Complementary relations of entanglement, coherence, steering, and Bell nonlocality inequality violation in three-qubit states

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(Received 24 October 2022; revised 28 January 2023; accepted 26 April 2023; published 9 May 2023)

We put forward complementary relations of entanglement, coherence, steering inequality violation, and Bell nonlocality for arbitrary three-qubit states. We show that two families of genuinely entangled three-qubit pure states with a single parameter exist, and they exhibit maximum coherence and steering inequality violation for a fixed amount of negativity, respectively. It is found that the negativity is exactly equal to the geometric mean of bipartite concurrences for the three-qubit pure states, although the negativity is always less than or equal to the latter for three-qubit mixed states. Moreover, the complementary relation between negativity and first-order coherence for tripartite entanglement states is established. Furthermore, we investigate the close relation between the negativity and the maximum steering inequality violation. In addition, the complementary relation between negativity and the maximum Bell-inequality violation for arbitrary three-qubit states is obtained. The results provide reliable evidence of fundamental connections among entanglement, coherence, steering inequality violation, and Bell nonlocality.

DOI: 10.1103/PhysRevA.107.052403

I. INTRODUCTION

Entanglement is considered to be the most fundamental feature of quantum mechanics. It has many potential applications in quantum information processing, including canonical ones: quantum cryptography [1], quantum teleportation [2], and dense coding [3]. It occurs when the state of a composite system cannot be written as a product of states of each subsystem. The information-theoretic quantification of entanglement is connected with its usefulness in terms of quantum computing and communication. There are a number of measures for quantifying the bipartite or multipartite entanglement, such as the concurrence [4], entanglement of formation [5], negativity [6,7], the geometric mean of bipartite concurrences (GBC) [8,9], and so on. Although these measures are regarded differently, there exist many evidences to show that they are in fact strongly related [10–12], and even potentially equivalent [13]. For example, Wootters found a functional relation between the entanglement of formation and concurrence [4].

Coherence, directly related to interference phenomena, describes the coherent superposition of states of interaction fields [14,15]. It has been regarded as a useful physical resource possessing different computable measures, such as l_1 norm [16], relative entropy [16,17], and skew information [18]. It plays an important role in quantum thermodynamics [19] and witnessing quantum correlations [20]. In addition, quantum steering can be captured as another effective quantum resource with local operations assisted by one-way classical communication as the free operations [21]. It shows a special phenomenon of quantum information that the correlation of a two-particle state allows one to steer the other party into an eigenstate of position or momentum by choosing the measurement. There are many criteria for the verification of steering violation, such as the linear steering criterion [22,23], the geometric Bell-like inequalities for steering [24], the steering criteria from entropic uncertainty relations [25–28], and so on. Quantum steering can be exploited to realize some quantum tasks for which the classical approach does not work, e.g., quantum information processing [29,30], quantum key distribution [31–33], and subchannel discrimination [34,35].

In particular, a successful and secure quantum information task requires knowing how quantum resources are shared and transformed over many sites. The question naturally arises of how to enhance one resource by modifying the other, and how much these resources can be converted in practical quantum tasks. Recently, the distribution and transformation of different quantum resources have stimulated a number of studies [36–47]. For example, Svozilík*et al.* found the conservations between first-order coherence and quantum correlations, including Bell nonlocality and the degree of entanglement, in the two-qubit state case [36]. In addition, Kalaga *et al.* investigated the complementary relations among entanglement, coherence, and steering parameter for bipartite subsystems of three-qubit states [48,49]. The complementary relations among different quantum resources enable one to estimate the degree of one quantum resource for a given degree of another resource, e.g., estimating entanglement from Bell nonlocality or vice versa [50]. Such complementary relations can lead to somehow counterintuitive but sound conclusions that mixed states can be relatively more entangled [51] or even more nonclassical [52]. However, it is worth noting that most of the related studies are related to the bipartite or three-qubit pure states. Moreover, many of the measures of entanglement chosen for these studies are difficult to

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analytically calculate for multipartite mixed states. This raises a significant issue: whether one can find the explicit relations among coherence, entanglement, and steering violation for the multipartite mixed states. In fact, the investigations of the intrinsic relations among various quantum resources in multipartite quantum systems are especially important for manipulating information transfer and flow in the context of quantum resource theories.

In this paper, we show that complementary relations among different measures of quantum resources exist, including negativity, GBC, first-order coherence, quantum steering, and Bell nonlocality, for arbitrary three-qubit states. The negativity is chosen as the measure of entanglement, because it can be calculated analytically in the multipartite mixed-state scenario. First of all, we find that the negativity is exactly the same as the GBC for the three-qubit pure states, although the negativity is always less than or equal to the GBC for the mixed states. In addition, the complementary relation between negativity and first-order coherence for tripartite entanglement states is established. For the three-qubit pure states, we obtain that a single-parameter family of state $|\psi\rangle_\alpha$ takes the maximum first-order coherence, although another family of state $|\psi\rangle_m$ takes the minimum first-order coherence for a given negativity. Note that state $|\psi\rangle_\alpha$ still takes the maximum first-order coherence for the mixed states. It is shown that the higher the rank of the density matrix of the state, the closer it is to the origin. Moreover, we study the complementary relation between the negativity and the maximum steering inequality violation. Interestingly, state $|\psi\rangle_m$ takes the maximum steering inequality violation for a given negativity. Finally, we study the complementary relation between negativity and the maximum Bell-inequality violation in three-qubit quantum systems. These relations quantify the intrinsic correlation among these quantum resources and show how they can be converted from one another.

This paper is organized as follows: In Sec. II, we briefly review some measures of coherence, entanglement, steering inequality violation, and Bell nonlocality. In Sec. III, we present the close relation between negativity and GBC. In Sec. IV, we give the complementary relation between the negativity and first-order coherence. The complementary relation between negativity and the maximum steering inequality violation is studied in Sec. V. The complementary relation between negativity and the maximum Bell-inequality violation is investigated in Sec. VI. The conclusion is provided in Sec. VII.

II. PRELIMINARIES

Here, we give a brief overview of entanglement, coherence, steering inequality violation, and Bell nonlocality to be used in the paper. The measure of entanglement is quantified by the negativity and GBC. The measure of coherence is given by the first-order coherence. We use the three-setting linear steering inequality and the Clauser-Horne-Shimony-Holt Bell's-like (Bell-CHSH) inequality as the measures of steering inequality and Bell nonlocality, respectively.

A. Negativity

The negativity can be calculated in the same way for pure and mixed states in arbitrary dimensions. In particular, the tripartite negativity is useful for distillability to a Greenberger-Horne-Zeilinger (GHZ) state in quantum computation [53]. For an arbitrary three-qubit state ρ, it is defined as [7]

$$\mathcal{N}_{ABC}(\rho) = (\mathcal{N}_{A|BC}\mathcal{N}_{B|AC}\mathcal{N}_{C|AB})^{\frac{1}{3}}, \tag{1}$$

where the bipartite negativity is given by [6]

$$\mathcal{N}_{I|JK} = -2\sum_i N_i(\rho^{T_I}), \tag{2}$$

with $\mathrm{I, J, K} \in \{A, B, C\}$, $\mathrm{I} \neq \mathrm{J} \neq \mathrm{K}$, and $N_i(\rho^{T_I})$ being the negative eigenvalues of the partial transpose ρ^{T_I} of the total state ρ with respect to the subsystem I, defined as $\langle h_I, j_{JK}|\rho^{T_I}|k_I, l_{JK}\rangle = \langle k_I, j_{JK}|\rho|h_I, l_{JK}\rangle$.

By the Schmidt decomposition theorem, for any bipartite pure state $|\phi\rangle$ in $d\otimes d'(d\leqslant d')$ quantum system, $\mathcal{H}_A\otimes\mathcal{H}_B$, an alternative form of negativity is written as [54]

$$\mathcal{N}(|\phi\rangle) = \frac{2}{d-1}\sum_{i<j}\sqrt{\lambda_i\lambda_j}, \tag{3}$$

where $\sqrt{\lambda_i}$ and $\sqrt{\lambda_j}$ are the Schmidt coefficients, with λ_i, λ_j being the eigenvalues of the reduced density matrix ρ_A. For example, if we take $d = 2$, then we have

$$\mathcal{N}(|\phi\rangle) = 2\sqrt{\lambda_1\lambda_2} = 2\sqrt{\det\rho_A}. \tag{4}$$

Therefore, the negativity of a three-qubit pure state $|\psi\rangle$ can be rewritten as

$$\mathcal{N}(|\psi\rangle) = \left(\prod_i 2\sqrt{\det\rho_i}\right)^{1/3} = 2\left(\prod_i \det\rho_i\right)^{1/6}, \tag{5}$$

where $i \in \{A, B, C\}$.

B. GBC

The GBC is introduced as a genuine multipartite entanglement measure [8], which should satisfy two conditions: it must be zero for all biseparable states and positive for any nonbiseparable state. The GBC relies on the concept of regularized bipartite concurrence [9]. The concurrence of a pure bipartite normalized state is given by

$$\mathcal{C}_{AB}(|\psi\rangle) = \sqrt{\frac{d_{\min}}{d_{\min}-1}\left[1-\mathrm{Tr}\left(\rho_A^2\right)\right]}, \tag{6}$$

where $d_{\min}$ denotes the dimension of the smaller subsystem. For an arbitrary n-partite pure state $|\Psi\rangle$, the GBC is defined as [8]

$$\mathcal{G}(|\Psi\rangle) = \sqrt[c(\alpha)]{\mathcal{P}(|\Psi\rangle)}, \tag{7}$$

where $\alpha = \{\alpha_i\}$ is the set of all possible bipartitions $\{A_{\alpha_i}|B_{\alpha_i}\}$ of the n parties, $c(\alpha)$ is the cardinality of α

$$c(\alpha) = \begin{cases} \sum_{m=1}^{(n-1)/2}\binom{n}{m}, & \text{if } n \text{ is odd} \\ \sum_{m=1}^{(n-2)/2}\binom{n}{m} + \frac{1}{2}\binom{n/2}{n}, & \text{if } n \text{ is even,} \end{cases} \tag{8}$$

and $\mathcal{P}(|\Psi\rangle)$ is the product of all bipartite concurrences

$$\mathcal{P}(|\Psi\rangle) = \prod_{\alpha_i \in \alpha} \mathcal{C}_{A_{\alpha_i} B_{\alpha_i}}(|\Psi\rangle). \tag{9}$$

Moreover, the GBC is generalized to mixed states ρ via the convex roof construction

$$\mathcal{G}(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \mathcal{G}(|\psi_i\rangle), \tag{10}$$

where the infimum is over all feasible decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

C. First-order coherence

The first-order coherence is extensively used in the optical systems for the measure of coherence. Its quantification is independent of the selection of the reference basis. For an arbitrary three-qubit state ρ_{ABC}, the first-order coherence of its subsystems A, B, and C are defined, in terms of the purity [55], as

$$\mathcal{D}(\rho_i) = \sqrt{2\,\mathrm{Tr}\left(\rho_i^2\right) - 1}, \tag{11}$$

where $i \in \{A, B, C\}$. When all subsystems can be considered independently, the first-order coherence of state ρ_{ABC} is given by [36]

$$\mathcal{D}(\rho_{ABC}) = \sqrt{\frac{\mathcal{D}(\rho_A)^2 + \mathcal{D}(\rho_B)^2 + \mathcal{D}(\rho_C)^2}{3}}, \tag{12}$$

where $0 \leqslant \mathcal{D}(\rho_{ABC}) \leqslant 1$.

D. The three-setting linear-steering inequality violation

Quantum steering is considered as a subset of entanglement and a superset of Bell nonlocality [56]. From the local hidden states model, some steering inequalities are derived to indicate steering phenomenon by the violation of them. As an example, the linear-steering inequality is formulated by Cavalcanti *et al.* to verify whether a bipartite state is steerable when Alice and Bob are both allowed to operate n dichotomic measurements on their own subsystems [22]:

$$F_n(\rho_{AB}, \mu) = \frac{1}{\sqrt{n}} \left| \sum_{k=1}^{n} \langle A_k \otimes B_k \rangle \right| \leqslant 1, \tag{13}$$

where $A_k = \hat{a}_k \cdot \vec{\sigma}$ and $B_k = \hat{b}_k \cdot \vec{\sigma}$, with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices; $\hat{a}_k, \hat{b}_k \in \mathbb{R}^3$ are unit and orthonormal vectors; $\langle A_k \otimes B_k\rangle = \mathrm{Tr}(\rho_{AB}(A_k \otimes B_k))$; and $\mu = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n\}$ is the set of measurement directions.

In general, an arbitrary two-qubit state can be denoted by the Hilbert-Schmidt representation

$$\rho_{AB} = \frac{1}{4}\left[I_2 \otimes I_2 + \vec{a} \cdot \vec{\sigma} \otimes I_2 + I_2 \otimes \vec{b} \cdot \vec{\sigma} + \sum_{i,j} t_{ij} \sigma_i \otimes \sigma_j \right], \tag{14}$$

where $\vec{a}$ and $\vec{b}$ are the local bloch vectors, and $T_{AB} = [t_{ij}]$ is the correlation matrix. The components t_{ij} are given by $t_{ij} = \mathrm{Tr}[\rho_{AB}(\sigma_i \otimes \sigma_j)]$. For the three measurement settings corresponding to $n = 3$ of Eq. (13), state ρ_{AB} is F_3 steerable if [23,46]

$$\mathcal{S}_{AB} = \mathrm{Tr}\left(T_{AB}^T T_{AB}\right) - 1 > 0, \tag{15}$$

where the superscript T represents the transpose of the correlation matrix T_{AB}. It can be shown that this steering inequality is a two-way steering criterion due to its invariance under qubit permutations. Among the three bipartite reduced states of a three-qubit state ρ_{ABC}, the maximum steering inequality violation is given by [46]

$$\mathcal{S}(\rho_{ABC}) = \max\{\mathcal{S}_{AB}, \mathcal{S}_{AC}, \mathcal{S}_{BC}\}. \tag{16}$$

E. Bell-inequality violation

In 1995, Horodecki *et al.* presented the necessary and sufficient condition for violating the Bell-CHSH inequality [57]. For an arbitrary two-qubit state ρ_{AB}, the maximum Bell-CHSH value $\mathcal{B}'_{AB}$ is given by

$$\mathcal{B}'_{AB} = 2\sqrt{M_{AB}}, \tag{17}$$

where $M_{AB} = m_1 + m_2$, with m_1 and m_2 being the largest two eigenvalues of $T_{AB}^T T_{AB}$, in which T_{AB} is the correlation matrix. $M_{AB} > 1$ implies the violation of the Bell-CHSH inequality. In this case, the Bell-inequality violation (i.e., the Bell-CHSH inequality violation) $\mathcal{B}_{AB}$ is defined as [58]

$$\mathcal{B}_{AB} = \max\{0, M_{AB} - 1\}. \tag{18}$$

Among the three pairwise reduced states of a three-qubit state ρ_{ABC}, the maximum Bell-inequality violation is obtained as [59]

$$\mathcal{B}(\rho_{ABC}) = \max\{\mathcal{B}_{AB}, \mathcal{B}_{BC}, \mathcal{B}_{AC}\}, \tag{19}$$

where only one of $\mathcal{B}_{AB}$, $\mathcal{B}_{AC}$, and $\mathcal{B}_{BC}$ is nonzero [60].

F. Two useful boundary states

In order to express the complementary relations of the above quantum resources for the arbitrary three-qubit states in a more explicit manner, here we introduce two boundary states with a single parameter. The first one is the generalized GHZ state, which can exhibit maximum first-order coherence value for a fixed amount of negativity,

$$|\psi\rangle_\alpha = \cos\alpha|i, j, k\rangle + \sin\alpha|\bar{i}, \bar{j}, \bar{k}\rangle, \tag{20}$$

where $i, j, k \in \{0, 1\}$ and the overbar means taking the opposite value. In the following, we take states with $i = j = k = 0$ as an example in the calculation:

$$|\psi\rangle_\alpha = \cos\alpha|000\rangle + \sin\alpha|111\rangle. \tag{21}$$

The second boundary state is a single-parameter family of three-qubit pure state

$$|\psi\rangle_m = \frac{|000\rangle + m(|010\rangle + |101\rangle) + |111\rangle}{\sqrt{2 + 2m^2}}, \tag{22}$$

where $m \in [0, 1]$. Note that the state is a GHZ-class state when $m \in [0, 1)$, and it is a W-class state when $m = 1$.

III. NEGATIVITY VERSUS THE GBC

Theorem 1. For the three-qubit pure states, the negativity is exactly equivalent to the GBC. However, for a three-qubit mixed state ρ, the negativity is always less than or equal to the GBC,

$$\mathcal{N}(\rho) \leqslant \mathcal{G}(\rho). \quad (23)$$

Proof. For a three-qubit pure state $|\psi\rangle$, the GBC is given by

$$\mathcal{G}(|\psi\rangle) = \left\{\prod_i \sqrt{2[1 - \mathrm{Tr}\,(\rho_i^2)]}\right\}^{1/3}, \quad (24)$$

where $i \in \{A, B, C\}$. Due to the trace condition of the reduced density matrices, $\lambda_1 + \lambda_2 = 1$, we can obtain that

$$\begin{aligned}\sqrt{2[1 - \mathrm{Tr}\,(\rho_A^2)]} &= \sqrt{2[1 - (\lambda_1^2 + \lambda_2^2)]}\\ &= 2\sqrt{\lambda_1\lambda_2} = 2\sqrt{\det \rho_A}.\end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned}\sqrt{2[1 - \mathrm{Tr}\,(\rho_B^2)]} &= 2\sqrt{\det \rho_B}\\ \sqrt{2[1 - \mathrm{Tr}\,(\rho_C^2)]} &= 2\sqrt{\det \rho_C}.\end{aligned} \quad (26)$$

Substituting Eqs. (25) and (26) into Eq. (5), we have

$$\mathcal{N}(|\psi\rangle) = \mathcal{G}(|\psi\rangle). \quad (27)$$

For a three-qubit mixed state ρ, since the bipartite negativity is a convex function [6], the tripartite negativity, as the geometric mean of three bipartite negativities, is also a convex function [61]. In addition, the GBC is defined as the minimum decomposition $\sum_j p_j|\psi_j\rangle\langle\psi_j|$ over all feasible decompositions. Thus, we obtain the following relation:

$$\mathcal{N}(\rho) \leqslant \sum_j p_j \mathcal{N}(\psi_j) = \sum_j p_j \mathcal{G}(\psi_j) = \mathcal{G}(\rho). \quad (28)$$

■

IV. NEGATIVITY VERSUS FIRST-ORDER COHERENCE

Theorem 2. If a three-qubit pure state $|\psi\rangle$ has the same value of negativity with states $|\psi\rangle_\alpha$ and $|\psi\rangle_m$, the first-order coherence of these three states satisfies the ordering $\mathcal{D}(|\psi\rangle_m) \leqslant \mathcal{D}(|\psi\rangle) \leqslant \mathcal{D}(|\psi\rangle_\alpha)$. The complementary relation of negativity and first-order coherence is expressed as

$$\begin{aligned}&\mathcal{N}(|\psi\rangle)^2 + \mathcal{D}(|\psi\rangle)^2 \leqslant 1,\\ &\mathcal{N}(|\psi\rangle)^6 + 3\mathcal{D}(|\psi\rangle)^2 \geqslant 1.\end{aligned} \quad (29)$$

Note that the first inequality is still valid for the three-qubit mixed states.

Proof. We will prove this theorem for the pure states first, and then extend it to the case of the mixed states.

Combining Eqs. (11) and (12), we have

$$\mathcal{D}(\rho)^2 = \frac{2}{3}\sum_i \mathrm{Tr}\,(\rho_i^2) - 1, \quad (30)$$

where $i \in \{A, B, C\}$. Then we can construct an equation

$$\sum_i [1 - \mathrm{Tr}\,(\rho_i^2)] + \sum_i \mathrm{Tr}\,(\rho_i^2) = 3, \quad (31)$$

and using the arithmetic-geometric mean value inequality, we have

$$\frac{1}{3}\sum_i 2[1 - \mathrm{Tr}\,(\rho_i^2)] \geqslant \left\{\prod_i 2[1 - \mathrm{Tr}\,(\rho_i^2)]\right\}^{1/3}. \quad (32)$$

By adding a term $2\sum_i \mathrm{Tr}(\rho_i^2)/3 - 1$ to both sides of the above equation, we have

$$\left\{\prod_i \sqrt{2[1 - \mathrm{Tr}\,(\rho_i^2)]}\right\}^{2/3} + \frac{2}{3}\sum_i \mathrm{Tr}\,(\rho_i^2) - 1 \leqslant 1. \quad (33)$$

For a three-qubit pure state $|\psi\rangle$, substituting Eqs. (24) and (30) into Eq. (33), and replacing $\mathcal{G}(|\psi\rangle)$ with $\mathcal{N}(|\psi\rangle)$, we get

$$\mathcal{N}(|\psi\rangle)^2 + \mathcal{D}(|\psi\rangle)^2 \leqslant 1. \quad (34)$$

To verify the second inequality in Eq. (29), we can construct a function $H(u, v, w)$ as

$$H(u, v, w) = 2^4 uvw - u - v - w + \tfrac{1}{2}, \quad (35)$$

where $u, v, w \in [0, \frac{1}{4}]$. It can be found that

$$\begin{aligned}\frac{\partial H}{\partial u} &= 2^4 vw - 1 \leqslant 0\\ \frac{\partial H}{\partial v} &= 2^4 uw - 1 \leqslant 0\\ \frac{\partial H}{\partial w} &= 2^4 uv - 1 \leqslant 0.\end{aligned} \quad (36)$$

Then the minimum of the function H can be calculated as

$$H\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right) = 0. \quad (37)$$

On the other hand, the trace conditions of the reduced density matrices ρ_A, ρ_B, and ρ_C are

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_3 + \lambda_4 = 1, \quad \lambda_5 + \lambda_6 = 1, \quad (38)$$

where λ_1 and λ_2, λ_3 and λ_4, and λ_5 and λ_6 are the eigenvalues of ρ_A, ρ_B, and ρ_C, respectively. We can see that

$$0 \leqslant \lambda_1\lambda_2 = \lambda_1(1 - \lambda_1) \leqslant \tfrac{1}{4}. \quad (39)$$

Similarly, we have

$$0 \leqslant \lambda_3\lambda_4 \leqslant \tfrac{1}{4}, \quad 0 \leqslant \lambda_5\lambda_6 \leqslant \tfrac{1}{4}, \quad (40)$$

Let $u = \lambda_1\lambda_2$, $v = \lambda_3\lambda_4$, and $w = \lambda_5\lambda_6$, and substituting them into Eq. (35), we obtain

$$2^4 \prod_{\mu=1}^{6} \lambda_\mu - \lambda_1\lambda_2 - \lambda_3\lambda_4 - \lambda_5\lambda_6 + \frac{1}{2} \geqslant 0. \quad (41)$$

Then, by using the trace conditions of the reduced density matrices, we get

$$2^6 \prod_{\mu=1}^{6} \lambda_\mu + 2\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2\right) - 3 \geqslant 1. \quad (42)$$

Finally, we can find that

$$\left[2\left(\prod_i \det \rho_i\right)^{1/6}\right]^6 + 3\left[\frac{2}{3}\sum_i \mathrm{Tr}\,(\rho_i^2) - 1\right] \geqslant 1. \quad (43)$$

Substituting Eqs. (5) and (30) into Eq. (43), we can obtain that

$$\mathcal{N}(|\psi\rangle)^6 + 3\mathcal{D}(|\psi\rangle)^2 \geqslant 1. \quad (44)$$

On the other hand, the negativity and first-order coherence of the boundary states $|\psi\rangle_\alpha$ and $|\psi\rangle_m$, from Eqs. (5) and (12), are given by

$$\mathcal{N}(|\psi\rangle_\alpha) = |\sin 2\alpha|, \quad (45)$$

$$\mathcal{D}(|\psi\rangle_\alpha) = |\cos 2\alpha|, \quad (46)$$

$$\mathcal{N}(|\psi\rangle_m) = \left(\frac{1-m^2}{1+m^2}\right)^{1/3}, \quad (47)$$

$$\mathcal{D}(|\psi\rangle_m) = \frac{2m}{\sqrt{3}(1+m^2)}. \quad (48)$$

We can find that

$$\mathcal{N}(|\psi\rangle_\alpha)^2 + \mathcal{D}(|\psi\rangle_\alpha)^2 = 1, \quad (49)$$

$$\mathcal{N}(|\psi\rangle_m)^6 + 3\mathcal{D}(|\psi\rangle_m)^2 = 1, \quad (50)$$

which imply that states $|\psi\rangle_\alpha$ and $|\psi\rangle_m$ are the upper and lower boundary states, respectively.

It can be found that the first-order coherence of subsystems of a three-qubit mixed state ρ_{ABC} are convex functions. The first-order coherence of state ρ_{ABC}, as the vector composition of $\mathcal{D}(\rho_i)$ and $h(x_1, x_2, x_3) = [(x_1^2 + x_2^2 + x_3^2)/3]^{1/2}$, is also a convex function [62]. On the other hand, Eq. (34) can be rewritten as

$$\mathcal{D}(|\psi\rangle) \leqslant \sqrt{1 - \mathcal{N}(|\psi\rangle)^2}. \quad (51)$$

Let $U[\mathcal{N}(|\psi\rangle)] = \sqrt{1 - \mathcal{N}(|\psi\rangle)^2}$, then we can see that the function U is concave function in regard to $\mathcal{N}(|\psi\rangle)$. Using the convexity of negativity, we have

$$\begin{aligned}\mathcal{D}(\rho) &\leqslant \sum_i p_i \mathcal{D}(\psi_i) \leqslant \sum_i p_i \sqrt{1 - \mathcal{N}(\psi_i)^2} \\ &\leqslant \sqrt{1 - \left[\sum_i p_i \mathcal{N}(\psi_i)\right]^2} \\ &\leqslant \sqrt{1 - \mathcal{N}(\rho)^2}, \end{aligned} \quad (52)$$

i.e.,

$$\mathcal{N}(\rho)^2 + \mathcal{D}(\rho)^2 \leqslant 1. \quad (53)$$

■

In Fig. 1, we plot how the square of first-order coherence changes with respect to the square of negativity for 10^5 Haar randomly generated three-qubit pure states [63]. The magenta squares donating state $|\psi\rangle_\alpha$ are located at the upper boundary, which satisfies the relation between negativity and first-order coherence in Eq. (49). The blue circles at the lower boundary show that the two quantum resources of state $|\psi\rangle_m$ fulfill the relation in Eq. (50). The 10^5 Haar randomly generated three-qubit pure states are included in the range constrained by states $|\psi\rangle_\alpha$ and $|\psi\rangle_m$, meaning that their negativity and first-order coherence obey the inequalities in Eq. (29). Moreover, we find that the first-order coherence increases (decreases) with the decrease (increase) of the negativity, showing a complementary relation.

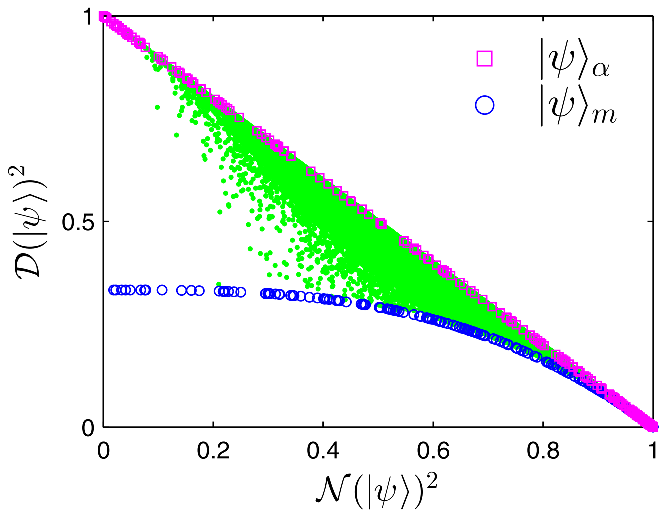

FIG. 1. Complementary relation between the negativity $\mathcal{N}(|\psi\rangle)$ and the first-order coherence $\mathcal{D}(|\psi\rangle)$ for 10^5 Haar randomly generated three-qubit pure states. The magenta squares are located at the upper boundary with state $|\psi\rangle_\alpha$, and state $|\psi\rangle_m$ represented by blue circles lies at the lower boundary. Both axes are dimensionless.

Figure 2 plots the relation between negativity and first-order coherence for 10^5 Haar randomly generated three-qubit mixed states. The magenta squares are still located at the upper boundary with the state $|\psi\rangle_\alpha$. The 10^5 Haar randomly generated three-qubit mixed states are under the boundary line, which means that their negativity and first-order co-

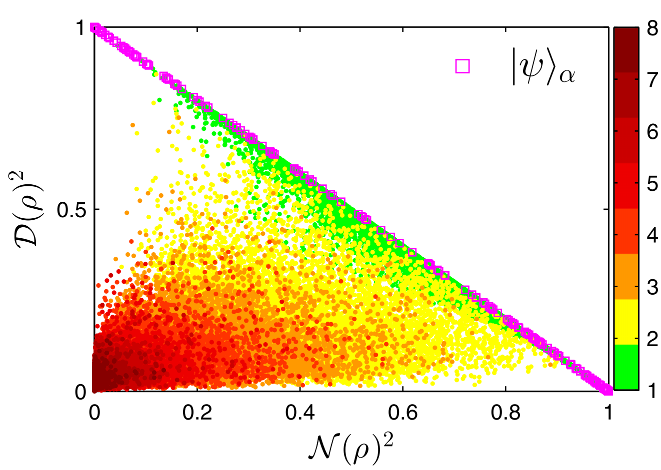

FIG. 2. Complementary relation between the negativity $\mathcal{N}(\rho)$ and the first-order coherence $\mathcal{D}(\rho)$ for 10^5 Haar randomly generated three-qubit mixed states. The magenta squares are located at the upper boundary with state $|\psi\rangle_\alpha$. The numbers on the color bar represent different ranks of the density matrices of the random states. Both axes are dimensionless.

herence satisfy the inequality in Eq. (53). In addition, we can see that the higher the rank of the density matrix of the random state, the closer it is to the origin. Also, it shows that a complementary relation between the negativity and first-order coherence for arbitrary three-qubit states exists.

V. NEGATIVITY VERSUS MAXIMUM STEERING INEQUALITY VIOLATION

Theorem 3. If an arbitrary three-qubit state ρ has the same value of negativity as state $|\psi\rangle_m$, the maximum steering inequality violations of these two states satisfy the ordering $\mathcal{S}(\rho) \leqslant \mathcal{S}(|\psi\rangle_m)$. The complementary relation of the negativity and the maximum steering inequality violation is given by

$$2\mathcal{N}(\rho)^6 + \mathcal{S}(\rho) \leqslant 2. \tag{54}$$

Proof. An arbitrary three-qubit state ρ_{ABC} can be written as

$$\rho_{ABC} = \frac{1}{8}\Bigg[\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \vec{A}\cdot\vec{\sigma}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes\vec{B}\cdot\vec{\sigma}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{I}\otimes\vec{C}\cdot\vec{\sigma} + \sum_{ij} t_{ij}^{AB}\sigma_i\otimes\sigma_j\otimes\mathbb{I} + \sum_{ik} t_{ik}^{AC}\sigma_i\otimes\mathbb{I}\otimes\sigma_k + \sum_{jk} t_{jk}^{BC}\mathbb{I}\otimes\sigma_j\otimes\sigma_k + \sum_{ijk} t_{ijk}^{ABC}\sigma_i\otimes\sigma_j\otimes\sigma_k\Bigg]. \tag{55}$$

The purities of the the reduced density matrices ρ_A and ρ_{BC} are

$$\mathrm{Tr}\left(\rho_A^2\right) = \frac{1+\vec{A}^2}{2}, \quad \mathrm{Tr}\left(\rho_{BC}^2\right) = \frac{1}{4}(2+\vec{B}^2+\vec{C}^2+\mathcal{S}_{BC}). \tag{56}$$

Similarly, we obtain

$$\mathrm{Tr}\left(\rho_B^2\right) = \frac{1+\vec{B}^2}{2}, \quad \mathrm{Tr}\left(\rho_{AC}^2\right) = \frac{1}{4}(2+\vec{A}^2+\vec{C}^2+\mathcal{S}_{AC}),$$
$$\mathrm{Tr}\left(\rho_C^2\right) = \frac{1+\vec{C}^2}{2}, \quad \mathrm{Tr}\left(\rho_{AB}^2\right) = \frac{1}{4}(2+\vec{A}^2+\vec{B}^2+\mathcal{S}_{AB}). \tag{57}$$

In the following, we will give the proof for pure states first, and then extend the theorem to mixed states. If ρ_{ABC} is a pure state with $\rho_{ABC} = |\psi\rangle\langle\psi|$, based on the Schmidt decomposition, we have $\mathrm{Tr}(\rho_i^2) = \mathrm{Tr}(\rho_{jk}^2)$ for $i \neq j \neq k,\ i, j, k \in \{A, B, C\}$. By Eqs. (56) and (57), the linear steering inequality violation of the bipartite reduced states of ρ_{AB}, $\mathcal{S}_{AB}$, can be written as a function of purities of subsystems of state $|\psi\rangle$

$$\mathcal{S}_{AB} = 2\big[2\,\mathrm{Tr}\left(\rho_C^2\right) - \mathrm{Tr}\left(\rho_A^2\right) - \mathrm{Tr}\left(\rho_B^2\right)\big]. \tag{58}$$

Assuming that $\mathcal{S}(|\psi\rangle) = \mathcal{S}_{AB}$, then let us construct a function with the form

$$R(u, v, w) = 2^6uvw + 2u + 2v - 4w, \tag{59}$$

where $u, v, w \in [0, \frac{1}{4}]$. We can show that

$$\frac{\partial R}{\partial u} = 2^6vw + 2 \geqslant 0$$
$$\frac{\partial R}{\partial v} = 2^6uw + 2 \geqslant 0$$
$$\frac{\partial R}{\partial w} = 2^6uv - 4 \leqslant 0. \tag{60}$$

Thus, the maximum of the function R is given by

$$R\left(\tfrac{1}{4}, \tfrac{1}{4}, 0\right) = 1. \tag{61}$$

Substituting relations $u = \lambda_1\lambda_2$, $v = \lambda_3\lambda_4$, and $w = \lambda_5\lambda_6$ into $R(u, v, w)$ in Eq. (59), we can obtain an inequality with respect to the eigenvalues of the reduced density matrices as

$$2^6\prod_{\mu=1}^{6}\lambda_\mu + 2\lambda_1\lambda_2 + 2\lambda_3\lambda_4 - 4\lambda_5\lambda_6 \leqslant 1. \tag{62}$$

The above inequality can be rewritten as

$$2^6\prod_{\mu=1}^{6}\lambda_\mu + 2(1-2\lambda_5\lambda_6) - (1-2\lambda_1\lambda_2) - (1-2\lambda_3\lambda_4) \leqslant 1. \tag{63}$$

By using the trace conditions of the reduced density matrices, we have

$$2^6\prod_{\mu=1}^{6}\lambda_\mu + 2\left(\lambda_5^2+\lambda_6^2\right) - \left(\lambda_1^2+\lambda_2^2\right) - \left(\lambda_3^2+\lambda_4^2\right) \leqslant 1. \tag{64}$$

Finally, we can get

$$2\left[2\left(\prod_i \det\rho_i\right)^{1/6}\right]^6 + 2\big[2\,\mathrm{Tr}\left(\rho_C^2\right) - \mathrm{Tr}(\rho_A^2) - \mathrm{Tr}\left(\rho_B^2\right)\big] \leqslant 2. \tag{65}$$

Substituting Eqs. (5) and (58) into Eq. (65), we have

$$2\mathcal{N}(|\psi\rangle)^6 + \mathcal{S}(|\psi\rangle) \leqslant 2. \tag{66}$$

The complementary relation also holds if $\mathcal{S}(|\psi\rangle) = \mathcal{S}_{AC}$ or $\mathcal{S}(|\psi\rangle) = \mathcal{S}_{BC}$.

Moreover, the maximum steering inequality violation of the boundary state $|\psi\rangle_m$, from Eq. (16), can be calculated as

$$\mathcal{S}(|\psi\rangle_m) = \frac{8m^2}{(1+m^2)^2}. \tag{67}$$

Together with Eq. (47), we can obtain

$$2\mathcal{N}(|\psi\rangle_m)^6 + \mathcal{S}(|\psi\rangle_m) = 2, \tag{68}$$

which imply that state $|\psi\rangle_m$ is the upper boundary states.

On the other hand, Eq. (66) can be rewritten as

$$\mathcal{S}(|\psi\rangle) \leqslant 2[1 - \mathcal{N}(|\psi\rangle)^6]. \tag{69}$$

Let $L[\mathcal{N}(|\psi\rangle)] = 2[1 - \mathcal{N}(|\psi\rangle)^6]$, we can show that L is a concave function with respect to $\mathcal{N}(|\psi\rangle)$. If ρ_{ABC} is a mixed state, both its negativity and the maximum steering inequality violation are convex functions [46]. Similar to the derivation in Eq. (52), we can obtain the complementary relation

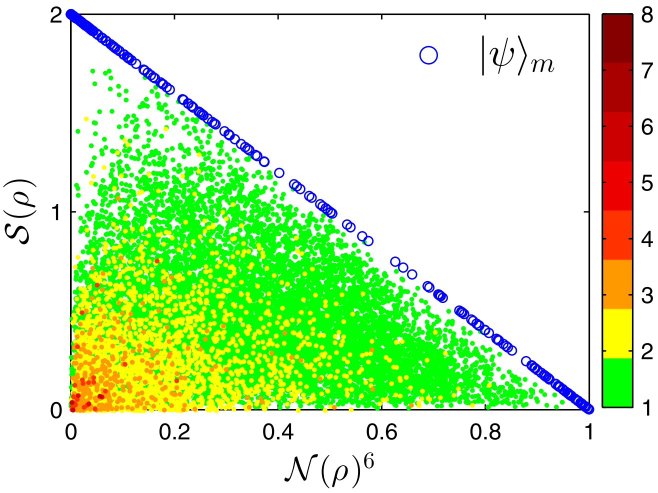

FIG. 3. Complementary relation between the negativity $\mathcal{N}(\rho)$ and the maximum steering inequality violation $\mathcal{S}(\rho)$ for 10^5 Haar randomly generated three-qubit mixed states of $\mathcal{S}(\rho) \geqslant 0$. The blue circles are located at the upper boundary with state $|\psi\rangle_m$. The numbers on the color bar represent different ranks of the density matrices of the random states. Both axes are dimensionless.

between the negativity and the maximum steering inequality violation for the three-qubit mixed states as

$$2\mathcal{N}(\rho)^6 + \mathcal{S}(\rho) \leqslant 2. \tag{70}$$

■

In Fig. 3, we plot how the maximum steering inequality violation changes in regard to negativity to the sixth power for 10^5 Haar randomly generated three-qubit mixed states when $\mathcal{S}(\rho) \geqslant 0$. We can see that state $|\psi\rangle_m$ is located at the upper boundary (blue circles), suggesting that its negativity and maximum steering inequality violation satisfy the relation in Eq. (68). The random states are under the boundary line, which means that their negativity and maximum steering inequality violation satisfy the inequality in Eq. (70). The results show that a complementary relation between negativity and the maximum steering inequality violation exists for arbitrary three-qubit states. Also, the higher the rank of the density matrix of the random state, the closer it is to the origin.

It is worth mentioning that this complementary relation is obtained under the conditions of tripartite entanglement and the maximum pairwise steering inequality violation. Alternatively, is there an exact relation between pairwise steering inequality violation and bipartite entanglement measure in arbitrary three-qubit states? In the following, we take $\mathcal{S}_{AC}$ as an example, and investigate its relations with the bipartite entanglement measures $\mathcal{N}_{A|BC}$, $\mathcal{N}_{C|AB}$, and $\mathcal{N}_{B|AC}$. In Fig. 4, we plot how $\mathcal{S}_{AC}$ changes with respect to $\mathcal{N}_{A|BC}$, $\mathcal{N}_{C|AB}$, and $\mathcal{N}_{B|AC}$, respectively. We find that the maximum of $\mathcal{S}_{AC}$ increases as $\mathcal{N}_{A|BC}$ ($\mathcal{N}_{C|AB}$) increases. However, there exists a complementary relation between $\mathcal{S}_{AC}$ and $\mathcal{N}_{B|AC}$.

Corollary 1. If an arbitrary three-qubit state ρ_{ABC} has the same value of bipartite negativity $\mathcal{N}_{I|JK}(\rho)$ (I, J, K $\in \{A, B, C\}$, I $\neq$ J $\neq$ K) with state $|\psi\rangle_m$ (may need qubit permutations), the pairwise steering inequality violations of these two states satisfy the ordering $\mathcal{S}_{JK}(\rho) \leqslant \mathcal{S}_{JK}(|\psi\rangle_m)$. The complementary relation of $\mathcal{N}_{I|JK}(\rho)$ and $\mathcal{S}_{JK}(\rho)$ is given by

$$2\mathcal{N}^2_{I|JK}(\rho) + \mathcal{S}_{JK}(\rho) \leqslant 2. \tag{71}$$

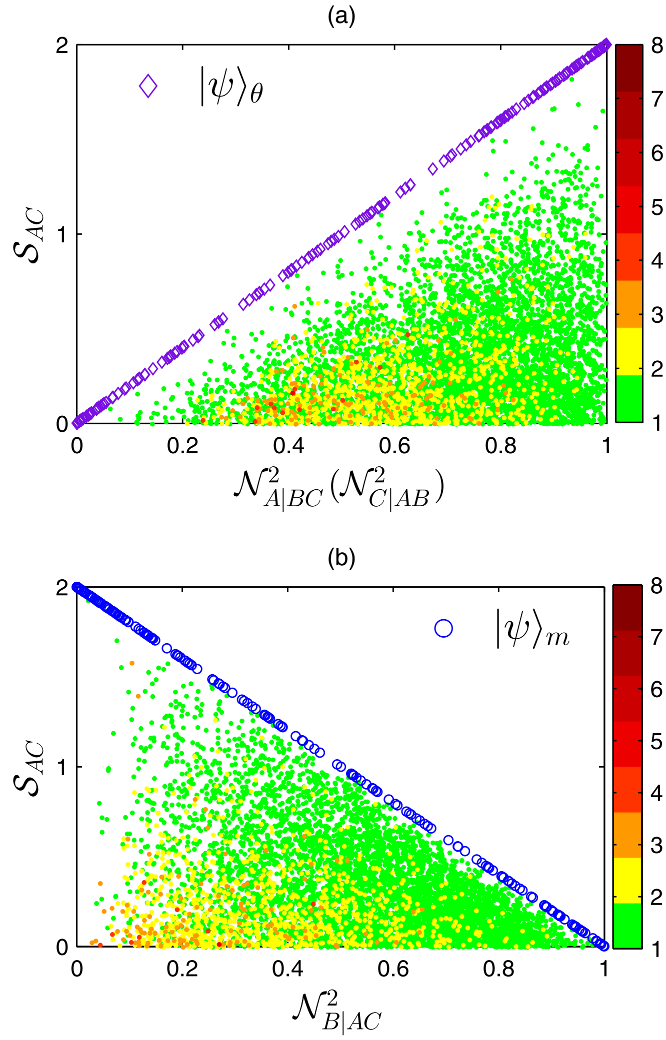

FIG. 4. . Relations between the pairwise steering inequality violation $\mathcal{S}_{AC}$ and three bipartite entanglement measures: (a) $\mathcal{N}_{A|BC}$ or $\mathcal{N}_{C|AB}$, (b) $\mathcal{N}_{B|AC}$. The numbers on the color bar represent different ranks of the random states. In (a), the boundary state is state $|\psi\rangle_\theta$ in Ref. [13]. Both axes are dimensionless.

The proof is similar to the proof of Theorem 3. The interpretation is that the increase of bipartite entanglement $\mathcal{N}_{I|JK}(\rho)$ decreases pairwise steering $\mathcal{S}_{JK}(\rho)$ by diminishing the entanglement of subsystem ρ_{JK}. Note that state $|\psi\rangle_m$ corresponds to maximum pairwise steering inequality violation $\mathcal{S}_{AC}$ for a fixed amount of bipartite negativity $\mathcal{N}_{B|AC}$. For the complementary relation between $\mathcal{S}_{AB}$ ($\mathcal{S}_{BC}$) and $\mathcal{N}_{C|AB}$ ($\mathcal{N}_{A|BC}$), the boundary state is the state after permutation of the latter (first) two qubits of state $|\psi\rangle_m$.

VI. NEGATIVITY VERSUS MAXIMUM BELL-INEQUALITY VIOLATION

Theorem 4. If an arbitrary three-qubit state ρ has the same value of negativity as state $|\psi\rangle_m$ (i, j, k), the maximum Bell-inequality violation of these two states satisfies the ordering

$\mathcal{B}(\rho) \leqslant \mathcal{B}(|\psi\rangle_m)$. The complementary relation of negativity and the maximum Bell-inequality violation is given by

$$\mathcal{N}(\rho)^6 + \mathcal{B}(\rho) \leqslant 1. \tag{72}$$

Proof. To begin with, we assume that $\mathcal{B}(|\psi\rangle) = \mathcal{B}_{AB}$, where $|\psi\rangle$ is a three-qubit pure state. For bipartite subsystem ρ_{AB} of $|\psi\rangle$, there is a complementary relation between first-order coherence and maximum Bell-CHSH value [36]

$$\frac{\mathcal{D}_{AB}^2}{2} + \left(\frac{\mathcal{B}'_{AB}}{2\sqrt{2}}\right)^2 \leqslant \mathrm{Tr}\left(\rho_{AB}^2\right) - 2(\varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3), \tag{73}$$

where

$$\mathcal{D}_{AB} = \sqrt{\frac{\left(\mathcal{D}_A^2 + \mathcal{D}_B^2\right)}{2}}. \tag{74}$$

$\mathcal{D}_{AB}$ is the bipartite first-order coherence, and $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \varepsilon_3 \geqslant \varepsilon_4$ are the eigenvalues of ρ_{AB}. Here $\varepsilon_3 = \varepsilon_4 = 0$ since ρ_{AB} has the same eigenvalues as ρ_C, another subsystem of $|\psi\rangle$. If $\mathcal{B}_{AB} = 0$, Eq. (72) obviously holds. If $\mathcal{B}_{AB} > 0$, from Eq. (18), we have

$$\mathcal{B}_{AB} = M_{AB} - 1. \tag{75}$$

Using Eqs. (17) and (75), Eq. (73) can be rewritten as

$$\mathcal{D}_{AB}^2 + \mathcal{B}_{AB} + 1 \leqslant 2\,\mathrm{Tr}\left(\rho_C^2\right). \tag{76}$$

Then, from Eqs. (11), (74), and (76), we get

$$\mathcal{B}_{AB} \leqslant 2\,\mathrm{Tr}\left(\rho_C^2\right) - \mathrm{Tr}\left(\rho_A^2\right) - \mathrm{Tr}\left(\rho_B^2\right). \tag{77}$$

On the other hand, from Ref. [64], we can know that for any pure three-qubit state, the triple (M_{AB}, M_{AC}, M_{BC}) has the same ordering as $(s_{\mathrm{iso}}^{AB}, s_{\mathrm{iso}}^{AC}, s_{\mathrm{iso}}^{BC})$ of pairwise isotropic strengths, which happen to be a third of the corresponding pairwise steering inequality violations. That means that the triple $(\mathcal{B}_{AB}, \mathcal{B}_{AC}, \mathcal{B}_{BC})$ has the same ordering as $(\mathcal{S}_{AB}, \mathcal{S}_{AC}, \mathcal{S}_{BC})$. From Eq. (64), we can obtain

$$2\,\mathrm{Tr}\left(\rho_C^2\right) - \mathrm{Tr}\left(\rho_A^2\right) - \mathrm{Tr}\left(\rho_B^2\right) \leqslant 1 - 2^6 \prod_{\mu=1}^{6} \lambda_\mu. \tag{78}$$

Therefore, it gives

$$\mathcal{B}_{AB} \leqslant 1 - \left[2\left(\prod_i \det \rho_i\right)^{1/6}\right]^6. \tag{79}$$

Substituting Eq. (5) into Eq. (79), we obtain

$$\mathcal{N}(|\psi\rangle)^6 + \mathcal{B}(|\psi\rangle) \leqslant 1. \tag{80}$$

Similarly, the above complementary relation also holds if $\mathcal{B}_{AC}$ or $\mathcal{B}_{BC}$ is the largest one among $\mathcal{B}_{AB}$, $\mathcal{B}_{AC}$, and $\mathcal{B}_{BC}$.

The maximum Bell-inequality violation of state $|\psi\rangle_m$, from Eq. (19), is given by

$$\mathcal{B}(|\psi\rangle_m) = \frac{4m^2}{(1+m^2)^2}. \tag{81}$$

Using Eqs. (47) and (81), we have

$$\mathcal{N}(|\psi\rangle_m)^6 + \mathcal{B}(|\psi\rangle_m) = 1, \tag{82}$$

which implies that state $|\psi\rangle_m$ is the upper boundary states.

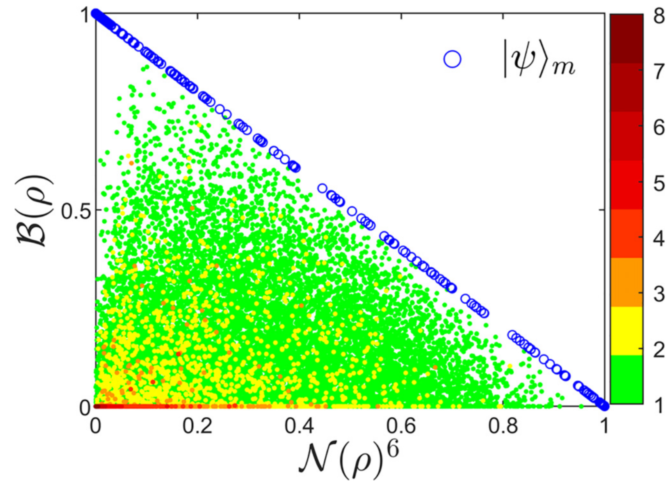

FIG. 5. Complementary relation between negativity $\mathcal{N}(\rho)$ and the maximum Bell-inequality violation $\mathcal{B}(\rho)$ for 10^5 Haar randomly generated three-qubit mixed states. The blue circles lie at the upper boundary with state $|\psi\rangle_m$. The numbers on the color bar represent different ranks of the random states. Both axes are dimensionless.

Furthermore, for an arbitrary three-qubit state ρ, the maximum Bell-inequality violation is also a convex function [59], so we can extend the complementary relation between negativity and the maximum Bell-inequality violation to mixed states by a similar derivation to Eq. (52). Thus, we have

$$\mathcal{N}(\rho)^6 + \mathcal{B}(\rho) \leqslant 1. \tag{83}$$

■

In Fig. 5, we plot the relation between negativity and the maximum Bell-inequality violation for 10^5 Haar randomly generated three-qubit mixed states. We can see that state $|\psi\rangle_m$ is located at the upper boundary (blue circles), suggesting that its negativity and maximum Bell-inequality violation satisfy the complementary relation in Eq. (83). In particular, we find that the three-qubit state is hard to violate the Bell inequality when its rank is greater than three.

Also, if considering three pairwise Bell-inequality violations separately, we can obtain the complementary relations between pairwise Bell-inequality violations and bipartite entanglement in the tripartite system.

Corollary 2. If an arbitrary three-qubit state ρ_{ABC} has the same value of bipartite negativity $\mathcal{N}_{I|JK}(\rho)$ (I, J, K $\in \{A, B, C\}$, I $\neq$ J $\neq$ K) as state $|\psi\rangle_m$ (may need qubit permutations), the pairwise Bell-inequality violations of these two states satisfy the ordering $\mathcal{B}_{JK}(\rho) \leqslant \mathcal{B}_{JK}(|\psi\rangle_m)$. The complementary relation of $\mathcal{N}_{I|JK}(\rho)$ and $\mathcal{B}_{JK}(\rho)$ is obtained as

$$\mathcal{N}_{I|JK}^2(\rho) + \mathcal{B}_{JK}(\rho) \leqslant 1. \tag{84}$$

VII. CONCLUSION

In this paper, we found that exact complementary relations among entanglement, coherence, steering inequality violation, and Bell nonlocality exist for arbitrary three-qubit states. First of all, it was shown that the negativity is exactly the same as the GBC for the three-qubit pure states, although the negativity was always less than or equal to the GBC for three-qubit mixed states. Then the complementary relation between negativity and first-order coherence was established. For the three-qubit pure states, the first-order coherence is constrained to a range formed by two inequalities for a fixed amount of negativity. The upper boundary state of the complementary relation is state $|\psi\rangle_\alpha$, and $|\psi\rangle_m$ is the lower boundary state. For the three-qubit mixed states, the upper boundary is still valid while the lower boundary is ineffective. We can obtain that the higher the rank of the density matrix of the random state, the closer it is to the origin. Moreover, we investigated the complementary relation between the negativity and the maximum steering inequality violation for the three-qubit states. Interestingly, the $|\psi\rangle_m$ state takes the maximum steering inequality violation for a given negativity. At last, we obtained that state $|\psi\rangle_m$ is also the upper boundary state of the complementary relation between negativity and the maximum Bell-inequality violation. These results show that these three quantum resources are closely related and can be transformed from one another. In particular, our boundaries are useful for quantifying the maximum value of one resource that can be converted from the other.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grants No. 12004006, No. 12075001, and No. 12175001), Anhui Provincial Key Research and Development Plan (Grant No. 2022b13020004), and Anhui Provincial Natural Science Foundation (Grant No. 2008085QA43).

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