


Bipartite quantum state discrimination and decomposable entanglement witnessDonghoon Ha  and Jeong San Kim **Department of Applied Mathematics and Institute of Natural Sciences, Kyung Hee University, Yongin 17104, Republic of Korea*

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We consider bipartite quantum state discrimination using positive-partial-transpose measurements and show that minimum-error discrimination by positive-partial-transpose measurements is closely related to entanglement witness. By using the concept of decomposable entanglement witness, we establish conditions on minimum-error discrimination by positive-partial-transpose measurements. We also provide conditions on the upper bound of the maximum success probability over all possible positive-partial-transpose measurements. Finally, we illustrate our results using examples of multidimensional bipartite quantum states.

DOI: [10.1103/PhysRevA.107.052410](https://doi.org/10.1103/PhysRevA.107.052410)**I. INTRODUCTION**

Quantum nonlocality is a fascinating phenomenon that occurs in multipartite quantum systems [1–3]. In discriminating multipartite quantum states, quantum nonlocality occurs when optimal state discrimination, which gives the maximum success probability overall possible measurements, cannot be realized only by *local operations and classical communication* (LOCC) [4–8]. Orthogonal quantum states can always be perfectly discriminated using appropriate global measurement. However, there exist some multipartite orthogonal quantum states that cannot be perfectly discriminated only by LOCC measurements [9,10]. Moreover, some multipartite nonorthogonal quantum states cannot be optimally discriminated using only LOCC measurements [11,12]. Nonetheless, characterizing local discrimination of quantum states is still a hard task due to the lack of good mathematical structure for LOCC.

The first nonlocal phenomenon in quantum state discrimination was shown through nine $3 \otimes 3$ pure orthogonal product states [9]. On the other hand, it was shown that nonlocality does not occur in discriminating any two multipartite pure states [13,14]. Since then, there have been several studies focused on local indistinguishability or local distinguishability of mutually orthogonal states [15–18]. In particular, the local distinguishability of two $2 \otimes 2$ pure orthogonal entangled states has been experimentally demonstrated [15]. It was also shown that the nonlocality of the N -fold quantum state ensemble can disappear asymptotically [16]. Recently, an upper bound of the maximum success probability overall separable measurements was established in the optimization for local unambiguous discrimination of multipartite quantum states [17]. Moreover, necessary conditions for perfect discrimination by asymptotic LOCC were established in discriminating orthogonal pure states [18].

The quantum nonlocal phenomenon also arises in the correlations of a multipartite quantum system. Entanglement is a

well-known quantum correlation that cannot be realized using only LOCC [1]. The nonlocal property of entanglement can be used as a useful resource in various quantum information processing tasks such as quantum cryptography, teleportation, and local discrimination of multipartite quantum states [19–22]. For this reason, much attention has been shown for characterizing quantum entanglement [23–25].

An important task of characterizing quantum entanglement is to design ways to detect the existence of entanglement. *Entanglement witness* (EW) is an observable having non-negative mean value for any separable state, but negative for some entangled states [26–29]. As EW detects the existence of entanglement that is an important quantum nonlocality, it is natural to ask whether EW can also be used to characterize the limit on local discrimination of quantum states, another important quantum nonlocality.

Here, we provide an answer to the question by establishing a specific relation between the properties of EW and *positive-partial-transpose* (PPT) measurements, a mathematically well-structured set of measurements having LOCC measurements as special cases. By using the concept of decomposability of operators, we show that the minimum-error discrimination of bipartite quantum states using PPT measurements strongly depends on the existence of *decomposable entanglement witness* (DEW). More precisely, we establish conditions on minimum-error discrimination by PPT measurements in terms of DEW. We also provide conditions on the maximum success probability over all possible PPT measurements. Finally, we illustrate our results using examples of multidimensional bipartite quantum states.

Because PPT measurements have LOCC measurements as special cases, our results provide a useful method to detect the nonlocality arising in quantum state discrimination. Moreover, our results can be applied to any ensemble of bipartite states in an arbitrary dimension, whereas the previous results [15,16,18] are only valid for some restricted cases with certain conditions. We also note that our results provide a systematic way to construct a bipartite quantum state ensemble showing nonlocality in quantum state discrimination.

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This paper is organized as follows. In Sec. II, we first recall the definitions and some properties about PPT measurements and DEW. We also recall the definition of minimum-error discrimination as well as some useful properties of the optimal measurements. In Sec. III, we provide conditions on the maximum success probability over all possible PPT measurements (Theorems 1 and 2). In Sec. IV, we provide conditions on minimum-error discrimination by PPT measurements in terms of DEW (Theorems 3 and 4). Finally, we illustrate our results using examples of multidimensional bipartite quantum states. In Sec. V, we summarize our results with possible applications and future works. We also discuss a systematic way to construct a bipartite quantum state ensemble showing nonlocality in quantum state discrimination.

II. PRELIMINARIES

For a bipartite Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, let \mathbb{H} be the set of all Hermitian operators acting on \mathcal{H} . A bipartite quantum state is expressed by a density operator that is $\rho \in \mathbb{H}$ with positive semidefiniteness $\rho \geq 0$ and unit trace $\text{Tr}\rho = 1$. A measurement is represented by a positive operator-valued measure that is $\{M_i\}_i \subseteq \mathbb{H}$ satisfying positive semidefiniteness $M_i \geq 0$ for all i and the completeness relation $\sum_i M_i = \mathbb{1}$, where $\mathbb{1}$ is the identity operator in \mathbb{H} . When a measurement $\{M_i\}_i$ is performed on the quantum state ρ , the probability of obtaining the measurement outcome corresponding to M_j is $\text{Tr}(\rho M_j)$.

Definition 1. $E \in \mathbb{H}$ is called *PPT* if $E^{\text{PT}} \geq 0$, where the superscript PT is to indicate the partial transposition [30,31]. Similarly, we say that $\{E_i\}_i \subseteq \mathbb{H}$ is *PPT* if E_i is PPT for all i .

We denote the set of all positive-semidefinite PPT operators in \mathbb{H} as

$$\mathbb{PPT}_+ = \{E \in \mathbb{H} \mid E \geq 0, E^{\text{PT}} \geq 0\}, \quad (1)$$

and its dual set as \mathbb{PPT}_+^* , that is,

$$\mathbb{PPT}_+^* = \{E \in \mathbb{H} \mid \text{Tr}(EF) \geq 0 \forall F \in \mathbb{PPT}_+\}. \quad (2)$$

A measurement is called a *LOCC measurement* if it can be implemented by LOCC, and a measurement $\{M_i\}_i$ is called a *PPT measurement* if M_i is PPT for all i . We note that every LOCC measurement is a PPT measurement [4].

A. Decomposable entanglement witness

Definition 2. $W \in \mathbb{H}$ is called *decomposable* if it can be written as a sum of a positive-semidefinite operator and a PPT operator in \mathbb{H} , that is,

$$W = P + Q^{\text{PT}} \quad (3)$$

for some $P \geq 0$ and $Q \geq 0$.

The following proposition provides an equivalent condition of the decomposability in Eq. (3) [28,29].

Proposition 1. $W \in \mathbb{H}$ is decomposable if and only if

$$W \in \mathbb{PPT}_+^*, \quad (4)$$

where \mathbb{PPT}_+^* is defined in Eq. (2).

A positive-semidefinite operator $E \in \mathbb{H}$ is called *separable* if it can be expressed as a sum of positive-semidefinite product

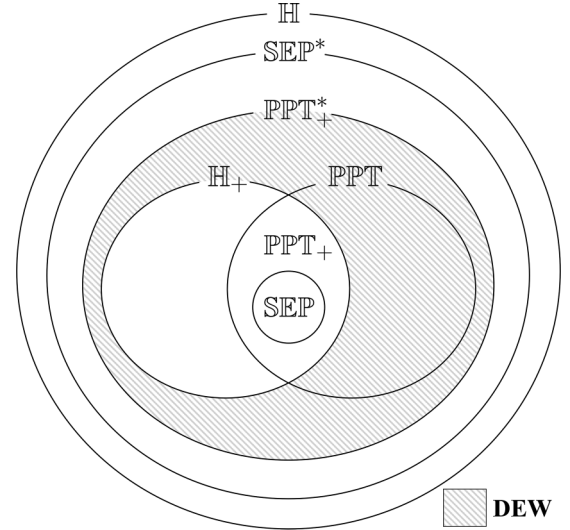


FIG. 1. The relationship of the subsets of \mathbb{H} . \mathbb{H}_+ is the set of all positive-semidefinite operators and \mathbb{PPT} is the set of all PPT operators. \mathbb{PPT}_+ is the intersection of \mathbb{H}_+ and \mathbb{PPT} . The shaded area $\{W \in \mathbb{PPT}_+^* \mid W \notin \mathbb{H}_+\}$ is the set of all DEWs.

operators, that is,

$$E = \sum_l A_l \otimes B_l, \quad (5)$$

where A_l and B_l are positive-semidefinite operators on \mathbb{C}^{d_1} and \mathbb{C}^{d_2} of \mathcal{H} , respectively. We denote the set of all positive-semidefinite *separable* operators in \mathbb{H} as

$$\text{SEP} = \{E \in \mathbb{H} \mid E \geq 0, E : \text{separable}\}. \quad (6)$$

We also denote the dual set of SEP as

$$\text{SEP}^* = \{E \in \mathbb{H} \mid \text{Tr}(EF) \geq 0 \forall F \in \text{SEP}\}. \quad (7)$$

Definition 3. $W \in \mathbb{H}$ is called an *EW* if it is in SEP^* but not positive semidefinite, that is,

$$W \in \text{SEP}^*, W \not\geq 0. \quad (8)$$

In particular, an EW $W \in \mathbb{H}$ is called a *DEW* if it is decomposable.

From Proposition 1, we can see that $W \in \mathbb{H}$ is a DEW if and only if

$$W \in \mathbb{PPT}_+^*, W \not\geq 0. \quad (9)$$

We note that $\mathbb{PPT}_+^* \subseteq \text{SEP}^*$ since $\text{SEP} \subseteq \mathbb{PPT}_+$. Figure 1 illustrates the relationship of the subsets of \mathbb{H} .

B. Minimum-error discrimination of bipartite quantum states

For a *bipartite* quantum state ensemble,

$$\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n, \quad (10)$$

where the state ρ_i is prepared with the probability $\eta_i \in [0, 1]$, let us consider the situation of discriminating the states from \mathcal{E} using a measurement $\{M_i\}_{i=1}^n$. Here, the detection of M_i means that we guess the prepared state as ρ_i .

The *minimum-error discrimination* of \mathcal{E} is to achieve the optimal success probability,

$$p_G(\mathcal{E}) = \max_{\text{measurement}} \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i), \quad (11)$$

where the maximum is taken over all possible measurements [32]. We note that a measurement $\{M_i\}_{i=1}^n$ gives the optimal success probability $p_G(\mathcal{E})$ if and only if it satisfies the following condition [33–36]:

$$\sum_{j=1}^n \eta_j \rho_j M_j - \eta_i \rho_i \geq 0 \quad \forall i = 1, \dots, n. \quad (12)$$

When the available measurements are limited to PPT measurements, we denote the maximum success probability by

$$p_{\text{PPT}}(\mathcal{E}) = \max_{\text{PPT measurement}} \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i), \quad (13)$$

where the maximum is taken over all possible PPT measurements. Similarly, we denote

$$p_L(\mathcal{E}) = \max_{\text{LOCC measurement}} \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i), \quad (14)$$

where the maximum is taken over all possible LOCC measurements.

Because $p_{\text{PPT}}(\mathcal{E})$ is the maximum value over the set of all PPT measurements, which is a proper subset of the set of all measurements, $p_{\text{PPT}}(\mathcal{E})$ is a lower bound of $p_G(\mathcal{E})$. Moreover, $p_L(\mathcal{E})$ is a lower bound of $p_{\text{PPT}}(\mathcal{E})$ since every LOCC measurement is a PPT measurement [4]. Thus, we have

$$p_L(\mathcal{E}) \leq p_{\text{PPT}}(\mathcal{E}) \leq p_G(\mathcal{E}). \quad (15)$$

As $p_L(\mathcal{E})$ and $p_{\text{PPT}}(\mathcal{E})$ have the same objective function to maximize, the first inequality in (15) becomes an equality if and only if there exists a LOCC measurement realizing $p_{\text{PPT}}(\mathcal{E})$. Similarly, the second inequality in (15) becomes an equality if and only if there exists a PPT measurement realizing $p_G(\mathcal{E})$.

III. PPT MEASUREMENTS AND QUANTUM STATE DISCRIMINATION

In this section, we provide the first main result of our paper. We first consider an upper bound of $p_{\text{PPT}}(\mathcal{E})$ in Eq. (13), and show that this upper bound is equal to $p_{\text{PPT}}(\mathcal{E})$. We further provide conditions on the maximum success probability over all possible PPT measurements. The results will be used to obtain necessary and/or sufficient conditions for minimum-error discrimination by PPT measurements in the next section.

For a bipartite quantum state ensemble \mathcal{E} in Eq. (10), we define $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ as

$$\mathbb{H}_{\text{PPT}}(\mathcal{E}) = \{H \in \mathbb{H} \mid H - \eta_i \rho_i \in \mathbb{PPT}_+^* \text{ for any } i \in \{1, \dots, n\}\}, \quad (16)$$

where \mathbb{PPT}_+^* is defined in Eq. (2). In other words, $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ is the set of all H such that $H - \eta_i \rho_i$ is decomposable in \mathbb{H} , for all $i = 1, \dots, n$. We further define

$$\mathbb{H}_{\text{DEW}}(\mathcal{E}) = \{H \in \mathbb{H}_{\text{PPT}}(\mathcal{E}) \mid H - \eta_j \rho_j \text{ is a DEW for some } j \in \{1, \dots, n\}\}, \quad (17)$$

that is, a subset of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ satisfying $H - \eta_j \rho_j \not\geq 0$ for some $j \in \{1, \dots, n\}$. From the argument after Definition 3, we can see that

$$H \in \mathbb{H}_{\text{PPT}}(\mathcal{E}) \setminus \mathbb{H}_{\text{DEW}}(\mathcal{E}) \quad (18)$$

if and only if

$$H - \eta_i \rho_i \geq 0 \quad \forall i = 1, \dots, n. \quad (19)$$

Now, let us consider the minimum quantity

$$q_{\text{PPT}}(\mathcal{E}) = \min_{H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})} \text{Tr} H, \quad (20)$$

which is an upper bound of $p_{\text{PPT}}(\mathcal{E})$ [37], that is,

$$p_{\text{PPT}}(\mathcal{E}) \leq q_{\text{PPT}}(\mathcal{E}). \quad (21)$$

The following theorem shows that $p_{\text{PPT}}(\mathcal{E})$ in Eq. (13) is equal to $q_{\text{PPT}}(\mathcal{E})$ in Eq. (20). The proof of Theorem 1 is given in Appendix A.

Theorem 1. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$,

$$p_{\text{PPT}}(\mathcal{E}) = q_{\text{PPT}}(\mathcal{E}). \quad (22)$$

For a given ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$, the following theorem provides a necessary and sufficient condition for a PPT measurement $\{M_i\}_{i=1}^n$ and $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ to realize $p_{\text{PPT}}(\mathcal{E})$ and $q_{\text{PPT}}(\mathcal{E})$, respectively.

Theorem 2. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$, a PPT measurement $\{M_i\}_{i=1}^n$, and $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$, $\{M_i\}_{i=1}^n$ realizes $p_{\text{PPT}}(\mathcal{E})$ and H provides $q_{\text{PPT}}(\mathcal{E})$ if and only if

$$\text{Tr}[M_i(H - \eta_i \rho_i)] = 0 \quad \forall i = 1, \dots, n. \quad (23)$$

Proof. Let us suppose that $\{M_i\}_{i=1}^n$ and H give $p_{\text{PPT}}(\mathcal{E})$ and $q_{\text{PPT}}(\mathcal{E})$, respectively. From $M_i \in \mathbb{PPT}_+$ and $H - \eta_i \rho_i \in \mathbb{PPT}_+^*$ for all $i = 1, \dots, n$, we have

$$\text{Tr}[M_i(H - \eta_i \rho_i)] \geq 0 \quad \forall i = 1, \dots, n. \quad (24)$$

We note that

$$\begin{aligned} \sum_{i=1}^n \text{Tr}[M_i(H - \eta_i \rho_i)] &= \text{Tr} H - \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) \\ &= q_{\text{PPT}}(\mathcal{E}) - p_{\text{PPT}}(\mathcal{E}) = 0, \end{aligned} \quad (25)$$

where the first equality is from $\sum_{i=1}^n M_i = \mathbb{1}$, the second equality is due to the assumption of H and $\{M_i\}_{i=1}^n$, and the last equality is by Theorem 1. Inequality (24) and Eq. (25) lead us to condition (23).

Conversely let us assume that $\{M_i\}_{i=1}^n$ and H satisfy condition (23). This assumption implies

$$\begin{aligned} q_{\text{PPT}}(\mathcal{E}) &= p_{\text{PPT}}(\mathcal{E}) \geq \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) \\ &= \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) + \sum_{i=1}^n \text{Tr}[M_i(H - \eta_i \rho_i)] \\ &= \text{Tr}H \geq q_{\text{PPT}}(\mathcal{E}), \end{aligned} \quad (26)$$

where the first equality follows from Theorem 1, the second equality is from condition (23), the last equality is due to $\sum_{i=1}^n M_i = \mathbb{1}$, and the first and second inequalities are from the definitions of $p_{\text{PPT}}(\mathcal{E})$ and $q_{\text{PPT}}(\mathcal{E})$, respectively. Inequality (26) leads us to

$$\sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) = p_{\text{PPT}}(\mathcal{E}), \quad \text{Tr}H = q_{\text{PPT}}(\mathcal{E}). \quad (27)$$

Thus, $\{M_i\}_{i=1}^n$ and H give $p_{\text{PPT}}(\mathcal{E})$ and $q_{\text{PPT}}(\mathcal{E})$, respectively. ■

We note that $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ giving $q_{\text{PPT}}(\mathcal{E})$ is generally not unique (see Example 2 in Sec. IV B). However, the following corollary states the case that $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ providing $q_{\text{PPT}}(\mathcal{E})$ is unique.

Corollary 1. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$, we have

$$p_{\text{PPT}}(\mathcal{E}) = \eta_1, \quad (28)$$

if and only if

$$\eta_1 \rho_1 - \eta_i \rho_i \in \mathbb{PPT}_+^* \quad \forall i = 2, \dots, n. \quad (29)$$

In this case, $\eta_1 \rho_1$ is the only element of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ providing $q_{\text{PPT}}(\mathcal{E})$.

Proof. Let $\{M_i\}_{i=1}^n$ be the measurement that $M_1 = \mathbb{1}$ and M_2, \dots, M_n are the zero operator in \mathbb{H} . We first assume Eq. (28) and consider $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ providing $q_{\text{PPT}}(\mathcal{E})$. Since $\{M_i\}_{i=1}^n$ is obviously a PPT measurement giving $p_{\text{PPT}}(\mathcal{E})$, it follows from Theorem 2 that $\text{Tr}(H - \eta_1 \rho_1) = 0$. From $H - \eta_1 \rho_1 \in \mathbb{PPT}_+^*$ and Lemma 1 in Appendix A, we have $H = \eta_1 \rho_1$. Thus, $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ together with the definition of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ leads us to condition (29).

Conversely, let us suppose condition (29) and consider $H = \eta_1 \rho_1$. Condition (29) implies $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$. The PPT measurement $\{M_i\}_{i=1}^n$ and $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ satisfy condition (23). Therefore, we have

$$p_{\text{PPT}}(\mathcal{E}) = q_{\text{PPT}}(\mathcal{E}) = \text{Tr}H = \eta_1, \quad (30)$$

where the first equality is from Theorem 1 and the second equality follows from Theorem 2. ■

When Eq. (28) of Corollary 1 holds, the maximum success probability $p_{\text{PPT}}(\mathcal{E})$ can be achieved without the help of measurement, simply by guessing ρ_1 is prepared. As we can check in the proof of Corollary 1, the choice of ρ_1 in Corollary 1 can be arbitrary. That is, any of $\{\rho_i\}_{i=1}^n$ can be used to play the role of ρ_1 in Corollary 1.

IV. MINIMUM-ERROR DISCRIMINATION BY PPT MEASUREMENTS

In this section, we provide another main result of our paper showing that the minimum-error discrimination of bipartite quantum states using PPT measurements strongly depends on the existence of DEW. More precisely, we establish necessary and/or sufficient conditions for minimum-error discrimination by PPT measurements in terms of DEW.

For a bipartite quantum state ensemble \mathcal{E} in Eq. (10), the minimum-error discrimination can be realized by PPT measurements if and only if

$$p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E}), \quad (31)$$

where $p_G(\mathcal{E})$ and $p_{\text{PPT}}(\mathcal{E})$ are defined in Eqs. (11) and (13), respectively. Here, we provide necessary and/or sufficient conditions for Eq. (31) in terms of DEW.

A. Necessary condition for $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$

Theorem 3. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$, if $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$, then there does not exist PPT measurement $\{M_i\}_{i=1}^n$ satisfying

$$\sum_{i=1}^n \eta_i \rho_i M_i \in \mathbb{H}_{\text{DEW}}(\mathcal{E}), \quad (32)$$

where $\mathbb{H}_{\text{DEW}}(\mathcal{E})$ is defined in Eq. (17). Moreover, if $\{M_i\}_{i=1}^n$ is a PPT measurement satisfying condition (32), we have

$$p_{\text{PPT}}(\mathcal{E}) = \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i). \quad (33)$$

Proof. Let us suppose $\{M_i\}_{i=1}^n$ is a PPT measurement satisfying condition (32), and consider

$$H = \sum_{i=1}^n \eta_i \rho_i M_i. \quad (34)$$

Equation (33) holds because

$$\begin{aligned} \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) &\leq p_{\text{PPT}}(\mathcal{E}) = q_{\text{PPT}}(\mathcal{E}) \\ &\leq \text{Tr}H = \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i), \end{aligned} \quad (35)$$

where the first and second inequalities follow from the definitions of $p_{\text{PPT}}(\mathcal{E})$ and $q_{\text{PPT}}(\mathcal{E})$, respectively, the first equality is due to Theorem 1, and the second equality is from the definition of H . This proves the second statement of our theorem.

Now, assume $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$. If there exists a PPT measurement $\{M_i\}_{i=1}^n$ satisfying condition (32), the second statement of our theorem implies that $\{M_i\}_{i=1}^n$ gives the optimal success probability $p_G(\mathcal{E})$ due to Eq. (33). From the optimality condition in Eq. (12), we have

$$\sum_{j=1}^n \eta_j \rho_j M_j' - \eta_i \rho_i \geq 0 \quad \forall i = 1, \dots, n, \quad (36)$$

which contradicts condition (32). Thus, there does not exist PPT measurement satisfying condition (32). This proves the first statement of our theorem. ■

Example 1. For any integer $d \geq 2$, let us consider the two-qudit state ensemble $\mathcal{E} = \{\eta_{i,j}^{(k,l)}, \rho_{i,j}^{(k,l)}\}_{i,j,k,l}$ consisting of $2d(d-1)$ mixed states with equal prior probabilities,

$$\eta_{i,j}^{(k,l)} = \frac{1}{2d(d-1)}, \rho_{i,j}^{(k,l)} = \frac{\lambda}{3}(\Psi_{i,j}^{(k)} + \Pi_{i,j}^{(l)}) + (1-\lambda)\sigma, \quad (37)$$

$i, j \in \{0, 1, \dots, d-1\}$ with $i < j, k, l \in \{1, 2\}$,

where σ is an arbitrary two-qudit state, $0 < \lambda \leq 1$, and

$$\begin{aligned} \Psi_{i,j}^{(1)} &= \frac{1}{2}(|ii\rangle + |jj\rangle)(\langle ii| + \langle jj|), \\ \Psi_{i,j}^{(2)} &= \frac{1}{2}(|ii\rangle - |jj\rangle)(\langle ii| - \langle jj|), \\ \Pi_{i,j}^{(1)} &= |ii\rangle\langle ii| + |jj\rangle\langle jj|, \\ \Pi_{i,j}^{(2)} &= |ij\rangle\langle ij| + |ji\rangle\langle ji|. \end{aligned} \quad (38)$$

For a PPT measurement $\{M_{i,j}^{(k,l)}\}_{i,j,k,l}$ with

$$\begin{aligned} M_{i,j}^{(1,1)} &= M_{i,j}^{(2,1)} = \frac{1}{2(d-1)}\Pi_{i,j}^{(1)}, \\ M_{i,j}^{(1,2)} &= M_{i,j}^{(2,2)} = \frac{1}{2}\Pi_{i,j}^{(2)}, \end{aligned} \quad (39)$$

we show that condition (32) holds with respect to the ensemble in Eq. (37) and the measurement in Eq. (39). It is straightforward to verify that

$$\begin{aligned} &\left[\sum_{i',j',k',l'} \eta_{i',j'}^{(k',l')} \rho_{i',j'}^{(k',l')} M_{i',j'}^{(k',l')} - \eta_{i,j}^{(k,1)} \rho_{i,j}^{(k,1)} \right]^{\text{PT}} \\ &= \frac{\lambda}{4d(d-1)} \left[\Psi_{i,j}^{(5-k)} + \frac{1}{3}\Psi_{i,j}^{(k+2)} + \hat{\Pi}_{i,j}^{(1)} + \frac{2}{3}\hat{\Pi}_{i,j}^{(2)} \right] \geq 0, \\ &\times \sum_{i',j',k',l'} \eta_{i',j'}^{(k',l')} \rho_{i',j'}^{(k',l')} M_{i',j'}^{(k',l')} - \eta_{i,j}^{(k,2)} \rho_{i,j}^{(k,2)} \\ &= \frac{\lambda}{4d(d-1)} \left[\Psi_{i,j}^{(3-k)} + \frac{1}{3}\Psi_{i,j}^{(k)} + \hat{\Pi}_{i,j}^{(1)} + \frac{2}{3}\hat{\Pi}_{i,j}^{(2)} \right] \geq 0, \\ &\forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \forall k \in \{1, 2\}, \end{aligned} \quad (40)$$

where

$$\begin{aligned} \Psi_{i,j}^{(3)} &= \frac{1}{2}(|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|), \\ \Psi_{i,j}^{(4)} &= \frac{1}{2}(|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|), \\ \hat{\Pi}_{i,j}^{(1)} &= \sum_{i'=0}^{d-1} |i'i\rangle\langle i'i| - \Pi_{i,j}^{(1)}, \\ \hat{\Pi}_{i,j}^{(2)} &= \sum_{i',j'=0, i' \neq j'}^{d-1} |i'j'\rangle\langle i'j'| - \Pi_{i,j}^{(2)}. \end{aligned} \quad (41)$$

From the positivity in (40), we have

$$\begin{aligned} &\sum_{i',j',k',l'} \eta_{i',j'}^{(k',l')} \rho_{i',j'}^{(k',l')} M_{i',j'}^{(k',l')} - \eta_{i,j}^{(k,l)} \rho_{i,j}^{(k,l)} \in \mathbb{PPT}^*_+ \\ &\forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \forall k, l \in \{1, 2\}. \end{aligned} \quad (42)$$

Furthermore, a straightforward calculation leads us to

$$\begin{aligned} &\text{Tr} \left[\left(\sum_{i',j',k',l'} \eta_{i',j'}^{(k',l')} \rho_{i',j'}^{(k',l')} M_{i',j'}^{(k',l')} - \eta_{i,j}^{(k,1)} \rho_{i,j}^{(k,1)} \right) \Psi_{i,j}^{(k)} \right] \\ &= -\frac{\lambda}{12d(d-1)} \\ &\forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \forall k \in \{1, 2\}. \end{aligned} \quad (43)$$

From Eq. (43), we have

$$\begin{aligned} &\sum_{i',j',k',l'} \eta_{i',j'}^{(k',l')} \rho_{i',j'}^{(k',l')} M_{i',j'}^{(k',l')} - \eta_{i,j}^{(k,1)} \rho_{i,j}^{(k,1)} \not\geq 0 \\ &\forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \forall k \in \{1, 2\}. \end{aligned} \quad (44)$$

From Eqs. (42) and (44), the ensemble in Eq. (37) and the measurement in Eq. (39) satisfy condition (32).

The second statement of Theorem 3 implies that $p_{\text{PPT}}(\mathcal{E})$ is the expectation of the measurement in Eq. (39) about the ensemble in Eq. (37):

$$p_{\text{PPT}}(\mathcal{E}) = \frac{6 + \lambda(2d-3)(d+2)}{12d(d-1)}. \quad (45)$$

From the first statement of Theorem 3, we have

$$p_{\text{PPT}}(\mathcal{E}) < p_G(\mathcal{E}). \quad (46)$$

We also note that the measurement in Eq. (39) is a LOCC measurement because it can be implemented by performing the same local measurement $\{|l\rangle\langle l|\}_{l=0}^{d-1}$ on each party. Thus, we have

$$p_L(\mathcal{E}) = p_{\text{PPT}}(\mathcal{E}) = \frac{6 + \lambda(2d-3)(d+2)}{12d(d-1)}. \quad (47)$$

B. Necessary and sufficient condition for $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$

Theorem 4. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$, $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$ if and only if there exists $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E})$ such that it provides $q_{\text{PPT}}(\mathcal{E})$ but does not satisfy

$$H \in \mathbb{H}_{\text{DEW}}(\mathcal{E}), \quad (48)$$

or equivalently, there is $H \in \mathbb{H}$ satisfying condition (18) and $\text{Tr}H = q_{\text{PPT}}(\mathcal{E})$.

Proof. Let $\{M_i\}_{i=1}^n$ be a PPT measurement providing $p_{\text{PPT}}(\mathcal{E})$. We first suppose $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$ and consider

$$H = \sum_{i=1}^n \eta_i \rho_i M_i. \quad (49)$$

Since the measurement $\{M_i\}_{i=1}^n$ gives the optimal success probability $p_G(\mathcal{E})$ due to the assumption of $\{M_i\}_{i=1}^n$ and $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$, we have from the optimality condition in Eq. (12) that

$$H - \eta_i \rho_i = \sum_{j=1}^n \eta_j \rho_j M_j - \eta_i \rho_i \geq 0 \quad \forall i = 1, \dots, n. \quad (50)$$

Therefore, H satisfies condition (18). Moreover, we have

$$\text{Tr}H = \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) = p_{\text{PPT}}(\mathcal{E}) = q_{\text{PPT}}(\mathcal{E}), \quad (51)$$

where the first equality is from Eq. (49), the second equality follows from the assumption of $\{M_i\}_{i=1}^n$, and the last equality is due to Theorem 1.

Conversely, let us assume H is an element of \mathbb{H} satisfying condition (18) and $\text{Tr}H = q_{\text{PPT}}(\mathcal{E})$. Thus, the positivity in (19) is satisfied in terms of H . For each $i = 1, \dots, n$, the positive-semidefinite operators M_i and $H - \eta_i \rho_i$ are orthogonal since they satisfy condition (23) from Theorem 2.

The optimality condition in Eq. (12) holds for the measurement $\{M_i\}_{i=1}^n$ because

$$\begin{aligned} & \sum_{j=1}^n \eta_j \rho_j M_j - \eta_i \rho_i \\ &= \sum_{j=1}^n \eta_j \rho_j M_j + \sum_{k=1}^n (H - \eta_k \rho_k) M_k - \eta_i \rho_i \\ &= H - \eta_i \rho_i \geq 0 \quad \forall i = 1, \dots, n, \end{aligned} \tag{52}$$

where the first equality is from the orthogonality of M_i and $H - \eta_i \rho_i$ for each $i = 1, \dots, n$ and the second equality is from $\sum_{i=1}^n M_i = \mathbb{1}$. Thus, we have

$$p_G(\mathcal{E}) = \sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) = p_{\text{PPT}}(\mathcal{E}), \tag{53}$$

where the second equality is due to the assumption of $\{M_i\}_{i=1}^n$. ■

If $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$, Theorem 4 implies that there must exist $H \in \mathbb{H}_{\text{PPT}}(\mathcal{E}) \setminus \mathbb{H}_{\text{DEW}}(\mathcal{E})$ giving $q_{\text{PPT}}(\mathcal{E})$. In this case, there possibly exists another Hermitian operator $H' \in \mathbb{H}_{\text{DEW}}(\mathcal{E})$ and $\text{Tr}H' = q_{\text{PPT}}(\mathcal{E})$, which is illustrated in the following example.

Example 2. Let us consider the two-qubit state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^3$ consisting of three pure orthogonal states with equal prior probabilities,

$$\begin{aligned} \eta_1 &= \frac{1}{3}, \rho_1 = |0\rangle \langle 0| \otimes |0\rangle \langle 0|, \\ \eta_2 &= \frac{1}{3}, \rho_2 = |1\rangle \langle 1| \otimes |1\rangle \langle 1|, \\ \eta_3 &= \frac{1}{3}, \rho_3 = \Psi_+, \end{aligned} \tag{54}$$

where

$$\Psi_{\pm} = |\Psi_{\pm}\rangle \langle \Psi_{\pm}|, |\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{55}$$

Since the states ρ_1 and ρ_2 are product states, we have

$$p_L(\mathcal{E}) = 1. \tag{56}$$

We note that three $2 \otimes 2$ pure orthogonal states can be perfectly discriminated by a finite-round LOCC if and only if at least two of them are product states [38]. Since $p_G(\mathcal{E})$ is bounded above by 1, it follows from inequality (15) and Eq. (56) that

$$p_L(\mathcal{E}) = p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E}) = 1. \tag{57}$$

Furthermore, we have

$$q_{\text{PPT}}(\mathcal{E}) = p_{\text{PPT}}(\mathcal{E}) = 1, \tag{58}$$

where the first equality follows from Theorem 1 and the second equality is due to Eq. (57).

Now, let us consider the Hermitian operator

$$H = \frac{1}{3}(\Phi_+ + \Phi_-) + \frac{1+t}{6}\Psi_+ + \frac{1-t}{6}\Psi_-, \tag{59}$$

where Ψ_{\pm} is defined in Eq. (55), $0 \leq t \leq 1$, and

$$\Phi_{\pm} = |\Phi_{\pm}\rangle \langle \Phi_{\pm}|, |\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \tag{60}$$

Equations (58) and (59) imply

$$\text{Tr}H = 1 = q_{\text{PPT}}(\mathcal{E}). \tag{61}$$

A straightforward calculation leads us to

$$\begin{aligned} H - \eta_1 \rho_1 &= \frac{1}{3}\rho_2 + \frac{1+t}{6}\Psi_+ + \frac{1-t}{6}\Psi_- \geq 0, \\ H - \eta_2 \rho_2 &= \frac{1}{3}\rho_1 + \frac{1+t}{6}\Psi_+ + \frac{1-t}{6}\Psi_- \geq 0, \\ H - \eta_3 \rho_3 &= \frac{1+t}{6}(\Phi_+ + \Phi_-) + \frac{1-t}{3}\Phi_-^{\text{PT}} \in \mathbb{PPT}^*_+. \end{aligned} \tag{62}$$

Due to Eqs. (61) and (62), H is an element of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ giving $q_{\text{PPT}}(\mathcal{E})$ regardless of $t \in [0, 1]$. Moreover, $H \in \mathbb{H}_{\text{DEW}}(\mathcal{E})$ for all $t \in [0, 1)$ because

$$\text{Tr}[(H - \eta_3 \rho_3)\Psi_+] = -\frac{1-t}{6}. \tag{63}$$

For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$ where H is the only element of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ giving $q_{\text{PPT}}(\mathcal{E})$, Theorem 4 tells us that $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$ if and only if there is no DEW in $\{H - \eta_i \rho_i\}_{i=1}^n$. From Corollary 1, $\eta_1 \rho_1$ is the only element of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ providing $q_{\text{PPT}}(\mathcal{E})$ when condition (29) holds. Thus, we have the following corollary.

Corollary 2. For a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$ with condition (29), $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$ if and only if there is no DEW in $\{\eta_1 \rho_1 - \eta_i \rho_i\}_{i=2}^n$.

Example 3. For any integer $d \geq 2$, let us consider the two-qudit state ensemble $\mathcal{E} = \{\eta_1, \rho_1\} \cup \{\eta_{i,j}^{(k)}, \rho_{i,j}^{(k)}\}_{i,j,k}$ consisting of $1 + 2d(d-1)$ states,

$$\begin{aligned} \eta_1 &= \frac{d}{5d-4}, \rho_1 = \frac{1}{d^2}\mathbb{1}, \\ \eta_{i,j}^{(k)} &= \frac{2}{d(5d-4)}, \rho_{i,j}^{(k)} = (1-\lambda)\Psi_{i,j}^{(k)} + \frac{\lambda}{d^2}\mathbb{1}, \\ i, j &\in \{0, 1, \dots, d-1\} \text{ with } i < j, k = 1, 2, 3, 4, \end{aligned} \tag{64}$$

where $0 \leq \lambda < 1$ and $\Psi_{i,j}^{(k)}$ is defined in Eqs. (38) and (41).

From a straightforward calculation, we can verify that

$$\begin{aligned} & \eta_1 \rho_1 - \eta_{i,j}^{(k)} \rho_{i,j}^{(k)} \\ &= \frac{1}{d^3(5d-4)} [(d^2 - 2\lambda)(\mathbb{1} - \mathbb{1}_{i,j}) + (d^2 - 2\lambda)\mathbb{1}_{i,j} \\ &\quad - 2d^2(1-\lambda)\Psi_{i,j}^{(k)}] \\ &= \frac{1}{d^3(5d-4)} [(d^2 - 2\lambda)(\mathbb{1} - \mathbb{1}_{i,j}) + \lambda(d^2 - 2)\mathbb{1}_{i,j} \\ &\quad + 2d^2(1-\lambda)\Psi_{i,j}^{(5-k)\text{PT}}] \in \mathbb{PPT}^*_+ \\ &\forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \forall k \in \{1, 2, 3, 4\}, \end{aligned} \tag{65}$$

where

$$\mathbb{1}_{i,j} = (|i\rangle\langle i| + |j\rangle\langle j|) \otimes (|i\rangle\langle i| + |j\rangle\langle j|), \quad (66)$$

and the second equality in Eq. (65) is due to

$$\begin{aligned} \frac{1}{2}\mathbb{1}_{i,j} - \Psi_{i,j}^{(k)} &= \Psi_{i,j}^{(5-k)PT} \\ \forall i, j \in \{0, 1, \dots, d-1\} \text{ with } i < j, \quad \forall k \in \{1, 2, 3, 4\}. \end{aligned} \quad (67)$$

From Corollary 1 and Eq. (65), we have

$$p_{\text{PPT}}(\mathcal{E}) = \eta_1 = \frac{d}{5d-4}. \quad (68)$$

For each i, j , and k , we can easily see from the first equality in Eq. (65) that $\eta_1 \rho_1 - \eta_{i,j}^{(k)} \rho_{i,j}^{(k)} \geq 0$ if and only if

$$\text{Tr}\left[\left(\eta_1 \rho_1 - \eta_{i,j}^{(k)} \rho_{i,j}^{(k)}\right) \Psi_{i,j}^{(k)}\right] = \frac{2\lambda d^2 - 2\lambda - d^2}{d^3(5d-4)} \geq 0. \quad (69)$$

Therefore, it follows from Corollary 2 that $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$ if and only if

$$\lambda \geq \frac{d^2}{2(d^2-1)}. \quad (70)$$

V. DISCUSSION

We have considered bipartite quantum state discrimination and shown that the minimum-error discrimination of bipartite quantum states using PPT measurements strongly depends on the existence of DEW. We have established conditions on minimum-error discrimination by PPT measurements, that is, $p_{\text{PPT}}(\mathcal{E}) = p_G(\mathcal{E})$, in terms of DEW (Theorems 3 and 4). We have also provided conditions on the maximum success probability over all possible PPT measurements (Theorems 1 and 2). Our results have been illustrated by examples of multidimensional bipartite quantum states.

Quantum nonlocality is a genuine phenomenon of multipartite quantum systems without any classical counterpart. Quantum nonlocality is a key ingredient making quantum states outperform the classical ones in various quantum information processing tasks such as quantum teleportation and quantum cryptography [19,20]. It is also known that this quantum nonlocality plays an important role in quantum algorithms which are more powerful than any classical ones [39,40]. Quantum nonlocality also occurs in discriminating multipartite quantum states; in discriminating states from a bipartite quantum state ensemble \mathcal{E} , quantum nonlocality occurs when optimal state discrimination cannot be realized only by LOCC measurement, that is, $p_L(\mathcal{E}) < p_G(\mathcal{E})$.

Our results here can be used to detect the occurrence of nonlocality in quantum state discrimination. Violation of the condition in Theorems 3 or 4 means $p_{\text{PPT}}(\mathcal{E}) < p_G(\mathcal{E})$, which consequently implies $p_L(\mathcal{E}) < p_G(\mathcal{E})$. Thus, condition (32) of Theorem 3 and condition (48) of Theorem 4 can be used as sufficient conditions on $p_L(\mathcal{E}) < p_G(\mathcal{E})$.

Corollary 2 provides a useful method to make a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$ showing nonlocality, that is, $p_L(\mathcal{E}) < p_G(\mathcal{E})$, by means of DEW: For a given DEW W , let us consider the bipartite quantum state ensemble

$\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^2$ where

$$\begin{aligned} \eta_1 &= \frac{\text{Tr}(P+W)}{\text{Tr}(2P+W)}, \rho_1 = \frac{P+W}{\text{Tr}(P+W)}, \\ \eta_2 &= \frac{\text{Tr}P}{\text{Tr}(2P+W)}, \rho_2 = \frac{P}{\text{Tr}P}, \end{aligned} \quad (71)$$

with any positive-semidefinite operator P satisfying

$$P+W \geq 0. \quad (72)$$

Since $\eta_1 \rho_1 - \eta_2 \rho_2$ is proportional to the DEW W , $p_{\text{PPT}}(\mathcal{E}) < p_G(\mathcal{E})$ holds from Corollary 2. Thus, inequality (15) leads us to $p_L(\mathcal{E}) < p_G(\mathcal{E})$. In other words, Corollary 2 provides a systematic way to construct a bipartite quantum state ensemble showing nonlocality in quantum state discrimination.

Corollary 2 can also be used to construct a bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$ with $n > 2$ showing nonlocality in quantum state discrimination. For a set of DEWs $\{W_i\}_{i=2}^n$, let us consider the bipartite quantum state ensemble $\mathcal{E} = \{\eta_i, \rho_i\}_{i=1}^n$ where

$$\begin{aligned} \eta_1 &= \frac{\text{Tr}\mathbb{1}}{\text{Tr}(n\mathbb{1} - \sum_{j=2}^n \lambda_j W_j)}, \rho_1 = \frac{\mathbb{1}}{\text{Tr}\mathbb{1}}, \\ \eta_i &= \frac{\text{Tr}(\mathbb{1} - \lambda_i W_i)}{\text{Tr}(n\mathbb{1} - \sum_{j=2}^n \lambda_j W_j)}, \rho_i = \frac{\mathbb{1} - \lambda_i W_i}{\text{Tr}(\mathbb{1} - \lambda_i W_i)}, \quad (73) \\ &i = 2, \dots, n, \end{aligned}$$

with any set of positive real numbers $\{\lambda_i\}_{i=2}^n$ satisfying

$$\mathbb{1} - \lambda_i W_i \geq 0 \quad \forall i = 2, \dots, n. \quad (74)$$

Because $\eta_1 \rho_1 - \eta_i \rho_i$ is proportional to W_i for any $i \in \{2, \dots, n\}$, $p_{\text{PPT}}(\mathcal{E}) < p_G(\mathcal{E})$ holds from Corollary 2. Thus, inequality (15) leads us to $p_L(\mathcal{E}) < p_G(\mathcal{E})$.

As our results establish a specific relation between the properties of EW and PPT measurements, it is natural to investigate the relationship between EW and other measurements such as the set of all separable measurements. It is also an interesting future work to investigate if EW can be used for the optimization of other state discrimination strategies besides minimum-error discrimination.

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APPENDIX A: PROOF OF THEOREM 1

As we already have inequality (21), it is enough to show that

$$p_{\text{PPT}}(\mathcal{E}) \geq q_{\text{PPT}}(\mathcal{E}). \quad (A1)$$

Lemma 1. If $E \in \mathbb{PPT}_+^*$ and $E \neq 0_{\mathbb{H}}$, then $\text{Tr}E > 0$, where $0_{\mathbb{H}}$ is the zero operator in \mathbb{H} .

Proof. The proof is by contradiction. We first note that $\text{Tr}E = \text{Tr}(\mathbb{1}E) \geq 0$ because $\mathbb{1} \in \text{PPT}_+$ and $E \in \text{PPT}_+^*$. Thus, let us suppose $\text{Tr}E = 0$.

For an arbitrary orthonormal product basis $\{|e_i\rangle\}_{i=1}^D$ of the bipartite Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we have

$$\sum_{i=1}^D \text{Tr}(E|e_i\rangle\langle e_i|) = \text{Tr}(E\mathbb{1}) = \text{Tr}E = 0, \quad (\text{A2})$$

where $D = d_1d_2$. From $E \in \text{PPT}_+^*$ and $|e_i\rangle\langle e_i| \in \text{PPT}_+$ for all $i = 1, \dots, D$, we have

$$\text{Tr}(E|e_i\rangle\langle e_i|) \geq 0 \quad \forall i = 1, \dots, D. \quad (\text{A3})$$

Equation (A2) and inequality (A3) lead us to

$$\text{Tr}(E|e_i\rangle\langle e_i|) = 0 \quad \forall i = 1, \dots, D. \quad (\text{A4})$$

Since the choice of $\{|e_i\rangle\}_{i=1}^D$ can be arbitrary,

$$\text{Tr}(E|e\rangle\langle e|) = 0 \quad (\text{A5})$$

for any product vector $|e\rangle \in \mathcal{H}$, therefore

$$\text{Tr}(EF) = 0 \quad \forall F \in \text{SEP}, \quad (\text{A6})$$

where SEP is defined in Eq. (6).

We note that SEP spans \mathbb{H} . To see this, we first note that the set of all positive-semidefinite operators on \mathbb{C}^{d_k} spans the set of all Hermitian operators on \mathbb{C}^{d_k} for each $k = 1, 2$. Moreover, every $F \in \mathbb{H}$ can be represented as a summation of product Hermitian operators,

$$F = \sum_l A_l \otimes B_l, \quad (\text{A7})$$

where A_l and B_l are Hermitian operators on \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively. Therefore, Eq. (A6) leads us to

$$\text{Tr}(EF) = 0 \quad \forall F \in \mathbb{H}. \quad (\text{A8})$$

Equation (A8) means $E = 0_{\mathbb{H}}$ which contradicts the assumption of the lemma. Thus, $\text{Tr}E > 0$. \blacksquare

Let us consider the set

$$\mathcal{S}(\mathcal{E}) = \left\{ \left(\sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) - p, \mathbb{1} - \sum_{i=1}^n M_i \right) \in \mathbb{R} \times \mathbb{H} \mid \right. \\ \left. p > p_{\text{PPT}}(\mathcal{E}), M_i \in \text{PPT}_+ \forall i = 1, \dots, n \right\}, \quad (\text{A9})$$

where \mathbb{R} is the set of all real numbers. We note that $\mathcal{S}(\mathcal{E})$ is a convex set due to the convexity of PPT_+ in Eq. (1). Moreover, $\mathcal{S}(\mathcal{E})$ does not have the origin $(0, 0_{\mathbb{H}})$ of $\mathbb{R} \times \mathbb{H}$, otherwise there is a PPT measurement $\{M_i\}_{i=1}^n$ with

$$\sum_{i=1}^n \eta_i \text{Tr}(\rho_i M_i) > p_{\text{PPT}}(\mathcal{E}), \quad (\text{A10})$$

and this contradicts the optimality of $p_{\text{PPT}}(\mathcal{E})$ in Eq. (13). We also note that the Cartesian product $\mathbb{R} \times \mathbb{H}$ can be considered as a real vector space with an inner product defined as

$$\langle (a, A), (b, B) \rangle = ab + \text{Tr}(AB) \quad (\text{A11})$$

for $(a, A), (b, B) \in \mathbb{R} \times \mathbb{H}$.

Since $\mathcal{S}(\mathcal{E})$ and the single-element set $\{(0, 0_{\mathbb{H}})\}$ are disjoint convex sets, it follows from the separating hyperplane theorem [41,42] that there is $(\gamma, \Gamma) \in \mathbb{R} \times \mathbb{H}$ satisfying

$$(\gamma, \Gamma) \neq (0, 0_{\mathbb{H}}), \quad (\text{A12})$$

$$\langle (\gamma, \Gamma), (r, G) \rangle \leq 0 \quad \forall (r, G) \in \mathcal{S}(\mathcal{E}). \quad (\text{A13})$$

Suppose

$$\text{Tr}\Gamma \leq \gamma p_{\text{PPT}}(\mathcal{E}), \quad (\text{A14})$$

$$\Gamma - \gamma \eta_i \rho_i \in \text{PPT}_+^* \quad \forall i = 1, \dots, n, \quad (\text{A15})$$

$$\gamma > 0. \quad (\text{A16})$$

From conditions (A15) and (A16), the Hermitian operator $H = \Gamma/\gamma$ is an element of $\mathbb{H}_{\text{PPT}}(\mathcal{E})$ in Eq. (16). Thus, the definition of $q_{\text{PPT}}(\mathcal{E})$ in Eq. (20) leads us to

$$q_{\text{PPT}}(\mathcal{E}) \leq \text{Tr}H. \quad (\text{A17})$$

Moreover, condition (A14) implies

$$\text{Tr}H \leq p_{\text{PPT}}(\mathcal{E}). \quad (\text{A18})$$

Inequalities (A17) and (A18) complete the proof of inequality (A1).

The rest of this section is to prove conditions (A14), (A15), and (A16).

Proof of (A14). From Eq. (A11), inequality (A13) can be rewritten as

$$\text{Tr}\Gamma - \sum_{i=1}^n \text{Tr}[M_i(\Gamma - \gamma \eta_i \rho_i)] \leq \gamma p \quad (\text{A19})$$

for all $p > p_{\text{PPT}}(\mathcal{E})$ and all $\{M_i\}_{i=1}^n \subseteq \text{PPT}_+$. If $M_i = 0_{\mathbb{H}}$ for all $i = 1, \dots, n$, inequality (A19) becomes inequality (A14) by taking the limit of p to $p_{\text{PPT}}(\mathcal{E})$. \blacksquare

Proof of (A15). For each $j \in \{1, \dots, n\}$, let us consider an arbitrary $M_j \in \text{PPT}_+$ and $M_i = 0_{\mathbb{H}}$ for all $i = 1, \dots, n$ with $i \neq j$. In this case, $\{M_i\}_{i=1}^n$ is clearly a subset of PPT_+ , and inequality (A19) becomes

$$\text{Tr}\Gamma - \text{Tr}[M_j(\Gamma - \gamma \eta_j \rho_j)] \leq \gamma p_{\text{PPT}}(\mathcal{E}) \quad (\text{A20})$$

by taking the limit of p to $p_{\text{PPT}}(\mathcal{E})$.

Suppose $\Gamma - \gamma \eta_j \rho_j \notin \text{PPT}_+^*$, then there is $M \in \text{PPT}_+$ with $\text{Tr}[M(\Gamma - \gamma \eta_j \rho_j)] < 0$. We note that $M \in \text{PPT}_+$ implies $tM \in \text{PPT}_+$ for any $t > 0$. Thus, $\{M_i\}_{i=1}^n$ consisting of $M_j = tM$ for $t > 0$ and $M_i = 0$ for all $i = 1, \dots, n$ with $i \neq j$ is also a subset of PPT_+ .

Now, inequality (A20) can be rewritten as

$$\text{Tr}\Gamma - \text{Tr}[tM(\Gamma - \gamma \eta_j \rho_j)] \leq \gamma p_{\text{PPT}}(\mathcal{E}). \quad (\text{A21})$$

Since inequality (A21) is true for arbitrary large $t > 0$, $\gamma p_{\text{PPT}}(\mathcal{E})$ can also be arbitrarily large. However, this contradicts that both γ and $p_{\text{PPT}}(\mathcal{E})$ are finite. Thus, $\Gamma - \gamma \eta_j \rho_j \in$

$\mathbb{P}\mathbb{P}\mathbb{T}_+^*$, which completes the proof of (A15). ■

Proof of (A16). To show $\gamma \geq 0$, we assume $\gamma < 0$. Let us consider $\{M_i\}_{i=1}^n$ with $M_i = 0_{\mathbb{H}}$ for all $i = 1, \dots, n$. Since $\{M_i\}_{i=1}^n \subseteq \mathbb{P}\mathbb{P}\mathbb{T}_+^*$, inequality (A19) becomes

$$\text{Tr}\Gamma \leq -\infty \quad (\text{A22})$$

by taking the limit of p to ∞ . This contradicts that Γ is bounded. Thus $\gamma \geq 0$.

Now, let us suppose $\gamma = 0$. In this case, conditions (A14) and (A15) become

$$\text{Tr}\Gamma \leq 0, \quad \Gamma \in \mathbb{P}\mathbb{P}\mathbb{T}_+^*. \quad (\text{A23})$$

From Lemma 1 together with condition (A23), we have

$$\Gamma = 0_{\mathbb{H}}, \quad (\text{A24})$$

which contradicts condition (A12). Thus, $\gamma > 0$. ■

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