RETRACTED: Non-Hermiticity of tunneling time in a spacetime-symmetric extension of nonrelativistic quantum mechanics

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Time continues to be an intriguing physical property in the modern era. On the one hand, we have the classical and relativistic notions of time, where space and time have the same hierarchy, which is essential in describing events in spacetime. On the other hand, in quantum mechanics time appears as a classical parameter, meaning that it does not have an uncertainty relation with its canonical conjugate. In this work, we use a recent spacetime-symmetric proposal [Phys. Rev. A **95**, 032133 (2017)] that tries to solve the unbalance in nonrelativistic quantum mechanics by extending the usual Hilbert space, having the time parameter t and the position operator $\hat{X}$ in one subspace and the position parameter x and time operator $\mathbb{T}$ in the other subspace. Time as an operator is better suitable for describing tunneling processes. We then solve the $1/2$-fractional integrodifferential equation for a particle subjected to strong and weak potential limits and obtain an analytical expression for the tunneling time through a rectangular barrier. Using a Gaussian energy distribution, we demonstrate that, for wavepackets well resolved in time, the expectation value of the operator $\mathbb{T}$ is the energy average of the classical time $T_{\text{class}} = \partial S/\partial E$, where S is the classical action, which can be real or imaginary. For wavepackets not well resolved in time, the contribution of T_{class} consistently vanishes, and solely properties of the energy distribution contribute to $\mathbb{T}$. We show that the time of travel for nontunneling particles is purely real. When tunneling is involved, complex arrival times emerge, becoming a signature of tunneling. Furthermore, we apply our results to a constant energy distribution, obtaining pure imaginary times for energies below the barrier while obtaining complex times for particles with a wavepacket spreading energies below and above the barrier, and show a comparison to previous works.

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I. INTRODUCTION

Time in quantum mechanics (QM) has always been a point of discussion [1–13], mostly in connection to the time of arrival in QM [13–27], including for the unification of QM and general relativity [28–37]. Contrary to what happens in relativity, where spacetime is a single entity [38], position and time are different kinds of numbers: while the position is a q-number, time is a c-number [39–41]. This hierarchy incompatibility between said quantities has led physicists to search for ways to include a time operator in QM. Though Pauli argued [42] that a bounded-from-below Hamiltonian was incompatible with a time operator canonically conjugated to it (both the Hamiltonian and the time operator must possess completely continuous spectra spanning the entire real line), there were ways to overcome it (see, for example, Refs. [1,16,20,21]). One of the most famous related works is the Page and Wootters (PaW) mechanism, together with its recent interpretations [43–46], in which the universe is in a stationary state, consistent with a Wheeler-DeWitt equation [19,47], and the apparent dynamical evolution that systems undergo is relative to the degrees of freedom of the rest of the universe that acts like a clock.

Not considering time as an operator, interpretations of the relation between time and energy were also made by Mandelstam and Tamm [48] and Margolus and Levitin [49]. Any Δt appearing in those works must be interpreted as a time interval, not as an operator uncertainty. In both cases, Δt is considered the smallest time interval for a system to evolve into an orthogonal state. However, in the first, the system has an energy *spread* ΔE, which bounds the interval by $\Delta t \Delta E \sim \hbar$. In contrast, in the second, the system has an *average* energy $\langle E\rangle$, bounding the interval from below by $\pi\hbar/2\langle E\rangle$.

Using the idea of quantum events [50,51], it is possible to give meaning to the usual time-energy "uncertainty" relations and relate the uncertainty of a quantum measurement of time to its energy uncertainty. By requiring consistency with the way that time enters the fundamental laws of physics, one can also draw a picture where it is shown that there is only one time: both classical and quantum times are manifestations of entanglement [52].

Höhn *et al.* showed that there is an equivalence between the relational quantum dynamics, (i) the relational observables in the clock-neutral picture of Dirac quantization, (ii) the PaW mechanism, and (iii) the relational Heisenberg picture obtained via symmetry reduction using quantum reduction maps [53,54].

The spacetime-symmetric proposal [55,56] that we present and use in this paper uses similar ideas to the PaW mechanism. The system has a Hilbert space with an *operator* time,

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implying an extended state for the system that depends on variables both in the usual position Hilbert space and variables in the additional temporal Hilbert space. One key difference to the PaW formalism is that this additional Hilbert space is as intrinsic to the system as the position Hilbert space, extending regular QM, and no auxiliary systems are required. This provides a clear interpretation of the time-energy uncertainty relation and different types of experiments, where predictions of the positions of particles, the time of arrival, or both, can be obtained. It is a natural subject then to examine the tunneling times in the spacetime-symmetric proposal. This proposal shows promising results compared to the Büttiker-Landauer and Phase-Time [13,57–59] approaches to time-of-arrival problems and is the main reason we use this formulation. Worth calling attention to is that $1/2$-fractional time derivatives and integrations appear in the equations to be solved.

Our goal in this paper is to study weak and strong potential barriers, providing connection formulas for the wave functions in the *extended Hilbert space* and examine results for the tunneling times by comparing them with tunneling times obtained using the usual QM formalism. The tunneling time through a barrier is an old problem which goes back to MacColl [60]. The exact definition, applicability, and measure of tunneling times change according to the circumstances and interest: dwell times, arrival times, asymptotic phase times, delay times, and jump times, among others. It is impossible to furnish a fair review of all these works here, but we refer the readers to Refs. [13,61,62]. Some particular cases will be mentioned later for comparison. It is worth noting that some works (e.g., Ref. [63]) argued that, when one considers a wavepacket that includes more than one momentum component, it is not possible to talk about tunneling time, except when considering square barriers, and that the concept of tunneling time comes from a classical interpretation of a quantum phenomenon. The spacetime-symmetric proposal does not have this distinction since it asks the question "what is the average time a particle takes to travel from one point to the other?" and this includes both tunneling and traveling above the barrier. Using a Gaussian distribution in energy, we demonstrate analytically that the time expectation (tunneling or not) of a wavepacket well localized in time equals the energy average of the classical time $T_{\text{class}} = \partial S/\partial E$, where S is the action, which can be real or imaginary. In the case of time-delocalized wavepackets, the main contribution to the time expectation comes from the energy distribution properties. Application is shown for the rectangular barrier with a constant energy distribution, which describes a wavepacket relatively well localized in time.

This paper is organized as follows. While in Sec. II we summarize the spacetime-symmetric proposal, in Sec. III we obtain and solve the approximated $1/2$-fractional equations for the weak and strong potential limits. In Sec. IV, we apply the results obtained in Sec. III to the tunneling time and show that, for a flat energy distribution with energies ranging from 0 to a maximum energy, it is purely imaginary for the largest energies below the barrier. The tunneling time has a fast decaying imaginary part for the largest energies above the barrier, a special case of one of our main results. We also obtain an average of classical times-of-flight plus a quantum correction up to first order. Final comments are presented in Sec. V.

II. SPACETIME-SYMMETRIC PROPOSAL

We begin our discussion by revisiting the proposal used in this paper. The spacetime-symmetric extension of QM proposed by Dias and Parisio (hereafter the STS proposal) [55,56] uses a similar idea to the PaW mechanism [43,44], in which the entire Hilbert space is divided into one subspace that refers to the *system of interest* and another one that refers to the *clock*. The main difference between the two is that the complete Hilbert space in the STS proposal

$$\mathcal{H} = \mathcal{H}_{\text{pos}} \otimes \mathcal{H}_{\text{time}} \tag{1}$$

(here $\mathcal{H}_{\text{pos}}$ is the usual Hilbert space of QM and $\mathcal{H}_{\text{time}}$ is a temporal *extension* of the regular theory) refers entirely to the system: $\mathcal{H}_{\text{time}}$ is as intrinsic to the system as $\mathcal{H}_{\text{pos}}$ in this approach.

In this additional space we define the time operator $\mathbb{T}$ with eigenkets $|t\rangle$ as

$$\mathbb{T}|t\rangle = t|t\rangle, \tag{2}$$

where t is the eigenvalue associated with $|t\rangle$. The set of eigenkets $\{|t\rangle\}$ resolve an identity $\mathbb{I} = \int_{-\infty}^{\infty} dt\, |t\rangle\langle t|$. We then define the energy operator $\mathbb{H}$ through the commutation relation [64]

$$[\mathbb{T}, \mathbb{H}] = -i\hbar, \tag{3}$$

which gives us naturally the time-energy uncertainty relation $\Delta\mathbb{T}\,\Delta\mathbb{H} \geqslant \hbar/2$. We want to emphasize that, since the STS proposal considers an extension of the Hilbert space of the system of interest, this uncertainty relation relates the energy *of the system* and a time operator that acts *on the system*. This does not happen when you consider, for example, the PaW mechanism, where an auxiliary system takes the role of a clock [43]. The price paid for this in the STS proposal is that we do not have the commutation relation $[x, \mathbb{P}] \propto i\hbar$, since in this Hilbert space x is a classical parameter.

The complete state of the system is given by

$$||\Psi\rangle = \iint dx\, dt\, \Psi(x\,\&\,t)|x\rangle \otimes |t\rangle. \tag{4}$$

The double ket notation indicates that this state belongs to both Hilbert spaces. The way we write the argument of $\Psi(x\,\&\,t) \equiv (\langle x| \otimes \langle t|)||\Psi\rangle$ is such as to remind us that, in this proposal, position and time will be on equal footing, but with some caveats that will be made clear later.

The square modulus of $\Psi(x\,\&\,t)$ is related to the wave functions in their respective spaces as

$$\begin{aligned} \mathcal{P}(x,t)\,dx\,dt &= |\Psi(x\,\&\,t)|^2\,dx\,dt \\ &= |\psi(x|t)|^2 f(t)\,dx\,dt \\ &= |\phi(t|x)|^2 g(x)\,dx\,dt, \end{aligned} \tag{5}$$

where $\mathcal{P}(x,t)\,dx\,dt$ is the probability of finding a particle in the length interval $[x, x+dx]$ and the time interval $[t, t+dt]$. The notation in $\psi(x|t)$, the usual wave function, means that $|\psi(x|t)|^2dx$ is the probability of finding the particle in the length interval $[x, x+dx]$ *given that* the clock reads t.

Through Bayes' rule [65], it implies that $f(t)\,dt$ is the probability of finding the particle in the time interval $[t, t+dt]$; analogously for $\phi(t|x)$ and $g(x)$. The functions $f(t)$ and $g(x)$ cannot be given through the equations of the systems alone; these depend on the type of experiment, the settings of the laboratory, and so on [55].

The STS proposal, then, tells us that if the experiment does not require predictions on time (for instance, the fringes at the end of the run of a double-slit experiment), all we need is the usual wave function $\psi(x|t)$. If we need only predictions of time (e.g., tunneling through a potential barrier), all that is required is $\phi(t|x)$. In cases where predictions of both position *and* time are needed, the complete wave function $\Psi(x\,\&\,t)$ should be used.

The "dynamics" in $\mathcal{H}_{\text{time}}$ is given by

$$\mathbb{P}|\phi(x)\rangle = -i\hbar\frac{\partial}{\partial x}|\phi(x)\rangle, \tag{6}$$

with $\mathbb{P}$ the Momentum operator in $\mathcal{H}_{\text{time}}$, defined as

$$\mathbb{P} \equiv \sigma_z\sqrt{2m[\mathbb{H} - V(x,\mathbb{T})]}, \tag{7}$$

$\sigma_z = \text{diag}(1,-1)$. When projected on $\langle t|$, this leads us to

$$\sigma_z\sqrt{2m\left(i\hbar\frac{\partial}{\partial t} - V(x,t)\right)}\phi(t|x) = -i\hbar\frac{\partial}{\partial x}\phi(t|x), \tag{8}$$

where $\phi(t|x) = \langle t|\phi(x)\rangle$. The quotation marks in "dynamics" mean that $\mathbb{P}$ generates variations not in time, as the usual QM does, but in the (classical) position parameter. Compare this to the usual QM, where the Hamiltonian is the generator of variations in the (classical) parameter time. Because of the presence of σ_z, $\phi(t|x)$ is a pseudospinor with components

$$\phi(x|t) = \begin{pmatrix}\phi^+(x|t)\\ \phi^-(x|t)\end{pmatrix}. \tag{9}$$

As such, the square modulus then is given by $|\phi(t|x)|^2 = \phi^\dagger(t|x)\phi(t|x)$.

A comment about this proposal is necessary. In the usual QM, $\hat{X}$, $\hat{P}$, and $\hat{H}$ are operators acting on $\mathcal{H}_{\text{pos}}$ and t is a classical parameter. The point of the STS proposal is that the extended Hilbert space *symmetrizes* operators and parameters. On the one hand, we have position and momentum as operators, and the generator of the dynamics, the Hamiltonian, as a function of these two, with the label t acting as a parameter. On the other hand, we have *time* and *energy* as operators and the *momentum* is the generator of the "dynamics," while still having a classical parameter: in this case, the position x of the particle. This is why, for time-of-flight experiments, all we need is $\phi(t|x)$: the measuring devices are classical objects, meaning that we have, in principle, an arbitrary precision of *where* the device is located. Then x has to act as a classical parameter.

Of course, if we consider the detectors to be lightweight and behave quantum mechanically [66], the uncertainty in the detector's position would be significant, and we would not be able to apply only the STS formalism. Since this will not be the case in the present work, we do not have to worry.

Expectation values in the spacetime-symmetric proposal

In the usual QM formalism, experimental results from measuring a quantity that has an operator $\hat{A}$ related to it are compared to the expectation value via

$$\langle\hat{A}\rangle(t) = \frac{\langle\psi(t)|\hat{A}|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle}, \tag{10}$$

which corresponds to averaging measurements of $\hat{A}$ in an ensemble of identically prepared systems, given that we measure at time t. We usually do not write the denominator because the wave function is normalized to the unit and its normalization is a constant:

$$i\hbar\frac{\partial}{\partial t}\langle\psi(t)|\psi(t)\rangle = \langle\psi(t)|(\hat{H} - \hat{H}^\dagger)|\psi(t)\rangle = 0, \tag{11}$$

because of the hermiticity of the Hamiltonian [39–41].

Now, consider the expectation value in $\mathcal{H}_{\text{time}}$. We have, as in $\mathcal{H}_{\text{pos}}$,

$$\langle\mathbb{B}\rangle(t) = \frac{\langle\phi(x)|\mathbb{B}|\phi(x)\rangle}{\langle\phi(x)|\phi(x)\rangle}, \tag{12}$$

having a similar interpretation of the average of measurements of $\mathbb{B}$, given that the measurement happened at the position x. However, in contrast to what happens in $\mathcal{H}_{\text{pos}}$, the denominator is generally not constant.

The physical interpretation, given by the authors of Ref. [56], is that, in the usual QM, the particle is expected to exist in some position, regardless of the instant of the measurement. This is different in the extended space, in general. Consider the double-slit experiment: there are points in space where the particle never arrives, independent of how long we wait. If we mirror the interpretation, the difference is clear: the particle *should* exist in *some* instant of time, *independent* of the position of the measurement. This does not happen in general; the dark regions on the fringes illustrate this. Some regions are forbidden no matter how long we wait for the particle to arrive. This means that whenever we use the STS expectation values, we have to carry the factor $\langle\phi(x)|\phi(x)\rangle$ throughout the calculations.

III. WEAK AND STRONG POTENTIAL APPROXIMATIONS

To obtain the wave function in the extended space, we need to solve Eq. (8). This is difficult because of the appearance of a derivative operator inside the square root. We can, however, consider the two extreme cases of weak and strong potentials, which enables us to obtain approximate equations in these limits that can be applied, for instance, to scattering and tunneling problems.

A. Weak potential

Since the generator of the "dynamics" in $\mathcal{H}_{\text{time}}$ is a function of the operators $\mathbb{T}$ and $\mathbb{H}$, we expand the momentum operator $\mathbb{P}$ in a Taylor series up to first order. For this, we consider the actuation of $\mathbb{H}$ to be greater than that of $V(x,\mathbb{T})$ in the sense that the particle rarely will have significant potential energy. Mathematically,

$$\begin{aligned}\mathbb{P} &= \sigma_z\sqrt{2m[\mathbb{H}-V(x,\mathbb{T})]}\\ &\simeq \sigma_z\sqrt{2m\mathbb{H}}\left[1-\frac{1}{4}\left(\frac{1}{\mathbb{H}}V(x,\mathbb{T})+V(x,\mathbb{T})\frac{1}{\mathbb{H}}\right)\right], \end{aligned} \quad (13)$$

where we use $\sqrt{1+\lambda}\simeq 1+\lambda/2$ for sufficiently small λ and since $\mathbb{H}$ does not commute with $\mathbb{T}$, we symmetrize the expansion. For simplicity, from now on, we will consider the potential to be independent of time, which gives us

$$\mathbb{P}\simeq\sigma_z\sqrt{2m}\left[\mathbb{H}^{1/2}-\frac{1}{2}\frac{V(x)}{\mathbb{H}^{1/2}}\right]. \quad (14)$$

When projected on $\langle t|$, the operators $\mathbb{H}^{1/2}$ and $1/\mathbb{H}^{1/2}\equiv\mathbb{H}^{-1/2}$ produces $1/2$-*fractional* derivatives and integrals, which can be defined as the Caputo fractional derivative [67] and the Riemann-Liouville fractional integral, respectively [68–72]. Then, we will have

$$\begin{aligned}-i\hbar\partial_x\phi(t|x) &= \sigma_z\sqrt{2mi\hbar}\partial_t^{1/2}\phi(t|x)\\ &\quad -\sigma_z\sqrt{\frac{m}{2i\hbar}}V(x)\partial_t^{-1/2}\phi(t|x).\end{aligned} \quad (15)$$

This fractional partial differential equation can be, in principle, solved through different methods, for instance, the Laplace transform of fractional derivatives and integrals [68–72]. For now, we will focus on the case of a constant potential $V(x)=V_0$. It is then possible to separate the equation onto temporal and spatial parts if we consider $\phi(t|x)=F(t)G(x)$:

$$pG(x) = -\sigma_z i\hbar\partial_x G(x), \quad (16a)$$

$$pF(t)=\sqrt{2m}\left[\sqrt{i\hbar}\partial_t^{1/2}-\frac{V_0}{2\sqrt{i\hbar}}\partial_t^{-1/2}\right]F(t), \quad (16b)$$

with p being the constant of separation, and we made use of the linearity of the fractional derivatives and integrals [68–72]. We use the ansatz

$$\begin{aligned}G^{\pm}(x) &= \exp\left[\pm\frac{i}{\hbar}px\right],\\ F(t) &= \exp[-i\omega t],\end{aligned} \quad (17)$$

where $G^{\pm}(x)$ are the $\pm$ spatial components of the spinor, together with the fractional derivative property [55,68,69]

$$\partial_t^{\alpha}\exp[\beta t]=\beta^{\alpha}\exp[\beta t], \quad (18)$$

to obtain

$$p=\sqrt{2m\hbar\omega}\left(1-\frac{V_0}{2\hbar\omega}\right), \quad (19)$$

that is, the first-order approximation of a particle with momentum $p=\sqrt{2m(E-V_0)}$ and energy $E=\hbar\omega$. Thus, the momentum in the STS proposal is consistent with the known results from classical mechanics (CM) and QM, at least in the weak and constant potential approximation.

Using this approximation, we can solve for E and arrive at

$$E=\frac{p^2}{2m}+V_0. \quad (20)$$

If we apply this to the case $V_0=0$, we obtain the solution for the free particle obtained in Ref. [55]

$$\phi^{\pm}(t|x)=\exp\left[-\frac{i}{\hbar}\frac{p^2}{2m}t\pm\frac{i}{\hbar}px\right], \quad (21)$$

as expected.

B. Strong potential

Considering a Taylor series expansion of the momentum operator with a strong, time-independent potential, we can write

$$\mathbb{P}\simeq\sigma_z\sqrt{-2mV(x)}\left[1-\frac{\mathbb{H}}{2V(x)}\right], \quad (22)$$

which leads us, in a similar way to Eq. (15), to

$$\sqrt{-2mV(x)}\left[1-\frac{i\hbar\partial_t}{2V(x)}\right]\phi(t|x)=-\sigma_z i\hbar\partial_x\phi(t|x). \quad (23)$$

Curiously, in the strong potential approximation, the order of the derivatives is the same, losing the fractional properties. Separating this equation enables us to write

$$i\hbar\partial_t F(t)=EF(t), \quad (24a)$$

$$\sigma_z i\hbar\partial_x G(x)=\sqrt{\frac{-m}{2V(x)}}[E-2V(x)]G(x), \quad (24b)$$

where, as before, $\phi(t|x)=F(t)G(x)$, and E is the separation constant. Equation (24a) is trivial, giving us

$$F(t)=\exp\left(-\frac{i}{\hbar}Et\right), \quad (25)$$

compatible with the known results from the usual QM [39–41]. Since we are considering a strong potential, we notice that the term on the right-hand side of Eq. (24b), multiplying $G(x)$, is a Taylor series expansion for small $E/V(x)$, and we can rewrite it as

$$\sqrt{\frac{-m}{2V(x)}}[E-2V(x)]\simeq-\sqrt{2m[E-V(x)]}, \quad (26)$$

as can be checked, giving us

$$\begin{aligned}G^{\pm}(x) &= \exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x}dx'\sqrt{2m[E-V(x')]}\right]\\ &= \exp\left(\pm\frac{i}{\hbar}S(E,x)\right),\end{aligned} \quad (27)$$

where $S(E,x)$ is the classical action and x_0 depends on the boundary conditions. $S(E,x)$ is also called the abbreviated action functional, and is related to the usual action by a Legendre transformation $\tilde{S}(x,t)=S(E,x)-Et$ (see Ref. [73] for details). Note that $\partial S(E,x)/\partial E=t=T_{\text{class}}$ provides the classical time, which we will use later. The constant potential is trivial and gives us, up to a multiplication constant,

$$G(x)=\exp\left[\pm\frac{i}{\hbar}px\right], \quad (28)$$

with $p=\sqrt{2m(E-V_0)}\in\mathbb{C}$ being the momentum of the system, which again coincides with the CM and QM momenta relations, subject to a constant potential with intensity V_0.

Notice that we obtained the relation $p = \sqrt{2m(E - V_0)}$ without any ad hoc hypothesis; the momentum was obtained through the dynamics of the STS proposal, as opposed to that in Ref. [56]. Our results confirm their findings.

IV. RESULTS

A. Tunneling time

As in Ref. [56], we define the time of travel (or, in the specific case we want to tackle in this section, tunneling time) as the difference between the expectation values

$$T_{\rm STS}(x_{\rm i} \to x_{\rm f}) = \langle \mathbb{T} \rangle(x_{\rm f}) - \langle \mathbb{T} \rangle(x_{\rm i}), \tag{29}$$

with $\langle \mathbb{T} \rangle(x)$ given by Eq. (12). A comment is in order. As argued in Ref. [63], the time a quantum object takes to tunnel through a barrier is ill posed since it is not generally possible to demarcate the tunneling and nontunneling regions, except for the rectangular barrier. The authors argued that the correct question is "How long does it take a quantum particle to cross a barrier?". Equation (29) is even more generic since it asks "How long, on average, does it take for a particle to move from $x_{\rm i}$ to $x_{\rm f}$?" and this includes smooth potentials like a Gaussian barrier [74]. In addition, expressions like the phase time and the Larmor times apply for monochromatic waves [13,55,58,61,63], while Eq. (29) and the tunneling flight time from Ref. [63] can be used for wavepackets.

The solutions that led to Eqs. (17), (20), (25), and (27) are eigenfunctions of $\mathbb{P}$, with eigenvalues $p = \sqrt{2mE}$ outside the barrier or $p = \sqrt{2m(E - V_0)}$ inside the barrier. When we prepare systems for experiments in the usual QM, we generally consider a wave packet, which is a linear combination of eigenfunctions of the Hamiltonian $\hat{H}$. In the same manner, since $\mathbb{P}$ is a linear operator, linear combinations of solutions of Eq. (8) are also solutions of the same equation. In this manner, the wavepacket is written as

$$\phi^{\pm}(t|x) = \int_{E_{\rm min}}^{E_{\rm max}} dE\, C_E^{\pm} \exp\left(-\frac{i}{\hbar}Et\right) G^{\pm}(E, x), \tag{30}$$

where the limits $E_{\rm min}$ and $E_{\rm max}$ (which will be supressed in the text from now on) must be chosen such that we meet the conditions of strong and/or weak potential, depending on the region, and $C_E^{\pm}$ is the energy distribution for the wave packet. The discrete case is straightforward. The correct way of writing the wavepacket should be in terms of eigenfunctions and eigenvalues of $\mathbb{P}$. Since we know the relation between p and E [e.g., $p = p(E)$], this is, at heart, just a change of variables in the integration, with the distributions $C_E^{\pm}$ having to change accordingly [55]. We also change the notation from $G^{\pm}(x)$ to $G^{\pm}(E, x)$ to emphasize the energy dependence of the spatial part. We are making an abuse of notation using the same $\phi^{\pm}(t|x)$ as before, but since from now on we will only work with the wavepacket, there should be no confusion.

Using the completeness relation $\int_{-\infty}^{\infty} dt\, |t\rangle\langle t| = \mathbb{I}$, we can write the expectation value of $\mathbb{T}$ as

$$\begin{aligned}\langle \mathbb{T} \rangle(x) &= \frac{\langle \phi(x)|\mathbb{T}|\phi(x)\rangle}{\langle \phi(x)||\phi(x)\rangle} \\ &= \frac{\int_{-\infty}^{\infty} dt\, t\, \rho(t|x)}{\int_{-\infty}^{\infty} dt\, \rho(t|x)},\end{aligned} \tag{31}$$

where

$$\begin{aligned}\rho(t|x) &= \phi^{\dagger}(t|x)\phi(t|x) \\ &= \left|\int dE\, C_E^{+} \exp\left(-\frac{i}{\hbar}Et\right) G^{+}(E, x)\right|^2 \\ &\quad + \left|\int dE\, C_E^{-} \exp\left(-\frac{i}{\hbar}Et\right) G^{-}(E, x)\right|^2,\end{aligned} \tag{32}$$

with $\langle t|\phi(x)\rangle = \phi(t|x)$, as used in Eq. (8). Equation (31) is very similar, for instance, to Eq. (4) combined with Eq. (3) from Ref. [63]. However, with some differences. First, the position Y of the "screen" is located far away from the interaction region. Equation (29), together with Eq. (31), can be used right at the interfaces since the STS considers the position x to be a classical parameter. Second, in Eq. (31), the limits in the integration are $(-\infty, +\infty)$ instead of $(0, +\infty)$ in Eqs. (3) and (4) of Ref. [63].

To calculate the expectation value in Eq. (31), we can write the numerator as

$$\begin{aligned}N &\equiv \int_{-\infty}^{\infty} dt\, t\, \rho(t|x) \\ &= \sum_{r=\pm} \int_{-\infty}^{\infty} dt\, t \left[\int dE\, C_E^{r} F(t) G^{r}(E, x)\right] \\ &\quad \times \left[\int dE'\, C_{E'}^{r} F'(t) G^{r}(E', x)\right]^*,\end{aligned} \tag{33}$$

where the prime denotes that we need to substitute $E \to E'$ in the argument of the second integral. The temporal integral can be rewritten as

$$\int_{-\infty}^{\infty} dt\, t\, \exp\left[-\frac{i}{\hbar}(E - E')t\right] = -2\pi i\hbar^2 \partial_{E'} \delta(E' - E), \tag{34}$$

where we make use of

$$t\, \exp\left[-\frac{i}{\hbar}(E - E')t\right] = -i\hbar \partial_{E'} \exp\left[-\frac{i}{\hbar}(E - E')t\right],$$

and the integral representation of the Dirac delta [75]

$$2\pi\delta(x - a) = \int_{-\infty}^{\infty} dp\, \exp\left[ip(x - a)\right].$$

Then, the numerator becomes

$$N = 2\pi i\hbar^2 \sum_{r=\pm} \int dE\, C_E^{r} G^{r}(E, x) \partial_E \left[C_E^{r} G^{r}(E, x)\right]^*. \tag{35}$$

Similarly, we can write the denominator as

$$\begin{aligned}D &\equiv \int_{-\infty}^{\infty} dt\, \rho(t|x) \\ &= 2\pi\hbar \sum_{r=\pm} \int dE\, \left|C_E^{r} G^{r}(E, x)\right|^2,\end{aligned} \tag{36}$$

which finally brings us to

$$\langle \mathbb{T} \rangle(x) = i\hbar \frac{\sum_{r=\pm} \int dE\, C_E^{r} G^{r}(E, x) \partial_E \left[C_E^{r} G^{r}(E, x)\right]^*}{\sum_{r=\pm} \int dE\, \left|C_E^{r} G^{r}(E, x)\right|^2}. \tag{37}$$

Using Eq. (27), the numerator of the above equation can be rewritten as

$$N = i\hbar \sum_{r=\pm} \int dE\, e^{-2r\mathrm{Im}[S(E,x)]/\hbar} \times \left[C_E^r \partial_E (C_E^r)^* - \frac{ir}{\hbar}|C_E^r|^2 T^*_{\mathrm{class}}(E,x)\right], \tag{38}$$

where $T_{\mathrm{class}}(E,x) = \partial S(E,x)/\partial E$ is the classical time, which is real for energies above the barrier and imaginary for energies below the barrier. Therefore, the expectation value $\langle \mathbb{T} \rangle(x)$ is proportional to an energy average of the (real or imaginary) classical time $T_{\mathrm{class}}(E,x)$, weighted by the energy distribution. This is our first main result. The imaginary classical time is obtained by inverting the potential.

For particles with energies above the barrier, the first integral in Eq. (38) is independent of x, as is the denominator of Eq. (37), $\sum_{r=\pm}\int dE\, |C_E^r G^r(E,x)|^2 = \sum_{r=\pm}\int dE\, |C_E^r|^2$ (since $\mathrm{Im}[S(E,x)] = 0$). Therefore, when using Eq. (29) to obtain $T_{\mathrm{STS}}(x_{\mathrm{i}} \to x_{\mathrm{f}})$, this contribution vanishes. In summary, we have the following three cases.

(1) For real momenta (energies above the barrier), the average time of flight $T_{\mathrm{STS}}(x_{\mathrm{i}} \to x_{\mathrm{f}})$ is real. Note that only the time *difference* becomes real.

(2) For imaginary momenta (energies below the barrier), the average time of flight $T_{\mathrm{STS}}(x_{\mathrm{i}} \to x_{\mathrm{f}})$ becomes imaginary. Essentially it is a contribution of the first integral in Eq. (38), which is independent of $T_{\mathrm{class}}(E,x)$, together with the second integral, which involves $T_{\mathrm{class}}(E,x)$.

(3) When the integration includes energies below and above the barrier, the average time of flight $T_{\mathrm{STS}}(x_{\mathrm{i}} \to x_{\mathrm{f}})$ becomes complex.

This provides a clear interpretation and contributions of each term to the time of flight. In weak measurements, while the real part of the conditional expectation value has been interpreted as the pointer, the imaginary part is a change in the momentum (kickback of the measurement) [76]. This is consistent with our interpretation coming from Eq. (38), except that in our interpretation the imaginary part also involves the complex classical time. For the particular case of $C_E^r = \text{const.}$, only the second integral survives and we can have either purely imaginary (as we will see in Sec. IV C) or purely real (as we will see in Sec. IV D) times of arrival.

Before discussing an application, let us provide additional general statements about Eqs. (36) and (38) using a Gaussian wavepacket for the energy distribution. For simplicity, we assume $C_E^- = 0$ and

$$C_E^+ = \left(\frac{\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} \exp\left(-\frac{\sigma^2}{\hbar^2}E^2\right), \tag{39}$$

which is the Gaussian wave packet and $\hbar/\sigma$ is the width in the energy. Therefore, while the numerator becomes

$$N = \frac{i\sigma}{\hbar\sqrt{\pi}} \int dE\, \exp\left(-\frac{2\sigma^2 E^2}{\hbar^2} - \frac{2\mathrm{Im}[S(E,x)]}{\hbar}\right) \times \left[-\frac{2\sigma^2 E}{\hbar} - i\,T^*_{\mathrm{class}}(E,x)\right], \tag{40}$$

the denominator, for energies above the barrier, becomes

$$D = \frac{1}{2\sqrt{2}}\left[\mathrm{erf}\left(\frac{\sqrt{2}E_{\max}\sigma}{\hbar}\right) - \mathrm{erf}\left(\frac{\sqrt{2}E_{\min}\sigma}{\hbar}\right)\right], \tag{41}$$

since $|G(E,x)| = 1$ for this case, and $\mathrm{erf}(\alpha)$ is the error function. We discuss two limiting physical situations.

(i) For $\sigma \to 0$ (the wavepacket well localized in time and delocalized in energy), the integral proportional to $T^*_{\mathrm{class}}(E,x)$ is the leading factor for N. It demonstrates that localization in time is directly related to the appearance of $T^*_{\mathrm{class}}(E,x)$. In this limit, for energies above the barrier, we have

$$\langle \mathbb{T} \rangle^{(\sigma\to 0)}(x) = \frac{N^{(\sigma\to 0)}}{D^{(\sigma\to 0)}} \simeq \frac{\int dE\, T_{\mathrm{class}}(E,x)}{\Delta E} = \frac{\Delta S}{\Delta E}, \tag{42}$$

where we use the approximation $\mathrm{erf}(ax) \simeq \frac{2ax}{\sqrt{\pi}}$. Clearly the quantum time is the energy average of the classical time, namely, $\overline{T}_{\mathrm{class}} = \Delta S/\Delta E$. For energies below the barrier, the above equation becomes

$$\begin{aligned}\langle \mathbb{T} \rangle^{(\sigma\to 0)}(x) &\simeq \frac{\int dE\, \exp\left(-\frac{2\mathrm{Im}[S(E,x)]}{\hbar}\right) T^*_{\mathrm{class}}(E,x)}{\Delta E} \\ &= i\frac{\int dS\, \exp\left(-\frac{2\mathrm{Im}[S(E,x)]}{\hbar}\right)}{\Delta E} \\ &= -\frac{i\hbar}{2}\frac{[e^{-\frac{2}{\hbar}S_f} - e^{-\frac{2}{\hbar}S_i}]}{\Delta E} \simeq i\frac{\Delta S}{\Delta E},\end{aligned} \tag{43}$$

with $\Delta S = S_f - S_i$, $S_f = S(E_{\max},x)$, and $S_i = S(E_{\min},x)$. In the last step we assume increasing values of $\hbar$ (deeper in the quantum limit) and the approximation leads again to $\overline{T}_{\mathrm{class}} = \Delta S/\Delta E$. For the semiclassical limit $\hbar \to 0$, the tunneling time expectation tends to zero.

(ii) For increasing σ (the wavepacket well localized in energy and delocalized in time), the integral proportional to $T^*_{\mathrm{class}}(E,x)$ becomes irrelevant. In this limit, for energies above and below the barrier we have

$$\begin{aligned}N^{(\sigma\to\infty)} &\sim i\hbar \int dE\, \frac{\sigma}{\hbar}\frac{1}{\sqrt{\pi}} \exp\left(-\frac{2\sigma^2 E^2}{\hbar^2}\right)\left[-\frac{2\sigma^2 E}{\hbar}\right] \\ &= \frac{i\sigma\hbar}{2\sqrt{\pi}} \exp\left(-\frac{2\sigma^2 E^2_{\max}}{\hbar^2}\right) \\ &\quad -\frac{i\sigma\hbar}{2\sqrt{\pi}} \exp\left(-\frac{2\sigma^2 E^2_{\min}}{\hbar^2}\right),\end{aligned} \tag{44}$$

which has *solely* characteristics of the energy envelope, independent of $S(E,x)$ and the classical time $T(E,x)$.

For the later application we consider the limit of a constant distribution in the energy interval. From the above results in (i) we can argue that we have a good time resolution, depending on ΔE.

B. Toy model: Rectangular potential barrier

The toy model we use for our main result is the textbook potential barrier

$$V(x) = \begin{cases} V_0 = \text{const.}, & 0 < x < L, \\ 0, & \text{everywhere else.} \end{cases} \tag{45}$$

V_0 is such that we can use the strong potential limit of Sec. III B for this region and L is the length of the barrier.

We want the wave function to be continuous in the interfaces $x = 0$ and $x = L$ for all instants of time, following the same principles as in the usual QM [39–41]. Since $F(t)$ has the same form for all regions, the temporal connection is trivial and implies that the energies $E = \hbar\omega$ must be equal in all regions. Then, for the spatial part, we consider

$$G^{\pm}(x) = \begin{cases} A_1^{\pm} \exp\left[\pm\frac{i}{\hbar}p_1 x\right], & x < 0, \\ A_2^{\pm} \exp\left[\pm\frac{i}{\hbar}p_2 x\right], & 0 < x < L, \\ A_3^{\pm} \exp\left[\pm\frac{i}{\hbar}p_1 x\right], & L < x, \end{cases} \tag{46}$$

with

$$p_1 = \sqrt{2mE},$$
$$p_2 = \sqrt{2m(E - V_0)}. \tag{47}$$

Connecting the wave function at the interfaces, we have

$$A_2^{\pm} = A_1^{\pm},$$
$$A_3^{\pm} = A_1^{\pm} \exp\left[\pm\frac{i}{\hbar}(p_2 - p_1)L\right]. \tag{48}$$

Combining Eqs. (46) and (48), together with $F(t) = \exp(-iEt/\hbar)$, we have the total wave function for the rectangular barrier.

Equation (37) allows us to predict tunneling times and dwell times whenever the potential is sufficiently strong and constant. Because, in principle, we can position the probes with arbitrary precision in this treatment, the time it takes for the particle to tunnel, on average, is given by

$$T_{\mathrm{STS}}(0 \to L) = \langle \mathbb{T} \rangle(L) - \langle \mathbb{T} \rangle(0) \tag{49}$$

for a potential barrier located between $x = 0$ and $x = L$ [56].

C. Application of tunneling time: Constant distribution for a wave packet moving to the right inside the barrier

For an application of Eq. (37), we consider the following: $C_E^+ = C =$ constant and $C_E^- = 0$. This second equality means that, since the expressions in Eq. (46) are plane waves, the wave traveling from right to left on the real x axis is discarded. The first equation, the flat distribution, indicates that we have a large energy spread (but with limits such that we obbey the strong or weak potential limits in the respective regions), such that we can analyze the tunneling or traveling times not only for a narrow energy bandwidth [63] (the limiting case being a monochromatic wave), but also a large one. We would like to point out that, at first glance, this could seem unphysical since we could have arbitrarily high energies in the wavepacket, but since they are limited by the potential (weak or strong approximations), this is not a problem. Also, a flat energy distribution means the particle is well localized in time, as discussed in the end of Sec. IV A.

Figure 1 displays $\rho(t|x)$ for a constant distribution and the rectangular potential barrier located between $x = 0 \to 1$. The wavepacket moves from left to right (that is, from most negative x to most positive x). Notice that since we have a nonnormalized wave function, the absolute values can be very high. For increasing times, $\rho(t|x)$ oscillates with diminishing amplitude, which certainly decreases the tunneling probabilities for larger times.

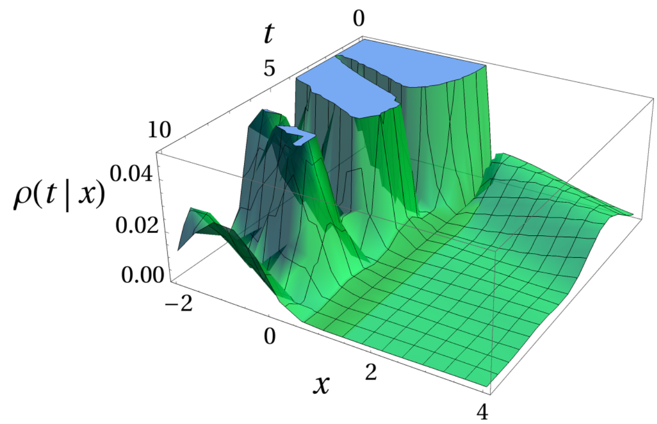

FIG. 1. $\rho(t|x)$ for $m = \hbar = L = 1$ and $V_0 = 5$. $C_E^+ = 1$ and $C_E^- = 0$, meaning that the plane wave moves initially from left to right for integration limits ranging from $E = 0$ to $E = 2$. Quantities are in arb. units.

For the constant distribution case, the expectation value of $\mathbb{T}$ can be simplified as

$$\langle \mathbb{T} \rangle(x) = i\hbar \frac{\int dE\, G(E,x)\partial_E[G(E,x)]^*}{\int dE\, |G(E,x)|^2}, \tag{50}$$

where we drop the $r = \pm$ indexes since we only have the $r = +$ component. The derivative inside the integral requires some attention. The numerator, following Eq. (38), and making the substitution $u = \sqrt{2m(V_0 - E)}$, $\implies du = -m/\sqrt{2m(V_0 - E)}\, dE$, can be rewritten as

$$\begin{aligned} i\hbar \int dE \quad & G(E,x)\partial_E[G(E,x)]^* \\ &= \int dE\, T^*_{\mathrm{class}} \exp\left[-\frac{2\mathrm{Im}[S(E,x)]}{\hbar}\right] \\ &= \frac{i\hbar}{2}[e^{-\frac{2}{\hbar}p_E x} - e^{-\frac{2}{\hbar}p_0 x}], \end{aligned} \tag{51}$$

where we apply the limits 0 and $E_{\max}$ (satisfying the strong potential approximation), $p_0 = \sqrt{2mV_0}$ and $p_E = \sqrt{2m(V_0 - E_{\max})}$. This is exactly the numerator of a wavepacket well localized in time, Eq. (43). The denominator follows from the same substitution, giving

$$\begin{aligned} \int_0^{E_{\max}} dE\, e^{-\frac{2\mathrm{Im}[S(E,x)]}{\hbar}} &= +\frac{e^{-\frac{2}{\hbar}p_E x}\hbar(\hbar + 2p_E x)}{4mx^2} \\ &\quad -\frac{e^{-\frac{2p_0 x}{\hbar}}\hbar(\hbar + 2p_0 x)}{4mx^2}. \end{aligned} \tag{52}$$

Together, both equations give us our second main result

$$\begin{aligned} T_{\mathrm{STS}}(0 \to L) &= \langle \mathbb{T} \rangle(L) - \langle \mathbb{T} \rangle(0) \\ &= \frac{2imL^2[1-\gamma]}{\hbar[1-\gamma] + 2L[p_E - p_0\gamma]}, \end{aligned} \tag{53}$$

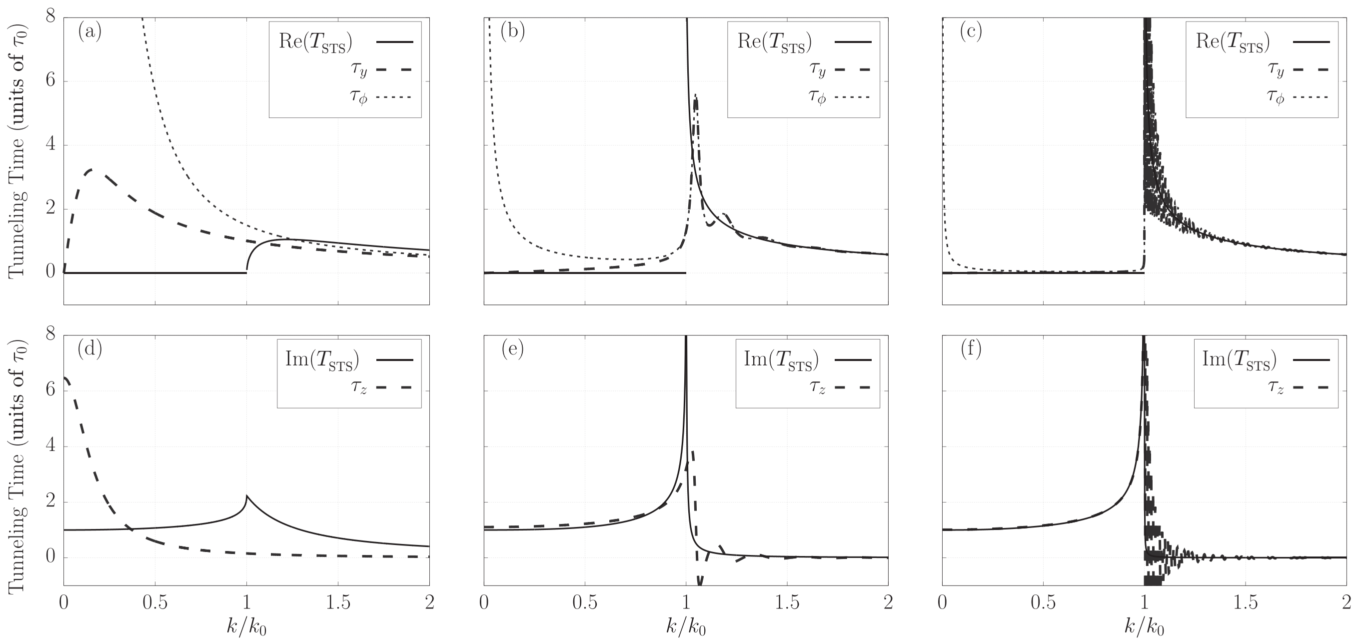

FIG. 2. Comparison between Eq. (53) and the travel times τ_z, $\tau_y = \tau_{\rm D}$, and τ_ϕ, as obtained in Ref. [58], in units of the characteristic time $\tau_0 = mL/\hbar k_0$ of the barrier. Here, we have $k = \sqrt{2mE_{\rm max}}/\hbar$ for $T_{\rm STS}$ and $k = \sqrt{2mE}/\hbar$ for the other times. In all the cases $k_0 = \sqrt{2mV_0}/\hbar$. The quantity k_0L gives us the strength of the barrier and we have $k_0L = \pi/10$ in (a) and (d), $k_0L = 3\pi$ in (b) and (e), and $k_L = 30\pi$ in (c) and (f). We notice that, for a weak barrier [Figs. 2(a) and 2(b)], our result differs greatly. As we increase the barrier strength, $T_{\rm STS}$ starts to agree more for large k while acting as an average for $k > k_0$.

where we use $\lim_{x\to 0}\langle \mathbb{T}\rangle(x) = 0$ and $\gamma = e^{-\frac{2}{\hbar}(p_0 - p_E)L}$. In the classical limit $\hbar \to 0$ we obtain

$$T_{\rm STS}^{(\rm class)}(0 \to L) = \frac{imL}{p_E}, \tag{54}$$

while for $E_{\rm max} \to 0$ (no wavepacket) we obtain

$$T_{\rm STS}^{(E_{\rm max}=0)}(0 \to L) = \frac{imL}{p_0}. \tag{55}$$

Equation (53) can be compared to the characteristic times obtained from the precession of spin in an infinitesimal field in the $\hat{z}$ direction (for more details check Ref. [58]), also known as Larmor times τ_z, τ_y (which coincides with the dwell time $\tau_{\rm D}$) and the phase time τ_ϕ [61]. The tunneling time, in units of the characteristic barrier time $\tau_0 = mL/\hbar k_0$, is shown in Fig. 2 as a function of k/k_0. For $k/k_0 < 1$, we have energies below the barrier and for $k/k_0 > 1$, energies above the barrier. Distinct strengths of the barrier $k_0L \equiv p_0L/\hbar$ are used. The top row of Fig. 2 compares the real part of $T_{\rm STS}(0 \to L)$ with τ_y, τ_ϕ, and the bottom row the imaginary part of $T_{\rm STS}(0 \to L)$ with τ_z. For small strengths, $k_0L = \pi/10$, Figs. 2(a) and 2(d) show that, except for values of $k/k_0 \gtrsim 1.5$, the curves do not match at all. This is expected since Eq. (53) is obtained from a strong potential approximation. For stronger barriers, we observe that the imaginary part of Eq. (53) starts to have a good agreement with τ_z inside the barrier while behaving as an average of the oscillations outside, as shown in Figs. 2(e) and 2(f).

We also observe that $\text{Im}[T_{\rm STS}(0 \to L)]$ always begins at $\tau_0 = mL/\hbar k_0 = mL/p_0$, which is the imaginary part of $T_{\rm STS}(0 \to L)$ for $E_{\rm max} \to 0$, as seen in Eq. (55), growing linearly with L (so no superluminal signaling occurs). Important to remember is that our results are *averages*; there could be particles that tunneled with faster-than-light speeds, but since this is due to the randomness and uncontrollability of the collapse of the wavepacket, this is useless for signaling purposes [76]. For the real part, on the other hand, since Eq. (53) is imaginary for $k < k_0$, it always vanishes inside the barrier. Outside the barrier, as shown in Figs. 2(b) and 2(c), we recognize that the real part is again an average of the times τ_y and τ_ϕ, which coincide for $k > k_0$ and oscillates very rapidly for stronger barriers. We also notice that, for increasing k, $\text{Im}[T_{\rm STS}(0 \to L)]$ decays very fast, going to 0 almost immediately for stronger barriers, while $\text{Re}[T_{\rm STS}(0 \to L)]$ remains considerable. Remembering that our findings come from the expectation values of the operator $\mathbb{T}$, it may imply that the presence of the barrier makes $\mathbb{T}$ act like a Hermitian operator (with only real eigenvalues) for energies above the barrier or an anti-Hermitian operator (with only imaginary eigenvalues) for energies below the barrier. Worth noticing also that since Eq. (51) has the real $\sqrt{2m(E - V_0)}$ condition included, there is no problem in applying Eq. (53) for energies outside the barrier.

Figure 3 compares with the experimental results from Ref. [74]. Even though the authors used a Gaussian barrier in the experiment, we see that $\text{Im}[T_{\rm STS}(0 \to L)]$ and $\text{Re}[T_{\rm STS}(0 \to L)]$ agree for energies above the barrier with the measure of τ_z and τ_y, respectively. Since near the barrier is not a region covered by the approximations from Eqs. (15) and (23), it is expected to behave differently. On the other hand, we can see that for nonrectangular barriers, as used by the authors in the experiment, we obtain real parts for energies "inside" (for a Gaussian barrier, it is difficult to define what "inside" means [63]) the barrier (through τ_y) as well as imaginary

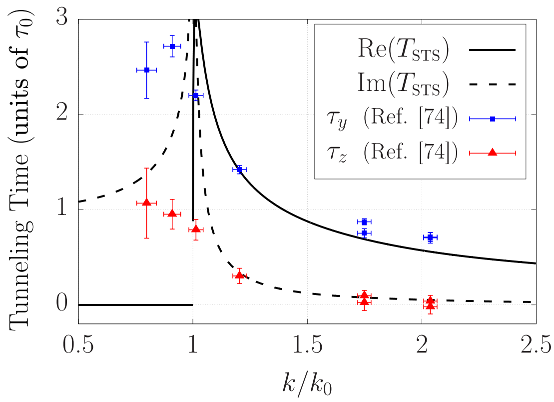

FIG. 3. Comparisons between Eq. (53) and the experimental data of Fig. 4(c) from Ref. [74], in units of τ_0. The barrier intensity is π (for a barrier with corresponding velocity 5.1 mm/s) and barrier length of 1.3 μm, such that $\tau_0 = mL/\hbar k_0 = L/v \simeq 2.5 \times 10^{-4}$ s. Blue squares are measurements of τ_y, while red triangles are measurements of τ_z. We notice that near the barrier our results disagree, as expected, since we consider approximations away from $k/k_0 \sim 1$ in Eqs. (15) and (23).

components (through τ_z), in agreement with our discussion at the end of Sec. IV A.

Though Eq. (53) is already obtained from a Taylor expansion, we may use further approximations to understand the underlying physical properties of the tunneling time: up to second order in $E_{\max}/V_0$, we have

$$T_{\rm STS}(0 \to L) \simeq \frac{imL}{p_0}\left(1 + \frac{E_{\max}}{4V_0}\right) + \left(\frac{imL}{8p_0} + \frac{imL^2}{24\hbar}\right)\left(\frac{E_{\max}}{V_0}\right)^2. \tag{56}$$

The first two terms in this expansion, the first line of Eq. (56), can be rewritten approximately as

$$\frac{imL}{p_0}\left(1 + \frac{E_{\max}}{4V_0}\right) \simeq \frac{imL}{\sqrt{2m(V_0 - E_{\max}/2)}}. \tag{57}$$

One can understand this result in two ways. First, this is the first-order expansion of the time of travel mL/p of a classical particle with energy $E_{\max}/2$ and momentum $p = \sqrt{2m(V_0 - E_{\max}/2)}$ inside the barrier (except for a multiplicative i). Note that $E_{\max}/2$ is the mean energy between 0 and $E_{\max}$.

Second, when we expand the momentum $\sqrt{2m(V_0 - E)}$, we obtain the same result through the energy average of times-of-arrival of classical particles, except for a multiplicative i:

$$\begin{aligned}\langle t\rangle_{\rm class} &\equiv \frac{1}{E_{\max}}\int_0^{E_{\max}} dE\, \frac{mL}{\sqrt{2m(V_0 - E)}} \\ &\simeq \frac{mL}{E_{\max}p_0}\int_0^{E_{\max}} dE\left(1 + \frac{E}{2V_0}\right) \\ &= \frac{mL}{p_0}\left(1 + \frac{E_{\max}}{4V_0}\right) \\ &\simeq -iT_{\rm STS}(0 \to L).\end{aligned} \tag{58}$$

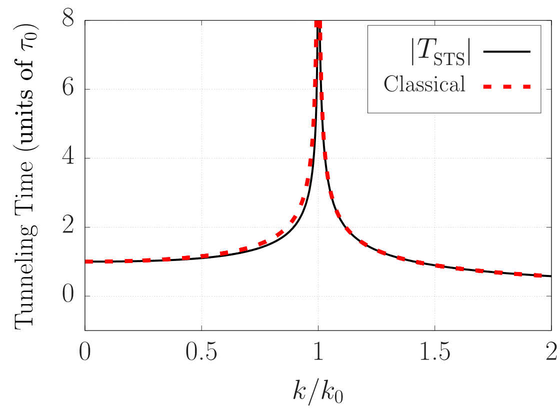

FIG. 4. Comparisons between Eq. (53) in absolute values and the "classical" time $|mL/\sqrt{2m(E_{\max} - V_0)}|$ in units of $\tau_0 = mL/\hbar k_0$, for a barrier intensity $k_0 L = 3\pi$.

We see that the time the particle takes to travel through the barrier is a classical-like contribution plus a quantum correction. We compare Eq. (53) with a classical particle with energy $E_{\max}$ and time of travel $mL/\sqrt{2m(E_{\max} - V_0)}$ in Fig. 4. We see that, outside the barrier, our result agrees very well with the classical time and is slightly different near $k \equiv \sqrt{2mE_{\max}} = k_0 \equiv \sqrt{2mV_0}$, as expected since Eq. (53) is obtained through a strong potential approximation. Thus, our results signal that the classical time is the most probable time.

The traveling time from Eqs. (49), (53), and (56) can also be compared to other approximated tunneling time expressions, which are summarized in Table I. Agreements are found in the limit $V_0 \gg E$ with the Larmor time (except for the factor i) and the complex time (except for the signal). The agreement is also obtained compared to the imaginary part of the tunneling time τ_S obtained in the telegrapher's equation. In this case, an additional term proportional to $(mL/p)^2$ is observed, where a is the friction coefficient due to dissipation, that may be related to the second line in Eq. (56): we write

$$\frac{imL^2}{24\hbar}\left(\frac{E_{\max}}{V_0}\right)^2 = \frac{iE_{\max}^2}{12\hbar V_0}\left(\frac{mL}{p_0}\right)^2, \tag{59}$$

in the strong potential limit (we discard the term proportional to $E_{\max}^2/V_0^3$), and identify $a = iE_{\max}^2/12\hbar V_0$.

TABLE I. Specific tunneling times expressions through a rectangular box with width L, $\kappa^2 = k_0^2 - k^2$, with k and k_0 as given in Fig. 2. The parameter a is related to the friction coefficient that enters the telegrapher's equation and $v = p/m$ is the particle's velocity through the barrier. Extracted from Refs. [13,55,61].

Quantity	Expression	Limit $V_0 \gg E$
Phase time	$\tau_P = \tau_\phi \simeq \frac{2m}{\hbar k\kappa}$	$\tau_P \to \frac{2m}{\hbar k k_0}$
Dwell time	$\tau_D = \tau_y \simeq \frac{2mk}{\hbar\kappa k_0^2}$	$\tau_D \to 0$
Larmor time	$\tau_L = \tau_z \simeq \frac{mL}{\hbar\kappa}$	$\tau_L \to \frac{mL}{\hbar k_0}$
Complex time	$\mathrm{Im}[\tau_C] = -\tau_L$	$\mathrm{Im}[\tau_C] \to -\frac{mL}{\hbar k_0}$
Stochastic model	$\tau_S \simeq a\left(\frac{mL}{\hbar\kappa}\right)^2 + i\frac{mL}{\hbar\kappa}$	$\frac{imL}{\hbar\kappa}$
STS model	Eq. (53)	$imL/\hbar k_0$

Not shown in Fig. 2 is the Büttiker-Landauer time [58,61] $\tau_{\text{B-L}} \equiv \tau_x = \sqrt{\tau_z^2 + \tau_y^2}$, but we can see that, for large $k_0 L$, only τ_z is important for $k < k_0$, while only τ_y contributes for $k > k_0$, meaning that our results agree in such limits.

D. Application of traveling time: Constant distribution for a wave packet moving to the right in a weak potential

In the case of a weak potential, we apply Eqs. (17) and (20) to Eq. (37) to obtain the traveling time for this situation. Similarly to Sec. IV C, we consider a wavepacket with $C_E^- = 0$ and $C_E^+ = C = \text{const}$. However, contrary to Sec. IV C, $G(E, x)$ is a complex exponential $|G(E, x)|^2 = 1$, leading us to

$$\langle \mathbb{T} \rangle(x) = \frac{\int_{E_i}^{E_f} dE\, T_{\text{class}}(E, x)}{\int_{E_i}^{E_f} dE}, \tag{60}$$

E_f and E_i are such as to allow us to use the weak potential approximation of Sec. III A. Then, the time of traveling is written as

$$\begin{aligned} T_{\text{travel}}(0 \to L) &= \langle \mathbb{T} \rangle(L) - \langle \mathbb{T} \rangle(0) \\ &= \frac{1}{E_f - E_i} \int_{E_i}^{E_f} dE\, T_{\text{class}}(E, L) \\ &= \frac{L}{\Delta E}[\sqrt{2m(E_f - V_0)} - \sqrt{2m(E_i - V_0)}] \\ &= \frac{S_f - S_i}{\Delta E} = \frac{\Delta S}{\Delta E} = \overline{T}_{\text{class}}, \end{aligned} \tag{61}$$

with $\Delta E = E_f - E_i$ and $\langle \mathbb{T} \rangle(0) = 0$. Equation (61) says that if a particle has an energy greater than the barrier and the wavepacket is well localized in time, the expected time of arrival is an average (in the energies) of the *classical* time-of-arrival $\tau_{\text{class}} = mL/p_E$, as discussed in Sec. IV A.

For the free particle, $V_0 = 0$, we then have

$$T_{\text{travel}}(0 \to L) = \frac{L}{\Delta E}[\sqrt{2mE_f} - \sqrt{2mE_i}]. \tag{62}$$

Consider the case in which $E_i = 0$ and $E_f = E_{\max}$, the same energies as the tunneling case in Sec. IV C. We obtain

$$T_{\text{travel}}^{\text{free}}(0 \to L) = 2\frac{mL}{\sqrt{2mE_{\max}}}. \tag{63}$$

We then compare Eq. (63) to the absolute value of the approximated tunneling time, given by Eq. (57)

$$\begin{aligned} \frac{T_{\text{travel}}^{\text{free}}}{T_{\text{STS}}} &\simeq 2\frac{mL}{\sqrt{2mE_{\max}}}\left(\frac{mL}{\sqrt{2m(V_0 - E_{\max})}}\right)^{-1} \\ &= 2\sqrt{\frac{V_0 - E_{\max}}{E_{\max}}} \\ &\simeq 2\sqrt{V_0/E_{\max}} \gg 1, \end{aligned} \tag{64}$$

where we use the fact that $V_0 \gg E_{\max}$. This is compatible with the known results from Ref. [77] and references therein, where the tunneling time is shorter than the time a free particle would take to cross the same region.

V. FINAL REMARKS

This work summarizes the main ideas of a recent proposal that tries to include and understand a time operator in QM. The proposal is spacetime-symmetric (STS) and allows for predicting times-of-flight and tunneling times. Using a Gaussian energy distribution, we demonstrate that, for wavepackets well resolved in time, the expectation value for the operator $\mathbb{T}$ is the energy average of the classical time $T_{\text{class}} = \partial S/\partial E$, which can be real or imaginary depending on the action S. The physical interpretation of the real and imaginary parts of the expectation value of $\mathbb{T}$ becomes clear.

We apply the proposal for a particle with energy E under weak and strong constant potentials, namely, a rectangular barrier with length L and intensity V_0. Connection formulas between distinct regions of motion are provided to obtain an explicit expression for the tunneling time through a barrier. Using a wavepacket with a constant distribution of energies, we show that the tunneling time in the STS proposal is in agreement with previous times, such as τ_z, $\tau_y = \tau_{\text{D}}$, τ_ϕ from Ref. [58]. Furthermore, we provide, in first order, the average of classical times-of-flight for an ensemble of particles with momenta $\sqrt{2m(V_0 - E)}$, except for a complex multiplicative unit. We show that for the rectangular barrier, complex times-of-arrival emerge, possibly bringing to light the discussion of the non-Hermiticity of a time operator for the tunneling case. The appearance of an imaginary time is consistent with a similar mechanism to the one that appears in Büttiker's model using a Larmor clock [13,58], which has a real part τ_ϕ and an imaginary part τ_z. Our procedure differs from those obtained in Ref. [55] when considering the connection formulas between allowed and prohibited regions, allowing us to furnish analytical expressions for the tunneling times.

The STS proposal is promising. It encompass the times of travel for both classically forbidden and clasically allowed regions, giving average times even for wavepackets and arbitrary potentials. Apart from helping our general understanding of the time in QM, it could assist in using fractional derivatives and integrals in physics and their interpretations in QM [69,78–82] or in other areas. They can be used to model power-law nonlocality, power-law long-term memory, or fractal properties (see Ref. [83] and references therein), anomalous diffusion processes in complex media [84], and the propagation of acoustical waves in biological tissue [85], to name a few applications. We can especially see, when comparing the dynamical equations for the weak *versus* strong potentials, that the order of the time derivative varies from 1/2 to 1, respectively, an artifact of the Taylor series expansions. In addition, it could motivate further studies giving more insights into the symmetries between space and time and energy and momentum. Furthermore, the present results demonstrate that the time of arrival can be used as a signature of tunneling.

Generally speaking, solving Eq. (8) is the main challenge. One possible way to do it is using the Fourier transform of the square-root operator. In Ref. [72], Sec. 28.2 gives us the treatment for powers of the operator $-\Delta x + \partial_t$, but for different integrodifferential operators. In principle, this could be expanded to the momentum operator in the STS proposal and give us solutions beyond the scope of constant potentials. We could then compare predictions with, for instance, the toy model for the Stark problem of Ref. [86]. Possible problems of the inverse Fourier transform convergence could be avoided by limiting the integration frequencies, the barrier acting as a filter, as justified by Eq. (12.5) of Ref [13].

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