Perturbative model of noisy quantum signal processing

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(Received 22 February 2023; accepted 6 April 2023; published 24 April 2023; corrected 27 July 2023)

Recent progress in quantum signal processing (QSP) and its generalization, quantum singular value transformation, has led to a grand unification of quantum algorithms. However, inherent experimental noise in quantum devices severely limits the length of realizable QSP sequences. We consider a model of QSP with generic perturbative noise in the signal processing basis and present a diagrammatic notation useful for analyzing such errors. To demonstrate our technique, we study a specific coherent error, that of under- or overrotation of the signal processing operator parametrized by $\epsilon \ll 1$. For this coherent error model, it is shown that while Pauli Z errors are not recoverable without additional resources, Pauli X and Y errors can be arbitrarily suppressed by coherently appending a noisy recovery QSP without the use of additional resources or ancillas. Furthermore, through a careful accounting of errors using our diagrammatic tools, we provide an upper and lower bound on the length of this recovery QSP operator. We anticipate that the perturbative technique and the diagrammatic notation proposed here will facilitate future study of generic noise in QSP and quantum algorithms.

DOI: 10.1103/PhysRevA.107.042429

I. INTRODUCTION

Quantum signal processing (QSP) [1,2] and its generalization, the quantum singular value transform (QSVT), have provided a framework unifying many important quantum algorithms [3,4]. Under this framework, Grover's search [5,6], the quantum Fourier transform [7] (the basis of Shor's factoring algorithm [8]), and quantum simulation algorithms [4,9–12] are all described by interleaving sequences of block-encoded signal rotations and single-qubit signal processing rotations.

However, unless the QSVT operators are constructed on top of a fault-tolerant quantum computer [13–17], inherent experimental noise in quantum devices limits the length of realizable QSP and QSVT sequences. Even with a fault-tolerant quantum computer, errors may still arise due to inherent approximations and truncation made in constructing the block encoding of the subsystem of interest [18,19]. These observations lead to an important question: How does one correct errors in a typical QSVT sequence? Of course one may employ existing gate-level error correction methods to every gate in a QSVT circuit, but the unifying perspective of representing quantum operations as polynomial transformations offers an entirely new possibility for studying error correction at the level of the algorithm.

To motivate the study of error correction at the algorithm level, we introduce a noise model for QSP describing a generic perturbative noise on the signal processing operation. Our paper [20] expands on the concept of algorithm-level error correction, using the specific coherent error of Sec. IV as an example, and provides numerical results including an application to a modified Grover fixed-point amplification algorithm. Here we provide a full derivation of the results stated in [20] using a diagrammatic notation and introduce a broader class of errors where this notation is useful. This general error model has the advantage of being able to additionally capture incoherent errors.

The rest of the paper is organized as follows. In Sec. II we review the QSP framework for quantum algorithms, introduce our general error model, and set up some useful nomenclature. In Sec. III we introduce our diagrammatic notation. To demonstrate the utility of our diagrammatic notation, we introduce a specific model of coherent error in Sec. IV using the notation to construct a scheme for error recovery. We conclude the paper with a discussion of incoherent errors and directions for future work in Sec. V.

II. PRELIMINARIES

We start in Sec. II A with a brief review of QSP and with a specification of the conventional choices made in this paper. This is followed in Sec. II B with the introduction of our model of signal processing noise. Finally, we define the error channel in Sec. II C and introduce notation for the perturbative decomposition of its Kraus operators in Sec. II D.

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A. Quantum signal processing

A number of conventions exist in the literature surrounding QSP primarily differing in the choice of bases and signal operators. We specify our choices below, which correspond to the "Wx" convention of [4].

A length-d QSP operator is parametrized by phases $\vec{\phi} = (\phi_0, \ldots, \phi_d) \in \mathbb{R}^{d+1}$,

$$U_0(\theta; \vec{\phi}) = \text{QSP}(\theta; \vec{\phi}) \equiv e^{i\phi_0 Z} \prod_{j=1}^{d} W(\theta) e^{i\phi_j Z}, \quad (1)$$

where the signal operator is a rotation in the X basis

$$W(\theta) \equiv e^{i\theta X} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \quad (2)$$

and X, Y, and Z are the Pauli matrices. The subscript 0 on U_0 is used to indicate that the QSP is noiseless. A general length-d QSP sequence takes the form

$$U_0(\theta; \vec{\phi}) = \begin{pmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{pmatrix}, \quad (3)$$

where $a \equiv \cos\theta$ and $P, Q \in \mathbb{C}[a]$ are polynomials such that (see Theorem 4 in [3]) (i) $\deg(P) \leqslant d$ and $\deg(Q) \leqslant d-1$, (ii) P has parity (d mod2) and Q has parity ($d-1$ mod2), and (iii) $|P(a)|^2 + (1-a)^2|Q(a)|^2 = 1 \,\forall\, a \in [-1, 1]$. We write $U = [\![P_U, Q_U]\!]$ as a shorthand for Eq. (3), dropping the subscripts when the QSP unitary is clear by context.

B. Noise model

We consider a generic noise in the signal processing basis. For length-d QSP with signal processing rotations indexed by $0 \leqslant j \leqslant d$, the error of a single signal processing rotation j is characterized by a set of Kraus operators $\{N_\epsilon^{(j,i_j)}\}$ for $i_j \in \{1, \ldots, M_j\}$, where M_j is the number of Kraus operators describing the channel at site j. In this work we only consider errors in the signal processing basis with Kraus operators of the the form

$$N_\epsilon^{(j,i_j)} = w_\epsilon^{(j,i_j)}(\phi_j) I + i z_\epsilon^{(j,i_j)}(\phi_j) Z \quad (4)$$

for complex functions $w_\epsilon^{(j,i_j)}$ and $z_\epsilon^{(j,i_j)}$ satisfying the completeness condition $\sum_{i_j} N_\epsilon^{(j,i_j)\dagger} N_\epsilon^{(j,i_j)} = I$ for all j.

Note that, in general, we allow the Kraus operators to depend on ϕ_j, but they are assumed to be independent of θ. Further, to allow a perturbative analysis, we assume each Kraus operator also depends on a parameter $\epsilon \ll 1$ and that all $N_\epsilon^{(j,i_j)}$ approach a value proportional to identity as $\epsilon \to 0$.

Given input ρ, such a noisy QSP produces an output state

$$\rho'_\epsilon = \sum_{i_0, \ldots, i_d} \left(e^{i\phi_0 Z} N_\epsilon^{(0,i_0)} \prod_{j=1}^{d} W(\theta) e^{i\phi_j Z} N_\epsilon^{(j,i_j)} \right)^\dagger \rho \left(e^{i\phi_0 Z} N_\epsilon^{(0,i_0)} \prod_{j=1}^{d} W(\theta) e^{i\phi_j Z} N_\epsilon^{(j,i_j)} \right), \quad (5)$$

which is depicted in Fig. 1.

C. Error channel

We express the result of applying the entire noisy QSP sequence as a channel

$$\mathcal{U}_\epsilon(\theta; \vec{\phi})(\rho) = \rho'_\epsilon, \quad (6)$$

with Kraus operators

$$M_\epsilon^{(i_0,\ldots,i_d)} = e^{i\phi_0 Z} N_\epsilon^{(0,i_0)} \prod_{j=1}^{d} W(\theta) e^{i\phi_j Z} N_\epsilon^{(j,i_j)}. \quad (7)$$

In order to isolate the effect of the error, we define the error channel that produces an erroneous state ρ'_ϵ from the ideal result ρ'_0,

$$\mathcal{E}_\epsilon(\rho'_0) = \mathcal{E}_\epsilon(U_0^\dagger \rho U_0) \equiv \rho'_\epsilon. \quad (8)$$

Combining Eqs. (6)–(8), we find that the error channel can be written with Kraus operators

$$E_\epsilon^{(i_0,\ldots,i_d)} = U_0^\dagger e^{i\phi_0 Z} N_\epsilon^{(0,i_0)} \prod_{j=1}^{d} W(\theta) e^{i\phi_j Z} N_\epsilon^{(j,i_j)}. \quad (9)$$

The error channel has the benefit of being nearly the identity channel in the perturbative regime, that is, as $\epsilon \to 0$, we have $\mathcal{E} \to \text{id}$. As a result, all of its its Kraus operators approach a value proportional to the identity matrix and we can write its Kraus operators in the form

$$\alpha^{(i_0,\ldots,i_d)} I + \epsilon A^{(i_0,\ldots,i_d)} + O(\epsilon^2) \quad (10)$$

for $\alpha \in \mathbb{R}$, where A is an operator of the following form.

Definition 1 (standard form, first order). We say an operator A is in first-order standard form of degree $2d$ if it is written as a weighted sum over QSP operators generated by conjugation of $e^{i(\pi/2)Z}$,

$$\begin{aligned} A = {} & \beta_d \times \text{QSP}(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \ldots, -\phi_2, -\phi_1, \pi, \phi_1, \phi_2, \ldots, \phi_{d-1}, \phi_d)) \\ & + \beta_{d-1} \times \text{QSP}(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \ldots, -\phi_2, \pi, \phi_2, \ldots, \phi_{d-1}, \phi_d)) + \cdots \\ & + \beta_1 \times \text{QSP}(\theta; (-\phi_d - \pi/2, \pi, \phi_d)) + \beta_0 \times \text{QSP}(\theta; (\pi/2)). \end{aligned} \quad (11)$$

where $\beta_i \in \mathbb{R}$ and $\phi_i \in \mathbb{R}$.

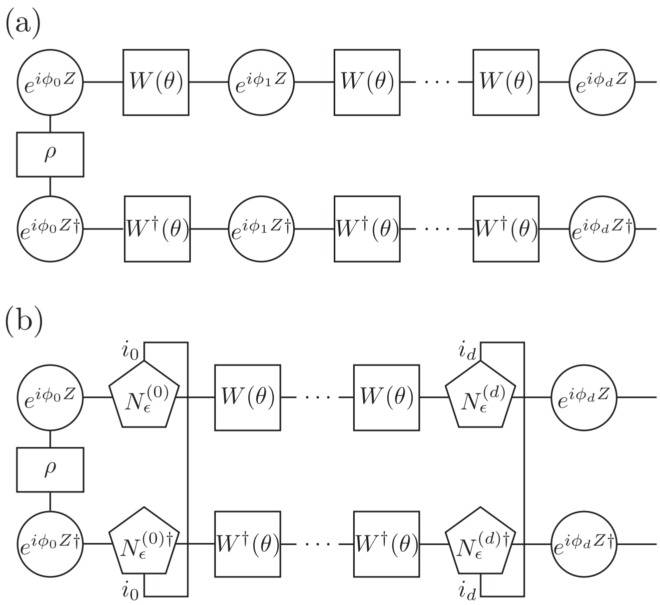

FIG. 1. Tensor diagram depicting the result of (a) a noiseless and (b) a noisy length-d QSP operator parametrized by phases $\phi_0, \dots, \phi_d \in \mathbb{R}$ on input state ρ. The noise on the signal processing rotations is characterized by Kraus operators $N_\epsilon^{(j,i_j)}$ for $0 \leqslant j \leqslant d$ and is contracted over the Kraus operator index i_j. In anticipation of the notation of Sec. III, we use circles to denote signal processing rotations and squares to denote signal rotations; additionally, pentagons are used to denote Kraus operators and rectangles are used for density matrices.

The component QSPs in the sum of Definition 1 take on a special form.

Definition 2 (error component, first order). Let U be length-d QSP operator parametrized by real phases $(\phi_0, \dots, \phi_d)$. Then a length-$2r$ QSP V with $1 \leqslant r \leqslant d$ is said to be a first-order error component of QSP U if it can be written in the form

$$V = \mathrm{QSP}(\theta; (-\phi_d - \pi/2, \dots, -\phi_{d-r+1}, \pi, \phi_{d-r+1}, \dots, \phi_d)). \tag{12}$$

Furthermore, it is assumed that no ϕ_i for $i < d$ is a half-integer multiple of π; otherwise, we can perform elision to simplify the diagram.

A generic error channel Kraus operator is depicted in Fig. 1 using the notation of Sec. III.

D. Canonical profile

It will often be useful to write an operator in the basis of Pauli matrices. The coordinates of noisy QSP operators in the Pauli basis will generically be polynomials of $\cos\theta$.

Certain operators, such as the Kraus operators of noisy QSP error channels subject to noise of the form in Eq. (4), can be written in a special form. We say that an operator $U_\epsilon(\theta)$ admits a canonical expansion if we can write $U_\epsilon = w_\epsilon(\theta)I + i[x_\epsilon(\theta)X + y_\epsilon(\theta)Y + z_\epsilon(\theta)Z]$ for functions w_ϵ, x_ϵ, y_ϵ, and z_ϵ of the form

$$w_\epsilon(\theta) = \cos^2\theta \sum_{k=0}^{\infty} \epsilon^k \sum_{j=-1}^{\infty} \mathcal{P}_j^{(0,k)} \cos^{2j}(\theta), \tag{13}$$

$$x_\epsilon(\theta) = \sin(2\theta) \sum_{k=0}^{\infty} \epsilon^k \sum_{j=0}^{\infty} \mathcal{P}_j^{(x,k)} \cos^{2j}(\theta), \tag{14}$$

$$y_\epsilon(\theta) = \sin(2\theta) \sum_{k=0}^{\infty} \epsilon^k \sum_{j=0}^{\infty} \mathcal{P}_j^{(y,k)} \cos^{2j}(\theta), \tag{15}$$

$$z_\epsilon(\theta) = \cos^2\theta \sum_{k=0}^{\infty} \epsilon^k \sum_{j=-1}^{\infty} \mathcal{P}_j^{(z,k)} \cos^{2j}(\theta), \tag{16}$$

and $\mathcal{P}_j^{(\sigma,k)} \in \mathbb{R}$ for all $\sigma \in \{0, x, y, z\}$ and $j, k \in \mathbb{Z}$. We call $\mathcal{P}$ the canonical profile of U_ϵ. For convenience, we allow $j \in \mathbb{Z}$ and define $\mathcal{P}_j^{(x,k)} = \mathcal{P}_j^{(y,k)} = \mathcal{P}_{j-1}^{(z,k)} = 0$ for all $j < 0$ and k.

Our parametrization, particularly the choice of factoring out $\sin(2\theta)$ from the X and Y components, $\cos^2\theta$ from the Z component, and starting the sum of the Z component at $j = -1$, is tailored to the diagrams which appear in the Kraus operators of QSP error channels (we leave the proof of this to Appendix A). Note that for unitary U_ϵ, the functions satisfy the completeness relationship $w_\epsilon(\theta)^2 + x_\epsilon(\theta)^2 + y_\epsilon(\theta)^2 + z_\epsilon(\theta)^2 = 1$, which holds for all θ and ϵ.

III. DIAGRAMMATIC NOTATION

In this section we develop a diagrammatic notation for visualizing quantum signal processing unitaries and demonstrate their utility for reasoning about the Kraus operators of noisy QSP channels. First, we prove a number of results to motivate the notation and provide a number of basic manipulations.

For QSP U of length-d parametrized by $\vec{\phi} \equiv (\phi_0, \dots, \phi_d)$, we will use $\vec{\phi}_{i:j}$ to denote the subsequence $(\phi_i, \dots, \phi_j)$ and $U^{(j)}$ to denote the length-j QSP parametrized by $\vec{\phi}_{0:j}$.

Lemma 1 (unit steps). Let $U_0 = [\![P, Q]\!]$ be a length-d QSP unitary and $k = \deg(P) \leqslant d$. The unitary $U_0' = U_0 e^{i\phi_0 Z} W e^{i\phi_1 Z} = [\![P', Q']\!]$ is a length-$(d+1)$ QSP unitary with either $\deg(P') = k - 1$ or $\deg(P') = k + 1$.

Proof. Computing the product, we find

$$P'(a) = e^{i(\phi_0+\phi_1)}[aP(a) - (1 - a^2)Q(a)e^{-2i\phi_0}]. \tag{17}$$

Since $\deg(P) = k$ by assumption and $\deg(Q) = k - 1$, it must be that $\deg(P') \leqslant k + 1$.

Next we prove that $\deg(P') \geqslant k - 1$ by contradiction. Assume that $\deg(P') = w < k - 1$. We can then iterate our construction above choosing $U_0'' = U_0' e^{-i[\phi_1 - (\pi/2)]Z} W e^{-i[\phi_0 - (\pi/2)]Z}$ and, by the above argument, we have $\deg(P'') \leqslant w + 1 < k$. However, we have chosen the additional phases such that $U_0'' = U_0 I_0$, where $I_0 = \mathrm{QSP}(\theta; (\phi_0, \pi/2, -\phi_0 + \pi/2))$ is an unbiased operator, i.e., $I_0 = I$. Therefore, $\deg(P'') = k$. This is a contradiction and so it must be that $\deg(P') \geqslant k - 1$.

Furthermore, we have that $\deg(P') \neq k$ by parity constraints. Therefore, Lemma 1 follows. ■

Definition 3 (QSP degree peak). Let R be an unbiased QSP sequence of length $d \geqslant 2$ parametrized by $(\phi_0, \dots, \phi_d) \in$

$\mathbb{R}^{d+1}$. Suppose for some $0 < i < d$ we have $\deg(P_{R^{(i)}}) = r + 1$ and $\deg(P_{R^{(i-1)}}) = \deg(P_{R^{(i+1)}}) = r$. We will call position i a degree peak of R.

Lemma 2 (QSP elision). Let R be a QSP operator of length $d \geqslant 2$ parametrized by $(\phi_0, \dots, \phi_d)$. Position i is a degree peak of R if and only if $\phi_i = \pi(n + \frac{1}{2})$ for some $n \in \mathbb{Z}$.

Additionally, R is equivalent to a length-$(d-2)$ QSP parametrized by phases

$$(\phi_0, \dots, \phi_{i-2}, \phi_{i-1} + \phi_i + \phi_{i+1}, \phi_{i+2}, \dots, \phi_d). \tag{18}$$

We refer to this transformation as QSP elision.

Proof. Writing out the product, we find the following relationship between QSP polynomials of $R^{(i-1)}$ and $R^{(i)}$:

$$P_{R^{(i)}} = a e^{i(\phi_{i-1}+\phi_i)}(P_{R^{(i-1)}} + e^{-2i\phi_{i-1}} Q_{R^{(i-1)}}) + \Theta(a^{r-1}). \tag{19}$$

By assumption $\deg(P_{R^{(i)}}) = r + 1$ and therefore $\deg(P_{R^{(i-1)}} + e^{-2i\phi_{i-1}} Q_{R^{(i-1)}}) = \deg(P_{R^{(i)}}) - 1 = r$.

Writing out the product for $R^{(i+1)}$, we find

$$P_{R^{(i+1)}} = 2a^2 \cos(\phi_i) e^{i(\phi_{i-1}+\phi_{i+1})}(P_{R^{(i-1)}} + e^{-2i\phi_{i-1}} Q_{R^{(i-1)}}) + \Theta(a^r). \tag{20}$$

Looking at the equation above and comparing with the $P_{R^{(i)}}$, we find that if $\deg(P_{R^{(i+1)}}) = r$, it must be that $\cos(\phi_i) = 0$ or equivalently $\phi_i = \pi(n + \frac{1}{2})$ for some $n \in \mathbb{Z}$. The converse is also true.

Assuming $\phi_i = \pi(n + \frac{1}{2})$, we find

$$P_{R^{(i+1)}} = e^{i(\phi_{i-1}+\phi_i+\phi_{i+1})} P_{R^{(i-1)}}. \tag{21}$$

This transformation is equivalent to a Z rotation by $\phi_{i-1} + \phi_i + \phi_{i+1}$ (a length-0 QSP). We can therefore elide the original QSP sequence by replacing the three phases ϕ_{i-1}, ϕ_i, and ϕ_{i+1} with a single phase $\phi_{i-1} + \phi_i + \phi_{i+1}$, thus proving Lemma 2. ∎

A number of useful corollaries follow from Lemma 2 including the construction of inverse QSP operators.

Corollary 1 (inverse QSPs). Let U be length-d QSP operator parametrized by phases $(\phi_0, \dots, \phi_d)$. The length-d QSP U' parametrized by $(-\phi_d + \frac{\pi}{2}, -\phi_{d-1}, \dots, -\phi_1, -\phi_0 - \frac{\pi}{2})$ is the inverse QSP sequence in the sense that $UU' = U'U = I$.

Additionally, Lemma 2 gives us the following uniqueness result for QSP parametrization.

Corollary 2 (uniqueness of QSP parametrization). Let $U = \mathrm{QSP}(\theta; \vec{\phi})$ be a length-d QSP and let $V = \mathrm{QSP}(\theta; \vec{\psi})$ be a length-d' QSP. Further assume such that no phase ϕ_i or ψ_j is a half-integer multiple of π. Then $U = e^{i\chi} V$ for some global phase $\chi \in [0, 2\pi)$ if and only if $d = d'$ and for all $0 \leqslant i \leqslant d$, $\psi_i - \phi_i = \pi n_i$ for some $n_i \in \mathbb{Z}$. Furthermore, either $\chi = 0$ or $\chi = \pi$.

Proof. The $\Longleftarrow$ direction is a straightforward consequence of $e^{i\pi Z} = -I$ and so we focus on $\Rightarrow$.

Since by assumption neither $\vec{\phi}$ nor $\vec{\psi}$ contains a half-integer multiple of π, Lemma 2 implies that neither contains any degree peaks and therefore $\deg(P_U) = d$ and $\deg(P_V) = d'$. As a result, the QSP unitaries must be of the same length $U = V \Rightarrow P_U = P_V \Rightarrow \deg(P_U) = \deg(P_V) \Rightarrow d = d'$. We can therefore limit our consideration to the case of $d = d'$.

Now we show that the phases must be equivalent up to an integer multiple of π inductively. First consider the case where $d = 0$. In this case, $U = e^{i\chi} V \Rightarrow e^{i\phi_0 Z} = e^{i\chi} e^{i\psi_0 Z} \Rightarrow \phi_0 = \psi_0 + \pi n_0$ for some $n_0 \in \mathbb{Z}$; furthermore, $\chi = 0$ for n_0 even and $\chi = \pi$ for n_0 odd. Thus Corollary 2 is satisfied for $d = 0$.

Assuming Corollary 2 for QSP unitaries of length d, we show that it holds for QSP unitaries of length $(d + 1)$. Consider QSPs $U = \mathrm{QSP}(\theta; \vec{\phi})$ and $V = \mathrm{QSP}(\theta; \vec{\psi})$, each of length $(d + 1)$ satisfying the conditions of Corollary 2. Given that $U = V$, we reduce U to a length-d QSP by right multiplying both sides by a QSP inverse (Corollary 1) of its final signal processing step $(We^{i\phi_{d+1}Z})^{-1} = e^{-i[\phi_{d+1}-(\pi/2)]Z} W e^{-i(\pi/2)Z}$. The result is

$$U = V, \tag{22}$$

$$\Rightarrow \mathrm{QSP}(\theta; \vec{\phi}_{0:d+1}) = \mathrm{QSP}(\theta; \vec{\psi}_{0:d+1}), \tag{23}$$

$$\Rightarrow \mathrm{QSP}(\theta; \vec{\phi}_{0:d+1}) e^{-i[\phi_{d+1}-(\pi/2)]Z} W e^{-i(\pi/2)Z} = \mathrm{QSP}(\theta; \vec{\psi}_{0:d+1}) e^{-i[\phi_{d+1}-(\pi/2)]Z} W e^{-i(\pi/2)Z}, \tag{24}$$

$$\Rightarrow \mathrm{QSP}(\theta; \vec{\phi}_{0:d}) = \mathrm{QSP}(\theta; \{\psi_0, \dots, \psi_d, \psi_{d+1} - \phi_{d+1} + \pi/2, -\pi/2\}). \tag{25}$$

On the left-hand side of Eq. (25) is a QSP unitary of degree d; by the inductive hypothesis, it must be the case that the QSP unitary on the right-hand side, which is of length $(d + 2)$, is also of degree d. This is only possible if we can perform elision at the next-to-last position. By Lemma 2 this requires $\psi_{d+1} - \phi_{d+1} + \frac{\pi}{2} = \pi(n_{d+1} + \frac{1}{2})$ for some $n_{d+1} \in \mathbb{Z}$, which implies $\psi_{d+1} - \phi_{d+1} = n_{d+1}\pi$. Furthermore, we find $\chi \in \{0, \pi\}$ again by noting that $e^{i\pi Z} = -I$, thus proving the inductive step $\phi_{d+1} - \psi_{d+1} = 2\pi m_{d+1}$ and by extension Corollary 2. ∎

We can summarize the results of this section using the following diagrammatic notation. An example of such a plot is given in Fig. 2 and has several notable features.

(i) Arbitrary Z rotations are represented by open circles and we use open triangles to plot QSP phases that are a half-integer multiple of π to distinguish degree peaks.

(ii) Closed markers, as in Figs. 4 and 5, are used to indicate additional rotations by $(\pi/2)$. Markers with checkerboard fill are used to indicate (ϵ)-noisy rotations.

(iii) Signal operators are represented by solid lines.

(iv) The vertical axis is used to plot the degree of the polynomial $P_{U^{(i)}}$ at position i.

(v) By Lemma 1, each layer of signal processing either increases or decreases the degree of the polynomial $P_i(a) = \langle 0| U(\theta, \vec{\phi}_{0:i}) |0\rangle$ by exactly one.

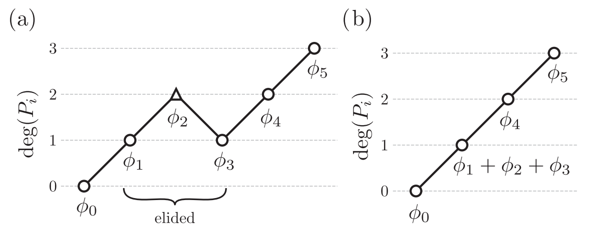

FIG. 2. Visualization of (a) a length-5 QSP sequence parametrized by $(\phi_0, \ldots, \phi_5)$ and (b) its length-3 elided form by the result of Lemma 2. We will refer to such plots in general as QSP degree plots, often omitting the vertical axis labels to improve legibility.

(vi) Finally, Lemma 2 provides us with a way of simplifying QSP diagrams with degree peaks through elision. This is depicted in Fig. 2.

A more involved application of elision can be found in Fig. 3, where the diagrammatic notation is used to represent a Kraus operator [Eq. (10)] of the error channel of Sec. II C.

IV. MODEL OF COHERENT ERROR

We consider a model of coherent errors to demonstrate the utility of the notation for reasoning about error correction.

In this error model, we assume that the signal processing operators under- or overrotate by a fixed multiplicative factor ϵ: $\phi \mapsto \phi(1+\epsilon)$ for all ϕ. While ϵ is unknown *a priori*, we assume that it is constant throughout the application of the sequence and that it is small, $\epsilon \ll 1$, so that we may expand errors in orders of ϵ. In this case, the error can be characterized by a single Kraus operator

$$N_\epsilon^{(j,1)} = e^{i\epsilon\phi_j Z} \tag{26}$$

at each site $0 \leqslant j \leqslant d$. Such an error may be due to imperfections on the hardware control and is akin to models of systematic error traditionally mitigated using composite pulses [21–23].

Using the notation developed in Sec. III, we first perform a perturbative analysis of both the error channel under this model (Sec. IV A) and the error channel of possible recovery operations (Sec. IV B). Next we show that the most general form of recovery is impossible without additional resources (Sec. IV C). Working around this constraint, we show that a weaker form of recovery is possible and provide an explicit construction along with an upper bound on the length of the recovered operator (Sec. IV D). Finally, allowing an additional assumption, we argue a lower bound on the length of recovered operator, which is tight for first-order recovery (Sec. IV F).

This section complements our paper [20], providing the full derivation of stated results using the notation introduced in Sec. III. The numbering of theorems in this section is consistent with that in [20]: Theorems 1–4 correspond to Theorems 1–4 in [20].

A. Perturbative analysis of the error channel

For this simple model of coherent errors, the QSP error channel can be characterized by a single unitary Kraus operator, which we call its error operator $E_\epsilon \equiv U_0^\dagger U_\epsilon$ (we call the canonical profile of E_ϵ the error profile). We now perform a perturbative analysis of the error operator under this noise model. Expanding a noisy Z rotation in orders of ϵ,

$$e^{i\phi(1+\epsilon)Z} = e^{i\phi Z} \sum_{k=0}^{\infty} \frac{\epsilon^k \phi^k}{k!} e^{i(\pi k/2)Z}. \tag{27}$$

Substituting into Eq. (1), we obtain

$$U_\epsilon(\theta;\vec{\phi}) = \text{QSP}_\epsilon(\theta;\vec{\phi}) \equiv \left(e^{i\phi_0 Z} \sum_{k_0=0}^{\infty} \frac{\epsilon^{k_0}\phi^{k_0}}{k_0!} e^{i(\pi/2)k_0 Z} \right) \prod_{j=1}^{d} \left[W(\theta) \left(e^{i\phi_j Z} \sum_{k_j=0}^{\infty} \frac{\epsilon^{k_j}\phi^{k_j}}{k_j!} e^{i(\pi/2)k_j Z} \right) \right]. \tag{28}$$

Rewriting in orders of ϵ,

$$\begin{aligned} U_\epsilon(\theta;\vec{\phi}) = {} & U_0 + \epsilon(\phi_0 e^{i(\phi_0+\pi/2)Z} W e^{i\phi_1 Z} W \cdots W e^{i\phi_d Z} + \phi_1 e^{i\phi_0 Z} W e^{i(\phi_1+\pi/2)Z} W \cdots W e^{i\phi_d Z} + \cdots \\ & + \phi_d e^{i\phi_0 Z} W e^{i\phi_1 Z} W \cdots W e^{i(\phi_d+\pi/2)Z}) + O(\epsilon^2). \end{aligned} \tag{29}$$

The first-order term is a sum of $d+1$ QSP unitaries, each a copy of the noiseless QSP with a $\pi/2$ overrotation at location j weighted by ϕ_j for each index j. Likewise, the kth-order term is a sum of $(d+1)^k$ QSP unitaries corresponding to all possible ways to to insert k overrotations by $\frac{\pi}{2}$ (including multiple overrotations at the same index).

To obtain the error operator E_ϵ, we left multiply by $U_0^\dagger$, which can be written as a noiseless QSP per the construction of Corollary 1. Each of these sequences can be simplified using repeated application of Lemma 2. To first order, the result is

$$\begin{aligned} E_\epsilon = {} & I + \epsilon[\phi_0 \times \text{QSP}(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \ldots, -\phi_2, -\phi_1, \pi, \phi_1, \phi_2, \ldots, \phi_{d-1}, \phi_d)) \\ & + \phi_1 \times \text{QSP}(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \ldots, -\phi_2, \pi, \phi_2, \ldots, \phi_{d-1}, \phi_d)) + \cdots \\ & + \phi_{d-1} \times \text{QSP}(\theta; (-\phi_d - \pi/2, \pi, \phi_d)) + \phi_d \times \text{QSP}(\theta; (\pi/2))] + O(\epsilon^2). \end{aligned} \tag{30}$$

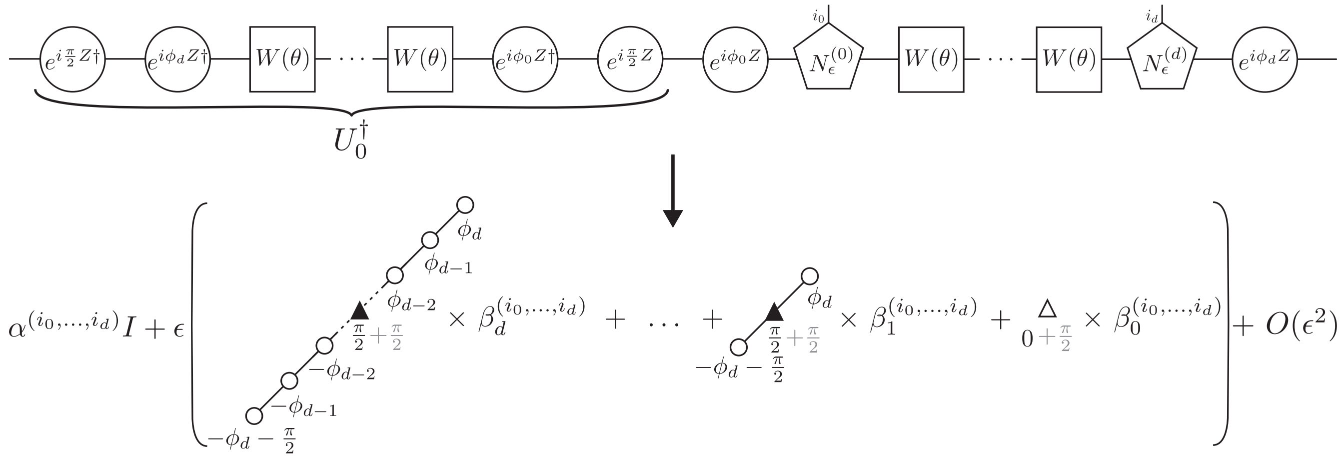

FIG. 3. QSP degree diagram decomposition of generic error channel Kraus operator for a length-d QSP parametrized by phases $\phi_0, \ldots, \phi_d \in \mathbb{R}$ in our noise model for $\alpha \in \mathbb{R}$ and all $\beta_j \in \mathbb{R}$.

Note that the first-order expansion in Eq. (30) consists of a weighted sum of even length QSP unitaries of a special form, which we generalize in Definition 1. Analogous calculations show that the higher-order terms in the expansion are likewise weighted sums over QSP unitaries of even length. Therefore, by Corollary 4, the error operator E_ϵ admits a canonical expansion. An example in diagrammatic form is provided for a general length-3 QSP in Fig. 4.

B. Perturbative analysis of recovery operators

Given a noisy QSP U_ϵ, we seek a recovery operator R_ϵ, itself a noisy QSP operator, such that their product $U_\epsilon R_\epsilon$ is "less noisy" in a sense that will be defined precisely in Sec. IV D. Since our recovery operation should leave the state unchanged (up to a global phase) as $\epsilon \to 0$, we define the natural class of degree-0 operators and perform a perturbative analysis using the diagrammatic notation of Sec. III.

Definition 4 (degree-0 operator). We call a QSP unitary U_ϵ degree 0 to order $k \geqslant 1$ if it can be written

$$U_\epsilon = e^{i[z_0 + z_1\epsilon + O(\epsilon^2)]Z + i\epsilon^k\{[x+O(\epsilon)]X + [y+O(\epsilon)]Y\}} \quad (31)$$

for some real x, y, and z_1 independent of ϵ but possibly functions of θ, and $z_0 \in \mathbb{R}$. Additionally, we call any QSP operator satisfying Eq. (31) for some $k \geqslant 1$ degree 0. Equivalently, a QSP operator U_ϵ is degree 0 if $U_0 = e^{iz_0 Z}$ for some $z_0 \in \mathbb{R}$.

Definition 5 (unbiased operator). We call a QSP unitary U_ϵ unbiased to order $k \geqslant 1$ if it is degree 0 and $U_0 = I$.

To this end, we study the properties of degree-0 QSP operators and in particular the properties of their irreducible building blocks.

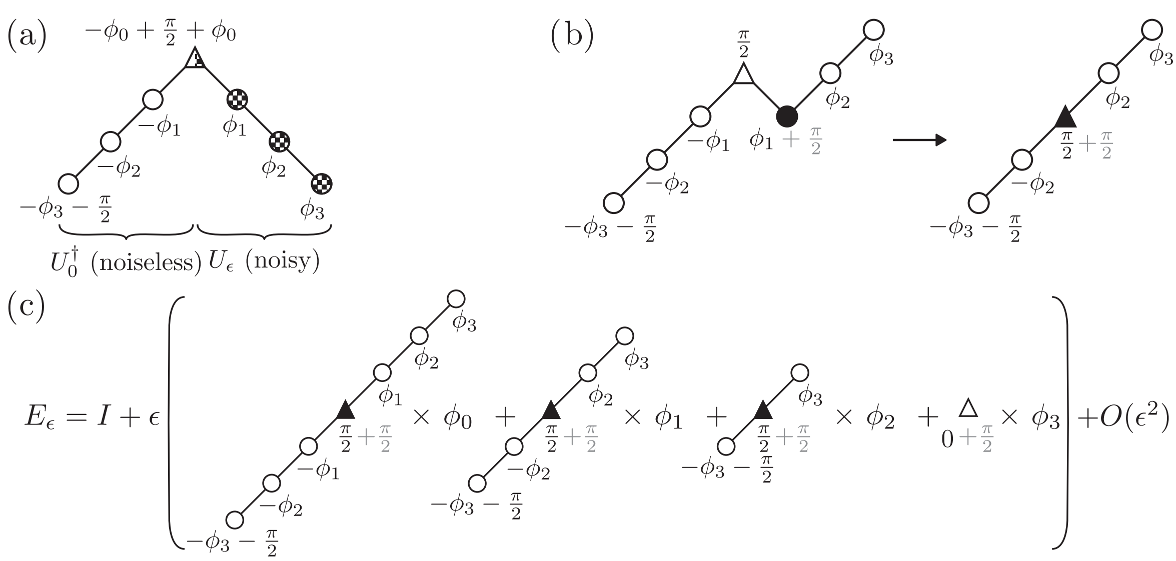

FIG. 4. Diagrammatic representation of an error operator for a length-3 QSP parametrized by $\phi_i \in \mathbb{R}$. (a) Error operator $E_\epsilon = U_0^\dagger U_\epsilon$. Checkerboard fill indicates ϵ-noisy rotations (note the peak phase is only partially noisy). (b) Analysis of one term in the first-order perturbative expansion of the error operator corresponding to an overrotation error of the ϕ_1 phase and elided form. The location of $\frac{\pi}{2}$ overrotation errors is marked by closed markers. (c) Expansion of the error operator showing all diagrams to first-order with corresponding weights.

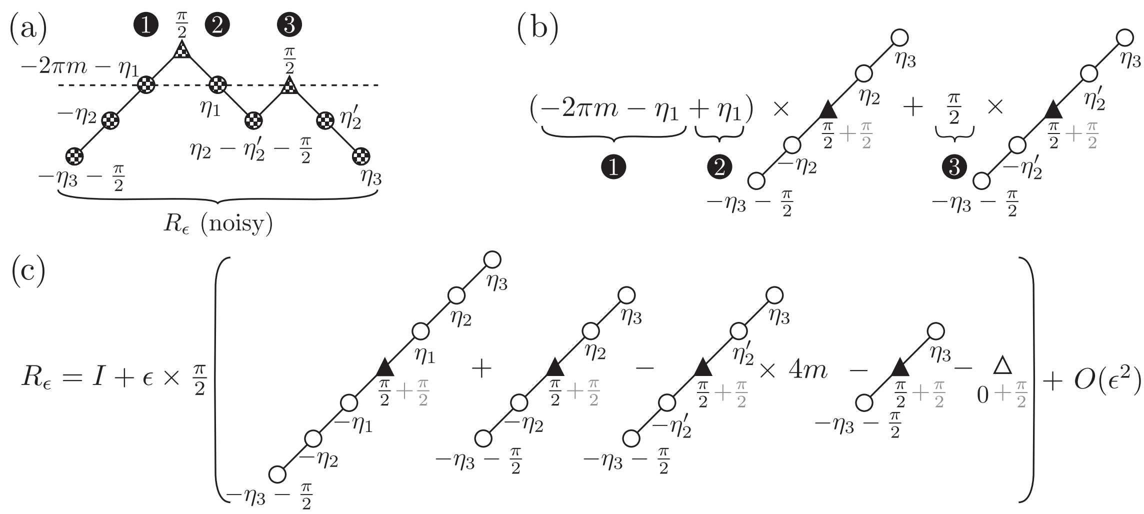

FIG. 5. Diagrammatic analysis of an irreducible degree-0 QSP parametrized by $\chi = 0$, $\eta_i \in \mathbb{R}$, and $m \in \mathbb{Z}$ (compare with Fig. 4). (a) Diagrammatic representation of an irreducible degree-0 QSP R_ϵ. A detailed analysis is performed for overrotation errors occurring at select locations (labels 1–3) corresponding to errors at degree 2 (dashed line). Checkerboard fill indicates ϵ-noisy rotations. (b) Analysis of diagrams resulting from overrotation errors at locations labeled in (a) after elision. The location of $\frac{\pi}{2}$ overrotation errors is marked by closed markers. (c) Expansion of the recovery operator showing all diagrams to first order with corresponding weights. Notice that weights are integer multiples of $\pi/2$.

Definition 6. A degree-0 QSP unitary of length d is called irreducible if $\deg(P^{(i)}) > 0$ for all $0 < i < d$; otherwise a degree-0 QSP is called reducible.

We start with the generic form of a degree-0 length-2 QSP unitary. Due to Lemma 1, all degree-0 QSP unitaries of length 2 are irreducible. Further, the following is a consequence of Lemma 2.

Corollary 3. A length-2 sequence $\text{QSP}(\theta; (\phi_0, \phi_1, \phi_2))$ is degree 0 if and only if

$$\phi_0 = \chi + \left[\phi + \pi\left(2m + n + \tfrac{1}{2}\right)\right], \tag{32}$$

$$\phi_1 = \pi\left(n + \tfrac{1}{2}\right), \tag{33}$$

$$\phi_2 = \phi \tag{34}$$

for some ϕ, $\chi \in \mathbb{R}$ and $n, m \in \mathbb{Z}$.

We can extend degree-0 QSP operations through the following operation.

Definition 7 (the conjugation superoperator). Given $\eta \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, we use $\mathcal{C}_{m,n,\eta}$ to denote the superoperator that maps a length-d sequence $\text{QSP}(\theta; \vec{\phi})$ to

$$\mathcal{C}_{m,n,\eta}\, \text{QSP}(\theta; \vec{\phi}) \equiv e^{-i[\eta + \pi(2m+n+1/2)]Z} W e^{i\pi(n+1/2)Z} \times \text{QSP}(\theta; \vec{\phi}) W e^{i\eta Z}, \tag{35}$$

which is a length-$(d+2)$ QSP sequence with phase angles $-[\eta + \pi(2m + n + \frac{1}{2})]$, $\pi(n + \frac{1}{2}) + \phi_0, \phi_1, \ldots, \phi_d$, and η.

Note that the irreducible length-2 degree-0 QSP of Corollary 3 can be written as

$$\text{QSP}(\theta; (\phi_0, \phi_1, \phi_2)) = e^{i\chi Z} \mathcal{C}_{m,n,\phi} I. \tag{36}$$

The conjugation $\mathcal{C}_{m,n,\eta}$ superoperator appears naturally in our analysis of the error operator and subsequent construction of the recovery sequence. The effect of conjugation on an operator's canonical profile is detailed in Remark 3.

The conjugation operation is unique in the following sense.

Lemma 3 (decomposition of irreducible degree-0 QSP unitary). An irreducible degree-0 QSP sequence R of length $d \geqslant 2$ parametrized by phases $\vec{\phi} \in \mathbb{R}^{d+1}$ can be written as $R = e^{i\chi Z} \mathcal{C}_{m,n,\phi_d} R'$ for some $\chi \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, and unbiased QSP R' of length $(d-2)$.

Proof. If $d = 2$, then $R = e^{i\chi Z} \mathcal{C}_{m,n,\phi_2} I$ for some $\chi \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ by Corollary 3.

For $d > 2$, we proceed by repeated application of the QSP elision operation (Lemma 2), each time reducing the length of R by 2. In particular, since the unbiased QSP R is irreducible, it has a degree peak at location $2 \leqslant i \leqslant d-2$. Performing elision about position i, we are left with the length-$(d-2)$ irreducible degree-0 QSP sequence. Notably, neither phases ϕ_0 nor ϕ_d are affected by performing elision at location $2 \leqslant i \leqslant d-2$. After $d/2 - 1$ elision steps, we are left with a length-2 QSP parametrized by $\text{QSP}(\theta; (\phi_0, \sum_{i=1}^{d-1} \phi_i, \phi_d))$, where by Lemma 2 we have that

$$\sum_{i=1}^{d-1} \phi_i = \pi\left(n + \tfrac{1}{2}\right) \tag{37}$$

for some $n \in \mathbb{Z}$.

Therefore, $\text{QSP}(\theta, \vec{\phi}_{1:d-1}) = e^{i\pi(n+1/2)Z}$ or equivalently $e^{-i\pi(n+1/2)Z}\, \text{QSP}(\theta, \vec{\phi}_{1:d-1}) = I$. Thus we can rewrite the original QSP in the desired form $R = e^{i\chi Z} \mathcal{C}_{m,n,\phi_d} R'$ for $\chi \in \mathbb{R}$ and unbiased length-$(d-2)$ QSP $R' \equiv \text{QSP}(\theta; (\phi_1 - \pi(n + \frac{1}{2}), \phi_2, \ldots, \phi_{d-1}))$ proving Lemma 3. ∎

A degree-0 operator is a rotated version of its own error operator. Therefore, R_ϵ can be written in a form similar to that of Definition 1. We aim to show that for the case of degree-0 R_ϵ, these weights are additionally integer multiples of $\frac{\pi}{2}$ save for the degree-0 term.

First, consider the case of irreducible degree-0 QSP $R = \text{QSP}(\theta; (\phi_0, \ldots, \phi_d))$. Consider the contributions to the first-order error from overrotations at the first and last positions (i.e., assume for now errors do not affect positions $0 < i < d$). By Lemma 3 we can write irreducible

$$R = e^{i\chi Z}\mathcal{C}_{m,n,\phi_d}R' = e^{i\phi_0 Z}We^{i\pi(n+1/2)}R'We^{i\phi_d Z} \tag{38}$$

for R' unbiased $\chi \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Overrotation at the first and last positions occurs at degree 0 and therefore both produce an degree-0 error term equivalent to $e^{i(\chi+\pi/2)Z}$ and the overall weight of the degree-0 diagram is $\phi_0 + \phi_d = \chi - \frac{\pi}{2}(4m + 2n + 1)$. The same analysis holds for the unbiased R'; however, $\phi_i + \phi_{d-i}$ must be an integer multiple of $\pi/2$ for $0 < i < d$ by Lemma 2 and therefore the error diagram must have weight that is an integer multiple of $\pi/2$. Furthermore, the weights are preserved by the linearity of error profile to first order under conjugation and product. Thus, the same holds for higher-degree error diagrams.

Now consider a general degree-0 QSP R decomposed into r constituent irreducible components

$$R = e^{i\chi_1 Z}J^{(1)}e^{i\chi_2 Z}J^{(2)}\cdots e^{i\chi_r Z}J^{(r)}, \tag{39}$$

where $\chi_1, \ldots, \chi_r \in \mathbb{R}$ and each $J^{(j)}$ is an irreducible unbiased QSP. We can in general write a degree-0 QSP R_ϵ up to first order in ϵ as

$$R_\epsilon = e^{i\chi Z}\left[I + \left(\sum_i c_i e^{i(\pi/2)Z} + \frac{\pi}{2}\sum_i d_i U_i\right) + O(\epsilon^2)\right], \tag{40}$$

for $\chi = \chi_1 + \cdots + \chi_r$, $c_i \in \mathbb{R}$, $d_i \in \mathbb{Z}$, and QSP unitaries U_i of even length. Additionally, each U_i is of the form

$$U_i = \mathcal{C}_{m_{i,d},n_{i,d},\eta_{i,d}}\cdots\mathcal{C}_{m_{i,1},n_{i,1},\eta_{i,1}}e^{i(\pi/2)}. \tag{41}$$

A diagrammatic analysis is provided for an example length-8 irreducible degree-0 QSP in Fig. 5.

C. Z error is not correctable in general

A natural question to ask is how one should define recovery and if it is possible, given access only to such noisy signal processing rotations. First, we show the impossibility of the most general form of error correction.

Theorem 1 (no correction of Z error). Let U_ϵ be a length-d noisy QSP unitary parametrized by $(\phi_0, \ldots, \phi_d) \in \mathbb{R}^{d+1}$. For general phases ϕ_i, no noisy QSP unitary U'_ϵ exists such that for any $k \geqslant 1$, for all states $|\psi\rangle$,

$$|\langle\psi|U'_\epsilon|\psi\rangle|^2 = |\langle\psi|U_0|\psi\rangle|^2 + O(\epsilon^{k+1}). \tag{42}$$

The condition given by Eq. (42) of Theorem 1 is equivalent to requiring

$$U'_\epsilon = U_0 e^{i\chi}e^{i\epsilon^{k+1}[xX+yY+zZ+O(\epsilon)]} \tag{43}$$

for some global phase $\chi \in \mathbb{R}$ and x, y, and z functions of θ.

We continue with a few results needed in our proof of the impossibility result.

Lemma 4 (bottom-degree term of degree-0 QSP). Let U_ϵ be a degree-0 QSP with $U_0 = e^{i\chi Z}$ for $\chi \in \mathbb{R}$ [as required by Eq. (43)]. Then the bottom-degree Z term in its error profile $\mathcal{P}$ to first order in ϵ is

$$\mathcal{P}_{-1}^{(z,1)} = (\chi + m\pi)\cos\chi \tag{44}$$

for some $m \in \mathbb{Z}$.

Proof. The expansion of a general degree-0 QSP U to first order is given by Eq. (40). Remark 3 gives us the lowest degree Z coefficients in the canonical expansion of each constituent diagram: The contribution to the lowest degree Z term is given by the bottom-left component of a product of the B matrices of Eq. (A13), which for all degree greater than or equal to 2 diagrams is -1 and for the degree-0 diagram $e^{i(\pi/2)Z}$ is -1. After a careful accounting of the weights, we find that the sum from all diagrams to this lowest-degree term is $\chi + m\pi$ for $m \in \mathbb{Z}$. Finally, the overall $e^{i\chi Z}$ rotation of the first-order term in Eq. (40) results in an overall multiplicative factor of $\cos\chi$ on the expansion by Remark 2. Together, this results in $\mathcal{R}_{-1}^{(z,1)}$ of the form required by Lemma 4. ■

We are now ready to prove Theorem 1.

Proof. First we show that we cannot fully recover a noisy length-0 QSP $U_\epsilon \equiv e^{i\phi_0(1+\epsilon)Z}$ to first order for general $\phi_0 \in \mathbb{R}$. For full recovery, the QSP must satisfy Eq. (43) and therefore either $U'_0 = e^{i\phi_0 Z}$ or $U'_0 = e^{i(\phi_0+\pi)Z} = -e^{i\phi_0 Z}$; therefore U' must be a degree-0 QSP. We see immediately that its bottom-degree term, given by Lemma 4, cannot be corrected in general (i.e., unless ϕ_0 is an integer multiple of $\pi/2$).

Now we generalize the result for QSPs of length $d > 0$. For contradiction, suppose that there exists an error correction function $\mathrm{EC} : \mathbb{R}^{d+1} \to \mathbb{R}^{d'+1}$ for some $d' > d$ that is capable of mapping an arbitrary length-d QSP sequence to one that is corrected to first order. That is, suppose for any $\vec{\phi} \in \mathbb{R}^{d+1}$ parametrizing QSP $U_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\phi})$ we have $\vec{\psi} = \mathrm{EC}(\vec{\phi})$ and $U'_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\psi})$ satisfying Eq. (43). We can simulate a length-0 QSP operator $\mathrm{QSP}(\theta, (\phi_0))$ by appending a recovered length-d QSP and a recovered version of its inverse (Corollary 1). For concreteness, we can choose phases $\vec{\phi}_1, \vec{\phi}_2 \in \mathbb{R}^{d+1}$,

$$\vec{\phi}_1 = (-\pi/2, 0, \ldots, 0, \pi/2), \tag{45}$$

$$\vec{\phi}_2 = (0, \ldots, 0, \phi_0). \tag{46}$$

Let $R_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\phi}_1)$ and $S_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\phi}_2)$. Further let $\vec{\psi}_1 = \mathrm{EC}(\vec{\phi}_1)$, $\vec{\psi}_2 = \mathrm{EC}(\vec{\phi}_2)$, $R'_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\psi}_1)$, and $S'_\epsilon = \mathrm{QSP}_\epsilon(\theta, \vec{\psi}_2)$. Note that, by construction, $R_0S_0 = e^{i\phi_0 Z}$, as desired and therefore $R'_0S'_0 = e^{i\phi_0 Z}$. Further, if both R'_ϵ and S'_ϵ satisfy Eq. (43) for $k \geqslant 1$, then resulting length-$2d'$ QSP $R'_\epsilon S'_\epsilon$ will also be fully corrected to order k. This contradicts our original result for $d = 0$ and therefore EC cannot exist for any $d \geqslant 0$, thus proving Theorem 1. ■

D. First-order recovery

In light of the impossibility result presented in Sec. IV C, we shift our attention to XY error recovery. We show that it is possible to perform this restricted form of recovery and make use of the tools developed in Sec. IV B to provide a general construction for XY recovery operators.

Theorem 2 (recoverability). Given any noisy QSP operator $U_\epsilon(\theta)$ of length d and an integer $k \geqslant 1$, there exists a recovery sequence $R_\epsilon(\theta)$ satisfying

$$|\langle 0|U_\epsilon R_\epsilon|0\rangle|^2 = |\langle 0|U_0|0\rangle|^2 + O(\epsilon^{k+1}) \tag{47}$$

for all θ.

The condition given by Eq. (47) of Theorem 2 is equivalent to requiring

$$U'_\epsilon = U_0 e^{i[\chi+O(\epsilon)]Z+\epsilon^{k+1}\{[x+O(\epsilon)]X+[y+O(\epsilon)]Y\}} \tag{48}$$

for some $\chi \in \mathbb{R}$ and x, y, and z functions of θ.

We make the following definition in light of Eq. (48).

Definition 8 (XY equivalence). We say that two operators $U = w(\theta)I + i[x(\theta)X + y(\theta)Y + z(\theta)Z]$ and $V = w'(\theta)I + i[x'(\theta)X + y'(\theta)Y + z'(\theta)Z]$ are XY equivalent if $x(\theta) = x'(\theta)$ and $y(\theta) = y'(\theta)$. We denote this by $U \sim V$.

We provide an explicit construction using unbiased recovery operators (i.e., $R_0 = I$). An upper bound on the length of the recovery operator R_ϵ will be a corollary of our construction.

Theorem 3 (upper bound on recovery length). Given any noisy QSP operator $U_\epsilon(\theta)$ of length d with c distinct phases (up to factors of 2π) and an integer $k \geqslant 1$, there exists a recovery sequence $R_\epsilon(\theta)$ satisfying

$$|\langle 0|U_\epsilon R_\epsilon|0\rangle|^2 = |\langle 0|U_0|0\rangle|^2 + O(\epsilon^{k+1}) \tag{49}$$

for all θ. Furthermore, there exists a QSP operator satisfying the above with length at most $O(2^k c^{k(k+1)/2} d)$.

To show Theorem 2, we provide an explicit construction for R_ϵ.

The irreducible components of our recovery operator will be length-$2r$ recovery operators constructed by conjugating the identity operator. We make use of an integral degree of freedom, namely, the freedom to overrotate by factors of 2π,

$$\begin{aligned}
&\mathrm{QSP}_\epsilon(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \ldots, -\phi_{d-r+1}, \pi/2, \phi_{d-r+1} + 2\pi m_{d-r+1}, \ldots, \phi_d + 2\pi m_d)) \\
&= I + \epsilon\Bigg(-\frac{n\pi}{2} \times \mathrm{QSP}(\theta; (\pi/2)) \\
&\quad + 2\pi m_d \times \mathrm{QSP}(\theta; (-\phi_d - \pi/2, \pi, \phi_d)) \\
&\quad + 2\pi m_{d-1} \times \mathrm{QSP}(\theta; (-\phi_d - \pi/2, -\phi_{d-1}, \pi, \phi_{d-1}, \phi_d)) \\
&\quad \vdots \\
&\quad + \frac{n\pi}{2} \times \mathrm{QSP}(\theta; (-\phi_d - \pi/2, \ldots, -\phi_{d-r+1}, \pi, \phi_{d-r+1}, \ldots, \phi_d))\Bigg) \\
&\quad + O(\epsilon^2)
\end{aligned} \tag{50}$$

for some $n \in \mathbb{Z}$ and $m_i \in \mathbb{Z}$ to be specified later.

There is a striking similarity between the error operator expansion in Eq. (30) and the recovery component of Eq. (50) which are visualized in Figs. 4 and 5, respectively. We take advantage of this fact to construct our recovery operator.

To be concrete, consider a noisy QSP $U_\epsilon = \mathrm{QSP}(\theta; (\phi_0, \ldots, \phi_d))$. Its error operator E_ϵ to first order can be decomposed into a sum of even-length QSPs from length $0, 2, \ldots, 2d$ [Eq. (30)]. The length-$2r$ diagram in general is

$$\phi_{d-r}\,\mathrm{QSP}(\theta; (-\phi_d - \pi/2, \ldots, -\phi_{d-r+1}, \pi, \phi_{d-r+1}, \ldots, \phi_d)). \tag{51}$$

The length-$2r$ recovery operator of the form in Eq. (50) can be chosen to match this by setting all $m_i = 0$. Since canonical profiles add to leading-order Remark 6, we can simply add an additional $\pi/2$ shift added to the final phase $\phi_d \mapsto \phi_d + \pi$ to negate the X and Y components of the recovery operator at first order; this can be verified using Remark 2.

The only remaining challenge is to match the ϕ_{d-r} weight of the degree-$2r$ term in Eq. (51). Here we make use of the trigonometric identity

$$\begin{pmatrix} \sin(\eta+\delta) \\ -\cos(\eta+\delta) \end{pmatrix} + \begin{pmatrix} \sin(\eta-\delta) \\ -\cos(\eta-\delta) \end{pmatrix} = 2\cos(\delta)\begin{pmatrix} \sin(\eta) \\ -\cos(\eta) \end{pmatrix}. \tag{52}$$

By duplicating the length-$2r$ sequence in Eq. (50) and counterrotating each copy by an amount $\delta/2$, we can construct a sequence that is XY equivalent to a rescaled version of the original (shown diagrammatically in Fig. 6). To fully cancel the length-$2r$ diagram in Eq. (51), we append two length-$2r$ recovery QSPs

$$\mathcal{C}_{0,n,\phi_d+\pi/2\pm\delta}\mathcal{C}_{0,n,\phi_{d-1}} \cdots \mathcal{C}_{0,n,\phi_{d-r+1}} I, \tag{53}$$

choosing $n \in \mathbb{Z}$ such that there is a solution to $\delta = \frac{1}{2}\cos^{-1}(\frac{\phi_{d-r}}{n\pi})$. This can be verified using Remark 2 and is represented diagrammatically in Fig. 6.

In summary, we have canceled the degree-$2r$ diagram using a length-$4r$ QSP operator. We repeat this for each diagram of length $2r$ in the first-order expansion of the error operator for $r \in \{2, 4, \ldots, 2d\}$; the length-0 term contributes only to the Z component of the error, which can be ignored. Overall, the recovery of each diagram takes a length-$\Theta(r)$ QSP and we need to correct $\Theta(d)$ diagrams, resulting in a final recovery operator of length $\Theta(d^2)$.

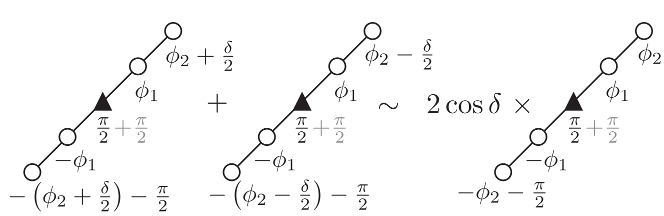

FIG. 6. Diagrammatic representations of the XY equivalence of counterrotated diagrams.

For a generic length-d QSP with all phases distinct, we cannot do better using this method, as we still need to perform counterrotation d times. However, for a length-d QSP operator with c distinct phases, we can group diagrams that are scaled by the same amount; each group can be corrected using a single length-$2r$ diagram by appropriately choosing m_i and n. Overall, if there are c distinct phases, there will be c distinct groups, each requiring a separate counterrotated diagram of length $\Theta(d)$. Note that we consider phases to be equivalent if they differ by an integer multiple of 2π as these can be matched within the same group by an appropriate choice of m_i and n. Thus the overall complexity using this scheme yields the improved $\Theta(cd)$ for QSP diagrams with high phase degeneracy. As an example, the first-order recovery phases for the special case of a QSP with one unique phase are provided in Remark 10. This shows Theorems 2 and 3 for $k = 1$.

E. Sketch of the recovery procedure

We now provide a high-level summary of the first-order recovery construction of the preceding sections and sketch out the proof of Theorems 2 and 3 for $k > 1$.

Expanding the error operator in orders of ϵ, we find that the contribution at each order can be written as a sum of QSP operators: The first-order components are of the form Definition 2. A similar expansion shows that irreducible degree-0 operators (Definition 6) can be expanded in a similar form except for the fact that the coefficients in a degree-0 operator's first-order expansion must be integer multiples of $\pi/2$ whereas the coefficients in the expansion of error operators are unconstrained. These expansions are shown diagrammatically for error operators in Fig. 4 and irreducible degree-0 operators in Fig. 5. A first-order recovery operator satisfying Eq. (48) can be constructed by concatenating irreducible degree-0 operators, making use of the counterrotation trick of Eq. (52) for continuous rescaling. One counterrotation is required for each unique phase in the original QSP with each counterrotated unit a QSP of length $\Theta(d)$. Generically this gives a $\Theta(d^2)$ first-order recovery procedure, but special cases, i.e., QSPs with high phase degeneracies, can admit shorter recovery operators. A QSP with c unique phases can be recovered to first order with a recovery operator of length $\Theta(cd)$. Grover's algorithm is a notable example of a QSP with high phase degeneracy (see Remark 10).

Subsequent recovery occurs order by order, making use of the additive property of leading-order terms (Remark 6). The higher-order expansions of both error and recovery operators can be written in terms of the generalized error components of Definition 9 (up to XY equivalence). Higher-order recovery units are defined in Remark 9 in analogy with the irreducible unbiased operators of Eq. (50) used for first-order recovery. The fact that we use unbiased operators for recovery places additional constraints on the coefficients of the recover operator's expansion which may be overcome through repetition of recovery units and judicious application of the counterrotation trick. The key bottleneck in the required length of the recovery sequence is again the number of required counter-rotations, which ultimately yields the result of Theorem 3. A more detailed analysis of the higher-order recovery construction is left to Appendix B.

F. Lower bound

We now show that, given an additional assumption, the length of our recovery sequence for first-order recovery is asymptotically optimal.

Theorem 4 (lower bound on recovery length). There exists a length-d QSP sequence U_ϵ such that for any XY recovery QSP R_ϵ of order $k \geqslant 1$ satisfying

$$U_0^\dagger U_\epsilon R_\epsilon = I + \epsilon f(a) e^{i(\pi/2)Z} + O(\epsilon^2)$$

for function $f(a) = O(a^0)$, R_ϵ has length $\Omega(d^2)$.

The assumption on the first-order Z component in Theorem 4 [i.e., $f(a) = O(a^0)$] is required for technical reasons but can also be seen as a desire to limit the complexity of the recovery sequence. While we conjecture that this assumption can be removed, it is important to point out that the condition for XY recovery [Eq. (48)] does not itself place any limits on $f(a)$ and in fact neither the recovery construction of Sec. IV D nor the construction of Appendix C presented satisfies this requirement, instead having $f(a) = \Omega(d)$.

First we introduce the inverse of the conjugation superoperator of Definition 7. We denote this operation $\mathcal{C}_{0,\eta}^{-1}$ such that $\mathcal{C}_{n,\eta}^{-1} \circ \mathcal{C}_{m,n,\eta} = \mathrm{id}$ for all $\eta \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Additional details can be found in Remark 5.

Lemma 5 (two error components cannot be combined, first order). Let U and V be first-order error components of degree $2r$ (i.e., of the form Definition 9 with all $b_i = 0$), parametrized by $\phi_{d-r+1}, \ldots, \phi_d$ and $\psi_{d-r+1}, \ldots, \psi_d$, respectively. Their weighted sum $\alpha U + \beta V$ for $\alpha, \beta \in \mathbb{R}$ can be written as a scaled single-error component if and only if $\psi_i - \phi_i = n_i \pi$ for $n_i \in \mathbb{Z}$ for all $d - r + 1 \leqslant i \leqslant d$.

Proof. The $\Longleftarrow$ direction follows directly from the fact that $e^{i\pi Z} = -I$. For the $\Rightarrow$ direction, consider that error components are QSP operators and therefore must be unitary. Therefore, we must have, for some $c \in \mathbb{R}$,

$$(\alpha U + \beta V)(\alpha U + \beta V)^\dagger = (\alpha^2 + \beta^2)I + \alpha\beta(UV^\dagger + VU^\dagger) \quad (54)$$

$$= (\alpha^2 + \beta^2)I - \alpha\beta(UV + VU) \quad (55)$$

$$= cI, \quad (56)$$

where we have used the fact that first-order error components are unitary as well as anti-Hermitian (i.e., $V^{-1} = V^\dagger = -V$).

Since U and V are of the form of Definition 9, their canonical expansions $\mathcal{P}$ and $\mathcal{P}'$ have $\mathcal{P}_j^{(0,1)} = \mathcal{P}'^{(0,1)}_j = 0$ for all j by Remark 3. Thus, in order for the final equality in Eq. (54) to hold, we must have $UV = VU = \pm I$ or equivalently $U^\dagger = -U = \pm V$. The result holds by application of Corollary 2. ■

We are now ready to prove Theorem 4.

Proof. Let $U_\epsilon = \mathrm{QSP}(\theta; (\phi_0, \ldots, \phi_d))$ be a noisy QSP of length $d > 1$ QSP with error operator E_ϵ and R_ϵ any recovery sequence satisfying Eq. (47) for $k \geqslant 1$.

From Eq. (40) we see that each error component scaled by an independent real value requires a separate

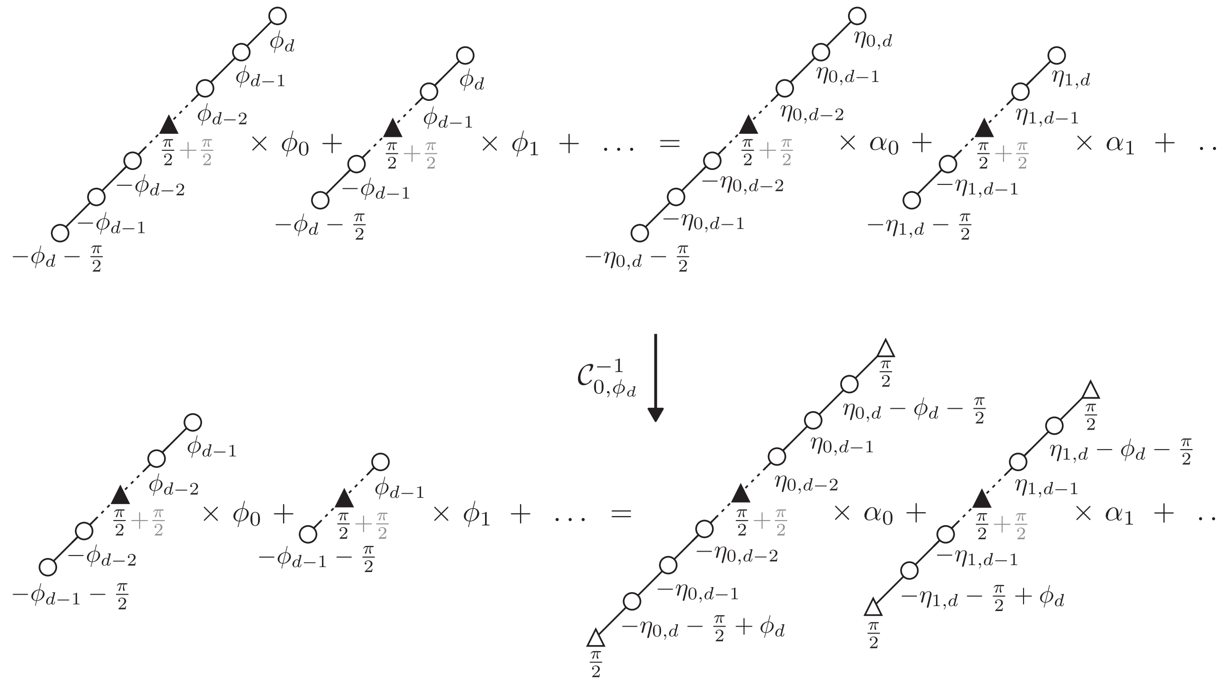

FIG. 7. Diagrammatic representation of one anticonjugation by step in the proof of Theorem 4 for a length-d QSP. The left-hand side depicts the first-order error terms and the right-hand side depicts a proposed set of recovery diagrams. Only the highest two degree terms in each sequence are shown as lower-degree diagrams cannot interfere assuming sufficiently large d and $f(a) = O(1)$. The anticonjugation by $\mathcal{C}_{0,\phi_d}^{-1}$ decreases the degree of all terms of the error operator by 2 (save for degree-0 term, which does not affect the analysis); in order for the right-hand side to match, it must be that $\eta_{i,d} = \phi_d$ for all recovery diagrams i.

irreducible recovery sequence of length $\Theta(d)$. To prove Theorem 4, we show that generically $\Omega(d)$ independently scaled error components are required. We argue that to approximate the first-order error operator of Eq. (30), we need a sequence of degree $2d, 2(d-1), 2(d-2), \ldots$ error components. In fact, given the restrictive condition of $f(a) = O(a^0)$, the only approximation is one that is identical to the error operator up to the π degrees of freedom allowed by Corollary 2.

Assume that we have found an approximation to first order for an error operator of degree $2d$. We proceed inductively, for the first $\Theta(d)$ diagrams by anticonjugating, thereby reducing the error operator to one of degree $2(d-1)$, neglecting the lowest-degree terms. Consider the first-order error terms of both E_ϵ and R_ϵ written in the form of Definition 2. Anticonjugating the error $\mathcal{C}_{0,\phi_d}^{-1} E_\epsilon$ results in all contributing diagrams decreasing in order by 2 (save for the degree-0 diagram) as the outermost phases of each diagram can be elided (Lemma 2). Therefore, anticonjugating the recovery operator $\mathcal{C}_{0,\phi_d}^{-1} R_\epsilon$ must likewise result in a two-degree reduction. One step of the procedure is depicted in Fig. 7. Since by assumption the difference in the Z component is $f(a) = O(a^0)$, it cannot interfere with the top two-degree diagrams for sufficiently large d and the two highest-degree diagrams in R must have outermost phase ϕ_d and be scaled by ϕ_0 and ϕ_1, respectively, as in E_ϵ. This can be seen by using Remark 5 and Lemma 5. We can iterate this procedure $\Theta(d)$ times, before $f(a) = O(a^0)$ becomes relevant, each time requiring the outermost phase of ϕ_{d-r+1} with scaling by ϕ_{r-1}.

Thus the top $\Theta(d)$-degree diagrams in the recovery operator must be identical to that in the error operator. If all ϕ_i are distinct, $\Theta(d)$ independently scaled error components are required, each of length $\Theta(d)$, thus showing the lower bound of $\Omega(d^2)$ for general length-d QSPs. ■

Quantum signal processing with phase degeneracies is able to circumvent this lower bound as in Theorem 3. This motivates the exploration of families of polynomials that can be generated (or approximated) by QSPs with $o(d)$ unique phases.

V. CONCLUSION

We have introduced a model of perturbative noise in the signal processing basis of QSP and provided a set of diagrammatic tools useful for reasoning about such noise. The utility of these techniques is demonstrated by application to a model of coherent noise, that of a multiplicative under- or overrotation, where we have developed and analyzed a method of ancilla-free recovery. We now discuss some directions for future work.

Strengthening the lower bound for the coherent error model. Comparing Theorem 3 to Theorem 4 reveals that our construction is optimal for $k = 1$. However, our lower bound is independent of k and is therefore loose for $k > 1$ and presents a direction for future work. An important limitation of our current construction is the recursive construction of higher-order recovery unitaries (Remark 9), which requires a doubling in length for each order in k. It remains an open question whether an order-k unbiased sequence can be constructed using subexponential resources.

Closing these gaps between the upper and lower bounds has an important implication for quantum computation, given

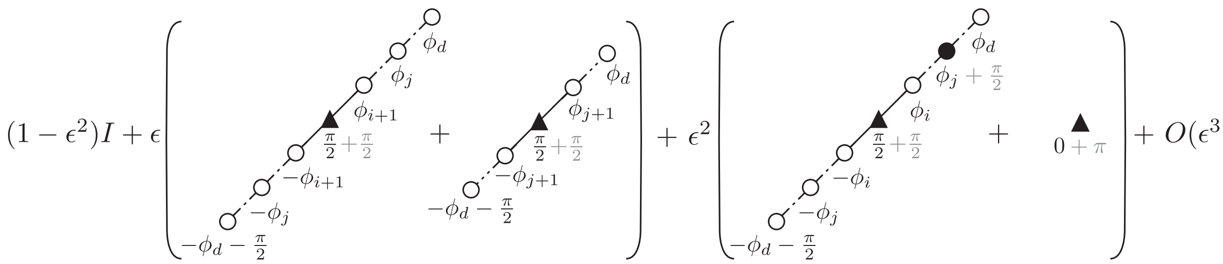

FIG. 8. Example Kraus operator for the error channel of a noisy QSP with phase damping at each site. The diagram corresponds to the Kraus operator with a phase flip at sites i and j only with $0 \leqslant i < j \leqslant d$.

that even the $O(d^2)$ scaling in our construction for first-order error correction would negate all quantum advantage (quadratic speedup) in most fixed-point quantum unstructured search [24].

Incoherent errors. The study of coherent errors has the benefit that one unitary Kraus operator is sufficient to describe both the noisy signal processing rotation and the QSP error channel; however, our formalism can also be applied to models of incoherent error. One significant limitation in the incoherent case is that the number of such Kraus operators grows exponentially with d.

In the case of incoherent noise, each Kraus operator of the error channel, rather than the error operator of Sec. IV A, can be written in the form of Definition 1. As an example, consider the noisy signal processing rotation corresponding to the phase damping channel with two Kraus operators

$$N_\epsilon^{(j,1)} = \sqrt{1-\epsilon} I, \tag{57}$$

$$N_\epsilon^{(j,2)} = \sqrt{\epsilon} Z, \tag{58}$$

at each site $0 \leqslant j \leqslant d$. An example Kraus operator of the error channel is shown diagrammatically in Fig. 8.

While it is already technically challenging to construct recovery sequences given the simple coherent error model that we consider, it is absolutely crucial in the future to analyze the recovery sequence in the presence of an extensive source of random errors. These random errors typically introduce entropy into the quantum circuit and often arise in various quantum algorithms and physical devices.

Generalizing the diagrammatic notation. To allow more complicated error sources, including errors in the signal basis, we anticipate further development of the diagrammatic perturbative expansion used in the present work as a formal tool to analyze error propagation in QSP. We hope such diagrammatic tools can serve as a complementary picture to aid in future development of noisy QSP recovery strategies.

Combining with standard quantum error correction. Whereas standard quantum error correction (QEC) techniques work by moving entropy into ancillary Hilbert spaces [16,25,26], one can view our construction as rotating errors into the Z component. The inability to correct the Z component of error proves a limitation of our method, as the Z error can be important for situations when the QSP sequence needs to be coherently concatenated with another quantum circuit [27]. However, our ancilla-free recovery technique can be concatenated with a standard QEC code to remove the remaining errors. For example, for incoherent errors, it may be possible to find a recovery channel that effectively standardizes the error channel, e.g., transforming the error channel into a phase damping channel; this can then be concatenated with standard QEC codes tailored for phase damping errors. One can envision that a combination of our ancilla-free recovery technique with standard QEC codes would provide a tunable trade-off between the required number of ancilla and gate depth.

ACKNOWLEDGMENTS

The work of A.K.T., Y.L., and I.L.C. on analysis and numerical simulation was supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Co-Design Center for Quantum Advantage under Contract No. DE-SC0012704. A.K.T. acknowledges support from the Natural Sciences and Engineering Research Council of Canada through Grant No. PGSD3-545841-2020. M.C.T. additionally acknowledges support from DARPA Grant No. 134371-5113608 and the Quantum Algorithms and Machine Learning grant from NTT (Grant No. AGMT DTD 9/24/20).

APPENDIX A: ADDITIONAL RESULTS RELATED TO THE CANONICAL EXPANSION

In this Appendix we demonstrate a class of operators, namely, linear combinations of noiseless even-length QSP unitaries, that admit the canonical expansion of Sec. II D. This class includes the diagrammatic components of QSP error channels, the Kraus operators studied in this paper.

Remark 1 (QSP operators of even length exhibit canonical expansion). Let $U_0(\theta) = \text{QSP}(\theta; \vec{\phi})$ be a noiseless QSP unitary of length $2d$; then $U_0(\theta)$ admits a canonical expansion at zeroth order in ϵ. Additionally, $\mathcal{P}_j^{(\sigma,k)} = 0$ for all $k > 0$ and for $k = 0$ with $\sigma \in \{0, x, y, z\}$, and $j \geqslant d$.

Proof. A QSP unitary of even length $2d$ can be written in the form (3) with polynomials $P, Q \in \mathbb{C}[a]$, where P has degree at most $2d$ and is of even parity in $\cos\theta$ and Q has degree at most $2d-1$ and is of odd parity [3]. Writing the polynomials as

$$P(\cos\theta) = \sum_{j=0}^{d} p_{2j} \cos^{2j}(\theta), \tag{A1}$$

$$Q(\cos\theta) = \sum_{j=0}^{d-1} q_{2j+1} \cos^{2j+1}(\theta), \tag{A2}$$

we have

$$w_0(\theta) = \mathrm{Re}[P(\cos\theta)] = \cos^2\theta \sum_{j=-1}^{d-1} \mathrm{Re}(p_{2j})\cos^{2j}(\theta), \quad \text{(A3)}$$

$$\begin{aligned} x_0(\theta) &= \sin\theta\, \mathrm{Re}[Q(\cos\theta)] \\ &= \sin(2\theta)\sum_{j=0}^{d-1} \tfrac{1}{2}\mathrm{Re}(q_{2j+1})\cos^{2j}(\theta), \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned} y_0(\theta) &= -\sin\theta\, \mathrm{Im}[Q(\cos\theta)] \\ &= -\sin(2\theta)\sum_{j=0}^{d-1} \tfrac{1}{2}\mathrm{Im}(q_{2j+1})\cos^{2j}(\theta), \end{aligned} \quad \text{(A5)}$$

$$z_0(\theta) = \mathrm{Im}[P(\cos\theta)] = \cos^2\theta \sum_{j=-1}^{d-1} \mathrm{Im}(p_{2j})\cos^{2j}(\theta). \quad \text{(A6)}$$

This is of the desired form. ■

Since the transformation from an operator to its canonical profile is linear, we have the following corollary.

Corollary 4 (Linear combinations of QSP operators of even length exhibit canonical expansion). If an operator A can be decomposed into

$$A = \sum_i \gamma_i U_i \quad \text{(A7)}$$

for $\gamma_i \in \mathbb{R}$ and QSP unitaries U_i of even length (i.e., A can be written as a linear combination of QSP unitaries of even length), then it admits a canonical expansion.

Next we show how the canonical expansion is transformed under Z rotation and conjugation. It will often be convenient to represent a canonical profile $\mathcal{P}$ in vector form. Assuming $\mathcal{P}_j^{(k)} = 0$ for all $j \geqslant d$, we can write the entire canonical profile as a vector in $\mathbb{R}^{4(d+1)}$,

$$\vec{\mathcal{P}}^{(k)} \equiv \begin{pmatrix} \vec{\mathcal{P}}^{(k)}_{d-1} \\ \vdots \\ \vec{\mathcal{P}}^{(k)}_{0} \\ \vec{\mathcal{P}}^{(k)}_{-1} \end{pmatrix}, \quad \text{(A8)}$$

where the vector $\vec{\mathcal{P}}_j^{(k)} \equiv (\mathcal{P}^{(0,k)}, \mathcal{P}_j^{(x,k)}, \mathcal{P}_j^{(y,k)}, \mathcal{P}_j^{(z,k)})$.

Remark 2 (canonical expansion of Z rotation). Let U_ϵ be an operator admitting canonical expansion with vector form $\vec{\mathcal{P}}^{(k)}$ at order $k \geqslant 0$. Then $V_\epsilon = e^{i\chi_0 Z} U_\epsilon e^{i\chi_1 Z}$ for $\chi_0, \chi_1 \in \mathbb{R}$ has a canonical profile at order k,

$$\vec{\mathcal{P}}^{(k)'} = O_z(\chi_0,\chi_1)\vec{\mathcal{P}}^{(k)} \equiv \begin{pmatrix} O_z(\chi_0,\chi_1) & & & & \\ & O_z(\chi_0,\chi_1) & & & \\ & & O_z(\chi_0,\chi_1) & & \\ & & & \ddots & \\ & & & & O_z(\chi_0,\chi_1) \end{pmatrix} \begin{pmatrix} \vec{\mathcal{P}}^{(k)}_{d-1} \\ \vdots \\ \vec{\mathcal{P}}^{(k)}_{0} \\ \vec{\mathcal{P}}^{(k)}_{-1} \end{pmatrix}, \quad \text{(A9)}$$

where

$$O_z(\chi_0,\chi_1) \equiv \begin{pmatrix} \cos(\chi_0+\chi_1) & 0 & 0 & -\sin(\chi_0+\chi_1) \\ 0 & \cos(\chi_0-\chi_1) & \sin(\chi_0-\chi_1) & 0 \\ 0 & -\sin(\chi_0-\chi_1) & \cos(\chi_0-\chi_1) & 0 \\ \sin(\chi_0+\chi_1) & 0 & 0 & \cos(\chi_0+\chi_1) \end{pmatrix}. \quad \text{(A10)}$$

We note several useful properties of the conjugation operation in the following remarks.

Remark 3 (recurrence under conjugation, first order). Given an unbiased operator U_ϵ with functions w, x, y, and z in its canonical expansion, the corresponding functions of conjugated operator $U'_\epsilon = \mathcal{C}_{m,n,\eta} U_\epsilon$ are given by

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) & 0 & 0 & 0 \\ 0 & \cos(2\eta) & -\cos(2\theta)\sin(2\eta) & \cos^2(\theta)\sin(2\eta) \\ 0 & \sin(2\eta) & \cos(2\theta)\cos(2\eta) & -\cos^2(\theta)\cos(2\eta) \\ 0 & 0 & 4\sin^2(\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}. \quad \text{(A11)}$$

From Eq. (A11) we can write the recurrence of the canonical profiles using a vector form of Eq. (A8). Suppose there exists some $d \geqslant 0$ such that the canonical profile of U_ϵ satisfies $\mathcal{P}_j^{(\sigma,1)} = 0$ for all $j \geqslant d$; then recurrence of the canonical profile of U'_ϵ satisfies

$$\begin{pmatrix} \vec{\mathcal{P}'}^{(1)}_{d} \\ \vec{\mathcal{P}'}^{(1)}_{d-1} \\ \vdots \\ \vec{\mathcal{P}'}^{(1)}_{0} \\ \vec{\mathcal{P}'}^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} A(\eta) & & & & \\ B(\eta) & A(\eta) & & & \\ & B(\eta) & A(\eta) & & \\ & & & \ddots & \\ & & & B(\eta) & A(\eta) \\ & & & & B(\eta) \end{pmatrix} \begin{pmatrix} \vec{\mathcal{P}}^{(1)}_{d-1} \\ \vdots \\ \vec{\mathcal{P}}^{(1)}_{0} \\ \vec{\mathcal{P}}^{(1)}_{-1} \end{pmatrix}, \quad \text{(A12)}$$

where the matrix on the right-hand side is a block-bidiagonal matrix of size $4(d+2) \times 4(d+1)$ with blocks

$$A(\eta) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\sin 2\eta & \sin 2\eta \\ 0 & 0 & 2\cos 2\eta & \cos 2\eta \\ 0 & 0 & -4 & 2 \end{pmatrix}, \tag{A13}$$

$$B(\eta) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\eta & \sin 2\eta & 0 \\ 0 & \sin 2\eta & -\cos 2\eta & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix}. \tag{A14}$$

Remark 4 (linearity of conjugation). Note that conjugation is linear. For operators U and V and coefficients $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{C}_{m,n,\eta}(\alpha U + \beta V) = \alpha \mathcal{C}_{m,n,\eta} U + \beta \mathcal{C}_{m,n,\eta} V. \tag{A15}$$

Similarly, we can write the effect of an anticonjugation operation on the canonical profile.

Remark 5 (recurrence under anticonjugation, first order). We can construct the inverse to the conjugation operation using Corollary 1,

$$\mathcal{C}_{n,\eta}^{-1}\,\mathrm{QSP}(\theta;\vec{\phi}) \equiv e^{i\pi(n+1/2)Z} W e^{i\eta Z}\,\mathrm{QSP}(\theta;\vec{\phi}) e^{-i(\eta+\pi/2)Z} \times W e^{i(\pi/2)Z}. \tag{A16}$$

Letting U_ϵ be a degree-d operator with canonical expansion $\mathcal{P}$, the canonical expansion of $\mathcal{C}_{n,\eta}^{-1} U_\epsilon$ is given by $\mathcal{P}'$ of the form (A12) with blocks

$$A(\eta) \equiv \begin{pmatrix} 0 & -4\sin 2\eta & -4\cos 2\eta & -2 \\ 0 & -2\sin 2\eta & 2\cos 2\eta & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A17}$$

$$B(\eta) \equiv \begin{pmatrix} 0 & -4\sin 2\eta & 4\cos 2\eta & 1 \\ 0 & \sin 2\eta & -\cos 2\eta & 0 \\ 0 & -\cos 2\eta & -\sin 2\eta & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A18}$$

Remark 6 (canonical profile of product of unbiased operators, leading order). If U_ϵ and V_ϵ are both unbiased to order $k \geqslant 1$ with canonical profiles $\mathcal{P}$ and $\mathcal{Q}$, respectively, then $W_\epsilon = U_\epsilon V_\epsilon$ is also unbiased to order k with an canonical profile $\mathcal{R}$ satisfying

$$\mathcal{R}_j^{(\sigma,k)} = \mathcal{P}_j^{(\sigma,k)} + \mathcal{Q}_j^{(\sigma,k)}, \quad \sigma \in \{x, y\}, \tag{A19}$$

$$\mathcal{R}_j^{(\sigma,1)} = \mathcal{P}_j^{(\sigma,1)} + \mathcal{Q}_j^{(\sigma,1)}, \quad \sigma \in \{0, z\}, \tag{A20}$$

for all j.

Remark 7 (equivalence of expansions of unbiased operators to leading order). Two different expansions in ϵ have been presented: The expansion in Remark 6 is performed in the exponent (i.e., the expansion is in the Hermitian generator of the unitary operator), while the canonical expansion is of the unitary operator itself. These expansions do not yield the same expansion coefficients in general; however, for unbiased operators, the coefficients are identical to the leading order for $k \geqslant 1$:

$$e^{i\epsilon[z+O(\epsilon)]Z + i\epsilon^k\{[x+O(\epsilon)]X+[y+O(\epsilon)]Y\}} = [1+O(\epsilon^2)]I + i\epsilon[z+O(\epsilon)]Z + i\epsilon^k\{[x+O(\epsilon)]X + [y+O(\epsilon)]Y\}. \tag{A21}$$

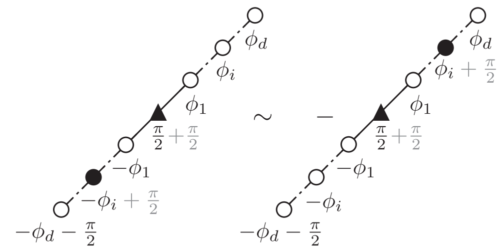

FIG. 9. Diagrammatic representations of Remark 8, i.e., the XY equivalence of conjugation by $\pi/2$.

As all of our analysis will be done recursively on the leading order, we will use these forms interchangeably.

APPENDIX B: PROOF OF THEOREMS 2 AND 3: HIGHER-ORDER COMPONENTWISE RECOVERY

In this Appendix we generalize the first-order componentwise recovery construction of Sec. IV D to all orders in k. First, we make the following observation, which is useful for simplifying higher-order operators.

Remark 8 ($\pi/2$-rotation identity). Let V' be a length-$2r$ QSP operator parametrized by phases

$$(-\phi_d - \pi/2, \ldots, -\phi_i + \pi/2, \ldots, -\phi_{d-r+1}, \pi, \phi_{d-r+1} + b_{d-r+1}\pi/2, \ldots, \phi_i, \ldots, \phi_d + b_d\pi/2), \tag{B1}$$

with $\phi_i \in \mathbb{R}$ and $b_i \in \{0, 1\}$. Using Remarks 2 and 3, we find that

$$V' \sim -V \tag{B2}$$

for the QSP operator V of the form of Definition 9 parametrized by phases

$$(-\phi_d - \pi/2, \ldots, -\phi_i, \ldots, -\phi_{d-r+1}, \pi, \phi_{d-r+1} + b_{d-r+1}\pi/2, \ldots, \phi_i + \pi/2, \ldots, \phi_d + b_d\pi/2). \tag{B3}$$

This is shown diagrammatically for an example QSP in Fig. 9.

The preceding remark motivates the following nomenclature useful in the analysis of the Kraus operators of error and recovery channels to higher order in ϵ:

Definition 9 (error component). Let U be length-d QSP operator parametrized by real phases $(\phi_0, \ldots, \phi_d)$. Then a length-$2r$ QSP V with $1 \leqslant r \leqslant d$ is said to be an error component of QSP U if it can be written in the form

$$V = \mathrm{QSP}(\theta; (-\phi_d - \pi/2, \ldots, -\phi_{d-r+1}, \pi, \phi_{d-r+1} + b_{d-r+1}\pi/2, \ldots, \phi_d + b_d\pi/2)), \tag{B4}$$

where $b_i \in \{0, 1\}$. Furthermore, it is assumed that no ϕ_i for $i < d$ is a half-integer multiple of π; otherwise, we can perform elision to simplify the diagram. This is a generalization of Definition 9.

Definition 10 (standard form, higher order). To simplify the analysis of error and recovery operators at any order, we generalize Definition 1 by writing the contribution at each order as an XY-equivalent linear combination of diagrams of the form of Definition 9. Note that the first-order analysis presented in Sec. IV A is already in this form. For all higher

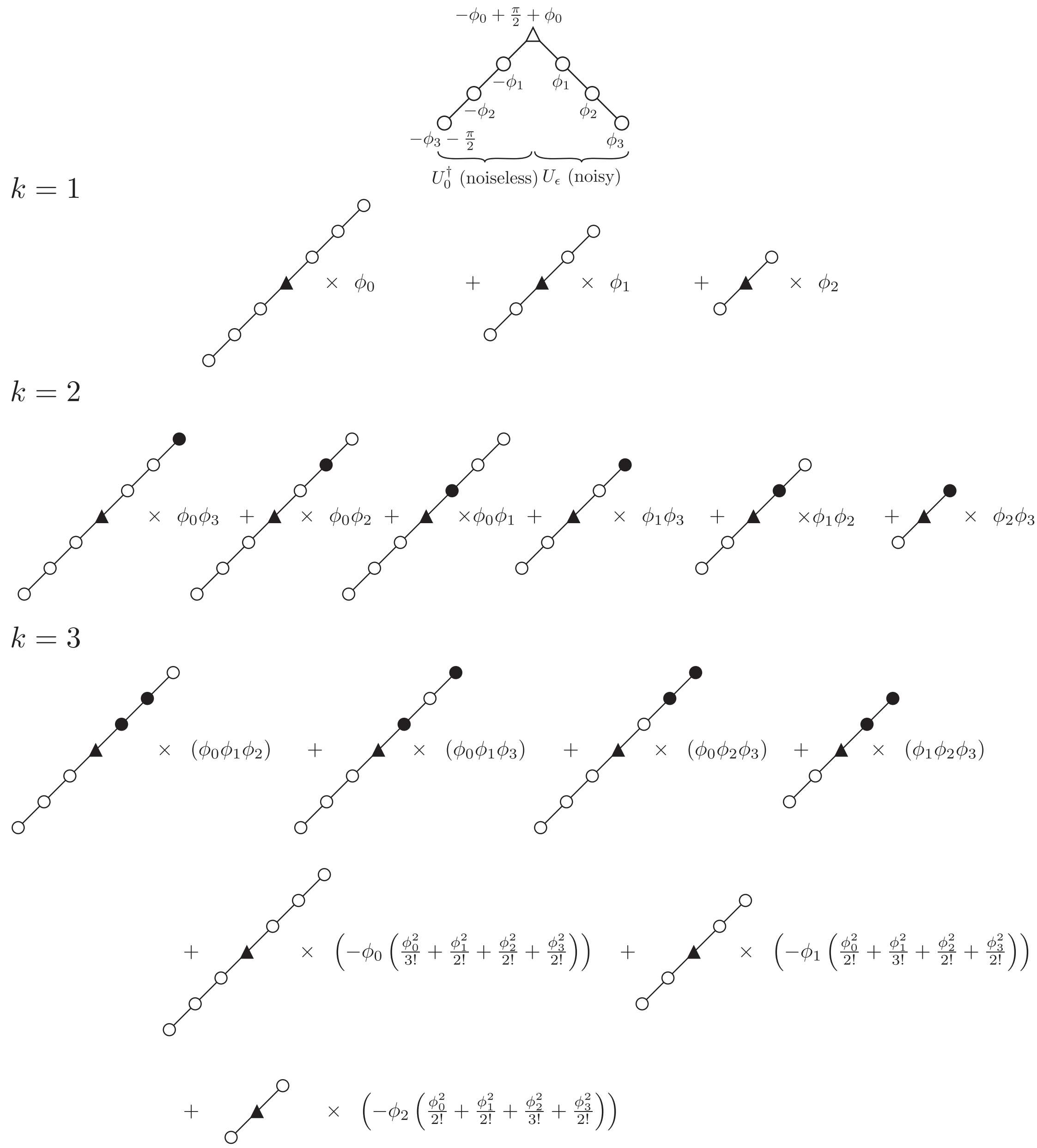

FIG. 10. Diagrammatic representation of error terms for a representative length-3 QSP parametrized by $\phi_i \in \mathbb{R}$ at (a) order 1, (b) order 2, and (c) order 3. To save space, we have omitted phase labels in the expansions. Diagrams are understood to be in the form of Definition 9 with open circles denoting $b_i = 0$ and closed circles denoting $b_i = 1$.

orders, this can be accomplished using repeated application of Remark 8 and the identity $e^{i\pi Z} = -I$, keeping track of factors of -1.

Diagrams to order $k = 3$ are shown for a generic length-3 QSP in Fig. 10.

Remark 9 (constructing higher-order recovery sequences). Let R_ϵ and $\bar{R}_\epsilon$ be kth-order unbiased sequences with canonical profiles $\mathcal{R}$ and $\bar{\mathcal{R}}$, respectively, with

$$\mathcal{R}_j^{(\sigma,k)} = -\bar{\mathcal{R}}_j^{(\sigma,k)} \quad (\sigma = x, y, z) \tag{B5}$$

for all j and

$$\mathcal{R}_j^{(z,k)} = \bar{\mathcal{R}}_j^{(z,k')} = 0 \tag{B6}$$

for all j for $k' < k$. Then the operator $S_\epsilon \equiv R_\epsilon \bar{R}_\epsilon$ with canonical profile $\mathcal{S}$ is an order-$(k+1)$ unbiased sequence with

$$\mathcal{S}_j^{(\sigma,k+1)} = \mathcal{R}_j^{(\sigma,k)} + \bar{\mathcal{R}}_j^{(\sigma,k)} \quad (\sigma = x, y, z) \tag{B7}$$

for all j.

We can make use of the above observation to generate general higher-order recovery sequences. For instance, to create a second-order recovery sequence, we may combine two first-order sequences. In general, this requires careful choice of m_i and n. Example recovery sequences up to order 3 are shown in Fig. 11.

Lemma 6 (higher-order componentwise recovery). Let $U_\epsilon^{(k-1)}$ be a noisy QSP unitary XY recovered to order $(k-1)$,

$$U_\epsilon^{(k-1)} = U_0 e^{i[\chi + O(\epsilon)]Z + \epsilon^k\{[x+O(\epsilon)]X + [y+O(\epsilon)]Y\}}. \tag{B8}$$

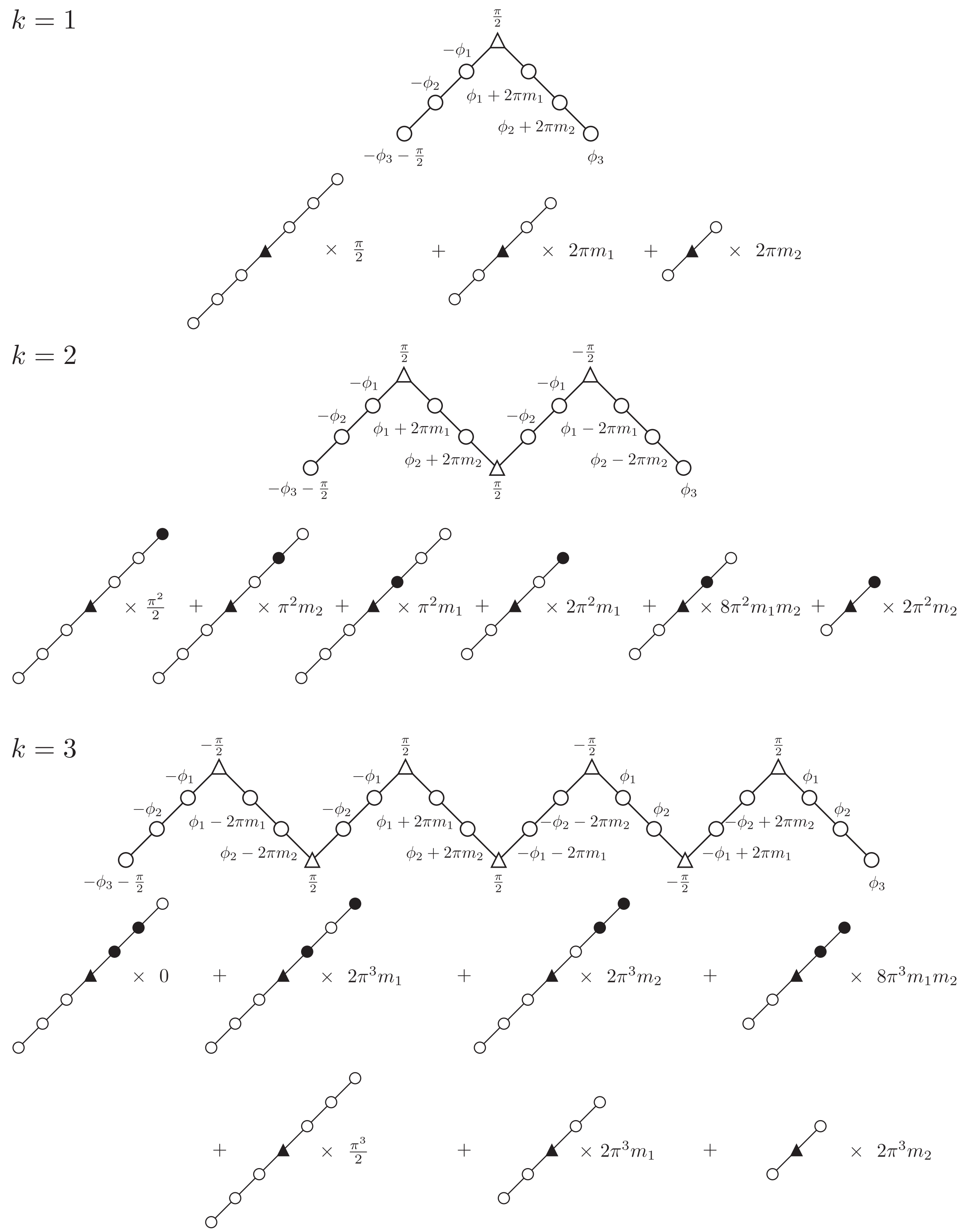

FIG. 11. Diagrammatic analysis of expansions for representative recovery diagrams of (a) order 1, (b) order 2, and (c) order 3. As in Fig. 10, phase labels are omitted and diagrams are understood to be in the form of Definition 9 with open circles denoting $b_i = 0$ and closed circles denoting $b_i = 1$.

If $U_\epsilon^{(0)}$ is of length d and with c unique phases (up to factors of 2π) then a recovery sequence $R^{(k)}$ of length $\Theta(2^k c^k d)$ exists to recover up to order k,

$$U_\epsilon^{(k-1)} R^{(k)} = U_0 e^{i[\chi+O(\epsilon)]Z+\epsilon^k\{[x+O(\epsilon)]X+[y+O(\epsilon)]Y\}}. \tag{B9}$$

Proof. The result for higher orders is similar to that for first-order recovery. The units of recovery at order k can be constructed using Remark 9 and are of length $\Theta(2^k d)$ (example recovery units are shown in Fig. 11).

As in the first-order case, we can correct groups of diagrams scaled by the same real coefficient using each recovery unit. In general, each recovery unit may need to be repeated a constant number of times with different values of m_i and n in order to attain the desired integral coefficients.

The main factor bounding recovery length is the number of distinct groups with different real coefficients. The real coefficients at order k consist of the k-tuples of the c distinct phases, of which there are $\binom{c}{k} = \Theta(c^k)$. Note that again we consider phases equivalent if they differ only by an integer multiple of 2π as these require only a constant number of additional recovery units. Thus the recovery to order k can be accomplished using a recovery sequence of length $\Theta(2^k c^k d)$. ■

We are now ready to prove Theorem 3.

Proof. Let U_ϵ be a length-d QSP sequence with c unique phases with error operator E_ϵ. We perform recovery order by order so that we can make use of the additive property of leading-order terms (Remark 6). At each order, from Eq. (48), it suffices to append a sequence of relatively negative XY-equivalent diagrams. Recovery to first order has been shown in Sec. IV D with a sequence $R_\epsilon^{(1)}$ of length $\Theta(cd)$.

After appending the first-order recovery sequence, we have $U_\epsilon R_\epsilon^{(1)}$, a length-$\Theta(cd)$ QSP sequence. The key to showing the desired scaling is to notice that this sequence has large phase redundancy, namely, the length-$\Theta(cd)$ QSP sequence is parametrized by the same phases, except for an additional $\Theta(c)$ new distinct phases used in the counterrotation. Thus, by Lemma 6, recovery to second order can be accomplished using a recovery sequence of length $\Theta(2^2 \times c^2 \times cd) = \Theta(2^2 c^3 d)$. Recovery to order k can be accomplished using a recovery sequence a factor $\Theta(c^k)$ longer than the previous order. Overall this construction requires a length $O(2^k c^{k(k+1)/2} d)$ sequence, thus proving Theorem 3. ■

We have Theorem 2 as a corollary.

Finally, we note that the scaling of the recovery length in Theorem 3 shows that QSPs with fewer unique phases require fewer resources for recovery. As an example, we provide an explicit choice of recovery phases for a QSP with a single unique phase.

Remark 10 (example for QSP with a single unique phase). Let U_ϵ be a length-d QSP with a single unique phase ϕ (e.g., Grover search). Using this construction, we can construct a first-order recovery operator by choosing phases

$$\vec{\eta}_1 = (-\phi - \pi - \delta, -\phi, \dots, -\phi, \pi/2, \phi + 2\pi m, \dots, \phi + 2\pi m, \phi + \pi/2 + \delta), \tag{B10}$$

$$\vec{\eta}_2 = (-\phi - 4n\pi - \delta, -\phi, \dots, -\phi, 4n\pi - \pi/2, \phi + 2\pi m, \dots, \phi + 2\pi m, \phi + \pi/2 + \delta), \tag{B11}$$

$$\vec{\eta}_3 = (-\phi - \pi + \delta, -\phi, \dots, -\phi, \pi/2, \phi + 2\pi m, \dots, \phi + 2\pi m, \phi + \pi/2 - \delta), \tag{B12}$$

$$\vec{\eta}_4 = (-\phi - 4n\pi + \delta, -\phi, \dots, -\phi, 4n\pi - \pi/2, \phi + 2\pi m, \dots, \phi + 2\pi m, \phi + \pi/2 - \delta), \tag{B13}$$

with appropriate $m, n \in \mathbb{Z}$ and δ depending on ϕ. Note crucially that each ellipse hides only $\Theta(d)$ phases.

Letting $V_\epsilon^{(1)} = \mathrm{QSP}(\theta; \vec{\eta}_1), \dots, V_\epsilon^{(4)} = \mathrm{QSP}(\theta; \vec{\eta}_1)$, $V_\epsilon^{(1)} V_\epsilon^{(2)} V_\epsilon^{(3)} V_\epsilon^{(4)}$ is a first-order recovery sequence for U_ϵ.

APPENDIX C: ALTERNATE PROOF OF THEOREM 2: DEGREEWISE RECOVERY

In this Appendix we present an alternate recovery construction for the coherent error model in Sec. IV. Though this construction is exponentially less efficient, generating a length-$\Theta(2^k d^{2^k})$ sequence, we find that it is sufficiently different to be worth discussion. Furthermore, despite being asymptotically worse, it can produce shorter recovery sequences in practice due to the large constants hidden in Lemma 6. Whereas the construction presented performs recovery componentwise, here we perform recovery degreewise.

1. First order

We now describe our construction of the recovery sequence that corrects the first-order error in a QSP sequence. Given a faulty QSP sequence U_ϵ and letting E_ϵ be its error operator and $\mathcal{P}$ its error profile, it suffices by Eq. (48) to construct a recovery sequence R_ϵ with an error profile $\mathcal{R}$ satisfying

$$\mathcal{R}_j^{(\sigma,1)} = -\mathcal{P}_j^{(\sigma,1)} \tag{C1}$$

for $\sigma = x, y$ and for all j.

The construction of R_ϵ is recursive. In the first iteration, we construct R_ϵ that satisfies Eq. (C1) only at $j_{\max}$, the largest j such that $\mathcal{P}_j^{(\sigma,1)} \neq 0$ ($\sigma = x, y$). At the end of this iteration, the appended QSP sequence has a modified error profile $\mathcal{P}'$ such that $\mathcal{P}_j'^{(\sigma,0)} = 0$ for all $j \geqslant j_{\max}$, resulting in a lower $j_{\max}$ for the next iteration. Repeating this procedure until $\mathcal{P}_j^{(\sigma,0)} = 0$ for all j, we arrive at the desired recovery sequence.

The building block for our recovery sequence is the conjugations in Eq. (C12). The following lemma gives the error profile for QSP sequence that results from the conjugations.

Lemma 7 (top-degree recovery term, first order). Let R be the length-$2d$ QSP sequence resulting from d conjugations in Eq. (C12) with $m_1 = \cdots = m_d = 0$ and $n_1 = \cdots = n_d = n$. Let $\mathcal{R}$ be the error profile of R_ϵ. We have

$$\begin{pmatrix} \mathcal{R}_{d-1}^{(x,1)} \\ \mathcal{R}_{d-1}^{(y,1)} \\ \mathcal{R}_{d-1}^{(z,1)} \end{pmatrix} = \pi 2^{2d-3}(2n+1) \prod_{j=1}^{d-1} \cos^2(\eta_j) \begin{pmatrix} \sin(2\eta_d) \\ -\cos(2\eta_d) \\ 2 \end{pmatrix}. \tag{C2}$$

Proof. We prove Lemma 7 by induction. For $d = 1$, the error profile of the corresponding length-2 QSP satisfies Eq. (C2):

$$\begin{pmatrix} \mathcal{R}_{0}^{(x,1)} \\ \mathcal{R}_{0}^{(y,1)} \\ \mathcal{R}_{0}^{(z,1)} \end{pmatrix} = \pi 2^{-1}(2n+1) \begin{pmatrix} \sin(2\eta_1) \\ -\cos(2\eta_1) \\ 2 \end{pmatrix}. \tag{C3}$$

Suppose Lemma 7 holds for all length-$2d$ sequences. We will prove that it also holds for all length-$(2d+2)$ sequences.

Let R' be a length-$(2d+2)$ sequence satisfying the assumptions of Lemma 7 and R be a length-$2d$ sequence satisfying

$$R' = \mathcal{C}_{0,n,\eta_{d+1}} R. \tag{C4}$$

Let $\mathcal{R}'$ and $\mathcal{R}$ be the error profiles of R'_ϵ and R_ϵ, respectively. Using Eq. (A11), we have

$$\mathcal{R}'^{(x,0)}_d = \sin(2\eta_{d+1})\big(\mathcal{R}^{(z,0)}_{d-1} - 2\mathcal{R}^{(y,0)}_{d-1}\big), \tag{C5}$$

$$\mathcal{R}'^{(y,0)}_d = -\cos(2\eta_{d+1})\big(\mathcal{R}^{(z,0)}_{d-1} - 2\mathcal{R}^{(y,0)}_{d-1}\big), \tag{C6}$$

$$\mathcal{R}'^{(z,0)}_d = 2\big(\mathcal{R}^{(z,0)}_{d-1} - 2\mathcal{R}^{(y,0)}_{d-1}\big). \tag{C7}$$

Applying Lemma 7 on $\mathcal{R}$, we have

$$\mathcal{R}^{(z,0)}_{d-1} - 2\mathcal{R}^{(y,0)}_{d-1} = \pi 2^{2d-1}(2n+1)\prod_{j=1}^{d}\cos^2(\eta_j). \tag{C8}$$

Therefore,

$$\begin{pmatrix}\mathcal{R}'^{(x,1)}_d\\ \mathcal{R}'^{(y,1)}_d\\ \mathcal{R}'^{(z,1)}_d\end{pmatrix} = \pi 2^{2d-1}(2n+1)\prod_{j=1}^{d}\cos^2(\eta_j)\begin{pmatrix}\sin(2\eta_{d+1})\\ -\cos(2\eta_{d+1})\\ 2\end{pmatrix} \tag{C9}$$

and Lemma 7 holds for R'. By induction, Lemma 7 holds for length-$2d$ QSP sequences for all $d \geqslant 1$. ■

Also, recall from Remark 3 that $R^{(x,1)}_j = R^{(y,1)}_j = R^{(z,1)}_j = 0$ for all $j \geqslant d$.

Lemma 8 (first-order degreewise recovery). Let U be a length-d QSP sequence, E_ϵ the error operator for U, and $\mathcal{P}$ its error profile. Let $j_{\max}$ be the largest j such that either $\mathcal{P}^{(x,1)}_j \neq 0$ or $\mathcal{P}^{(y,1)}_j \neq 1$. There exists an unbiased recovery sequence R such that the error profile $\mathcal{P}'$ of $U_\epsilon R_\epsilon$ satisfies

$$\mathcal{P}'^{(x,1)}_j = \mathcal{P}'^{(y,1)}_j = 0 \tag{C10}$$

for all $j \geqslant j_{\max}$. In addition, the length of the recovery sequence is at most $2(j_{\max}+1)$ if $j_{\max} \geqslant 1$ and at most 4 if $j_{\max} = 0$.

Proof. First, we consider $j_{\max} \geqslant 1$. Let n be the smallest integer such that

$$\sqrt{\big(\mathcal{P}^{(x,1)}_{j_{\max}}\big)^2 + \big(\mathcal{P}^{(y,1)}_{j_{\max}}\big)^2} \leqslant \pi 2^{2j_{\max}-1}\big(n+\tfrac{1}{2}\big). \tag{C11}$$

Let R be the length-$(2j_{\max}+2)$ QSP sequence in the form

$$\mathcal{C}_{m_d,n,\eta_d}\cdots\mathcal{C}_{m_1,n,\eta_1} I, \tag{C12}$$

with $m_1 = \cdots = m_{j_{\max}+1} = 0$ and $n_1 = \cdots = n_{j_{\max}+1} = n$, and $\mathcal{R}$ be the error profile of R_ϵ. By Lemma 7 we have

$$\begin{pmatrix}\mathcal{R}^{(x,1)}_{j_{\max}}\\ \mathcal{R}^{(y,1)}_{j_{\max}}\end{pmatrix} = \pi 2^{2j_{\max}-1}\big(n+\tfrac{1}{2}\big) \times\prod_{j=1}^{j_{\max}}\cos^2(\eta_j)\begin{pmatrix}\sin(2\eta_{j_{\max}+1})\\ -\cos(2\eta_{j_{\max}+1})\end{pmatrix}. \tag{C13}$$

Next we choose $\eta_1 = \cdots = \eta_{j_{\max}-1} = 0$,

$$\eta_{j_{\max}} = \cos^{-1}\left(\frac{\sqrt{\big(\mathcal{P}^{(x,1)}_{j_{\max}}\big)^2 + \big(\mathcal{P}^{(y,1)}_{j_{\max}}\big)^2}}{\pi 2^{2j_{\max}-1}\big(n+\frac{1}{2}\big)}\right)^{1/2}, \tag{C14}$$

$$\eta_{j_{\max}+1} = -\frac{1}{2}\tan^{-1}\left(\frac{\mathcal{P}^{(x,1)}_{j_{\max}}}{\mathcal{P}^{(y,1)}_{j_{\max}}}\right) \leqslant 0. \tag{C15}$$

Substituting these phase angles into Eq. (C13), we obtain

$$\begin{pmatrix}\mathcal{R}^{(x,1)}_{j_{\max}}\\ \mathcal{R}^{(y,1)}_{j_{\max}}\end{pmatrix} = -\begin{pmatrix}\mathcal{P}^{(x,1)}_{j_{\max}}\\ \mathcal{P}^{(y,1)}_{j_{\max}}\end{pmatrix}. \tag{C16}$$

Thus, by Remark 6 we have Lemma 8 for $j_{\max} \geqslant 1$.

Finally, we consider the case $j_{\max} = 0$. Recall that for $j_{\max} \geqslant 1$ we have continuous control over the magnitude of the error profile provided by $\eta_{j_{\max}}$. For $j_{\max} = 0$, we use the counterrotation trick of Eq. (52). Accordingly, we choose $\eta = -\tan^{-1}(\mathcal{P}^{(x,1)}_0/\mathcal{P}^{(y,1)}_0) \leqslant 0$ and

$$\delta\eta = \frac{1}{2}\cos^{-1}\left(\frac{\sqrt{\big(\mathcal{P}^{(x,1)}_0\big)^2 + \big(\mathcal{P}^{(y,1)}_0\big)^2}}{\pi\big(n+\frac{1}{2}\big)}\right) \tag{C17}$$

to arrive at Eq. (C16) for $j_{\max} = 0$. This concludes the proof of Lemma 8. ■

Repeatedly applying Lemma 8, we incrementally lower $j_{\max}$. When $\mathcal{P}^{(x,0)}_j = \mathcal{P}^{(y,0)}_j = 0$ for all $j \geqslant 0$, we arrive at Theorem 2 for $k = 1$. Since the length of the recovery sequence in each iteration of Lemma 7 is $2(j_{\max}+1)$ and $j_{\max}$ is initially at most $d-1$, the total length of the recovery sequence is at most

$$4 + \sum_{j_{\max}=1}^{d-1} 2(j_{\max}+1) = d^2 + d + 2. \tag{C18}$$

Therefore, for a length-d QSP U_ϵ, there exists recovery sequence R_ϵ of length d^2+d+2 satisfying Theorem 2 for $k = 1$.

2. Higher order

We now generalize to higher orders.

First, we provide an explicit recursive construction of higher-order unbiased sequences.

Lemma 9 (top-degree recovery term, higher order). For all $k \geqslant 1$, there exists a kth-order unbiased QSP sequence of length-$(2^k d)$ R_ϵ, parametrized by $\eta_1, \ldots, \eta_d \in [-\pi, \pi)$ and $n \in \mathbb{Z}$ with an error profile $\mathcal{R}$ satisfying

$$\begin{pmatrix}\mathcal{R}^{(x,k)}_{d-1}\\ \mathcal{R}^{(y,k)}_{d-1}\end{pmatrix} = \pi^k 2^{2d-3}(2n+1)^k \times\prod_{j=1}^{d-1}\cos^2(\eta_j)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^{k-1}\begin{pmatrix}\sin(2\eta_d)\\ -\cos(2\eta_d)\end{pmatrix}, \tag{C19}$$

and $\mathcal{R}^{(x,k-1)}_j = \mathcal{R}^{(y,k-1)}_j = 0$ for all $j \geqslant d$.

Proof. We will provide a recursive construction for a length-$(2^k d)$ QSP satisfying Lemma 9.

Let R_ϵ be the length-$2d$ recovery sequence parametrized by $n \in \mathbb{Z}$ and $\eta_1, \ldots, \eta_d$ as in Lemma 7 and $\mathcal{R}$ its error profile. From Lemma 9 we have

$$\begin{pmatrix} \mathcal{R}_{d-1}^{(x,1)} \\ \mathcal{R}_{d-1}^{(y,1)} \end{pmatrix} = \pi 2^{2d-3}(2n+1) \times \prod_{j=1}^{d-1} \cos^2(\eta_j) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^0 \begin{pmatrix} \sin(2\eta_d) \\ -\cos(2\eta_d) \end{pmatrix} \tag{C20}$$

and $\mathcal{R}_j^{(x,0)} = \mathcal{R}_j^{(y,0)} = 0$ for all $j \geqslant d$. So R satisfies Lemma 9 for $k = 1$.

Suppose Lemma 9 holds for $k \geqslant 1$ and let R_ϵ be the kth-order unbiased QSP sequence satisfying Lemma 9. We define $\bar{R}_\epsilon$ to be the QSP sequence identical to R_ϵ except that $\eta_d \mapsto \eta_d + \frac{\pi}{2}$ and $n \mapsto -(n+1)$. Let $\mathcal{R}$ and $\bar{\mathcal{R}}$ be their respective error profiles. Using Remark 9, we can show that

$$\mathcal{R}_j^{(\sigma,k)} = \bar{\mathcal{R}}_j^{(\sigma,k)} \quad (\sigma = x, y), \tag{C21}$$

$$\mathcal{R}_j^{(z,k)} = -\bar{\mathcal{R}}_j^{(z,k)} \tag{C22}$$

for all j and $k \geqslant 1$.

Thus we we construct a sequence S using Remark 9 that is unbiased to order $k+1$ as

$$S_\epsilon \equiv e^{-i\pi(n+1/2)(1+\epsilon)Z} R_\epsilon e^{i\pi(n+1/2)(1+\epsilon)Z} \bar{R}_\epsilon, \tag{C23}$$

which is a length-$(2^{k+1}d)$ QSP unitary. By the result of Remark 9, the error profile $\mathcal{S}$ of S_ϵ satisfies

$$\begin{pmatrix} \mathcal{S}_{d-1}^{(x,k)} \\ \mathcal{S}_{d-1}^{(y,k)} \end{pmatrix} = \pi(2n+1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{R}_{d-1}^{(y,k-1)} \\ \mathcal{R}_{d-1}^{(x,k-1)} \end{pmatrix} \tag{C24}$$

$$= \pi^k 2^{2d-3}(2n+1)^k \prod_{j=1}^{d-1} \cos^2(\eta_j) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{k-1} \times \begin{pmatrix} \sin(2\eta_d) \\ -\cos(2\eta_d) \end{pmatrix}. \tag{C25}$$

Therefore, Lemma 9 holds for $k+1$ and by induction it holds for all k. ■

Lemma 10 (Higher-order degreewise recovery). Let U be a length-d QSP sequence, E_ϵ its error operator, and $\mathcal{P}$ its error profile. Suppose E_ϵ is unbiased to order k, that is, $\mathcal{P}_j^{(x,k')} = \mathcal{P}_j^{(y,k')} = 0$ for all $j \geqslant 0$ and $k' < k$. Let $j_{\max}$ be the largest j such that either $\mathcal{P}_j^{(x,k)} \neq 0$ or $\mathcal{P}_j^{(y,k)} \neq 0$. There exists an unbiased recovery sequence R such that the error profile $\mathcal{P}'$ of $U_\epsilon R_\epsilon$ satisfies

$$\mathcal{P}_j^{\prime(x,k)} = \mathcal{P}_j^{\prime(y,k)} = 0 \tag{C26}$$

for all $j \geqslant j_{\max}$ and $k \geqslant 0$. In addition, the length of the recovery sequence is at most $2^{k+1}(j_{\max}+1)$ if $j_{\max} \geqslant 1$ and at most 2^{k+2} if $j_{\max} = 0$.

Proof. The proof of Lemma 10 is nearly identical to that of Lemma 8. First, consider $k \geqslant 1$ and $j_{\max} \geqslant 1$. Let n be the smallest integer such that

$$\sqrt{\left(\mathcal{P}_{j_{\max}}^{(x,k)}\right)^2 + \left(\mathcal{P}_{j_{\max}}^{(y,k)}\right)^2} \leqslant \pi^k 2^{2j_{\max}-1}(2n+1)^k. \tag{C27}$$

Let R be the kth-order unbiased length-$2^{k+1}(j_{\max}+1)$ QSP sequence parametrized by $\eta_1, \ldots, \eta_{j_{\max}+1} \in [-\pi, \pi)$ and $n \in \mathbb{Z}$ that satisfies Lemma 9 and $\mathcal{R}$ be its error profile. From Lemma 9 we have

$$\begin{pmatrix} \mathcal{R}_{j_{\max}}^{(x,k)} \\ \mathcal{R}_{j_{\max}}^{(y,k)} \end{pmatrix} = \pi^k 2^{2j_{\max}-1}(2n+1)^k \times \prod_{j=1}^{j_{\max}} \cos^2(\eta_j) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{k-1} \begin{pmatrix} \sin(2\eta_{j_{\max}+1}) \\ -\cos(2\eta_{j_{\max}+1}) \end{pmatrix}. \tag{C28}$$

Next, we choose $\eta_1 = \cdots = \eta_{j_{\max}-1} = 0$, n from Eq. (C27), and

$$\eta_{j_{\max}} = \cos^{-1}\left(\frac{\sqrt{\left(\mathcal{P}_{j_{\max}}^{(x,k)}\right)^2 + \left(\mathcal{P}_{j_{\max}}^{(y,k)}\right)^2}}{\pi^k 2^{2j_{\max}-1}(2n+1)^k}\right)^{1/2}, \tag{C29}$$

$$\eta_{j_{\max}+1} = -\frac{1}{2}\tan^{-1}\left(\frac{\mathcal{P}_{j_{\max}}^{(x,k)}}{\mathcal{P}_{j_{\max}}^{(y,k)}}\right) - \frac{3\pi k}{4}. \tag{C30}$$

Substituting these phase angles into Eq. (C28), we obtain

$$\begin{pmatrix} \mathcal{R}_{j_{\max}}^{(x,k)} \\ \mathcal{R}_{j_{\max}}^{(y,k)} \end{pmatrix} = -\begin{pmatrix} \mathcal{P}_{j_{\max}}^{(x,k)} \\ \mathcal{P}_{j_{\max}}^{(y,k)} \end{pmatrix}. \tag{C31}$$

From Remark 6 we have Lemma 10 for $j_{\max} \geqslant 1$.

Finally, for the case $j_{\max} = 0$ we again use the counter-rotation trick of Eq. (52) setting $\eta_1 = \eta \pm \delta\eta$. We choose $\eta = -\tan^{-1}(\mathcal{P}_0^{(x,0)}/\mathcal{P}_0^{(y,0)}) - \frac{3\pi k}{4}$ and

$$\delta\eta = \frac{1}{2}\cos^{-1}\left(\frac{2\sqrt{\left(\mathcal{P}_0^{(x,k)}\right)^2 + \left(\mathcal{P}_0^{(y,k)}\right)^2}}{\pi^k(2n+1)^k}\right) \tag{C32}$$

to arrive at Eq. (C31) for $j_{\max} = 0$. This concludes the proof of Lemma 10. ■

This provides an alternate proof for Theorem 2.

Proof. To prove Theorem 2 in full generality for $k \geqslant 1$, we repeatedly apply Lemma 10. Let $R_\epsilon^{(k')}$ be the recovery operator accumulated from such repeated applications for order k'. We start by constructing $R_\epsilon^{(1)}$ such that $U_0^\dagger U_\epsilon R_\epsilon^{(1)}$ is unbiased to order 2. We then increment k', repeating the process up to $k' = k+1$ to obtain a $(k+1)$th-order unbiased operator $U_0^\dagger U_\epsilon R_\epsilon^{(1)} \ldots R_\epsilon^{(k)}$. We can therefore write

$$U_\epsilon R_\epsilon^{(1)} \cdots R_\epsilon^{(k)} \equiv U_0 e^{i\epsilon[z+O(\epsilon)]Z + i\epsilon^{k+1}\{[x+O(\epsilon)]X + [y+O(\epsilon)]Y\}}, \tag{C33}$$

as required by Eq. (48), thus providing an alternate proof of Theorem 2. ■

Recalling that for $k = 1$, the length of the recovery operation using this construction is $d^2 + d + 2 = \Theta(2^1 d^{2^1})$ [Eq. (C18)], we note that, given a noisy QSP corrected to order k of length d_k with $d_k = \Theta(2^k d^{2^k})$, we can perform correction to the $(k+1)$th order using at most d_k applications of Lemma 9 for each $0 \leqslant j_{\max} \leqslant d_k - 1$. From Lemma 10, each application adds length $2^k(j_{\max}+1)$ for $j_{\max} > 0$ and 2^{k+1} for $j_{\max} = 0$. The overall length of the resulting sequence therefore has length d_k satisfying

$$d_{k+1} = d_k + 2^{k+2} + \sum_{j_{\max}=1}^{d_k} 2^{k+1}(j_{\max} + 1) = \Theta(2^{k+1}d^{2^{k+1}}). \tag{C34}$$

The main reason for the exponentially worse performance compared with the construction in Appendix B is the fact that we do not make use of the phase redundancy in the recovered operators and thus recovery at each order is performed de novo.

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Correction: The omission of factors $1/k!$ in Eqs. (27) and (28) has been fixed. The previously published Figures 8 and 10 contained diagrams that reflected the error of the missing factors and have been replaced.