\delta}_{i_m}(h)|\}$ [56]. Similarly, for a stationary sequence $\{\mathbb{1}_{\mathsf{A}(h)}(i),\ i \in \mathbb{N}\}$ a sufficient condition for the existence of the limit in (21) as a RV is $\mathbb{E}\{\mathbb{1}_{\mathsf{A}(h)}(i)\} < \infty$, $i \in \mathbb{N}$; if in addition the sequence is ergodic, then the limit is deterministic and given by $\mathbb{E}\{\mathbb{1}_{\mathsf{A}(h)}(i)\}$.

Definition 4 (Statistical Consistency). Under a given statistical model for the point processes $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ and $\{\mathsf{e}_j,\ j \in \mathbb{N}_0\}$, a teleportation protocol is statistically consistent if all arriving ICQs (q_0 excluded) are teleported a.s., possibly after a transient of finite duration t_0 (in seconds), where

$$\mathsf{t}_0 \triangleq \min_{i\in\mathbb{N}_0} \left\{\mathsf{q}_i :\ i+k \in \mathsf{A}(h),\ \forall k \in \mathbb{N}\right\}. \tag{23}$$

□

For a statistically consistent protocol $h \in \mathcal{P}$, $\eta_t(h) = 1$ a.s.

B. Optimality of FID protocol

To prove the path-optimality of the FID protocol, the matched sequence $\{[\mathsf{e}_{j_m}\ \mathsf{q}_{i_m}],\ m \in \mathbb{I}_M\}$ resulting from applying the FID protocol to the superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ is characterized.

Definition 5 (Bridges and Covers). Let $[\mathsf{e}_{j_m}\ \mathsf{q}_{i_m}]$ be a matched vector. Then: (i) a *bridge* is said to connect the points e_{j_m} and q_{i_m}; and (ii) any point of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ in the interval $(\min\{\mathsf{e}_{j_m}, \mathsf{q}_{i_m}\}, \max\{\mathsf{e}_{j_m}, \mathsf{q}_{i_m}\})$ is said to be *covered* by the bridge connecting e_{j_m} and q_{i_m}. □

Proposition 1 (FID Bridges). Consider the superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ defined in (18). The bridges, resulting from using the FID protocol, do not cover a.s. any unmatched point. □

Proof. The proof employs the principle of contradiction. Consider the superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ in (18) and suppose that the FID protocol leaves at least one unmatched point of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ covered a.s. by the bridge connecting some points e_{j_m} and q_{i_m}. Let s_ν be the minimum among the unmatched points; it satisfies a.s. the inequalities $0 < \mathsf{e}_{j_m} < \mathsf{s}_\nu < \mathsf{q}_{i_m}$. The point s_ν can be either an ICQ or an EQP point. In the former case, the FID protocol would match s_ν to e_{j_m}, implying that e_{j_m} would be matched a.s. to at least two ICQs. In the latter case, the FID protocol would match q_{i_m} to s_ν, implying that q_{i_m} would be matched a.s. to at least two EQPs. Both cases lead to a contradiction. ⊞

Definition 6 (FID Clusters). A cluster induced by the FID protocol (henceforth simply *cluster*) is any collection of consecutive matched points of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ delimited to the left end by an unmatched point unless the collection contains s_1 and to the right end by an unmatched point unless the collection is right-unbounded. □

Definition 7 (FID Holes). A hole induced by the FID protocol (henceforth simply *hole*) is any collection of consecutive unmatched points of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ delimited to the left end by a matched point unless the collection contains s_1 and to the right end by a matched point unless the collection is right-unbounded. □

In both definitions, the delimitation points are not part of the cluster or hole. Holes can be made of a single point, while clusters contain at least one ICQ point and one EQP point. The following theorem gives the structure of clusters and holes.

Theorem 1 (FID Clusters and Holes). Consider the points $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$ of the superposed point process $\{\mathsf{s}_k,\ k \in \mathbb{N}_0\}$ defined in (18). With the FID protocol, the following assertions are a.s. true.

(i) For any cluster, the first element is an EQP point and the last element, if any, is an ICQ point.

(ii) There exists a one-to-one correspondence between ICQ points and EQP points belonging to the same cluster; the clusters contain an equal number of ICQ points and EQP points.

(iii) Any hole has one of the following structures: (a) all elements are ICQ points; (b) all elements are EQP points; (c) it consists of a sequence of all ICQ points followed by a sequence of all EQP points.

(iv) If there exists a hole containing an EQP point, then all successive holes consist of EQP points only. □

Proof. The arguments in the proof are intended in the a.s. sense. Suppose that, contrary to the assertion in (i), the first element of a cluster is an ICQ point. If such an ICQ point is s_1, then it cannot be matched to some EQP point to its left, leading to a contradiction. If such an ICQ point is not s_1, then it is matched necessarily to some EQP point to its left, forming a bridge. Furthermore, the point immediately to the left of the cluster is unmatched implying that the bridge covers this unmatched point. This contradicts Proposition 1. Similar arguments show that the last element of a cluster is an ICQ point. To prove (ii), note that clusters contain only matched points and the matching must be made necessarily within the same cluster as, otherwise, there exists a bridge connecting two clusters covering at least one unmatched point, which contradicts Proposition 1. Therefore, the matching defines a one-to-one correspondence between ICQ points and EQP points belonging to the same cluster. To prove (iii), note that in a hole an EQP point cannot be followed by an ICQ point as, otherwise, the ICQ point would be matched. The structures of holes asserted in (iii) (a)–(c) follow from this observation. To prove (iv), note that if an EQP point in a hole was followed by one or more unmatched ICQ points in successive holes, then the smallest of those unmatched ICQ points would be necessarily matched and cannot be part of a hole. ⊞

Corollary 1. With the FID protocol, all ICQ points appearing after the occurrence of an unmatched EQP point are a.s. teleported. □

Proof. This follows from the last assertion of Theorem 1. ⊞

The importance of the FID protocol is apparent in the next theorem. Note that the theorem is valid for general point processes without requiring specific statistical models for ICQ and EQP points.

Theorem 2 (Path Optimality of FID). For ICQ and EQP point processes, the FID protocol $h_\mathrm{F} \in \mathcal{I}$ is path-optimal in the class $\mathcal{I}$ of instantaneous protocols. □

Proof. Consider an instantiation $\{s_k,\ k \in \mathbb{N}\}$ of $\{\mathsf{s}_k,\ k \in \mathbb{N}\}$. For protocol $h \in \mathcal{I}$, let $d(h)$ and $\eta_t(h)$ denote the instantiations, corresponding to $\{s_k,\ k \in \mathbb{N}\}$, of $\mathsf{d}(h)$ and $\eta_t(h)$ defined respectively in (20) and (21). Any two protocols belonging to class $\mathcal{I}$ teleport the same collection of ICQs. The only difference among the protocols is the policy for matching those ICQs to EQPs. Therefore, $\eta_t(h)$ does not depend on $h \in \mathcal{I}$ and the second objective (22b) of the multiobjective optimization problem in (22) is maximized by any $h \in \mathcal{I}$.

Consider the first objective (22a) and recall that each of the clusters in $\{s_k,\ k \in \mathbb{N}\}$ induced by h_{F} contains an equal number of ICQs and EQPs. For any such cluster $\mathcal{C}$, let $q_{i_m}, q_{i_{m+1}}, \dots, q_{i_{m+\nu-1}}$ and $e_{j_m}, e_{j_{m+1}}, \dots, e_{j_{m+\nu-1}}$ denote its ICQ and EQP points, respectively, where 2ν is the cluster size, which can be finite or infinite.

Suppose first $\nu < \infty$. The contribution of the cluster $\mathcal{C}$ to the sum of the delays is

$$\sum_{k=m}^{m+\nu-1} \delta_{i_k}(h_{\mathrm{F}}) = \sum_{k=m}^{m+\nu-1} q_{i_k} - \sum_{k=m}^{m+\nu-1} e_{j_k}\,. \tag{24}$$

For any alternative protocol $h \in \mathcal{I}$, there are two possible cases. In the first case, all the ν ICQ points in $\mathcal{C}$ are matched to all the ν EQP points in $\mathcal{C}$, yielding $\sum_{k=m}^{m+\nu-1} \delta_{i_k}(h) = \sum_{k=m}^{m+\nu-1} \delta_{i_k}(h_{\mathrm{F}})$. This is true regardless of the specific matching as long as all matched points belong to $\mathcal{C}$. In the second case, at least one of the ICQ points in $\mathcal{C}$ is matched to an EQP point not belonging to $\mathcal{C}$. Such an EQP point is necessarily smaller than $\min\{e_{j_m}, e_{j_{m+1}}, \dots, e_{j_{m+\nu-1}}\}$, yielding $\sum_{k=m}^{m+\nu-1} \delta_{i_k}(h) > \sum_{k=m}^{m+\nu-1} \delta_{i_k}(h_{\mathrm{F}})$.

Combining the conclusions for the two cases, the contribution of the ICQ points $q_{i_m}, q_{i_{m+1}}, \dots, q_{i_{m+\nu-1}}$ to the sum of the delays at the numerator of (20) satisfies

$$\sum_{k=m}^{m+\nu-1} \delta_{i_k}(h) \geqslant \sum_{k=m}^{m+\nu-1} \delta_{i_k}(h_{\mathrm{F}})\,, \tag{25}$$

implying that the contribution of those ICQ points to the arithmetic mean of the delays for protocol $h \in \mathcal{I}$ is not smaller than the corresponding contribution for FID protocol $h_{\mathrm{F}} \in \mathcal{I}$. Repeating the argument for all clusters in $\{s_k,\ k \in \mathbb{N}\}$ proves that the FID protocol is an optimal solution to the first objective (22a) of the multiobjective optimization problem in (22).

Suppose now $\nu = \infty$. The size of cluster $\mathcal{C}$ is infinite and the arithmetic mean in (20) for FID protocol $h_{\mathrm{F}} \in \mathcal{I}$ can be computed considering only the ICQ and EQP points belonging to the cluster $\mathcal{C}$ as $d(h_{\mathrm{F}}) = \lim_{\mu\to\infty} \frac{1}{\mu} \sum_{k=m}^{m+\mu-1} \delta_{i_k}(h_{\mathrm{F}})$. Dividing both sides of (24) by μ and using similar arguments as in the case of finite clusters gives a scaled version of inequality (25) with scaling factor $1/\mu$. Finally letting $\mu \to \infty$ proves that $d(h) \geqslant d(h_{\mathrm{F}})$. ⊞

V. CHARACTERIZATION OF PROTOCOLS

This section presents the renewal process for the ICQ points, describes the functionality of the quantum node, and derives the renewal process for the EQP points. Based on the renewal process models for the ICQ and EQP points, the statistical consistency of the instantaneous protocols is proven and the operational characteristic of the FID protocol is derived.

A. Consistency theorems

The L ICQ streams $\{\mathsf{q}_i^{(\ell)},\ i \in \mathbb{N}_0\}$ for $\ell \in \mathbb{I}_L$ arriving at the quantum node are described by point processes, each generated by IID interarrivals $\{\mathsf{x}_k^{(\ell)},\ k \in \mathbb{N}\}$ with $\mathbb{E}\{\mathsf{x}_k^{(\ell)}\} = 1/\lambda_{\mathrm{q}}$, where $\lim_{t\to\infty} \mathsf{n}_{\mathrm{q}}^{(\ell)}(t)/t = \lambda_{\mathrm{q}}$ a.s. is the rate (in Hz) of each ICQ stream, with $\lambda_{\mathrm{q}} \in \mathbb{R}^+$.

EQPs represent the resources of the quantum node. From (8), the EQP establishment time is an exponential RV having expected value $1/\lambda_{\mathrm{e}}$, where $\lambda_{\mathrm{e}} \in \mathbb{R}^+$ is determined by setting $\lambda_{\mathrm{a}} \in \mathbb{R}^+$ in (16) compatible with the distances of the receivers. Our vision is that the node establishes EQPs with the receivers before the arrival of ICQs, rather than waiting for their arrival. Thus, it is natural to establish as many EQPs as possible with the receivers, in which case the intervals between adjacent EQP points form a sequence of IID exponential RVs.

To accommodate ICQ streams with the same arrival rate λ_{q} for L different receivers, the resources of the quantum node are equally apportioned among the L receivers in a round-robin fashion without considering the instantiation of the arriving ICQ streams. More specifically, the platform first makes attempts to establish an EQP with receiver $\ell = 1$ until the first occurrence of a success; this process is repeated with receiver $\ell = 2, 3, \dots, L$; and then a new round-robin cycle takes place.

With the round-robin EQP establishment procedure, the sequence $\{\mathsf{e}_j^{(\ell)},\ j \in \mathbb{N}_0\}$ is a delayed renewal process generated by independent interarrivals $\{\mathsf{y}_k^{(\ell)},\ k \in \mathbb{N}\}$ for each $\ell \in \mathbb{I}_L$:[13]

$$\mathsf{e}_j^{(\ell)} \sim \mathcal{E}(L(j-1)+\ell, \lambda_{\mathrm{e}}), \quad j \in \mathbb{N}\,, \tag{27a}$$

$$\begin{aligned} \mathsf{y}_k^{(\ell)} &\sim \mathcal{E}(\ell, \lambda_{\mathrm{e}}), \quad k = 1, \\ \mathsf{y}_k^{(\ell)} &\sim \mathcal{E}(L, \lambda_{\mathrm{e}}), \quad k = 2, 3, \dots\,. \end{aligned} \tag{27b}$$

The rate of the EQP point process is [51]

$$\lim_{t\to\infty} \frac{\mathsf{n}_{\mathrm{e}}^{(\ell)}(t)}{t} = \frac{\lambda_{\mathrm{e}}}{L}\,, \tag{28}$$

where the limit is in the a.s. sense. The rate λ_{q} of the ICQ point process together with the rate λ_{e}/L of the EQP point process define the excess ratio.

Definition 8 (Excess Ratio). The excess ratio is defined to be

$$\varepsilon_{\mathrm{r}} \triangleq \frac{\lambda_{\mathrm{e}}}{L\,\lambda_{\mathrm{q}}}\,. \tag{29}$$

□

The excess ratio serves as a fundamental parameter that characterizes the functionality of the quantum node with respect to arriving ICQ streams. It represents a quantifier of the node's resources (number of established EQPs) per arriving ICQ.

Consider a single receiver $L = 1$ and an arriving stream with exponentially distributed ICQ interarrivals $\{\mathsf{x}_k,\ k \in \mathbb{N}\}$; the superscript (ℓ) is superfluous because $\ell = 1$. Then the point processes $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ and $\{\mathsf{e}_j,\ j \in \mathbb{N}_0\}$ become two

[13] Recall that the RV z has the Erlang distribution with shape parameter $i \in \mathbb{N}$ and rate parameter $\lambda \in \mathbb{R}^+$, denoted by $\mathsf{z} \sim \mathcal{E}(i, \lambda)$, if its PDF is [57]

$$f_{\mathsf{z}}(z) = \frac{\lambda^i z^{i-1} e^{-\lambda z}}{\Gamma(i)}\, \mathrm{u}(z) \tag{26}$$

having expected value $\mathbb{E}\{\mathsf{z}\} = i/\lambda$. For $i = 1$, the Erlang distribution $\mathcal{E}(1, \lambda)$ coincides with the exponential distribution having expected value $1/\lambda$.

PPPs with rates λ_q and λ_e, respectively. If $\varepsilon_r > 1$, the next theorem shows that for any instantaneous protocol the probability $\mathbb{P}\{n_q(t) < n_e(t)\}$ is close to unity for large t. In other words, if $\lambda_q < \lambda_e$, ICQs that arrive at the quantum node at large times have a good chance of being teleported.

Theorem 3 (Excess of EQPs for $L = 1$). For a quantum node serving a single receiver, consider: (i) the ICQ point process $\{q_i,\ i \in \mathbb{N}_0\}$ generated by IID interarrivals $\{x_k,\ k \in \mathbb{N}\}$ with $x_k \sim \mathcal{E}(1, \lambda_q)$; (ii) the EQP point process $\{e_j,\ j \in \mathbb{N}_0\}$ generated by IID interarrivals $\{y_k,\ k \in \mathbb{N}\}$ with $y_k \sim \mathcal{E}(1, \lambda_e)$. Let the node employ an instantaneous protocol and let $\varepsilon_r = \lambda_e/\lambda_q > 1$. For any $0 < \epsilon \leqslant 1$, the following holds:

$$\mathbb{P}\big\{n_q(t) \geqslant n_e(t)\big\} \leqslant \epsilon \quad \text{for} \quad t \geqslant t_\epsilon\,, \tag{30}$$

where

$$t_\epsilon \triangleq \frac{-\log_{10}\epsilon}{\big(\sqrt{\lambda_e} - \sqrt{\lambda_q}\big)^2}\,. \tag{31}$$

□

Proof. For any fixed $t > 0$, the probability mass function (PMF) of the RV $n_q(t) - n_e(t)$ is the Skellam PMF

$$\mathbb{P}\big\{n_q(t) - n_e(t) = k\big\} = e^{-t(\lambda_q+\lambda_e)}\left(\frac{\lambda_q}{\lambda_e}\right)^{\frac{k}{2}} I_k\big(2t\sqrt{\lambda_q\lambda_e}\big)\,, \tag{32}$$

where $k \in \mathbb{Z}$ and $I_k(w)$, $w \in \mathbb{R}_0^+$, is the modified Bessel function of the first kind, see [58, 8.445, p. 919]. The moment generating function of the RV $n_q(t) - n_e(t)$ is [59]

$$\mathbb{E}\big\{e^{\xi\,[n_q(t)-n_e(t)]}\big\} = e^{t\left(-\lambda_q-\lambda_e+\lambda_q e^{\xi}+\lambda_e e^{-\xi}\right)} \tag{33}$$

with $\xi \in \mathbb{R}$. Using the Chernoff-Rubin[14] bound [63] gives

$$\mathbb{P}\big\{n_q(t) \geqslant n_e(t)\big\} \leqslant \inf_{\xi>0} M(\xi)\,. \tag{34}$$

The infimum in (34) can be found since the moment generating function $M(\xi)$ is a convex function [64]. Indeed, the right-hand side of (33) is a composition of an inner strictly convex function with an outer convex increasing function [65,66]. The upper bound in (34) can be computed by noting that the exponent on the right-hand side of (33) attains its minimum at $\xi^* = \frac{1}{2}\log\frac{\lambda_e}{\lambda_q} > 0$, yielding

$$\mathbb{P}\big\{n_q(t) \geqslant n_e(t)\big\} \leqslant e^{-t(\sqrt{\lambda_e}-\sqrt{\lambda_q})^2}\,. \tag{35}$$

Therefore, if $t \geqslant t_\epsilon$, then $\mathbb{P}\{n_q(t) \geqslant n_e(t)\} \leqslant \epsilon$. ⊞

While Theorem 3 considers $L = 1$, Theorems 4 and 5 refer to an arbitrary number $L \geqslant 1$ of receivers.

Theorem 4 (Statistical Consistency). For $\ell \in \mathbb{I}_L$, consider: (i) the ICQ point process $\{q_i^{(\ell)},\ i \in \mathbb{N}_0\}$ generated by IID interarrivals $\{x_k^{(\ell)},\ k \in \mathbb{N}\}$ having a PDF with respect to the Lebesgue measure; (ii) the EQP point process $\{e_j^{(\ell)},\ j \in \mathbb{N}_0\}$ generated by the IID interarrivals (27b), yielding $e_j^{(\ell)} \sim \mathcal{E}(L(j-1)+\ell, \lambda_e)$, $j \in \mathbb{N}$. If $x_k^{(\ell)} \succcurlyeq \tilde{x}_k$ and $\tilde{x}_k \sim \mathcal{E}(1, \tilde{\lambda}_q)$, $\forall k \in \mathbb{N}$, with $\tilde{\lambda}_q < \lambda_e/L$, then any instantaneous protocol is statistically consistent. □

Proof. Dropping the superscript (ℓ) for notational simplicity, consider the RV $q_i = \sum_{k=1}^{i} x_k$ that represents the ith ICQ point. The RV q_i has a PDF with respect to the Lebesgue measure since the RV x_k has a PDF with respect to the Lebesgue measure. Define the denumerable collection of events $E_i \triangleq$ {ICQ corresponding to the arrival time q_i is not teleported}, $i \in \mathbb{N}$. The following relationships between events hold for any instantaneous protocol: $E_i =$ {at q_i, no unmatched EQP established at time $\leqslant q_i$ exists} $\subseteq \{n_e(q_i) \leqslant n_q(q_i)\} = \{n_e(q_i) \leqslant i\}$. Thus, denoting the PDF of q_i by $f_{q_i}(t)$, the total probability law gives [52]

$$\begin{aligned}\mathbb{P}\{E_i\} &= \int_0^\infty \mathbb{P}\{E_i \,|\, q_i = t\}\, f_{q_i}(t)\, dt\\ &\leqslant \int_0^\infty \mathbb{P}\{n_e(q_i) \leqslant i \,|\, q_i = t\}\, f_{q_i}(t)\, dt\\ &= \int_0^\infty \mathbb{P}\{e_{i+1} > q_i \,|\, q_i = t\}\, f_{q_i}(t)\, dt\\ &= \int_0^\infty \mathbb{P}\{e_{i+1} > t\}\, f_{q_i}(t)\, dt\,,\end{aligned} \tag{36}$$

where the last equality follows from the independence of the RVs e_{i+1} and q_i. Thus, $\mathbb{P}\{E_i\} \leqslant \mathbb{E}\{g(q_i)\}$ with $g(t) \triangleq \mathbb{P}\{e_{i+1} > t\}$. Using $x_k \succcurlyeq \tilde{x}_k$, Theorem 1.A.3b in [67] gives $q_i = \sum_{k=1}^{i} x_k \succcurlyeq \sum_{k=1}^{i} \tilde{x}_k \triangleq \tilde{q}_i$. Since $g(t)$ is a nonincreasing function of $t \in \mathbb{R}_0^+$, Theorem 1.A.3a in [67] implies $g(q_i) \preccurlyeq g(\tilde{q}_i)$, resulting in $\mathbb{E}\{g(q_i)\} \leqslant \mathbb{E}\{g(\tilde{q}_i)\}$. Therefore

$$\mathbb{P}\{E_i\} \leqslant \int_0^\infty \mathbb{P}\{e_{i+1} > t\}\, f_{\tilde{q}_i}(t)\, dt \triangleq A_i\,, \tag{37}$$

where $f_{\tilde{q}_i}(t)$ is the PDF of $\tilde{q}_i$, which has an Erlang distribution. For $\tilde{q}_i \sim \mathcal{E}(i, \tilde{\lambda}_q)$ and $\tilde{\lambda}_q < \lambda_e/L$, Appendix A proves that $\sum_{i\in\mathbb{N}} A_i < \infty$. This implies $\sum_{i\in\mathbb{N}} \mathbb{P}\{E_i\} < \infty$. Then, by the first Borel-Cantelli Lemma [64, Thm. 4.3]

$$\mathbb{P}\{\limsup_{i\to\infty} E_i\} = 0\,,$$

which is equivalent to

$$\mathbb{P}\{\text{infinitely many events } E_i \text{ occur}\} = 0\,. \tag{38}$$

The equivalence follows from the set-theoretical definition $\limsup_{i\to\infty} E_i \triangleq \bigcap_{i\in\mathbb{N}} \bigcup_{k=i}^{\infty} E_k =$ {infinitely many events E_i occur}. Equation (38) implies that, a.s., only finitely many events E_i occur, namely, only finitely many ICQs are not teleported. Thus, there exists a finite time after which all arriving ICQs are teleported a.s., which proves the statistical consistency. ⊞

Theorem 4 proves the statistical consistency of instantaneous protocols for all renewal processes generated by interarrivals that are stochastically larger than exponential RVs having expected value $1/\tilde{\lambda}_q$. Note that $x_k^{(\ell)} \succcurlyeq \tilde{x}_k$ implies $\mathbb{E}\{x_k^{(\ell)}\} \geqslant \mathbb{E}\{\tilde{x}_k\}$, namely, the rate λ_q of the ICQ point process with interarrivals $\{x_k^{(\ell)},\ k \in \mathbb{N}\}$ is less than or equal to the rate $\tilde{\lambda}_q$ of the PPP with interarrivals $\{\tilde{x}_k,\ k \in \mathbb{N}\}$. Thus, the assumptions of the theorem imply $\varepsilon_r > 1$. If $\lambda_q = \tilde{\lambda}_q$, then

[14]Inequality (34) is referred to as Chernoff-Rubin bound, although often named as Chernoff bound, to reflect the contribution of Herman Rubin [60–63].

$\mathsf{x}_k^{(\ell)} \sim \mathcal{E}(1, \tilde{\lambda}_q)$ and the ICQ point process reduces to the PPP [67].

The next theorem establishes an upper bound on the ICQ arrival rate, above which no statistically consistent teleportation protocol can be found.

Theorem 5 (Excess of ICQs). For $\ell \in \mathbb{I}_L$, consider ICQ and EQP point processes, each with interarrivals forming a stationary and ergodic sequence of RVs having finite expected values $1/\lambda_q$ and L/λ_e, respectively. If $\varepsilon_r < 1$, then no statistically consistent teleportation protocol exists a.s. □

Proof. The definitions of ICQ and EQP point processes imply the following a.s. relationships [superscript (ℓ) is dropped]:

$$\sum_{k=1}^{\mathsf{n}_q(t)} \mathsf{x}_k = \mathsf{q}_{\mathsf{n}_q(t)} \leqslant t < \mathsf{q}_{\mathsf{n}_q(t)+1} = \sum_{k=1}^{\mathsf{n}_q(t)+1} \mathsf{x}_k \,, \tag{39a}$$

$$\sum_{k=1}^{\mathsf{n}_e(t)} \mathsf{y}_k = \mathsf{e}_{\mathsf{n}_e(t)} \leqslant t < \mathsf{e}_{\mathsf{n}_e(t)+1} = \sum_{k=1}^{\mathsf{n}_e(t)+1} \mathsf{y}_k \,. \tag{39b}$$

Dividing (39a) by $\mathsf{n}_q(t)$ and (39b) by $\mathsf{n}_e(t)$, lower and upper bounds for $t/\mathsf{n}_q(t)$ and $t/\mathsf{n}_e(t)$ can be obtained. From these bounds, it follows that

$$\underbrace{\left(\frac{\sum_{k=1}^{\mathsf{n}_e(t)} \mathsf{y}_k}{\mathsf{n}_e(t)}\right)}_{\xrightarrow{\text{a.s.}} L/\lambda_e} \underbrace{\left(\frac{\mathsf{n}_q(t)}{\sum_{k=1}^{\mathsf{n}_q(t)+1} \mathsf{x}_k}\right)}_{\xrightarrow{\text{a.s.}} \lambda_q} < \frac{\mathsf{n}_q(t)/t}{\mathsf{n}_e(t)/t} < \underbrace{\left(\frac{\sum_{k=1}^{\mathsf{n}_e(t)+1} \mathsf{y}_k}{\mathsf{n}_e(t)}\right)}_{\xrightarrow{\text{a.s.}} L/\lambda_e} \underbrace{\left(\frac{\mathsf{n}_q(t)}{\sum_{k=1}^{\mathsf{n}_q(t)} \mathsf{x}_k}\right)}_{\xrightarrow{\text{a.s.}} \lambda_q} . \tag{40}$$

Letting $t \to \infty$ yields $\mathsf{n}_q(t) \to \infty$ a.s. and $\mathsf{n}_e(t) \to \infty$ a.s. The pointwise ergodic theorem [56] then implies the a.s. convergences indicated in (40). Thus, letting $t \to \infty$ in (40) gives the a.s. limit

$$\lim_{t\to\infty} \frac{\mathsf{n}_q(t)}{\mathsf{n}_e(t)} = \frac{1}{\varepsilon_r} \,. \tag{41}$$

The proof now employs the principle of contradiction. Suppose that $\varepsilon_r < 1$ and there exists some statistically consistent protocol, even noninstantaneous. This implies

$$\frac{1}{\varepsilon_r} = \lim_{t\to\infty} \frac{\mathsf{n}_q(t)}{\mathsf{n}_e(t)} \overset{(a)}{=} \lim_{t\to\infty} \frac{\mathsf{n}_q(t) - \mathsf{n}_q(t_0)}{\mathsf{n}_e(t)} \overset{(b)}{\leqslant} 1 \,, \tag{42}$$

where the limits are in the a.s. sense, t_0 is defined in (23), equality (a) follows from the fact that $\mathsf{n}_q(t_0)$ is a.s. finite, and inequality (b) follows from the supposed statistical consistency. Thus, (42) contradicts that $\varepsilon_r < 1$. ⊞

B. Operational characteristic

Armed with the renewal models described at the beginning of Sec. V A, a simple closed-form expression for the operational characteristic of the FID protocol is derived in the following, assuming $M = \infty$.

The resources of the quantum node are the EQPs established with the L receivers. Part of these EQPs is eventually utilized to teleport arriving ICQs and the remaining part is eventually *dissipated*, i.e., never utilized for teleportation. Using a protocol $h \in \mathcal{P}$, the expected number of eventually utilized EQPs per unit time is $\mathbb{E}\{\eta_t(h)\} L\lambda_q$. By Little's law [68], the expected number of eventually utilized EQPs that are in the EQM is given by

$$N_d = \mathbb{E}\{\mathsf{d}(h)\}\, \mathbb{E}\{\eta_t(h)\}\, L\lambda_q \,. \tag{43}$$

The value of $\mathbb{E}\{\mathsf{d}(h)\}$ in (43) is the same for all $\ell \in \mathbb{I}_L$. Since $(\mathbb{E}\{\eta_t(h)\} L\lambda_q)^{-1}$ represents the interval between the arrival times of two consecutive teleported ICQs in the aggregate L arriving ICQ streams, N_d in (43) will also be referred to as the *normalized delay*. A large value of N_d indicates that the quantum node is inefficient in consuming its resources. When $\eta_t(h) = 1$ a.s., the relationship between N_d and ε_r represents the fundamental tradeoff between the inefficiency in consuming resources and the amount of resources supplied by the EQP establishment platform per arriving ICQ.

Consider the FID protocol $h_F \in \mathcal{I}$ operating on ICQ and EQP point processes as described in Theorem 4, yielding $\eta_t(h_F) = 1$ a.s. in view of the statistical consistency of the protocol. For receiver $\ell \in \mathbb{I}_L$ [superscript (ℓ) is dropped], let t denote the instantiation of the arrival time q_{i_m} corresponding to the mth teleported ICQ, $m \in \mathbb{N}$. For $\varepsilon_r \gg 1$ and $m \gg 1$, with high probability the ICQ point $\mathsf{q}_{i_m} = t$ is matched to the closest EQP point $\mathsf{e}_k < t$, because it is unlikely that the arrival time of the $(m-1)$th teleported ICQ satisfies $\mathsf{e}_k < \mathsf{q}_{i_{m-1}} < t$. If the ICQ point $\mathsf{q}_{i_m} = t$ is matched to the closest EQP point $\mathsf{e}_k < t$, then $j_m = k$ and $k = \mathsf{n}_e(t)$. The first equality is due to the matching of the FID protocol and the second equality is due to the observation that k is simply the number of EQP points up to time t. Therefore, for $\mathsf{q}_{i_m} = t$, the delay in (19) can be approximated as

$$\delta_{i_m}(h_F) = t - \mathsf{e}_{j_m} \approx t - \mathsf{e}_{\mathsf{n}_e(t)} \triangleq \mathsf{a}_e(t) \,, \tag{44}$$

where $\mathsf{a}_e(t)$ is known as the age at t of the renewal process $\{\mathsf{e}_j,\ j \in \mathbb{N}_0\}$. The conditions $\varepsilon_r \gg 1$ and $m \gg 1$ imply that t is large and from [51, Prop. 3.4.6]

$$\lim_{t\to\infty} \mathbb{E}\{\mathsf{a}_e(t)\} = \frac{\mathbb{E}\{\mathsf{y}_k^2\}}{2\,\mathbb{E}\{\mathsf{y}_k\}} = \frac{L+1}{2\lambda_e} \,, \tag{45}$$

where y_k is given in (27b) and $k > 1$ because $\varepsilon_r \gg 1$ and $m \gg 1$.

Two ages $\mathsf{a}_e(t_1)$ and $\mathsf{a}_e(t_2)$ can be considered independent if $|t_2 - t_1| \gg L/\lambda_e$ because they correspond, with high probability, to fractions of two distant interarrivals of the EQP point process established with receiver ℓ. When t_1 and t_2 correspond to arrival times of two ICQs teleported to receiver ℓ, the condition $|t_2 - t_1| \gg L/\lambda_e$ is implied by $\varepsilon_r \gg 1$. When, in addition to $|t_2 - t_1| \gg L/\lambda_e$, t_1 and t_2 are large, then both the CDFs characterizing $\mathsf{a}_e(t_1)$ and $\mathsf{a}_e(t_2)$ can be approximated by the same asymptotic CDF [51, Prop. 3.4.5]. Thus, the subsequence $\{\delta_{i_m}(h_F), m \in \mathbb{N}\}$ can be considered IID for $\varepsilon_r \gg 1$ and $m \gg 1$. For an IID sequence $\{\delta_{i_m}(h_F), m \in \mathbb{N}\}$ the strong law of large numbers implies $\mathsf{d}(h_F) = \mathbb{E}\{\delta_{i_m}(h_F)\}$ a.s. [52], thus $\mathbb{E}\{\mathsf{d}(h_F)\} = \mathsf{d}(h_F)$. This motivates the approximation

$$\mathbb{E}\{\mathsf{d}(h_F)\} \approx \lim_{m\to\infty} \mathbb{E}\{\delta_{i_m}(h_F)\} = \frac{L+1}{2\lambda_e} \,, \tag{46}$$

where the last equality is due to (44) and (45).

It will be apparent in Sec. VII that (46) is accurate for $\varepsilon_r \gg 1$. However, for moderate values of $\varepsilon_r > 1$, the probability that the ICQ point $\mathsf{q}_{i_m} = t$ is matched to the closest EQP point $\mathsf{e}_k \leqslant t$ is no longer close to unity and (46) requires a correction. A correction factor is given by the ratio between the number of established EQPs and the number of eventually dissipated EQPs, i.e., $\frac{\lambda_e/L}{\lambda_e/L-\lambda_q}$, thus yielding

$$\mathbb{E}\{\mathsf{d}(h_F)\} \approx \frac{L+1}{2}\frac{1}{\lambda_e - L\lambda_q}, \tag{47}$$

which is valid for any $\ell \in \mathbb{I}_L$. For a given L and λ_e, (47) characterizes the tradeoff between $\mathbb{E}\{\mathsf{d}(h_F)\}$ and λ_q. For example, fixed values of $L = 1$ and $\lambda_e = 10$ Hz imply that $\mathbb{E}\{\mathsf{d}(h_F)\} \leqslant 0.2$ s can be achieved for arriving ICQ streams having rate $\lambda_q \leqslant 5$ Hz. Similarly, fixed values of $L = 5$ and $\lambda_e = 10$ Hz imply that $\mathbb{E}\{\mathsf{d}(h_F)\} \leqslant 0.6$ s can be achieved for arriving ICQ streams having rate $\lambda_q \leqslant 1$ Hz.

The operational characteristic $N_d(\varepsilon_r)$ of the quantum node using the FID protocol operating on ICQ and EQP point processes as described in Theorem 4 is obtained for $\varepsilon_r > 1$ by substituting (47) and $\eta_t(h_F) = 1$ into (43) as

$$N_d(\varepsilon_r) \approx \frac{L+1}{2}\frac{1}{\varepsilon_r - 1}. \tag{48}$$

Node inefficiency in consuming its resources, N_d, is inversely proportional to the eventually dissipated resources per arriving ICQ, $\varepsilon_r - 1$, through the proportionality factor $(L + 1)/2$. No protocol of class $\mathcal{I}$ operating on ICQ and EQP point processes that are described in Theorem 4 can achieve $N_d(\varepsilon_r)$ smaller than the right-hand side of (48) and the approximation in (48) is tight for $\varepsilon_r \gg 1$.

VI. RECAPITULATION OF THE MAIN ANALYTICAL RESULTS

Before delving into the numerical investigations, a brief recapitulation of the main analytical results is provided.

(a) FID is path-optimal in the class $\mathcal{I}$ of instantaneous protocols in the sense that it achieves both the smallest delay and the largest teleportation probability.

(b) For arriving ICQ streams described by renewal processes with interarrivals stochastically larger than exponential RVs having expected value $1/\tilde{\lambda}_q > L/\lambda_e$, any instantaneous protocol is statistically consistent.

(c) For arriving ICQ streams described by point processes with stationary ergodic interarrivals having finite expected values, no statistically consistent teleportation protocol exists for $\varepsilon_r < 1$.

(d) A simple closed-form analytical expression describing the fundamental operational characteristic $N_d(\varepsilon_r)$ of the quantum node using the statistically consistent FID protocol has been derived for $\varepsilon_r > 1$.

VII. NUMERICAL EXPERIMENTS

Statistical consistency is a desirable property for the design of the quantum node and motivates the adoption of instantaneous protocols since they permit statistically consistency for $\varepsilon_r > 1$. This section presents the results of numerical experiments for the FID protocol h_F, which is optimal in the class $\mathcal{I}$ of instantaneous protocols. The BDD protocol h_B is also considered for comparison.

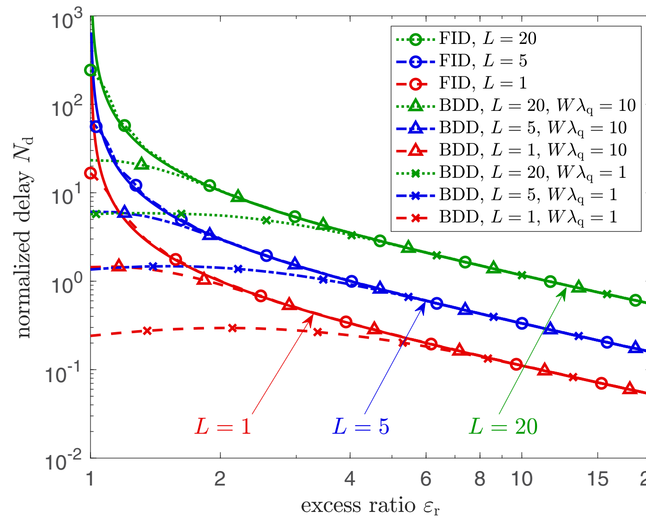

FIG. 1. Operational characteristic $N_d(\varepsilon_r)$ of the quantum node serving $L = 1, 5,$ and 20 receivers, using the FID and BDD protocols. The arriving ICQ streams are characterized by independent PPPs having a common rate λ_q. The solid lines refer to the closed-form expression (48).

The main parameters characterizing the functionality of the quantum node are L, λ_q, λ_e, r_q, and r_e. The numerical experiments consider: (i) a quantum node serving multiple receivers $L \geqslant 1$; (ii) arriving ICQ streams, having a common rate λ_q, characterized by independent point processes with interarrivals stochastically larger than exponential RVs; and (iii) an established EQP point process, having rate λ_e/L, characterized by (27). The role of λ_q and λ_e is manifested in the excess ratio ε_r defined by (29). The parameters r_q and r_e, characterizing IQM and EQM, span several orders of magnitude to reflect the large variability of the decoherence rates associated with different technologies for quantum memories. The numerical experiments are based on Monte Carlo simulations and these simulations show no appreciable difference for different values of $\ell \in \mathbb{I}_L$. This is expected because the value of ℓ affects only the first interarrival of the EQP point process, see (27b). The results shown in Figs. 1–5 are based on numerical experiments using 5×10^4 Monte Carlo runs with an observation window of length $2 \times 10^3/\lambda_q$ s.

For arriving ICQ streams characterized by independent PPPs having a common rate λ_q, Fig. 1 shows the operational characteristic $N_d(\varepsilon_r)$ of the quantum node using the FID protocol h_F, for different values of L. An excellent match between closed-form expression (48) and simulation results is observed, except for values of ε_r close to unity. Figure 1 also shows the results of numerical experiments for the BDD protocol h_B, assuming two values of $W \triangleq W_b = W_f$. It can be seen that the operational characteristic of the quantum node using the BDD protocol is not monotonic with respect to ε_r. Furthermore, the BDD and FID protocols yield similar normalized delays N_d for large ε_r. This is expected because for large ε_r, the probability that an ICQ arriving at t is immediately teleported by the BDD protocol utilizing an EQP

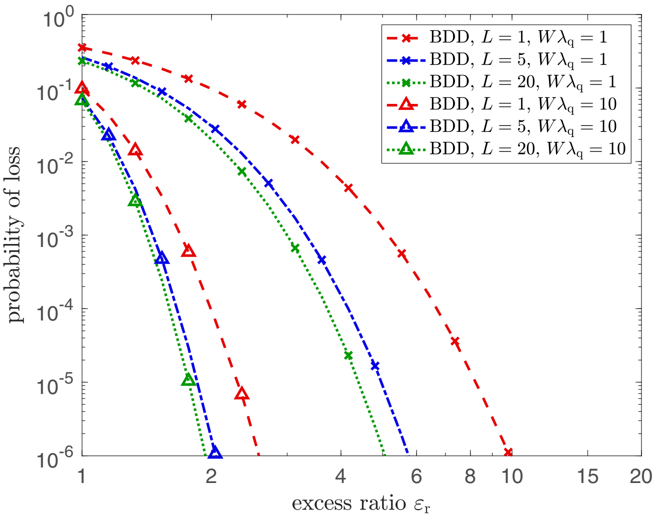

FIG. 2. Probability of loss as a function of the excess ratio ε_r of the quantum node using the BDD protocol h_B. The arriving ICQ streams are characterized by independent PPPs having a common rate λ_q.

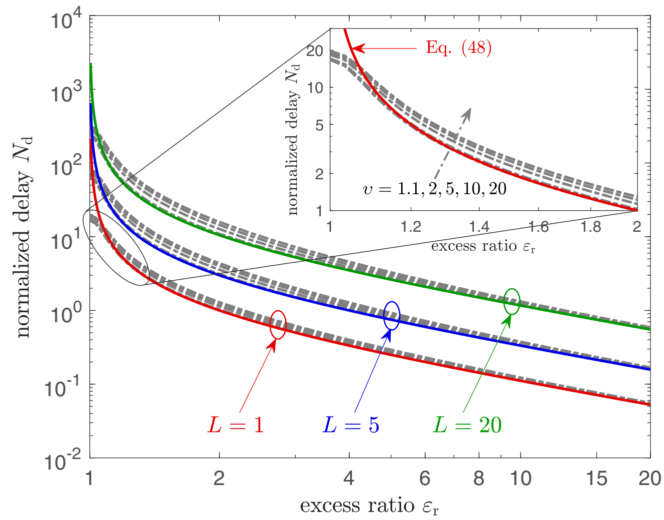

FIG. 3. Operational characteristic $N_\mathrm{d}(\varepsilon_\mathrm{r})$ of the quantum node serving $L = 1, 5$, and 20 receivers using the FID protocol h_F. The arriving ICQ streams are renewal point processes with interarrivals stochastically larger than exponential RVs. The parameter $\upsilon \geqslant 1$ serves as a quantifier of deviation from the PPP. The solid lines refer to the closed-form expression (48).

established between $t - W$ and t is close to one, yielding N_d similar to that obtained by the FID protocol.

The appearance that the BDD protocol outperforms the FID protocol for small ε_r should not be misinterpreted, since BDD protocol incurs ICQ losses as shown in Fig. 2, while the FID protocol is statistically consistent as proven by Theorem 4 in Sec. V. In particular, Fig. 2 shows that the probability of loss for $W\lambda_\mathrm{q} = 10$ is smaller than that for $W\lambda_\mathrm{q} = 1$. This is expected because with the BDD protocol, an ICQ is teleported by utilizing an available EQP established within a window of length $2W$ centered on the arrival time of the ICQ and increasing W gives more chances of finding an available EQP.

Consider next arriving ICQ streams characterized by independent renewal point processes with interarrivals stochastically larger than exponential RVs. Specifically, the IID interarrivals $\{\mathsf{x}_k,\ k \in \mathbb{N}\}$ of the ICQ point process $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ are given by $\mathsf{x}_k = \grave{\mathsf{x}}_k + \acute{\mathsf{x}}_k$, with $\grave{\mathsf{x}}_k$ and $\acute{\mathsf{x}}_k$ being independent Gamma[15] RVs,

$$\grave{\mathsf{x}}_k \sim \mathcal{E}\left(\frac{1}{2}, \tilde{\lambda}_\mathrm{q}\right), \quad \acute{\mathsf{x}}_k \sim \mathcal{E}\left(\frac{1}{2}, \frac{\tilde{\lambda}_\mathrm{q}}{\upsilon}\right), \tag{49}$$

where $\tilde{\lambda}_\mathrm{q} \in \mathbb{R}^+$ and $\upsilon \geqslant 1$. The rate of the ICQ point process $\{\mathsf{q}_i,\ i \in \mathbb{N}_0\}$ is $\lambda_\mathrm{q} = 2\tilde{\lambda}_\mathrm{q}/(1+\upsilon)$. The stochastic order relation $\mathsf{x}_k \succcurlyeq \tilde{\mathsf{x}}_k$, where $\tilde{\mathsf{x}}_k \sim \mathcal{E}(1, \tilde{\lambda}_\mathrm{q})$, follows by [69, Thm. 3], and Theorem 4 in Sec. V then proves that the FID protocol is statistically consistent for $\tilde{\lambda}_\mathrm{q} < \lambda_\mathrm{e}/L$. The parameter $\upsilon \geqslant 1$ in (49) serves as a quantifier of the largeness in the relation $\mathsf{x}_k \succcurlyeq \tilde{\mathsf{x}}_k$. For $\upsilon = 1$, x_k has the same distribution as $\tilde{\mathsf{x}}_k$ and the PPP with interarrivals $\mathsf{x}_k \sim \mathcal{E}(1, \lambda_\mathrm{q})$ is obtained for the ICQ interarrivals. Figure 3 shows the operational characteristic $N_\mathrm{d}(\varepsilon_\mathrm{r})$ of the quantum node using the FID protocol h_F. The figure confirms the accuracy of the closed-form expression (48) even when arriving ICQ streams are characterized by renewal processes that severely deviate from the PPP according to (49).

Figures 4 and 5 consider the same case study illustrated in Fig. 1 of arriving ICQ streams characterized by independent PPPs having a common rate λ_q. The fidelity obtained by the FID protocol h_F for the depolarizing model in (10) is considered in Fig. 4. The figure shows the arithmetic mean $\frac{1}{n}\sum_{m=1}^{n} \varrho_m$ as a function of the EQP decoherence rate r_e. Here, n is the number of Monte Carlo runs and ϱ_m is the fidelity (15) with τ replaced by the instantiation of the delay $\delta_{i_m}(h_\mathrm{F}) = \mathsf{q}_{i_m} - \mathsf{e}_{j_m}$, which is the duration that the EQP established at time e_{j_m} is stored in the EQM before it is utilized to teleport the ICQ arrived at time q_{i_m}. The fidelity of the EQP at the time of its establishment is 0.9. The two dashed asymptotes in the figure depict two extreme cases. Upper dashed line corresponds to the ideal EQM ($r_\mathrm{e} = 0$) in which the established EQP does not decohere regardless of the duration that it spends in the EQM. Lower dashed line corresponds to the worst-case EQM ($r_\mathrm{e} = \infty$) in which the state of the established EQP decoheres to $\boldsymbol{I}_4/4$ as soon as it is stored in the EQM. It can be seen from Fig. 4 that the arithmetic mean of the fidelity is a decreasing function of r_e. It can be also seen that the arithmetic mean of the fidelity is an increasing function of the excess ratio ε_r, which is expected because increasing ε_r reduces $\delta_{i_m}(h_\mathrm{F})$.

Finally, Fig. 5 refers to the case in which the teleportation of the ICQ arrived at time q_{i_m} is successful if $|\delta_{i_m}(h)| = |\mathsf{q}_{i_m} - \mathsf{e}_{j_m}|$ is less than or equal to a preassigned threshold value. Otherwise, if $|\delta_{i_m}(h)| = |\mathsf{q}_{i_m} - \mathsf{e}_{j_m}|$ is larger than the preassigned threshold value, a *failure* is said to occur because the teleported ICQ is severely degraded due to outdated EQP or ICQ. This fits the bistate models for ICQs and EQPs, respectively, described in Sec. II C and Sec. II E. In such scenario, the BDD protocol h_B may represent a valid alternative

[15]The Gamma RV $\mathsf{z} \sim \mathcal{E}(i, \lambda)$, $i \in \mathbb{R}^+$, is a generalization of the Erlang RV $\mathsf{z} \sim \mathcal{E}(i, \lambda)$ for which $i \in \mathbb{N}$; in both cases, the PDF of the RV is given by (26).

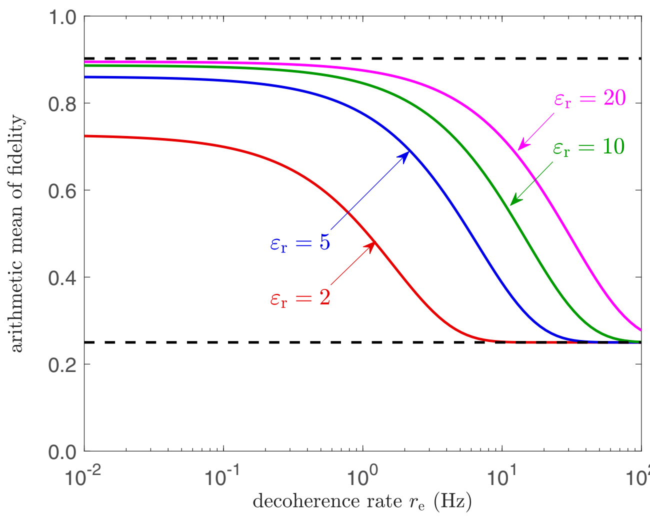

FIG. 4. Arithmetic mean of the fidelity (15) for the FID protocol as a function of the decoherence rate r_e of the EQPs, for several values of the excess ratio ε_r. Upper and lower dashed lines correspond to $r_e = 0$ and $r_e = \infty$, respectively. The parameter w_0 is equal to 0.87, and the quantum node serves $L = 5$ receivers. The arriving ICQ streams are characterized by independent PPPs having a common rate λ_q.

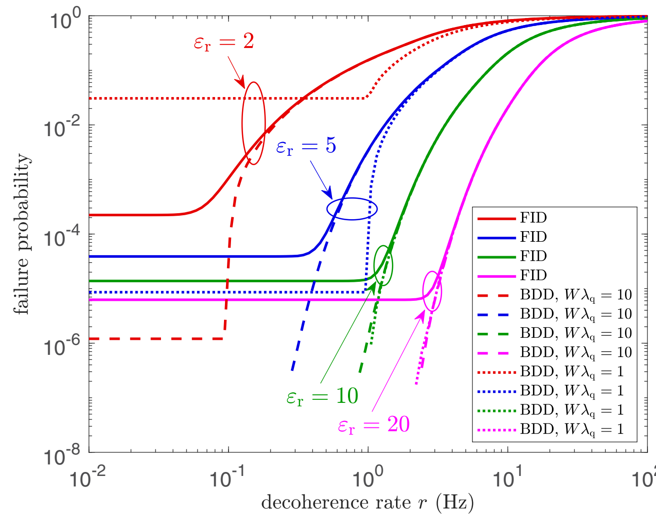

FIG. 5. Failure probability as a function of the memory decoherence rate $r = r_e = r_q$, for several values of the excess ratio ε_r. The quantum node serves $L = 5$ receivers. The arriving ICQ streams are characterized by independent PPPs having a common rate λ_q.

to the FID protocol h_F, because severely degraded teleported ICQs are equivalent to lost ICQs. Figure 5 considers $L = 5$, with various values of $W\lambda_q$, and ε_r. For simplicity of analysis, $r \triangleq r_e = r_q$ and the teleportation is successful if $|\delta_{i_m}(h)| \leqslant 1/r$. The figure shows the failure probability for the FID and the BDD protocols versus r. When $W\lambda_q = 10$, the failure probabilities of the two protocols are close for large values of r (severe decoherence) while the BDD outperforms the FID for small values of r (high-quality memories). As to the case $W\lambda_q = 1$, for large ε_r, the BDD outperforms the FID, while for small ε_r and small r the FID is superior. For instance, with $\varepsilon_r = 2$, FID exhibits smaller failure probability than BDD for $r \lesssim 0.34$ Hz.

VIII. SUMMARY

This paper puts forth a framework in which distributing information-carrying quantum states with finite lifetime is formulated as a problem of matching between two point processes. Performance degradation due to the decoherence of the arriving ICQs is mitigated by the utilization of EQPs established prior to their arrivals. We consider the class $\mathcal{I}$ of instantaneous matching protocols that guarantee zero waiting time for distributing the information-carrying quantum states. The excess ratio ε_r is identified as a fundamental parameter and $\varepsilon_r = 1$ as a critical threshold. With $\varepsilon_r > 1$: under qualifying conditions on the arriving ICQ streams, any protocol belonging to class $\mathcal{I}$ is statistically consistent. With $\varepsilon_r < 1$: no statistically consistent protocol, even not belonging to class $\mathcal{I}$, exists. This establishes the ultimate limit for distributing quantum state with finite lifetime. Within class $\mathcal{I}$, the FID protocol is introduced and its optimality is proven. The operational characteristic of the optimal FID protocol, describing the functional relationship between the normalized waiting time of the preestablished EQPs and the excess ratio, is derived. The proposed framework has been validated by numerical experiments which corroborate the theoretical analysis. The results obtained in this paper provide guidelines for the design of quantum nodes, paving the way for future quantum networks using NISQ technology.

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APPENDIX: PROOF OF $\sum_{i\in\mathbb{N}} A_i < \infty$

With reference to the proof of Theorem 4 in Sec. V A, this Appendix shows that $\sum_{i\in\mathbb{N}} A_i < \infty$, where

$$A_i \triangleq \int_0^{\infty} \mathbb{P}\{\mathsf{e}_{i+1} > t\}\, f_{\tilde{\mathsf{q}}_i}(t)\, dt\,. \tag{A1}$$

The moment generating function of an Erlang distribution $\mathcal{E}(i, \lambda_e)$ with shape parameter $i \in \mathbb{N}$ and rate parameter $\lambda_e \in \mathbb{R}^+$ is

$$M(\xi) = \left(1 - \frac{\xi}{\lambda_e}\right)^{-i}, \quad \xi < \lambda_e\,. \tag{A2}$$

Exploiting (A2), Chernoff-Rubin bound gives [60–63]

$$\begin{aligned}\mathbb{P}\big\{\mathsf{e}_{i+1} > t\big\} &\leqslant \inf_{0<\xi<\lambda_e} e^{-\xi t}\left(1 - \frac{\xi}{\lambda_e}\right)^{-(iL+\ell)} \\ &= \begin{cases} 1, & t < \frac{iL+\ell}{\lambda_e}, \\ e^{-t\lambda_e + iL + \ell}\left(\frac{t\,\lambda_e}{iL+\ell}\right)^{iL+\ell}, & t \geqslant \frac{iL+\ell}{\lambda_e}, \end{cases}\end{aligned} \tag{A3}$$

where the value of ξ attaining the infimum is $\xi^* = \max\{0, \lambda_e - (iL+\ell)/t\}$. Using (A3) in (A1),

$$A_i \leqslant \int_0^{\frac{iL+\ell}{\lambda_e}} f_{\tilde{q}_i}(t)\,dt + \int_{\frac{iL+\ell}{\lambda_e}}^{\infty} f_{\tilde{q}_i}(t) e^{-t\lambda_e + iL+\ell}\left(\frac{t\,\lambda_e}{iL+\ell}\right)^{iL+\ell} dt$$
$$\leqslant B_i + C_i\,, \tag{A4}$$

where

$$B_i \triangleq \int_0^{\frac{iL+\ell}{\lambda_e}} f_{\tilde{q}_i}(t)\,dt\,, \tag{A5a}$$

$$C_i \triangleq \int_0^{\infty} f_{\tilde{q}_i}(t) e^{-t\lambda_e + iL+\ell}\left(\frac{t\,\lambda_e}{iL+\ell}\right)^{iL+\ell} dt\,. \tag{A5b}$$

In order to show that $\sum_{i\in\mathbb{N}} A_i < \infty$, it is sufficient to show that $\sum_{i\in\mathbb{N}} B_i < \infty$ and $\sum_{i\in\mathbb{N}} C_i < \infty$.

Consider the integral B_i in (A5a). Using the expression for $f_{\tilde{q}_i}(t)$ given in (26) yields

$$B_i = \frac{\tilde{\lambda}_q^i}{\Gamma(i)} \int_0^{\frac{iL+\ell}{\lambda_e}} t^{i-1} e^{-t\tilde{\lambda}_q}\,dt = \frac{\gamma\left(i, \frac{iL+\ell}{\lambda_e}\tilde{\lambda}_q\right)}{\Gamma(i)}\,, \tag{A6}$$

where $\gamma(a,b) = \int_0^b t^{a-1}e^{-t}\,dt$ is the incomplete Gamma function. Since $\ell \leqslant L$ and $\gamma(a,b)$ is monotonically increasing in b,

$$B_i = \frac{\gamma\left(i, \frac{iL+\ell}{\lambda_e}\tilde{\lambda}_q\right)}{\Gamma(i)} \leqslant \frac{\gamma(i,(i+1)\kappa)}{\Gamma(i)} \triangleq \breve{B}_i\,, \tag{A7}$$

where $\kappa \triangleq L\tilde{\lambda}_q/\lambda_e \in (0,1)$. The condition $\sum_{i\in\mathbb{N}} B_i < \infty$ is implied by the condition $\sum_{i\in\mathbb{N}} \breve{B}_i < \infty$. In order to show that $\sum_{i\in\mathbb{N}} \breve{B}_i < \infty$, it is sufficient to show that $\lim_{i\to\infty} \breve{B}_{i+1}/\breve{B}_i < 1$, in view of the ratio test for the convergence of the series [70, Thm. 2, p. 117].

Consider the ratio

$$\frac{\breve{B}_{i+1}}{\breve{B}_i} = \frac{1}{i}\frac{\gamma(i+1,(i+2)\kappa)}{\gamma(i,(i+1)\kappa)}\,. \tag{A8}$$

Using the recurrence formula $\gamma(a+1,b) = a\gamma(a,b) - b^a e^{-b}$ [71, 6.5.22] yields

$$\gamma(i,(i+1)\kappa) = \frac{\gamma(i+1,(i+1)\kappa)}{i} + \frac{[(i+1)\kappa]^i e^{-(i+1)\kappa}}{i}\,. \tag{A9}$$

Since $\kappa \in (0,1)$, for $a\to\infty$, $\gamma(a, a\kappa)$ can be replaced with the following asymptotically equivalent expression [72, 8.11.6]:

$$(a\kappa)^a e^{-a\kappa} \sum_{m=0}^{\infty} \frac{(-a)^m c_m(\kappa)}{[a(1-\kappa)]^{2m+1}}\,, \tag{A10}$$

where $c_0(\kappa) = 1$. Eqs. (A9) and (A10) give an asymptotically equivalent expression for $i\gamma(i,(i+1)\kappa)$ as follows:

$$\left([(i+1)\kappa]^{i+1} e^{-(i+1)\kappa} \sum_{m=0}^{\infty} \frac{(-1)^m c_m(\kappa)}{(i+1)^{m+1}(1-\kappa)^{2m+1}}\right)$$
$$+ [(i+1)\kappa]^i e^{-(i+1)\kappa}\,. \tag{A11}$$

Retaining the dominant term in the series of (A11), namely, $[(i+1)(1-\kappa)]^{-1}$ corresponding to the $m=0$ term, expression (A11) reduces to $[(i+1)\kappa]^i e^{-(i+1)\kappa}/(1-\kappa)$. Using this asymptotically equivalent expression for $i\gamma(i,(i+1)\kappa)$ in (A8) and taking the limit yields

$$\lim_{i\to\infty} \frac{\breve{B}_{i+1}}{\breve{B}_i} = \lim_{i\to\infty} \frac{1}{i+1} \frac{[(i+2)\kappa]^{i+1} e^{-(i+2)\kappa}}{[(i+1)\kappa]^i e^{-(i+1)\kappa}}$$
$$= \lim_{i\to\infty} \kappa\, e^{-\kappa}\left(\frac{i+2}{i+1}\right)^{i+1} = \kappa\, e^{1-\kappa}\,. \tag{A12}$$

The value on the right-hand side of (A12) is less than unity for $\kappa \in (0,1)$, which proves that $\lim_{i\to\infty} \breve{B}_{i+1}/\breve{B}_i < 1$.

Consider the integral C_i in (A5b). Using the expression for $f_{\tilde{q}_i}(t)$ given in (26) yields

$$C_i = \frac{\tilde{\lambda}_q^i}{\Gamma(i)} e^{iL+\ell} \int_0^{\infty} t^{i-1} e^{-t(\lambda_e + \tilde{\lambda}_q)}\left(\frac{t\,\lambda_e}{iL+\ell}\right)^{iL+\ell} dt$$
$$= e^{iL+\ell}\left(\frac{\tilde{\lambda}_q}{\lambda_e+\tilde{\lambda}_q}\right)^i \left(\frac{\lambda_e}{\lambda_e+\tilde{\lambda}_q}\frac{1}{iL+\ell}\right)^{iL+\ell} \frac{\Gamma(i(L+1)+\ell)}{\Gamma(i)}\,. \tag{A13}$$

In order to apply the ratio test for the convergence of the series [70, Thm. 2, p. 117], consider the ratio

$$\frac{C_{i+1}}{C_i} = \frac{e^L}{i}\frac{\tilde{\lambda}_q}{\lambda_e+\tilde{\lambda}_q}\left(\frac{\lambda_e}{\lambda_e+\tilde{\lambda}_q}\right)^L$$
$$\times \frac{(iL+\ell)^{iL+\ell}}{(iL+\ell+L)^{iL+\ell+L}} \frac{\Gamma((i+1)(L+1)+\ell)}{\Gamma[i(L+1)+\ell]}\,. \tag{A14}$$

Using the asymptotic expansion $\Gamma(ia+b) \sim \sqrt{2\pi}e^{-ia}(ia)^{ia+b-\frac{1}{2}}$ valid for $a>0$ and $i\to\infty$ [71, 6.1.39], the ratio of Gamma functions on the right-hand side of (A14) can be replaced by the asymptotically equivalent expression $[i(L+1)]^{L+1}$. Thus, taking the limit of (A14) yields

$$\lim_{i\to\infty} \frac{C_{i+1}}{C_i} = \frac{\tilde{\lambda}_q}{\lambda_e+\tilde{\lambda}_q}\left(\frac{e\,\lambda_e}{\lambda_e+\tilde{\lambda}_q}\right)^L (L+1)^{L+1}$$
$$\times \lim_{i\to\infty} \frac{(iL+\ell)^{iL+\ell}}{(iL+\ell+L)^{iL+\ell+L}}\, i^L$$
$$= (L+1)\frac{\tilde{\lambda}_q}{\lambda_e+\tilde{\lambda}_q}\left(\frac{\lambda_e}{\lambda_e+\tilde{\lambda}_q}\frac{L+1}{L}\right)^L\,. \tag{A15}$$

For any integer $L \geqslant 1$ and any $\lambda_e > 0$, the function on the right-hand side of (A15) is nonnegative and strictly increasing for $0 \leqslant \tilde{\lambda}_q \leqslant \lambda_e/L$, taking value 0 at $\tilde{\lambda}_q = 0$ and value 1 at $\tilde{\lambda}_q = \lambda_e/L$. This shows that the right-hand side of (A15) is less than unity when $\tilde{\lambda}_q < \lambda_e/L$, which proves that $\lim_{i\to\infty} C_{i+1}/C_i < 1$.

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