Bell-nonlocality quantifiers and their persistent mismatch with the entropy of entanglement

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For some commonly used nonlocality quantifiers, the maximally nonlocal state may not coincide with the maximally entangled state, what became known as the anomaly of nonlocality. More recently, nonanomalous quantifiers have been defined, for which maximal entanglement and maximal nonlocality do coincide. In this work we investigate in detail how these quantifiers behave for nonmaximal-resource states and show that some mismatch remains between the studied figures of merit. Besides the nonlocal volume (a quantifier that counts the volume of the set of behaviors accessible to a specific state that gives rise to nonlocality), we present an alternative quantity referring to states and based on behaviors, the trace-weighted nonlocal volume. The construction is based on the nonlocal volume integral, but weighted by a quantifier of nonlocality for behaviors (the shortest distance between the behavior and the local set). Although in all investigated scenarios the anomaly is not present, the nonlocality quantifiers as compared to the entropy of entanglement are inversely related for some set of states within some Bell scenarios. In the investigated situations this phenomenon occurs when the considered states go through a change of rank. This fact is discussed in general and illustrated through computations for the (2,3,2) scenario.

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I. INTRODUCTION

Quantum nonlocality is an eccentric phenomenon when viewed through the eyes of a classical observer. Bell nonlocality is a consequence of entanglement and implies that the statistics of certain measurements performed on spatially separated entangled quantum systems cannot be explained by local hidden-variable models [1]. The debate around quantum nonlocality started during the development of the mathematical foundations of quantum theory [2] and gained the status of a theorem mainly with the works of Bell [1]. Besides its importance in foundations of quantum theory, Bell nonlocality has been raised to the status of a physical resource due to its intimate relationship with quantum information science, where it is necessary to enhance our computational and information processing capabilities in a variety of applications [3,4].

Since nonlocality has been identified as a resource, it is essential to formulate resource theories of nonlocality, allowing not only for operational interpretations but also for the precise quantification of nonlocality. Given its pervasive importance, the resource theory of entanglement [5] is arguably the most well understood and vastly explored and thus has become the archetypal example for the development of resource theories of other quantum phenomena [6–12]. Entanglement is a necessary ingredient for Bell nonlocality, but they refer to truly different resources [4], as there are entangled states that can only give rise to local correlations [13].

Recently, several nonlocality quantifiers for behaviors have been proposed [4,8,14–30] and a proper resource theory of nonlocality for behaviors has been developed [8,29]. Of special importance to this work is the nonlocality quantifier based on the trace distance presented in Ref. [31], which measures the l_1 distance between a given behavior and the local polytope. It is shown in this reference that, for the $(2, 2, d)$ scenario, the maximum numerical violation of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality grows with d and likewise the maximum value of the trace distance follows the same trend.

We also have several examples of nonlocality quantifiers for a resource theory where the objects are quantum states [15,19,23,32–34]. Different measures will have different operational meanings and do not necessarily have to agree on the ordering for the amount of nonlocality. For instance, a natural way of quantifying nonlocality is the maximum violation of a specific Bell inequality allowed by a given quantum state. However, we might also be interested in quantifying the nonlocality of a state by the amount of noise (e.g., detection inefficiencies) it can stand before becoming local [32,35,36]. Interestingly, these two measures can be inversely related as demonstrated by the fact that in the Clauser-Horn-Shimony-Holt (CHSH) scenario [37] the resistance against detection inefficiency increases as the entanglement of the state decreases [15], also reducing the violation of the CHSH inequality.

A curious feature of quantifiers based on Bell inequalities is that the highest violation value is not always achieved by

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the maximally entangled state, a property known as anomaly of nonlocality [32]. This is a counterintuitive property since it was initially expected that more entanglement would lead to more nonlocality, at least for maximum-resource states. This observation led to the investigation of whether the anomaly was a property of this specific quantifier or if it was a manifestation of the fact that nonlocality and entanglement are distinct resources. With this purpose, the authors of [23] proposed a nonlocality quantifier called volume of violation, which counts the volume of the set of measurements that give rise to violation of the chosen inequality when applied to this state. This quantifier removes the anomaly for qutrits and ququarts, since in this case maximally entangled states are maximally nonlocal (with respect to the volume of violation). In Ref. [24] the authors analyzed several scenarios, including up to five qubits and different entangled states. They showed that the probability of violation increases as we increase the number of measurements and the maximum violation for these dimensions is obtained by the maximally entangled state. Several properties of these quantifiers were proven in Ref. [38].

The disadvantage of the volume of violation is that it focuses on one inequality and for scenarios with more than two parties, measurements, or outcomes it is known that there are many nonequivalent families of Bell inequalities. Hence, a single inequality cannot capture the entire structure of the set of local behaviors. For example, considering only a single CGLMP inequality, the anomaly of nonlocality reappears for $d > 7$ [34].

Another nonlocality quantifier was proposed to solve this problem. The definition is similar to the definition of the volume of violation, but it takes the entire space of behaviors into consideration: It counts the volume of the set of measurements that give rise to nonlocal behaviors when applied to this state. The maximum nonlocal volume is obtained with the maximally entangled state for dimensions up to $d = 10$ [39], which indicates that the quantifiers based on the volume have a greater relevance when calculated over the polytopes than considering a specific inequality. Recently, a quantifier based on both the volume of violation and noise resistance was studied, where these two quantities are combined [40], and an experiment based on this combination has also been performed [41].

However, the nonlocal volume does not completely remove the anomaly of nonlocality. It was shown in Ref. [24] that this quantifier exhibits a weak anomaly for qutrits: The maximum volume is achieved for the maximally entangled state, but the nonlocal volume is not a monotonic function of the state's entanglement. We suggest that the appearance of this nonmonotonicity, despite the coincidence of the maxima, is an unavoidable manifestation of the intrinsic inequivalence between entanglement and nonlocality.

In this paper we investigate the relation between nonlocality and entanglement for families of pure states. While nonseparability is quantified with the entropy of entanglement, we use two figures of merit to investigate Bell nonlocality, namely, the nonlocal volume and an analogous quantity where the integration is weighted by the trace distance [31] (we call it the trace-weighted nonlocal volume). Initially, we reproduce some results in previous works for the scenarios (2,2,2) and (2,3,2). We also verify whether this measure of nonlocality satisfies the properties listed in [38]. Finally, we use these quantities to study the weak anomaly of nonlocality reported in [24] and show that some mismatch between entanglement and nonlocality remains, far from the maximally entangled and maximally nonlocal state. Particularly when the estate smoothly passes from a lower to a higher rank, the entropy of entanglement always increases whereas the employed nonlocality quantifiers may decrease slightly. Although the relation between entanglement and nonlocality, for arbitrary Bell scenarios and states, is a complex problem with several open questions, we hope that the present work may help in the understanding of some features of the general problem.

II. PRELIMINARIES: GEOMETRY OF THE SET OF BELL CORRELATIONS

We consider a bipartite Bell scenario in which Alice and Bob perform measurements labeled by variables A_x and B_y, obtaining measurement outcomes described by variables a and b, respectively. The description of Alice's and Bob's outcomes is given by a set of probability distributions $p(a, b|x, y)$, called a behavior, that gives the probability of outcomes a and b given inputs x and y.

The fact that Alice and Bob are spatially separated and cannot communicate with each other implies that the statistics of a measurement on one part is independent of the measurement choice of the other. These assumptions imply a set of linear constraints known as nonsignaling conditions

$$p(a|x) = \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y'), \tag{2.1}$$

an analogous expression being valid for the marginal $p(b|y)$, regarding summations over a, for inputs x and x'. A stronger constraint on the description of the experiment is that the statistics of Alice and Bob be consistent with the assumption of local causality, which implies that the probability distributions can be decomposed as

$$p(a, b|x, y) = \sum_\lambda p(\lambda)p(a|x, \lambda)p(b|y, \lambda). \tag{2.2}$$

For this type of behavior, correlations between Alice and Bob are mediated by the variable λ that thus suffices to compute the probabilities of each of the outcomes, that is, $p(a|x, y, b, \lambda) = p(a|x, \lambda)$, and similarly for b. The behaviors that can be decomposed in this way are called local behaviors.

Bell's theorem [1] states that Alice and Bob can perform measurements in a entangled quantum state to generate behaviors that cannot be decomposed in the form (2.2). These can be obtained by local measurements M_a^x and M_b^y on distant parts of a bipartite state ρ that, according to quantum theory, can be described by

$$p(a, b|x, y) = \mathrm{Tr}\big[\big(M_a^x \otimes M_b^y\big)\rho\big]. \tag{2.3}$$

In general, the set of local behaviors $\mathcal{L}$ is a strict subset of the quantum behaviors $\mathcal{Q}$ that in turn is a strict subset of nonsignaling behaviors $\mathcal{NS}$.

The local set is a polytope and hence any local behavior can be written as a convex sum of a finite set of extremal points. If we represent a behavior $p(a, b|x, y)$ as a vector $\mathbf{p}$ with

$|x||y||a||b|$ components, the condition (2.2) can be written succinctly as $\mathbf{p} = A \cdot \boldsymbol{\lambda}$, with $\boldsymbol{\lambda}$ a probability vector over the set of variables λ, with components $\lambda_i = p(\lambda = i)$, and A a matrix indexed by i and the multi-index variable $j = (x, y, a, b)$ with $A_{j,i} = \delta_{a,f_a(x,\lambda=i)}\delta_{b,f_b(y,\lambda=i)}$, where f_a and f_b are deterministic functions that give the values of measurements x and y when $\lambda = i$. Thus, checking whether $\mathbf{p}$ is local amounts to a simple feasibility problem that can be written as the linear program [4,24,25,42,43]

$$\begin{aligned} \min_{\lambda\in\mathbb{R}^m} \quad & \mathbf{v}\cdot\boldsymbol{\lambda} \\ \text{subject to} \quad & \mathbf{p} = A\cdot\boldsymbol{\lambda}, \\ & \lambda_i \geqslant 0, \\ & \sum_i \lambda_i = 1, \end{aligned} \tag{2.4}$$

where $\mathbf{v}$ represents an arbitrary vector with the same dimension $m = |x|^{|a|}|y|^{|b|}$ as the vector representing the variable $\boldsymbol{\lambda}$.

Since the local set is a polytope, alternatively, it can be characterized by a finite number of facet-defining Bell inequalities. Hence, testing membership in $\mathcal{L}$ can be done by testing if the behavior satisfies all the facet-defining Bell inequalities for that scenario. Although this is equivalent to the linear program (LP) formulation, it cannot be done in practice because finding the set of all facet-defining Bell inequalities is an extremely hard problem [44].

A. Quantifying nonlocality of a behavior

The LP (2.4) can be slightly modified to define a nonlocality quantifier based on the trace distance [31] between two probability distributions $\mathbf{q} = q(x)$ and $\mathbf{p} = p(x)$:

$$D(\mathbf{q},\mathbf{p}) = \frac{1}{2}\sum_x |q(x) - p(x)|. \tag{2.5}$$

To quantify nonlocality we use the distance between the probability distribution generated in a Bell experiment and the closest classical probability. We are then interested in the trace distance between $q(a, b, x, y) = q(a, b|x, y)p(x, y)$ and $p(a, b, x, y) = p(a, b|x, y)p(x, y)$, where $p(x, y)$ is the probability of the inputs that we choose to fix as the uniform distribution, that is, $p(x, y) = \frac{1}{|x||y|}$.

We define the quantifier $Q_{NL}(\mathbf{q})$ for the nonlocality of behavior $\mathbf{q} = q(a, b|x, y)$ as

$$\begin{aligned} Q_{NL}(\mathbf{q}) &= \frac{1}{|x||y|}\min_{\mathbf{p}\in\mathcal{L}} D(\mathbf{q},\mathbf{p}) \\ &= \frac{1}{2|x||y|}\min_{\mathbf{p}\in\mathcal{L}} \sum_{a,b,x,y} |q(a,b|x,y) - p(a,b|x,y)|. \end{aligned} \tag{2.6}$$

This is the minimum trace distance between the distribution under testing and the set of local distributions. Geometrically, it can be understood as how far we are from that set with respect to the ℓ_1-norm, as shown in Fig. 1.

B. Quantifying nonlocality of a quantum state

We now address the problem of quantifying nonlocality of a quantum state ρ. Since from ρ we can generate many different nonlocal behaviors p, this is not a trivial problem.

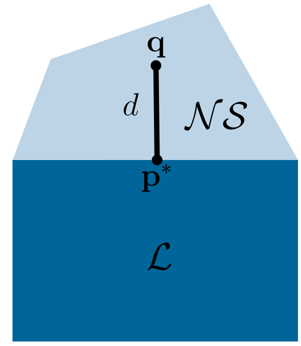

FIG. 1. Schematic drawing of a distribution $\mathbf{q} \in \mathcal{NS}$ and $d = Q_{NL}(\mathbf{q})$, the distance (with respect to the ℓ_1 norm) from $\mathbf{q}$ to the closest local distribution $\mathbf{p}^* \in \mathcal{L}$.

One way of defining a nonlocality quantifier for quantum states is to maximize the degree of violation of a Bell inequality over all possible measurements for Alice and Bob and to associate a greater numerical violation with a greater degree of nonlocality. This association has generated some debatable conclusions. For example, using the usual measure, the so-called anomaly of nonlocality appears [32,45]. Consider the CGLMP inequality in the (2,2,3) scenario [46] and a two-qutrit system with the state

$$|\psi(\gamma)\rangle = \frac{1}{\sqrt{2+\gamma^2}}(|00\rangle + \gamma|11\rangle + |22\rangle). \tag{2.7}$$

For the maximally entangled state $\gamma = 1$, the optimal choice of measurements leads to a value of the CGLMP function equals to $\frac{4(2\sqrt{3}+3)}{9} \simeq 2.873$ [46]. However, the authors in [32] found that, for the very same choice of settings, another state gives a higher violation. Specifically, the violation of the CGLMP inequality equal to $1 + \sqrt{\frac{11}{3}} \simeq 2.915$ is obtained for the nonmaximally entangled state with $\gamma = \frac{\sqrt{11}-\sqrt{3}}{2} \simeq 0.792$. This fact is known as anomaly of nonlocality.

In Ref. [23] the authors presented an alternative measure to quantify nonlocality, called volume of violation. While the previous measure takes only the settings that lead to the maximum violation, the volume of violation takes into account all the settings that produce a violation of a Bell inequality. For the calculation of this quantity, for a particular state, we perform an integration in the region that leads to the violation of a fixed Bell inequality [45]. In general, we can write

$$V_I(\rho) = \frac{1}{V_T}\int_\Gamma d^n x, \tag{2.8}$$

where Γ is the set of measurement choices for Alice and Bob that lead to a violation of the inequality I for state ρ and $V_T = \int_\Lambda d^n x$ is the volume of the set Λ of all possible measurement choices for Alice and Bob. Note that $d^n x$ will display the format that gives equal weights to any setting. Thus, state ρ is more nonlocal than state σ if and only if $V_I(\rho) > V_I(\sigma)$. Also, if for state ρ the volume of violation is $V_I(\rho) = 0$, we say that state ρ is local with respect to the given inequality. Following the same reasoning, $V_I(\rho) = 1$ indicates that ρ is maximally nonlocal with respect to that inequality.

This measure uses the relative volume of measurement choices that lead to violation of a particular Bell inequality to quantify nonlocality. Hence, $V_I(\rho)$ has a direct interpretation: It corresponds to the probability of violating a particular Bell inequality with state ρ when the measurement configuration is chosen randomly.

The volume of violation is a measure of nonlocality with many good properties, as already reported in Ref. [23]. Nevertheless, $V_I(\rho)$ does not take into account all the measurement configurations that lead to nonlocal behaviors, but only the ones that lead violations of a particular Bell inequality. Except for the simplest scenario (2,2,2), there are many nonequivalent families of Bell inequalities and hence $V_I(\rho)$ gives only a lower bound to the relative volume of measurement choices that lead to nonlocality.

In Ref. [24] the authors considered a modification of the volume of violation, called nonlocal volume, replacing the violation of a Bell inequality by membership of the corresponding behavior in the polytope of local correlations. Hence, the nonlocal volume takes into account all the settings that produce a nonlocal behavior, which is in general strictly larger than the set of behaviors violating a single Bell inequality. For the calculation of this quantity, for a particular state, we perform an integration in the region Δ of the set of measurement setups that lead to a nonlocal behavior. In general, we can write

$$V(\rho) = \frac{1}{V_T}\int_{\Delta} d^n x, \qquad (2.9)$$

where V_T is defined as before. Again we consider an integration that gives equal weights to any setting.

III. BALANCED CLASS OF QUANTIFIERS

Now we are in a position to define a hybrid class of figures of merit, which corresponds to a merge of the nonlocal volume and some nonlocality quantifier for individual behaviors. This amounts to an integration over the set of measurement choices that give a nonlocal behavior but using a nonlocality quantifier Q defined in the set of behaviors as a weight in the integral

$$V_Q(\rho) = \frac{1}{V_T}\int_{\Delta} Q(x) d^n x, \qquad (3.1)$$

where $Q(x)$ is the quantifier Q for the behavior generated by the state and measurement settings x and Δ and V_T are defined as before. We consider only faithful quantifiers: $Q(\mathbf{p}) > 0$ if and only if $\mathbf{p} \notin \mathcal{L}$.

This quantifier, which we call Q-weighted nonlocal volume, can be interpreted in a similar way. As the nonlocal volume, it takes into account all the settings that produce a nonlocal behavior for state ρ, but it sums with a higher weight the behaviors that are more nonlocal according to the quantifier Q. In particular, we are interested in the nonlocality quantifier for states obtained when we choose $Q = Q_{NL}$,

$$V_{NL}(\rho) = \frac{1}{V_T}\int_{\Delta} Q_{NL}(x) d^n x, \qquad (3.2)$$

where $Q_{NL}(x)$ is the trace distance for the behavior generated by the state and measurement settings x. We call this quantifier trace-weighted nonlocal volume.

A. Properties

In Ref. [38] the authors showed that the nonlocal volume is invariant under local unitaries and that it is strictly positive for entangled bipartite states. The proofs can be slightly modified to show that these properties are also true for V_Q.

Theorem 1. V_Q is invariant under local unitaries.

Proof. Let $\rho' = U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger$, where U_1 and U_2 are local unitaries for subsystems 1 and 2, respectively. The behavior generated with measurements $\{M_i\}$ and $\{N_i\}$ and state ρ is equal to the behavior generated with measurements $\{U_1^\dagger M_i U_1\}$ and $\{U_2^\dagger N_i U_2\}$ and state ρ'. Hence, the sets Δ are the same for ρ and ρ', which implies that $V_Q(\rho) = V_Q(\rho')$. ■

Theorem 2. If Q is a continuous function, for all pure bipartite entangled states, in a scenario with at least two choices of two-outcome measurements, V_Q is strictly positive, that is, $V_Q(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is a product state.

Proof. Since $|\psi\rangle$ is entangled, we know from [47] that there exist measurements $\{M_i\}$ and $\{N_j\}$ in the simplest scenario (2,2,2) such that the corresponding behavior is nonlocal. By continuity of the probabilities $p(ab|xy)$ as a function of the measurements and continuity of Q as a function of $\mathbf{p}$, there is a ball around $\{M_i\}$ and $\{N_j\}$ such that, for any choice of measurements inside this ball, Q is strictly positive. Since we are integrating a strictly positive function over a set that is of measure larger than zero, this implies that $V_Q(|\psi\rangle) > 0$.

On the other hand, if $|\psi\rangle$ is separable, every behavior generated with $|\psi\rangle$ is local and hence $Q(x) = 0$ for every x and $V_Q(|\psi\rangle) = 0$. ■

It is also easy to see that $V_Q(\rho)$ is always upper bounded by $V(\rho)$ for any Q satisfying $0 \leqslant Q \leqslant 1$. In fact, if Q is faithful, we can write

$$V_Q(\rho) = \frac{1}{V_T}\int_{\Lambda} Q(x) d^n x \qquad (3.3)$$

and

$$V(\rho) = \frac{1}{V_T}\int_{\Lambda} \chi(x) d^n x, \qquad (3.4)$$

where χ is the characteristic function of the nonlocal set, that is,

$$\chi(\mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{p} \notin \mathcal{L} \\ 0 & \text{if } \mathbf{p} \in \mathcal{L}. \end{cases} \qquad (3.5)$$

If $\mathbf{p}$ is local and Q is faithful, $Q(\mathbf{p}) = \chi(\mathbf{p}) = 0$. If $\mathbf{p}$ is nonlocal, $Q(\mathbf{p}) \leqslant 1 = \chi(\mathbf{p})$, which implies that $Q(\mathbf{p}) \leqslant \chi(\mathbf{p})$ for all $\mathbf{p}$ and hence $V_Q(\rho) \leqslant V(\rho)$ for all ρ.

Reference [38] also showed that the nonlocal volume goes to 1 in the limit where both parties have an infinite number of measurements. The proof in this case cannot be easily modified to show that this is also true for V_Q. Numerical results in Fig. 2 indicate that if we choose $Q = Q_{NL}$, V_{NL} seems to increase monotonically with the number of measurements n, which would be the desired behavior. However, it is not possible to claim its limit as n goes to infinity is 1 based on this evidence only. Since V_Q is always smaller than the

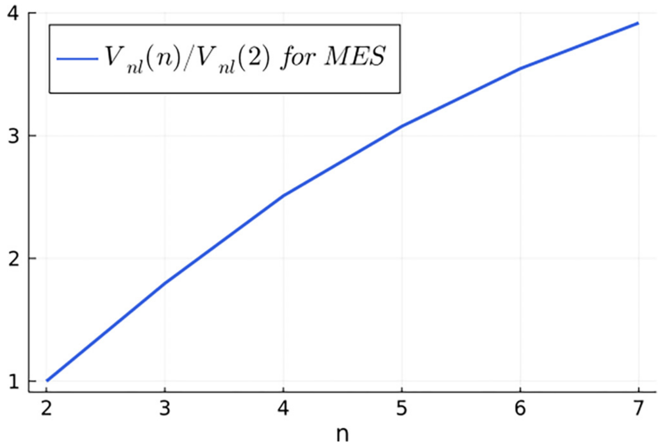

FIG. 2. Trace-weighted nonlocal volume for the maximally entangled two-qubit state as a function of the number of measurements each part can execute in the Bell scenario, i.e., scenarios of the type (2, n, 2). In this figure we normalized the (2,2,2) result to unity.

nonlocal volume, we cannot discard the possibility that V_Q goes to a as n goes to infinity, with $0 < a < 1$. Whether or not this property holds for V_{NL} is an open problem.

B. Revisiting simple scenarios

In this section we test the quantifier V_{NL}, analyzing its performance when compared to previous results in the literature. Behaviors are determined to be nonlocal in our simulations for trace distances with magnitude larger than 10^{-8}. We start by checking the effect of the weight on the nonlocal volume for the simplest situation, the (2,2,2) scenario. The local set in this case has a simple structure, since there is only one family of Bell inequalities, and we expect that the plots for the entropy of entanglement, the nonlocal volume, and the Q-weighted nonlocal volume all have the same comportment if Q is a continuous faithful nonlocality quantifier for behaviors. We consider the family of states parametrized by

$$|\psi(\alpha)\rangle = \alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle \qquad (3.6)$$

and plot the nonlocal volume (red dotted line), the trace-weighted nonlocal volume (blue solid line), and the entropy of entanglement (black dashed line) in Fig. 3(a). For both the nonlocal volume and the trace-weighted nonlocal volume, the maximum nonlocality is attained at the maximally entangled state. We can also see that the weighted version is smaller than or equal to the nonweighted version considering a normalization using the maximum value for each curve.

The next interesting case is the (2,2,3) scenario (two parts sharing two qubits with three measurements per part). In Fig. 3(b) the weighted and the nonweighted versions are compared with the entropy of entanglement for the same qubit states, Eq. (3.6). Despite the more complex scenario, the same qualitative features are observed as compared to the CHSH scenario.

Now we get to the scenario in which the anomaly was first observed. Since the nonlocal volume has been shown to be nonanomalous, we only compare V_{NL} and the entropy of entanglement for the family of states given in (2.7) (see

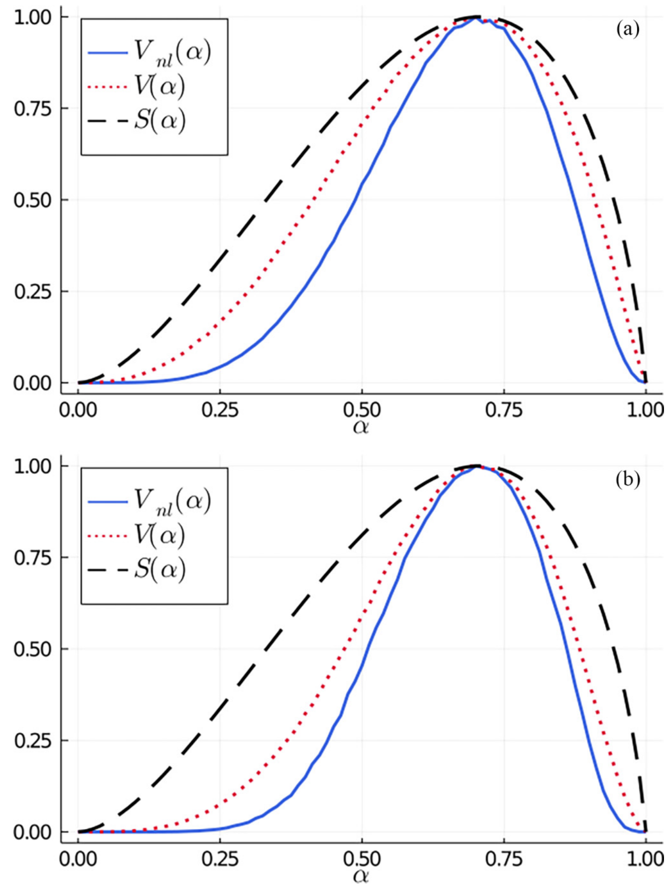

FIG. 3. In blue (solid line) our quantifier lies below its nonweighted version in red (dotted line), as expected, while maintaining the point of maximum, which coincides with the maximum of the dashed curve representing the entropy of entanglement, for (a) the CHSH scenario and (b) the (2,2,3) scenario (also known as the 3322 scenario).

Fig. 4). For this family of states, we see that the maximum nonlocality is achieved by the maximally entangled state, as in the previous scenarios with qubits. The same behavior is observed for the nonweighted version, as shown in Ref. [24]. However, in general, V_{NL}, as is the case for V, is not a monotonic function of the entropy of entanglement. This property is known as weak anomaly of nonlocality. Some persistent nonmonotonicity among several kinds of nonlocality measures and the entropy of entanglement S should in fact be expected. If it were possible to define a nonlocality quantifier corresponding to a convex function of S, for all states and scenarios, one would be forced to conclude that entanglement and Bell nonlocality would after all be the *same* resource. This would be hardly acceptable.

It is probably a reasonable conjecture to state that it is not possible to define a measure of nonlocality MN which is a monotonic function of the entropy of entanglement S for all Bell scenarios and arbitrary pure quantum states. As we saw, however, it is possible to provide quantifiers for which maximum-resource states do coincide. In the remainder of this work, we investigate how the arguably necessary weak anomaly manifests for these quantifiers. We will see that by pushing the mismatch between nonlocality and entanglement

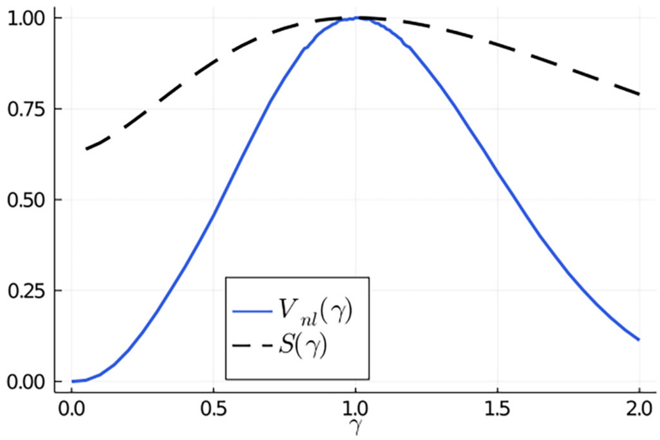

FIG. 4. The maximum of the trace-weighted nonlocal volume (blue solid line) is reached at the state exhibiting the maximum entanglement entropy (black dashed line).

away from maximum-resource states, the effect shows up for lower entanglement in the range of parameters corresponding to a continuous change of rank, when the state ceases (starts) to have a component in some subspace of $\mathcal{H}$.

IV. WEAK ANOMALY OF NONLOCALITY AND RANK TRANSITION

Still in the same system of qutrits, we consider the GHZ(α) states parametrized as

$$|\mathrm{GHZ}(\alpha)\rangle = \sin(\alpha)|00\rangle + \frac{\cos(\alpha)}{\sqrt{2}}(|11\rangle + |22\rangle). \tag{4.1}$$

Interestingly, nonlocality does not increase monotonically with α for small values of this parameter. Figure 5 shows the weighted and nonweighted versions of nonlocal volume as a function of α. The inset shows what is happening close to the local minima, emphasizing the different behavior of the quantifiers. While the entropy of entanglement S (not shown)

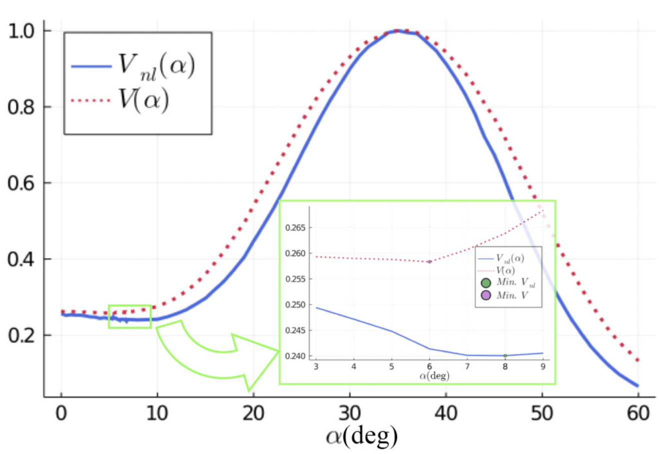

FIG. 5. Weak anomaly from the perspective of the nonlocal volume and trace-weighted nonlocal volume. We see how the weight compresses the normalized curve slightly. The close-up view of the region containing the local minima shows that each quantifier attains the minimum at a different angle (bold ticks).

grows monotonically until the maximally entangled state is attained, both the nonlocality quantifiers considered here are decreasing functions of α, between $\alpha = 0$ and some small $\alpha_{\mathrm{critical}}$. As in the previous cases, they reach the peak at the maximally entangled state, but have a shallow local minimum for some small α. Note that the passage from $\alpha = 0$ to $\alpha > 0$ corresponds to the transition between the rank-2 maximally entangled state $(|11\rangle + |22\rangle)/\sqrt{2}$ to the rank-3 state approximately equal to $\alpha|00\rangle + (1-\alpha^2)(|11\rangle + |22\rangle)/\sqrt{2}$. On the one hand, the state becomes full rank, but, on the other hand, it becomes a less symmetric state. This compromise has distinct effects on the entropy of entanglement and on the nonanomalous nonlocality measures addressed here.

The local minima are reached for different values of α for the weighted and nonweighted versions ($\alpha_{\mathrm{critical}} \approx 6^{\mathrm{o}}$ for V and $\alpha_{\mathrm{critical}} \approx 8^{\mathrm{o}}$ for V_{NL}). This shows that the detailed structure of the weak anomaly is not an intrinsic characteristic of the scenario, but is dependent on the quantifier.

To investigate further how entanglement and nonlocality can behave differently, we study how these quantities change when we go from a lower-rank state to a full-rank state. We first show that the entropy of entanglement never decreases in this situation.

Theorem 3. The entropy of entanglement always increases when making a continuous transition from a lower-rank state $|\psi\rangle$ to a higher-rank state $|\psi'\rangle$.

Proof. We will consider the higher-rank state to be full rank; the extension of the proof to the general case described above is immediate. Let $|\psi\rangle \in \mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$, $\dim \mathcal{H} = D$. Suppose that the rank of $|\psi\rangle$ is smaller than $\sqrt{D}$, so its Schmidt decomposition reads

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |ii\rangle, \tag{4.2}$$

where $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$ and $d < \sqrt{D}$, leading to the entropy of entanglement: $E = -\sum_{i=0}^{d-1} |\alpha_i|^2 \log_2 |\alpha_i|^2$ Ebits. Consider now another state $|\psi'\rangle$ which is full rank $\mathrm{rank}(|\psi'\rangle) = \sqrt{D}$, $|\psi'\rangle \propto \delta|\phi\rangle + |\psi\rangle$, where $|\phi\rangle = \sum_{i=d}^{\sqrt{D}-1} \beta_i |ii\rangle$, $\sum_{i=d}^{\sqrt{D}-1} |\beta_i|^2 = 1$, and $\delta \in \mathbb{C}$. Using that $\langle\phi|\psi\rangle = 0$,

$$|\psi'\rangle = \frac{1}{\sqrt{1+|\delta|^2}}(\delta|\phi\rangle + |\psi\rangle),$$

whose entropy of entanglement is given by

$$E' = -\frac{|\delta|^2}{1+|\delta|^2} \sum_{i=d}^{\sqrt{D}-1} |\beta_i|^2 \log\left(\frac{|\delta|^2|\beta_i|^2}{1+|\delta|^2}\right) - \frac{1}{1+|\delta|^2} \sum_{i=0}^{d-1} |\alpha_i|^2 \log\left(\frac{|\alpha_i|^2}{1+|\delta|^2}\right).$$

Thus

$$E' = \frac{1}{1+|\delta|^2}[-|\delta|^2 \log|\delta|^2 + |\delta|^2 E_\phi + |\delta|^2 \log(1+|\delta|^2) + E + \log(1+|\delta|^2)].$$

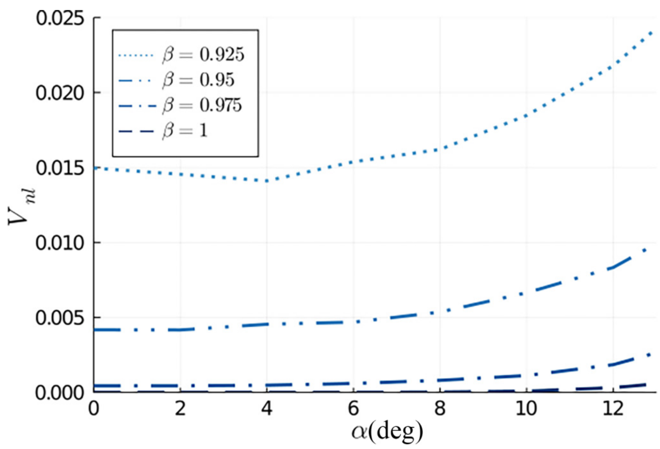

FIG. 6. Trace-weighted nonlocal volume for different families of states as a function of α; a family in this case is labeled by the value of its β parameter in GHZ(α, β). Lighter curves stand for greater entanglement between states $|11\rangle$ and $|22\rangle$.

Now we take $|\delta|^2 \ll 1$,

$$E' \to (1 - |\delta|^2)[-|\delta|^2 \log |\delta|^2 + |\delta|^2 E_\phi + E + |\delta|^2 + O(|\delta|^4)].$$

Since $-(|\delta|^2 \log |\delta|^2)/|\delta|^2 \to \infty$ when $\delta \to 0$, the leading-order expression is $E' \sim (1 - |\delta|^2)[-|\delta|^2 \log |\delta|^2 + E]$. So pure-state bipartite entanglement always increases when we make a continuous change from a lower-rank state $|\psi\rangle$ to a higher-rank state $|\psi'\rangle$. In addition, the leading-order entanglement gain is independent of the "direction" from which the new subspace is visited (independent of $|\phi\rangle$ and of E_ϕ):

$$\Delta E = E' - E \sim -|\delta|^2 \log |\delta|^2 > 0 \tag{4.3}$$

for $|\delta|^2$ small. ■

As we already know, nonlocality can behave differently when changing from a low-rank to a higher-rank state. Figure 6 shows an investigation within regions of states with partial entanglement between states $|11\rangle$ and $|22\rangle$ in the GHZ state with an extra parameter:

$$|\mathrm{GHZ}(\alpha, \beta)\rangle = \sin(\alpha)|00\rangle + \frac{\cos(\alpha)}{\sqrt{2}}(\beta|11\rangle + \sqrt{1 - \beta^2}|22\rangle). \tag{4.4}$$

It is clear that for $\beta = 1$, the state with $\alpha = 0$ is separable and thus Bell local. Therefore, by continuity, there must be some finite interval ending at $\beta = 1$ for which all Q-weighted nonlocal volumes become increasing functions of α, restoring, for this particular scenario and these states, the monotonicity between nonlocality and entanglement. Notice in Fig. 6 that the minimum of the anomaly moves towards zero degree as we increase the value of β, i.e., as we consider a starting point closer to a rank-1 state ($|11\rangle$). Somewhere between $\beta = 1$ and $\beta = 0.925$ the weak anomaly shows up again. This is again related to a change in the state's rank.

This illustrates how the relation between entanglement and nonlocality is complex, depending on many aspects of the scenario and on the tools we use to quantify these properties. Although this may appear counterintuitive at first sight, since nonlocality is a consequence of entanglement, we stress that entanglement and nonlocality are indeed different resources, which supports the conjecture that they cannot be described by figures of merit which are monotonic functions of each other, in general.

V. CONCLUSION

The contribution of this work is twofold. First, we have provided an alternative way of quantifying nonlocality of states based on the Bell nonlocality of behaviors. The key difference from preceding quantities was the introduction of a quantifier of nonlocality to weight each contribution from each behavior in the nonlocal volume. An additional degree of freedom was then introduced. Here we explored it by considering the simplest possibility among the ones at our disposal in the literature when the question is how to compare two quantum probability distributions: the trace distance. We proved that this quantifier has several good properties, including its formulation in terms of a linear programming.

Second, we used two nonanomalous nonlocality quantifiers to investigate the persistent weak anomaly. For the situations addressed here this phenomenon arises in the neighborhood of a change in the rank of the state. The local minimum for nonlocality with the trace-weighted nonlocal volume occurs at a different state as compared to the minimum for the non-weighted version. We conjecture that this nonmonotonicity, despite the coincidence of the maxima, is an unavoidable manifestation of the intrinsic inequivalence between entanglement and nonlocality. If no anomaly remains, that is, if nonlocality increases monotonically with entanglement, in general, we would conclude that nonlocality and entanglement are equivalent resources, which we know not to be true.

It is a topic for further investigation how the behavior of the weighted version would be for different quantifiers Q and whether we can prove that every nonlocality quantifier for states exhibits some kind of anomaly. Moreover, its robustness against noisy systems also remains to be tested.

It should also be noted that in the integral defining the nonlocal volume, we considered projective measurements only. The next natural generalization would be to consider the set of positive-operator-valued measures (POVMs), but two difficulties appear. First, there is no natural definition of uniform measure for POVMs, which makes the quantifier strongly dependent on the choice of measure we choose to sample the measurements. Second, numerical results show that the probability of finding a nonlocal behavior when sampling over the set of POVMs is very small. In fact, in the first attempt we made, using Neumark's dilation theorem and the Haar measure in the set of unitaries in the larger Hilbert space, typically a nonlocal behavior was observed in one out of 10×10^6 settings for the (2,2,2) scenario (even for the maximally entangled state), a simulation which, in addition, takes much more time to execute.

It would be interesting to know whether it is possible to define a quantifier of nonlocality that presents the monotonicity of the weighted nonlocal volume with entanglement for projective measurements. However, we consider that this is too strong a constraint, probably not possible to fulfill.

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[2] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935).
[3] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, Rev. Mod. Phys. **82**, 665 (2010).
[4] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).
[5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[6] G. Gour and R. W. Spekkens, The resource theory of quantum reference frames: Manipulations and monotones, New J. Phys. **10**, 033023 (2008).
[7] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource Theory of Quantum States Out of Thermal Equilibrium, Phys. Rev. Lett. **111**, 250404 (2013).
[8] J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A: Math. Theor. **47**, 424017 (2014).
[9] A. Winter and D. Yang, Operational Resource Theory of Coherence, Phys. Rev. Lett. **116**, 120404 (2016).
[10] B. Coecke, T. Fritz, and R. W. Spekkens, A mathematical theory of resources, Inf. Comput. **250**, 59 (2016).
[11] S. Abramsky, R. S. Barbosa, and S. Mansfield, Contextual Fraction as a Measure of Contextuality, Phys. Rev. Lett. **119**, 050504 (2017).
[12] B. Amaral, A. Cabello, Marcelo T. Cunha, and L. Aolita, Noncontextual Wirings, Phys. Rev. Lett. **120**, 130403 (2018).
[13] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).
[14] S. Popescu and D. Rohrlich, Quantum nonlocality as an axiom, Found. Phys. **24**, 379 (1994).
[15] P. H. Eberhard, Background level and counter efficiencies required for a loophole-free Einstein-Podolsky-Rosen experiment, Phys. Rev. A **47**, R747 (1993).
[16] B. F. Toner and D. Bacon, Communication Cost of Simulating Bell Correlations, Phys. Rev. Lett. **91**, 187904 (2003).
[17] S. Pironio, Violations of Bell inequalities as lower bounds on the communication cost of nonlocal correlations, Phys. Rev. A **68**, 062102 (2003).
[18] W. van Dam, R. D. Gill, and P. D. Grunwald, The statistical strength of nonlocality proofs, IEEE Trans. Inf. Theory **51**, 2812 (2005).
[19] A. Acín, R. Gill, and N. Gisin, Optimal Bell Tests Do Not Require Maximally Entangled States, Phys. Rev. Lett. **95**, 210402 (2005).
[20] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, and M. M. Wolf, Operator Space Theory: A Natural Framework for Bell Inequalities, Phys. Rev. Lett. **104**, 170405 (2010).
[21] M. J. W. Hall, Relaxed Bell inequalities and Kochen-Specker theorems, Phys. Rev. A **84**, 022102 (2011).
[22] R. Chaves, D. Cavalcanti, L. Aolita, and A. Acín, Multipartite quantum nonlocality under local decoherence, Phys. Rev. A **86**, 012108 (2012).
[23] E. A. Fonseca and F. Parisio, Measure of nonlocality which is maximal for maximally entangled qutrits, Phys. Rev. A **92**, 030101(R) (2015).
[24] A. de Rosier, J. Gruca, F. Parisio, T. Vértesi, and W. Laskowski, Multipartite nonlocality and random measurements, Phys. Rev. A **96**, 012101 (2017).
[25] R. Chaves, R. Kueng, J. B. Brask, and D. Gross, Unifying Framework for Relaxations of the Causal Assumptions in Bell's Theorem, Phys. Rev. Lett. **114**, 140403 (2015).
[26] M. Ringbauer, C. Giarmatzi, R. Chaves, F. Costa, A. G. White, and A. Fedrizzi, Experimental test of nonlocal causality, Sci. Adv. **2**, e160016 (2016).
[27] A. Montina and S. Wolf, Information-based measure of nonlocality, New J. Phys. **18**, 013035 (2016).
[28] J. B. Brask and R. Chaves, Bell scenarios with communication, J. Phys. A: Math. Theor. **50**, 094001 (2017).
[29] R. Gallego and L. Aolita, Nonlocality free wirings and the distinguishability between Bell boxes, Phys. Rev. A **95**, 032118 (2017).
[30] V. S. Gomes and R. M. Angelo, Nonanomalous measure of realism-based nonlocality, Phys. Rev. A **97**, 012123 (2018).
[31] S. G. A. Brito, B. Amaral, and R. Chaves, Quantifying Bell nonlocality with the trace distance, Phys. Rev. A **97**, 022111 (2018).
[32] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, Quantum nonlocality in two three-level systems, Phys. Rev. A **65**, 052325 (2002).
[33] N. Brunner, N. Gisin, and V. Scarani, Entanglement and nonlocality are different resources, New J. Phys. **7**, 88 (2005).
[34] A. Barasiński and M. Nowotarski, Volume of violation of Bell-type inequalities as a measure of nonlocality, Phys. Rev. A **98**, 022132 (2018).
[35] D. Kaszlikowski, P. Gnaciński, M. Żukowski, W. Miklaszewski, and A. Zeilinger, Violations of Local Realism by Two Entangled N-Dimensional Systems are Stronger Than for Two Qubits, Phys. Rev. Lett. **85**, 4418 (2000).
[36] W. Laskowski, J. Ryu, and M. Żukowski, Noise resistance of the violation of local causality for pure three-qutrit entangled states, J. Phys. A: Math. Theor. **47**, 424019 (2014).
[37] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[38] V. Lipinska, F. J. Curchod, A. Máttar, and A. Acín, Towards an equivalence between maximal entanglement and maximal quantum nonlocality, New J. Phys. **20**, 063043 (2018).
[39] A. Fonseca, A. de Rosier, T. Vértesi, W. Laskowski, and F. Parisio, Survey on the Bell nonlocality of a pair of entangled qudits, Phys. Rev. A **98**, 042105 (2018).
[40] A. de Rosier, J. Gruca, F. Parisio, T. Vértesi, and W. Laskowski, Strength and typicality of nonlocality in multisetting and multipartite Bell scenarios, Phys. Rev. A **101**, 012116 (2020).
[41] A. Barasiński, A. Černoch, W. Laskowski, K. Lemr, T. Vértesi, and J. Soubusta, Experimentally friendly approach towards nonlocal correlations in multisetting N-partite Bell scenarios, Quantum **5**, 430 (2021).
[42] M. Fitzi, E. Hänggi, V. Scarani, and S. Wolf, The non-locality of n noisy Popescu-Rohrlich boxes, J. Phys. A: Math. Theor. **43**, 465305 (2010).
[43] S. Abramsky, R. S. Barbosa, and S. Mansfield, in *Proceedings of the 13th International Conference on Quantum Physics and Logic, Glasgow, 2016*, edited by R. Duncan and C. Heunen (EPTCS, 2016).

[44] K. T. Goh, J. Kaniewski, E. Wolfe, T. Vértesi, X. Wu, Y. Cai, Y.-C. Liang, and V. Scarani, Geometry of the set of quantum correlations, Phys. Rev. A **97**, 022104 (2018).
[45] A. A. Méthot and V. Scarani, An anomaly of non-locality, Quantum Inf. Comput. **7**, 157 (2007).
[46] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Bell Inequalities for Arbitrarily High-Dimensional Systems, Phys. Rev. Lett. **88**, 040404 (2002).
[47] N. Gisin, Bell's inequality holds for all non-product states, Phys. Lett. A **154**, 201 (1991).