Inverse problem of photocount statistics: Applicability criterion for the inverse Bernoulli transform method

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It is shown that the applicability conditions for the inverse Bernoulli transform method to solve the inverse problem of photocount statistics are determined by the fulfillment of the associativity condition for multiplying the matrices included in this transformation. A general criterion for evaluating the photocount distributions Q_m in the case of few-photon light, which makes it possible to establish whether the solution to the inverse problem of photocount statistics by the inverse Bernoulli transform method is applicable for $\eta < 0.5$, is found. As an example of the application of the obtained criterion, the critical quantum efficiency η_{cr} is found for compound Poisson distribution, below which the solution of the inverse problem of photocount statistics becomes incorrect. Additionally it is shown that the normalization of Q_m is not sufficient to obtain a correct solution using the inverse Bernoulli transform.

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I. INTRODUCTION

One of the most widely used methods for determining the energy characteristics of light is based on measuring the photocount distribution [1], that is, the statistics of electrons emitted from the photocathode irradiated by the light beam. This method is based on detecting the optical radiation by single-photon counters [2]. To date, the photon counting method, which has a long history [3], is widely used in both applied [4–8] and fundamental research [9–12]. Nowadays, it is one of the key experimental methods used in quantum optics.

A. Semiclassical inverse problem of photocount statistics

Already at the initial stage of application of the photon counting method, in addition to the direct problem (finding the photocount distribution from the known light state) interest was aroused by the inverse problem, the determination of the light properties from the known photocount statistics. One of the most important properties of light is the energy distribution. The problem of reconstructing the energy distribution from the photocount one was considered in [13]; later, various approaches were developed to solve this problem [14–16]. These studies were based on Mandel's semiclassical formula for the photocount distribution Q_m [17], which is mathematically the averaged Poisson distribution over the energy distribution of the detected radiation (Poisson transformation of the energy distribution)

$$Q_m = \int_0^\infty \frac{(\eta\mathcal{E})^m}{m!} \exp(-\eta\mathcal{E}) w(\mathcal{E}) d\mathcal{E}, \qquad (1)$$

where $\mathcal{E} = \mathcal{E}(T) = \int_t^{t+T} \int_S I(\boldsymbol{r}, t) d^2 r dt$ is the light energy falling onto the detector area S during time T in a number of photons; η is the quantum efficiency of detection; $I(\boldsymbol{r}, t)$ is the light intensity in $(m^2 s)^{-1}$; $w(\mathcal{E})$ is the probability density of the fluctuating parameter $\mathcal{E}$; m is a number of photoelectrons, emitted during time T.

Within the framework of the semiclassical model, the inverse problem of photocount statistics [18] is the reconstruction of the distribution $w(\mathcal{E})$ from measured distribution Q_m. The problem of reconstructing the energy distribution has been solved more than once. Thus, in [13], the Poisson transform was inverted using the Fourier transform and the apparatus of characteristic functions. In [14], an expansion of the intensity distribution in terms of Laguerre polynomials was applied. The authors of [15] used Padé approximants to inverse the Poisson transform and cubic B-splines were used in [16] for the same purpose.

B. Quantum inverse problem of photocount statistics

For a correct description of the photodetection process, especially when applied to few-photon light, the field itself should also be considered as a quantum object. A consistent theory of such a process gives the following result [19]:

$$Q_m = \left\langle : \frac{1}{m!} [\eta\hat{\mathcal{E}}]^m \exp[-\eta\hat{\mathcal{E}}] : \right\rangle. \qquad (2)$$

Here $\hat{\mathcal{E}} = \hat{\mathcal{E}}(T) = \int_t^{t+T} \int_S \hat{E}^-(\boldsymbol{r}, t)\hat{E}^+(\boldsymbol{r}, t) d^2 r dt$ is the operator of the light energy; $\hat{E}^-(\hat{E}^+)$ is the negative- (positive-) frequency field operator, $\langle :: \rangle$ is the normally ordered averaging. $\langle : \hat{\mathcal{E}}(T) : \rangle$ is the light energy falling onto the detector area S during time T in a number of photons. As shown in [19], the expression (2) can be rewritten in the Fock basis

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as

$$Q_m = \sum_{n=m}^{\infty} C_n^m \eta^m (1-\eta)^{n-m} P_n, \tag{3}$$

where $P_n = \langle : \frac{1}{n!}\hat{\mathcal{E}}^n \exp[-\hat{\mathcal{E}}] : \rangle$ is the photon-number distribution, or the probability that n photons hits a detector during a time T, and $C_n^m = n!/[m!(n-m)!]$ is the binomial coefficient. The formula (3) is commonly called the Bernoulli transformation [20–22]. The distribution (3) allows us to give a clear physical interpretation. Since η is the probability of registering one photon during the measurement interval, then η^m is the probability of registering m photons, and $(1-\eta)^{(n-m)}$ is the probability of not registering $(n-m)$ photons when n photons arrive at the detector. The coefficient C_n^m takes into account the possible number of combinations of occurrence of m photoelectrons.

Thus, in the quantum approach, the photocount distribution is given by the convolution of the photon-number distribution with the Bernoulli one (3) and differs from the semiclassical Mandel formula (1). Therefore, at a low-intensity (few-photon) level, for a correct description of the photodetection process, one must proceed from the Bernoulli transformation (3).

The inverse problem of the photocount statistics in a few-photon mode is the finding P_n from the known distribution Q_m which are associated by the relation (3). The importance of this problem lies, in particular, in the fact that, currently, few-photon light sources are of significant interest in quantum technologies [2,23,24]. Photon-number distribution for few-photon radiation is an analog of the intensity for bright radiation; therefore, it can be considered as one of the most important characteristics of a light source. Experimentally, the photon counting method is apparently the simplest and cheapest method to obtain data about the photon-number distribution. If we had an ideal photon counter with $\eta = 1$ there would not be any problem with obtaining these data, as distributions Q_m and P_n coincide. However, the quantum efficiency of existing photon counters is typically less than 1, for example, for modern silicon photomultipliers (SiPM) detectors $\eta \leqslant 0.65$ [25]. This, as will be shown below, causes significant difficulties in inverting the formula (3).

At present, numerical statistical methods are most often used to solve the inverse problem. They give an approximate solution to the problem under some reasonable assumptions about its structure. One of the main methods of this kind is the maximum likelihood method [26], in which a solution is sought that maximizes the likelihood function. Combined with expectation [27,28] and entropy [29] maximization methods, it gives very good results. However, these methods are not universal. The problem is that they are approximate and use *a priori* information when searching for a solution, which limits their generality, making it impossible to solve the problem reliably in all cases.

Analytical methods [20–22,30,31] are free from this shortcoming. Analytical methods, in contrast to numerical ones, not only give an exact solution to the problem, but also allow us to understand its essence. The best known and most commonly used analytical method for solving the inverse problem for few-photon light, which takes into account the losses in photodetection due to $\eta < 1$, is the method based on the direct inversion of the Bernoulli transformation [20,21]. As is known, one of the main fundamental problems that arise in solving inverse problems is the problem of stability of their solutions. Apparently, the first work in which a fundamental study of the stability of solutions obtained by the inverse Bernouli transform was carried out is the work [21], where the authors managed to get an expression of statistical uncertainty for P_n values and to make the first base research of the solution convergence. They showed that for any finite Q_m the solutions are stable for arbitrary η. However, for infinite Q_m, the solutions are stable only if $\eta > 0.5$, and in the case of $\eta < 0.5$, one can find counterexamples when the solution turns out to be unstable. As an example, they gave the thermal distribution, for which the stable reconstruction is impossible below some critical quantum efficiency. The results obtained in this work caused a scientific discussion [21,22,32–34] about whether the threshold value identified in [21] is a fundamental limitation, i.e., whether it is due to any physical reasons or due to mathematical problems of the solution. However, the causes for the instability of solutions were not identified in the subsequent works, and the question remained open.

In this paper, we managed to find the reasons for the apparent instability of the solutions of the inverse Bernoulli transform. In addition, we found a general criterion for evaluating the distributions Q_m, which allows us to establish whether the solution of the inverse problem obtained by the inverse Bernoulli transform method is mathematically correct for a given value of η less then 0.5.

II. INVERSE BERNOULLI TRANSFORM METHOD

The inverse Bernoulli transform method is based on the fact that the formula (3) can be reversed [20]. The easiest way to see this is to represent this formula in matrix form $\mathbf{Q} = \hat{T}\mathbf{P}$. In the case of a finite P_n, when the number of members in the distribution is limited to N, the matrix representation of the formula (3) has the form

$$\begin{pmatrix} Q_0 \\ Q_1 \\ \cdot \\ \cdot \\ Q_{N-1} \\ Q_N \end{pmatrix} = \begin{pmatrix} 1 & (1-\eta) & (1-\eta)^2 & \cdot & \cdot & (1-\eta)^N \\ 0 & \eta & 2\eta(1-\eta) & \cdot & \cdot & N\eta(1-\eta)^{N-1} \\ 0 & 0 & \eta^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \eta^{N-1} & N\eta^N(1-\eta) \\ 0 & 0 & 0 & \cdot & 0 & \eta^N \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \cdot \\ \cdot \\ P_{N-1} \\ P_N \end{pmatrix}, \tag{4}$$

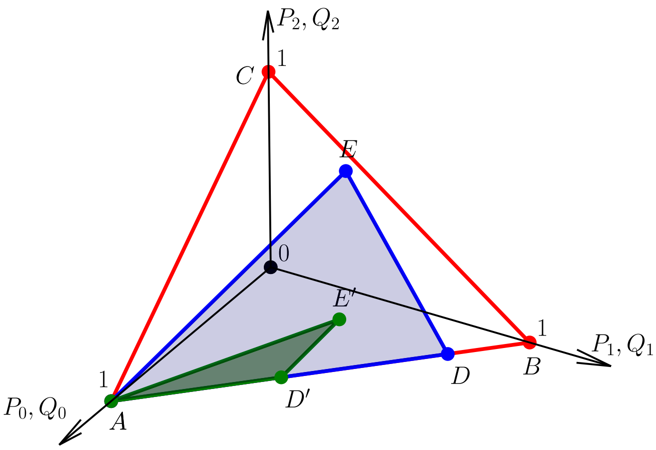

FIG. 1. Ranges of possible values of P_n (red) and Q_m for $\eta = 0.8$ (blue), and $\eta = 0.4$ (green), depicted in three-dimensional space.

or simply

$$\mathbf{Q} = \hat{T}\mathbf{P}, \tag{5}$$

where $\mathbf{Q}$ and $\mathbf{P}$ are the N-dimensional vectors. This matrix equation is easy to solve:

$$\mathbf{P} = \hat{T}^{-1}\mathbf{Q}, \tag{6}$$

where $\hat{T}^{-1}$ is an inverse matrix to $\hat{T}$.

Calculating the inverse matrix and passing back to analytical form, it is possible to show [20] that the solution to the inverse problem of photocount statistics can be represented as

$$P_n = \sum_{m=n}^{\infty} C_m^n \eta^{-n}\left(1 - \frac{1}{\eta}\right)^{m-n} Q_m. \tag{7}$$

As seen from the formula (7), the inverse matrix is triangular, like the original matrix $\hat{T}$. If the distribution Q_m is finite and no restrictions are imposed on it, then no problems with the solutions to the inverse problem arise for any η [21].

Before turning to the presentation of the main results related to the analysis of the inverse Bernoulli transform as applied to infinite distributions of photons and electrons, let us dwell on some important properties of this transform, which usually remain outside the scope of the standard consideration. First, note that vectors $\mathbf{P}$ and $\mathbf{Q}$ are the probability distributions, hence $P_n \geqslant 0$ for all n and $Q_m \geqslant 0$ for all m. Moreover, the normalization conditions must be met, i.e., $\sum_{n=0}^{N} P_n = 1$ and $\sum_{m=0}^{N} Q_m = 1$. The listed restrictions imposed on P_n and Q_m lead to the important feature of the solution (7), namely, that the transformation $\hat{T}$ turns out to be contracting (all distances shrink). If to interpret P_n and Q_m as projections of unit vectors to coordinate axes in N-dimensional space, it is convenient to illustrate a contraction action of $\hat{T}$ by depicting the ranges of possible values for P_n and Q_m in such a space. To illustrate this, refer to Fig. 1, which shows the ranges of valid values of P_n and Q_m for three-dimensional distributions. In Fig. 1 the distributions P_n and Q_m are interpreted as projections of unit vectors on the coordinate axes in three-dimensional space.

The domains of possible values of P_n and Q_m lie in the same plane, which intersects the coordinate axes at the points $A = \{1, 0, 0\}, B = \{0, 1, 0\}, C = \{0, 0, 1\}$. However, the sizes of these domains are different. Thus, the possible values of P_n fill the equilateral triangle ABC, while the possible values of Q_m lie inside the triangle ADE or $AD'E'$ of a smaller areas depending on η. Note that the coordinates D and E of the vertices of this triangle coincide with the columns of the matrix $\hat{T}$, and, therefore, depend on the η: $D = \{1 - \eta, \eta, 0\}, E = \{(1 - \eta)^2, 2\eta(1 - \eta), \eta^2\}$. If $\eta = 1$ then P and Q triangles coincide. In the case of N dimensions the reasoning is similar, but instead of triangles one should use multidimensional generalization of triangles: simplices.

This specified property of the transformation $\hat{T}$ causes potential incorrectness in the P_n reconstruction from experimentally obtained Q_m according to the formula (7). Indeed, in the case of a contraction mapping, normalization of the Q distribution does not guarantee the correctness of reconstruction since, although the normalization guarantees that the end of the vector $\mathbf{Q}$ falls on the ABC plane, it does not guarantee that it falls inside the Q-simplex. As a result, the reconstructed P_n distribution may be outside of P-simplex and you can obtain, in principle, arbitrary values of P_n, both greater than 1 and even negative values.

To illustrate this conclusion, consider an example of incorrect recovery of P_n from a prenormalized three-dimensional Q_m. Let us assume that $\mathbf{Q} = \{0, 0, 1\}$, then according to formula (7) the elements of the reconstructed distribution P_n will have the form: $P_0 = (1 - \frac{1}{\eta})^2$, $P_1 = 2\eta^{-1}(1 - \frac{1}{\eta})$, $P_2 = \eta^{-2}$. For example, if $\eta = 0.5$ then $\mathbf{P} = \{1, -4, 4\}$. Note that, despite the obtained negative value of P_1, the normalization of the distribution P_n remains.

This feature of the solution (7) becomes critical in the processing of experimental data and, no doubt, requires the development of special methods which would not allow the experimental values to go beyond the boundaries of the Q-simplex. However, this problem is beyond the scope of the questions discussed in this article and we will not touch on this issue further.

III. CRITERION FOR THE EXISTENCE OF A SOLUTION TO THE INVERSE PROBLEM OF PHOTOCOUNT STATISTICS FOR INFINITE DISTRIBUTIONS

In most physical problems, the photocount statistics is described by infinite distributions. As mentioned in the Introduction, when studying the properties of the solution to the inverse problem of photocounts by the method of inverse Bernoulli transformation in the case of infinite distributions, a possible instability of the behavior of the solution (7) was found for quantum efficiency values $\eta < 0.5$. At the same time, in the case of finite distributions, no peculiarities in the behavior of solutions were noted. If we trace the derivation of the formula (7), then at first glance, no mathematical reasons for the appearance of the instability threshold are visible. Indeed, let us describe in more detail the procedure for finding a solution to the inverse problem in matrix form (5). Multiplying both sides of the equation (5) on the left by $\hat{T}^{-1}$, we obtain $\hat{T}^{-1}\mathbf{Q} = \hat{T}^{-1}\hat{T}\mathbf{P}$. Considering that $\hat{T}^{-1}\hat{T} = 1$, we directly arrive at the solution of the inverse problem: $\mathbf{P} = \hat{T}^{-1}\mathbf{Q}$. However, if we look closely at the above derivation, we can see that the derivation implicitly assumed the associativity of matrix multiplication $\hat{T}^{-1}$, $\hat{T}$, and $\mathbf{P}$. However, as follows from functional analysis, the product of infinite matrices is generally not associative, i.e., $\hat{T}^{-1}(\hat{T}\mathbf{P})$ is not always equal to $(\hat{T}^{-1}\hat{T})\mathbf{P}$. It follows that the solution to the problem formulated in the introduction is not to find the stability regions of the solution (6), but to determine the ranges of η for which the matrix product $\hat{T}^{-1}\hat{T}\mathbf{P}$ is associative, i.e., when the solution of the inverse problem can in principle be written as (6).

As an associativity criterion, one can choose the conditions for the convergence of the series that determine the elements of the products of matrices. From a technical point of view, it is easier to examine not the associativity of matrices, but to examine for convergence the final solution (7), which from a mathematical point of view can be viewed as a countable set of series. Therefore, the solution exists if all series (7) converge for any n.

Below are the results of studying the existence of a solution to the inverse problem in the form (6) for various distributions of photons. Let us choose from the solution (7) an arbitrary series with number n and rewrite it in the equivalent form

$$P_n = (\eta - 1)^{-n} \sum_{m=n}^{\infty} (-1)^m \left(\frac{1}{\eta} - 1\right)^m C_m^n Q_m, \tag{8}$$

where it can be seen that it is an alternating series. Denoting $a_{nm} = (\frac{1}{\eta} - 1)^m C_m^n Q_m$, we can write it in a more compact form

$$P_n = (\eta - 1)^{-n} \sum_{m=n}^{\infty} (-1)^m a_{nm}. \tag{9}$$

An interesting feature of the solution (8) is the presence of a critical value of the quantum detection efficiency $\eta_{\mathrm{cr}} = 0.5$. You can see this by looking at the structure of the sequence a_{nm}, which can be considered as the product of two sequences $a_{nm}^{(1)} = (\frac{1}{\eta} - 1)^m C_m^n$ and $a_m^{(2)} = Q_m$. Because the series $\sum_{m=n}^{\infty} Q_m$ converges for any distributions Q_m due to the normalization condition, then by Abel convergence criterion it is sufficient for the convergence of the series (8) that the sequence $a_{nm}^{(1)}$ is monotonic and limited. The sequence $a_{nm}^{(1)}$ starting from some number m is monotonic, therefore, for its boundedness it is sufficient that it converges to 0. As follows from the explicit form of $a_{nm}^{(1)}$, for $\eta > 0.5$ $\lim_{m\to\infty} a_{nm}^{(1)} = 0$, and for $\eta < 0.5$ $\lim_{m\to\infty} a_{nm}^{(1)} = \infty$. This implies that for $\eta > 0.5$ the series (8) converges for any distribution Q_m, and for $\eta < 0.5$ the convergence of the series (8) depends on the type of distribution Q_m.

Now we can find the criterion that the distribution Q_m must satisfy to ensure the convergence of the series (8) for $\eta < 0.5$. The series (8) will converge for any n if the Leibniz criterion is fulfilled for partial series starting from some finite indexes $M_n \in \mathbb{N}, M_n < \infty$ possible dependent from n, i.e., for any n the sequence $a_{nm}, m \geqslant M_n$ monotonically tends to zero as $m \to \infty$. Noticing that

$$a_{n,m+1} = a_{nm} \frac{m+1}{m-n+1} \left(\frac{1}{\eta} - 1\right) \frac{Q_{m+1}}{Q_m}, \tag{10}$$

we obtain the condition of monotonicity for the sequence a_{nm} in the form

$$Q_{m+1} < Q_m \left(1 - \frac{n}{m+1}\right) \frac{\eta}{1-\eta}. \tag{11}$$

The relation (11) can be regarded as a criterion for the convergence of the series (7) for an arbitrary n. To prove its necessity we can form a proof by contradiction.

If the criterion is not necessary, we can find a converges series violated it. Let us assume that the series (9) converges for a fixed n. If the relation (11) is not met for a finite subset of indices m, we can fix the lower bound M_n outside of this subset. So the relation (11) must not met for all m. Obviously, that the mildest violation of the criterion is achieved if

$$Q_{m+1} = Q_m \left(1 - \frac{n}{m+1}\right) \frac{\eta}{1-\eta}. \tag{12}$$

Let us simplify the formula (9) using the equality (12) as a recursive relation

$$P_n = \eta^{-n} Q_n \sum_{m=n}^{\infty} (-1)^m C_m^n \left(1 - \frac{n}{m+1}\right)^{m-n}. \tag{13}$$

To test the divergence of the series (13) we can use nth-term test for absolute values of its terms. It is simple to calculate that

$$\lim_{m\to\infty} C_m^n \left(1 - \frac{n}{m+1}\right)^{m-n} = \infty. \tag{14}$$

So the series (13) diverges in contradiction to our initial guess and the relation (11) is a sufficient and necessary condition for the convergence of the series (7).

In relation to the question of the existence of the inverse problem solution in the form (7) this existence criterion should be understood as follows. If for all n for a given η it is possible to find a finite $m = M_n$, starting from which the condition (11) is satisfied, then the solution to the inverse problem exists. Note that it follows from the obtained criterion that for each m there exists some η_{cr} below which the solution in the form of the inverse Bernoulli transform does not exist.

IV. EXAMPLES OF APPLICATION OF THE EXISTENCE CRITERION

A. Poisson distribution

Let Q_m be the Poisson distribution

$$Q_m = \frac{(\overline{m})^m}{m!} e^{-\overline{m}}, \tag{15}$$

where $\overline{m}$ is the mean number of photocounts.

The stability criterion (11) for the distribution (15) is written as

$$\frac{(\overline{m})^{m+1}}{(m+1)!} e^{-\overline{m}} < \left(1 - \frac{n}{m+1}\right) \frac{\eta}{1-\eta} \frac{(\overline{m})^m}{m!} e^{-\overline{m}}. \tag{16}$$

From the relation (16) it follows that the inequality is satisfied if $m > M_n = n - 1 + (1-\eta)\eta^{-1}\overline{m}$. Hence it follows that for any given n and η there exists M_n, starting from which the existence criterion is fulfilled. It means that in the case of the Poisson distribution the solution obtained by inverse Bernoulli transform exists for arbitrary η.

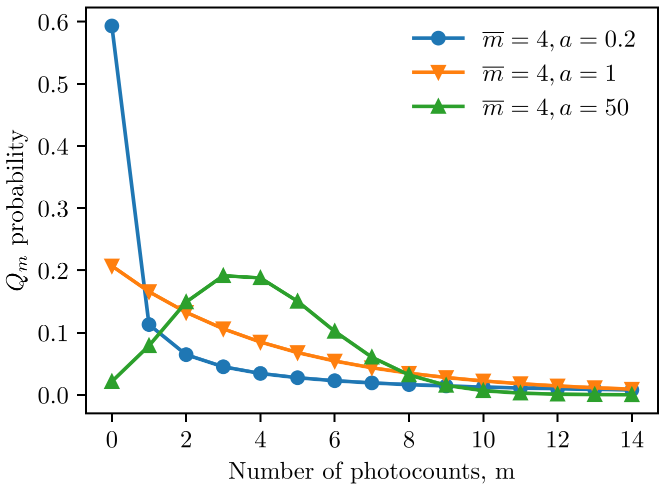

FIG. 2. Compound Poisson distributions with $\overline{m} = 4$, $a = 0.2, 1, 50$.

B. Compound Poisson distribution

Let Q_m be the compound Poisson distribution

$$Q_m = \frac{\Gamma(a+m)}{m!\Gamma(a)}\left(\frac{\overline{m}}{a}\right)^m \frac{1}{(1+\overline{m}/a)^{m+a}}, \quad (17)$$

where $\overline{m}$ is the mean number of photocounts, a is a clusterization (or bunching) parameter. Using (17) we can describe a wide class of photocount distributions [35]. As shown from Fig. 2, this distribution strongly depends on the a value. If $a \to \infty$ it goes to the Poisson distribution, if $a = 1$ it coincides with the thermal one. Also, it has a physical meaning if $0 < a < 1$ and for negative integers if $\overline{m} \leqslant -a$. However, for negative a the distribution becomes finite and as shown above all series (7) converge. So problems of convergence can arise only for $a > 0$.

Writing down the existence criterion (11) for the distribution (17) and taking into account that $\Gamma(a+m+1) = (a+m)\Gamma(a+m)$, we arrive to the inequality

$$m\left(\frac{\eta}{1-\eta} - \frac{\overline{m}}{a+\overline{m}}\right) > \frac{\eta(n-1)}{1-\eta} + \frac{\overline{m}a}{a+\overline{m}}. \quad (18)$$

Further transformation of inequality (18) depends on the sign of $\xi = \eta(1-\eta)^{-1} - \overline{m}(a+\overline{m})^{-1}$.

If $\xi > 0$, then the inequality (18) can be written as

$$m > M_n = \left(\frac{\eta(n-1)}{1-\eta} + \frac{\overline{m}a}{a+\overline{m}}\right)\left(\frac{\eta}{1-\eta} - \frac{\overline{m}}{a+\overline{m}}\right)^{-1}. \quad (19)$$

Note that for $n = 0$ the right-hand side of the inequality (19) for small a can be negative, but it will hold for any m and we can put $M_n = 0$.

If $\xi < 0$, then the inequality (18) leads to an upper constraint of m, which means that M_n does not exist. Thus, the condition for the existence of M_n is the fulfillment of the condition $\xi > 0$, from which it immediately follows that

$$\eta_{\rm cr} = \left(\frac{a}{\overline{m}} + 2\right)^{-1}. \quad (20)$$

As is known, the compound Poisson distribution transforms into the usual Poisson distribution for $a \to \infty$. With this a the critical quantum efficiency $\eta_{\rm cr} = 0$, i.e., the solution exists for any η, which coincides with the conclusion obtained above when analyzing the usual Poisson distribution.

The expression (20) also generalizes the special case of the thermal distribution given in [21] as an example of the possibility of the existence of unstable solutions. The compound Poisson distribution transforms into a thermal distribution at $a = 1$. Substituting this value into (20) we obtain $\eta_{\rm cr} = (\overline{m}^{-1} + 2)^{-1}$, which coincides with the value obtained in [21] from fundamentally different considerations. This analysis additionally confirms the correctness of the existence criterion obtained in this paper.

V. CONCLUSION

Nonideal quantum efficiency $\eta < 1$ leads to a number of problems when trying to reconstruct photon-number distribution P_n from the measured photocount distribution Q_m for an unbounded number of photons. These problems must be taken into account in practical implementation of the recovery procedure. In the present paper we showed that, for $\eta < 0.5$ in the case of an infinite distribution of photons, the problems associated with solving the inverse problem of photocount statistics in the form (6) do not reside in the instability of the behavior of its solution, but in the fact that a solution in the form of the inverse Bernoulli transform does not always exist.

Also, we found the criterion that allows us to determine whether or not there exists a solution to the inverse problem of photocount statistics in the form of the inverse Bernoulli transform for arbitrary types of infinite photocount distributions for any $\eta \in (0, 1]$. According to the obtained criterion, it becomes possible for each Q_m to determine the minimum possible detector quantum efficiency $\eta = \eta_{\rm cr}$, below which the reconstruction of the infinite P_n distribution by inverse Bernoulli transform becomes impossible. This conclusion is also important for experimental applications of the method. The found criterion limits a scope of good conditioning of the inverse problem in the case of experimentally obtained finite photocount statistics with statistical uncertainty. If the criterion is met, the problem is well conditioned, otherwise it is ill-conditioned.

Insights obtained here for finite distributions are also important since they should be useful for the development of stable numerical methods for solving the inverse problem of photocount statistics for finite photon-number distributions. The obtained results show that normalization of the experimentally obtained distribution of photocounts can be insufficient to obtain the correct solution using the inverse Bernoulli transform. It is necessary to additionally ensure that probabilities Q_m lie inside the Q-simplex. This condition indicates a different way of developing algorithms for solving the inverse problem of photocount statistics.

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