Complete monogamy of multipartite quantum mutual information

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Quantum mutual information not only displays the mutual information in the system but also demonstrates some quantum correlation beyond entanglement. We explore here two alternatives of the multipartite quantum mutual information (MQMI) based on the von Neumann entropy according to the framework of the *complete* measure of the multiparticle quantum system. We show that these two MQMIs are complete and are monogamous on pure states, and one of them (we call it type-1) is not only *completely monogamous* but also *tightly complete monogamous*, while the other one (we call it type-2) is not. Moreover, we present another two MQMIs by replacing the von Neumann entropy with the Tsallis q-entropy from the former two ones. It is proved that one of them displays some degree of "completeness" as a measure of the multiparticle quantum system, but the other one is not even non-negative and thus it cannot be an alternative of MQMI. We also discuss the triangle relation for these three alternatives of MQMI. It is shown that the triangle inequalities hold for the former two MQMIs as that of entanglement measure but the later one fails. We thus can conclude that the type-1 von Neumann entropy MQMI is a "fine" measure of multipartite quantum correlation.

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I. INTRODUCTION

Quantum correlation, such as entanglement [1], Einstein-Podolsky-Rosen steering [2,3], quantum discord [4,5], etc., has been shown to be an essential resource to achieve quantum advantages in various nonclassical information processing tasks [1,6–12]. One of the foremost issues in this area is to understand and quantify the various forms of quantum correlations, especially for the multiparticle quantum system. Consequently, a series of multipartite entanglement measures [13–18], multipartite quantum discord [19–21], and multipartite quantum mutual information (MQMI) [22,23] have been proposed.

From the information-theoretical point of view, another crucial issue for multiparticle quantum systems is the distribution of the correlation up to the given measure. The first contribution in this connection is the monogamy relation of entanglement [24], which states that, unlike classical correlations, if two parties A and B are maximally entangled, then neither of them can share entanglement with a third party C. Entanglement monogamy has many applications not only in quantum physics [25–27] but also in other area of physics, such as no-signaling theories [28,29], condensed matter physics [30–32], statistical mechanics [25], and even black-hole physics [33]. Particularly, it is the key feature that guarantees quantum key distribution is secure [24,34]. The fundamental matter in this context is to determine whether a given measure of quantum correlation is monogamous. Indeed, intense research has been undertaken in this direction. It has been proved that almost all the bipartite entanglement measures so far are monogamous [28,35–43]. However, these monogamy relations discussed via the bipartite measures (e.g., the entanglement measures) display certain drawbacks: only the relations among $A|BC$, AB, and AC are revealed, and the global correlation in ABC and the correlation contained in part BC are missing [44], where the vertical bar indicates the bipartite split across which we measure the (bipartite) correlation. In addition, the correlation between any partitions or any subsystem(s) with the coarsening relation cannot be compared with each other thoroughly by means of the bipartite measure. To address such a subject, the so-called *complete multipartite measures* and the *complete monogamy relation* has been explored for multipartite systems [45–47]. In such a context, a measure of multipartite correlation should be *complete* in the sense that the distribution of the correlation could be depicted exhaustively [45–47].

It has been shown that many complete multipartite entanglement measures are completely monogamous; i.e., any tripartite state (we take the tripartite case here) that satisfies [45]

$$E(ABC) = E(AB) \tag{1}$$

implies $E(AC) = E(BC) = 0$, which is equivalent to the assertion that

$$E^{\alpha}(ABC) \geqslant E^{\alpha}(AB) + E^{\alpha}(AC) + E^{\alpha}(BC) \tag{2}$$

holds for any state for some $\alpha > 0$ whenever E is continuous, where E is a tripartite entanglement measure. In Ref. [46], with the same strategy as the complete multipartite entanglement measure established in Ref. [45], the concept of complete multipartite quantum discord is investigated and is proved that the mulitipartite quantum discord is completely monogamous if it is complete.

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The mutual information, originally defined for the classical system, i.e., the reciprocal information that is common to or shared by two or more parties, has an authoritative stand in the arena of information theory. Quantum mutual information is well defined for bipartite quantum systems, i.e.,

$$\begin{aligned} I(A:B) &= S(A)+S(B)-S(AB) \\ &= S(AB\|A\otimes B)\geqslant 0, \end{aligned} \tag{3}$$

where $S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$ is the von Neumann entropy and $S(\rho\|\sigma)=\mathrm{Tr}(\rho\log_2\rho-\rho\log_2\sigma)$ is the quantum relative entropy, $S(X):=S(\rho^X)$, and $\rho^{A,B}=\mathrm{Tr}_{B,A}\rho^{AB}$ is the reduced state of ρ^{AB}. It reflects the total correlation between the two subsystems [48]. It can be generalized into the multipartite case in different ways [22,23]. A natural n-party quantum mutual information (QMI) defined in the literature is

$$\begin{aligned} &I(A_1:A_2:\cdots:A_n) \\ &\quad := \sum_{k=1}^{n} S(A_k) - S(A_1A_2\cdots A_n) \\ &\quad = S(A_1A_2\cdots A_n\|A_1\otimes A_2\otimes\cdots\otimes A_n). \end{aligned} \tag{4}$$

Another alternative is [23]

$$\begin{aligned} &I'(A_1:A_2:\cdots:A_n) \\ &\quad := \sum_{k=1}^{n} S(\overline{A_k}) - (n-1)S(A_1A_2\cdots A_n) \\ &\quad = S[(A_1A_2\cdots A_n)^{\otimes n-1}\|\overline{A_1}\otimes\overline{A_2}\otimes\cdots\otimes\overline{A_n}], \end{aligned} \tag{5}$$

where $\overline{X}$ denotes the subsystems complementary to those of X. It is clear that $I\geqslant 0$ and $I'\geqslant 0$ and that $I(A_1:A_2:\cdots:A_n)=0$ [or $I'(A_1:A_2:\cdots:A_n)=0$] if and only if $\rho^{A_1A_2\cdots A_n}$ is a product state, since the relative entropy is nonnegative and $S(\rho\|\sigma)=0$ iff $\rho=\sigma$. I coincides with I' for $n=2$ and this is the trivial case. Any nonproduct state contains some quantum correlation [49,50], and furthermore it seems that the mutual information increases when the entanglement increases [23]. So the QMI displays some quantum correlation beyond entanglement in the system, and thus it can be also regarded as a measure of some kind of quantum correlation.

The main purpose of this paper is to investigate whether the MQMI is a well-defined multipartite measure from the strategy in Refs. [45–47]. Namely, whether the MQMI is complete, monogamous, and completely monogamous. Throughout this paper, we let $\mathcal{H}^{A_1A_2\cdots A_n}$ be an n-partite Hilbert space with finite dimension and we let $\mathcal{S}^X$ be the set of density operators acting on $\mathcal{H}^X$. ρ^X (sometimes ρ_X) denotes the state in $\mathcal{S}^X$.

The rest of this paper is arranged as follows. In Sec. II, we review the notion of the coarser relation for multipartite partition of multiparticle states, which is convenient for discussing the complete measure of multipartite quantum correlation. Section III discusses whether the MQMIs I and I' are complete with the same spirit as the complete multipartite entanglement measure and the complete multipartite quantum discord put forward in the literature. In Sec. IV, we show that I is completely monogamous but I' is not, and that they are monogamous only on pure states. In Sec. V we establish the mutual information in terms of the Tsallis q-entropy and explore the complete monogamy accordingly. Furthermore, we explore the triangle inequality for these different MQMIs and the relation with entanglement in Sec. VI. Finally, we conclude with some discussions in Sec. VII.

II. COARSER RELATION OF MULTIPARTITE PARTITION

We recall the coarser relation of multipartite partition proposed in Ref. [47]. Let $X_1|X_2|\cdots|X_k$ be a k-partition of $A_1A_2\cdots A_m$, i.e., $X_s=A_{s(1)}A_{s(2)}\cdots A_{s(f(s))}$, $s(i)<s(j)$ whenever $i<j$, and $s(p)<t(q)$ whenever $s<t$ for any possible p and q, $1\leqslant s,t\leqslant k$. For instance, partition $AB|C|DE$ is a 3-partition of $ABCDE$. Let $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ be two partitions of $A_1A_2\cdots A_n$ or a subsystem of $A_1A_2\cdots A_n$.We denote by [47]

$$X_1|X_2|\cdots|X_k \succ^a Y_1|Y_2|\cdots|Y_l, \tag{6}$$

$$X_1|X_2|\cdots|X_k \succ^b Y_1|Y_2|\cdots|Y_l, \tag{7}$$

$$X_1|X_2|\cdots|X_k \succ^c Y_1|Y_2|\cdots|Y_l \tag{8}$$

if $Y_1|Y_2|\cdots|Y_l$ can be obtained from $X_1|X_2|\cdots|X_k$ by

(a) discarding some subsystem(s) of $X_1|X_2|\cdots|X_k$,

(b) combining some subsystems of $X_1|X_2|\cdots|X_k$,

(c) discarding some subsystem(s) of some subsystem(s) X_k, provided that $X_k=A_{k(1)}A_{k(2)}\cdots A_{k(f(k))}$ with $f(k)\geqslant 2$,

respectively. For example, $A|B|C|D\succ^a A|B|D\succ^a B|D$, $A|B|C|D\succ^b AC|B|D\succ^b AC|BD$, $A|BC\succ^c A|B$, and $A|BC\succ^c A|C$.

Furthermore, if $X_1|X_2|\cdots|X_k\succ Y_1|Y_2|\cdots|Y_l$, we denote by $\Xi(X_1|X_2|\cdots|X_k-Y_1|Y_2|\cdots|Y_l)$ [47] the set of all the partitions that are coarser than $X_1|X_2|\cdots|X_k$ and either exclude any subsystem of $Y_1|Y_2|\cdots|Y_l$ or include some but not all subsystems of $Y_1|Y_2|\cdots|Y_l$. We take the five-partite system $ABCDE$ for example: $\Xi(A|B|CD|E-A|B)=\{CD|E$, $A|CD|E$, $B|CD|E$, $A|CD$, $B|CD$, $B|C|E$, $B|D|E$, $A|D|E$, $A|C|E$, $A|E$, $B|E$, $A|C$, $A|D$, $B|C$, $B|D$, $C|E$, $D|E\}$.

III. COMPLETENESS OF MUTUAL INFORMATION

In Ref. [45], the complete multipartite entanglement measure is defined. With the same spirit in mind, we discuss the completeness of the MQMI as a measure of the multipartite quantum system. For more clarity, we recall the definition of the complete multipartite entanglement measure at first. A multipartite entanglement measure $E^{(n)}$ is called a *unified* multipartite entanglement measure if it also satisfies the *unification condition* [45]: i.e., $E^{(n)}$ is consistent with $E^{(k)}$ for any $2\leqslant k<n$. The unification condition should be comprehended in the following sense [45]:

$$\begin{aligned} &E^{(n)}(\rho^{A_1A_2\cdots A_k}\otimes\rho^{A_{k+1}\cdots A_n}) \\ &\quad = E^{(k)}(\rho^{A_1A_2\cdots A_k})+E^{(n-k)}(\rho^{A_{k+1}\cdots A_n}), \end{aligned} \tag{9}$$

$$E^{(n)}(\rho^{A_1A_2\cdots A_n})=E^{(n)}(\rho^{A_{\pi(1)}A_{\pi(2)}\cdots A_{\pi(n)}}) \tag{10}$$

for any $\rho^{A_1A_2\cdots A_n}\in\mathcal{S}^{A_1A_2\cdots A_n}$ and any permutation π, and

$$E^{(k)}(X_1|X_2|\cdots|X_k)\geqslant E^{(l)}(Y_1|Y_2|\cdots|Y_l) \tag{11}$$

for any $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$ whenever $X_1|X_2|\cdots|X_k \succ^a Y_1|Y_2|\cdots|Y_l$, where $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ are two partitions of $A_1A_2\cdots A_n$ or a subsystem of $A_1A_2\cdots A_n$. $E^{(n)}$ is called a *complete* multipartite entanglement measure [45] if it satisfies both the unification condition above and Eq. (11) holds for all $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$ whenever $X_1|X_2|\cdots|X_k \succ^b Y_1|Y_2|\cdots|Y_l$ additionally. For instance, $E_f^{(n)}$ is a complete multipartite entanglement measure, and there do exist unified multipartite entanglement measures that are not complete [45]. For the coarser relation of type (c), it is automatically true for any entanglement measure; i.e., Eq. (11) holds for all states that obey the coarser relation $\succ^c$, since the partial trace is a specific LOCC (local operation and classical communication) and entanglement is nonincreasing under LOCC.

With this scenario in mind, we now begin to investigate the completeness of the MQMIs I and I'. It is clear that

$$I(A_1 : A_2 : \cdots : A_n) = I(A_{\pi(1)} : A_{\pi(2)} : \cdots : A_{\pi(n)})$$

and

$$I'(A_1 : A_2 : \cdots : A_n) = I'(A_{\pi(1)} : A_{\pi(2)} : \cdots : A_{\pi(n)})$$

for any $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$ and any permutation π. Analogous to Eq. (9), we can prove that

$$I(A_1 : A_2 : \cdots : A_n) = I(A_1 : \cdots : A_k) + I(A_{k+1} : \cdots : A_n) \tag{12}$$

whenever $I(A_1A_2\cdots A_k : A_{k+1}\cdots A_n) = 0$; i.e., $\rho^{A_1A_2\cdots A_n} = \rho^{A_1A_2\cdots A_k} \otimes \rho^{A_{k+1}\cdots A_n}$ equivalently. In fact, $I(A_1{:}A_2{:}\cdots{:}A_n) = \sum_{i=1}^n S_{A_i} - S_{A_1A_2\cdots A_n} = \sum_{i=1}^n S_{A_i} - S_{A_1A_2\cdots A_k} - S_{A_{k+1}:\cdots:A_n} = I(A_1 : \cdots : A_k) + I(A_{k+1} : \cdots : A_n)$, which is straightforward, where $S_X := S(X)$. For I', we take $n = 4$ for example. If $I'(AB : CD) = 0$ (i.e., $\rho^{ABCD} = \rho^{AB} \otimes \rho^{CD}$), then $I'(A : B : C : D) = S_{ABC} + S_{BCD} + S_{ABD} + S_{ACD} - 3S_{ABCD} = [(S_{AB} + S_C) + (S_B + S_{CD}) + (S_A + S_{CD}) + (S_{AB} + S_D)] - 3(S_{AB} + S_{CD}) = (S_A + S_B - S_{AB}) + (S_C + S_D - S_{CD}) = I'(A : B) + I'(C : D)$. In general, we can get

$$I'(A_1 : A_2 : \cdots : A_n) = I'(A_1 : \cdots : A_k) + I'(A_{k+1} : \cdots : A_n) \tag{13}$$

for any state with $I'(A_1A_2\cdots A_k : A_{k+1}\cdots A_n) = 0$. As Eqs. (9), (12), and (13) point out, if there is no mutual information between subsystem $A_1A_2\cdots A_k$ and subsystem $A_{k+1}\cdots A_n$, then the global mutual information contained only in the system $A_1A_2\cdots A_k$ and the system $A_{k+1}\cdots A_n$ independently. Henceforward, we say the measure is *additive* if it satisfies relations such as Eqs. (9), (12), and (13). Namely, mutual information and the complete multipartite entanglement measure are additive.

We now discuss whether I and I' are decreasing under coarsening of the system. Namely, whether the counterparts of Eq. (11) for I and I' are valid under the coarser relations $\succ^a$, $\succ^b$, and $\succ^c$.

Proposition 1. Let $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ be two partitions of $A_1A_2\cdots A_n$ or subsystems of $A_1A_2\cdots A_n$. If $X_1|X_2|\cdots|X_k \succ Y_1|Y_2|\cdots|Y_l$, then

$$I(X_1 : X_2 : \cdots : X_k) \geqslant I(Y_1 : Y_2 : \cdots : Y_l) \tag{14}$$

and

$$I'(X_1 : X_2 : \cdots : X_k) \geqslant I'(Y_1 : Y_2 : \cdots : Y_l) \tag{15}$$

hold for any $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$.

The proof for Proposition 1 is provided in Appendix A. Proposition 1 indicates that I and I' are well-defined complete measures in the sense of Refs. [45–47]. Henceforward, we call such a measure a complete measure. Under this framework, the mutual information between different subsystems can be compared with each other in a clear hierarchic structure sense, from which we can discuss the distribution of the corresponding quantity thoroughly and comprehensively.

IV. COMPLETE MONOGAMY OF I

Having discussed the underlying concept of the complete MQMI, we now restrict attention to present the definition of the complete monogamy for MQMI with the same essence as that of the complete monogamy of the multipartite entanglement [45,47] and the complete monogamy of the multipartite quantum discord [46]. Let $J = I$ or $J = I'$. With the notations aforementioned, (i) we call J is monogamous if it satisfies the *discorrelated condition*; i.e., for any state $\rho \in \mathcal{S}^{ABC}$ that satisfies

$$J(A : BC) = J(A : B), \tag{16}$$

we have that

$$J(A : C) = 0. \tag{17}$$

(ii) J is said to be completely monogamous if it satisfies the *complete discorrelated condition*; i.e., for any state $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$ that satisfies

$$J(X_1|X_2|\cdots|X_k) = J(Y_1|Y_2|\cdots|Y_l), \tag{18}$$

we have that

$$J(\Gamma) = 0 \tag{19}$$

holds for all $\Gamma \in \Xi(X_1|X_2|\cdots|X_k - Y_1|Y_2|\cdots|Y_l)$, where $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ are arbitrarily given partitions of $A_1A_2\cdots A_m$ or subsystems of $A_1A_2\cdots A_m$, and where $X_1|X_2|\cdots|X_k \succ^a Y_1|Y_2|\cdots|Y_l$. (iii) J is said to be tightly complete monogamous if we replace $\succ^a$ by $\succ^b$ in the above item (ii), and the counterpart of Eq. (18) is called the *tightly complete discorrelated condition* instead.

In such a sense, according to the proof of Theorem 1 in Ref. [42], (i) if J is monogamous, then there exists $\alpha > 0$ such that

$$J^\alpha(A : BC) \geqslant J^\alpha(A : B) + J^\alpha(A : C)$$

holds for any state in $\mathcal{S}^{ABC}$, where α is related to the dimension of $\mathcal{H}^{ABC}$. We observe here that I and I' are continuous functions since the von Neumann entropy is continuous. (ii) If J is completely monogamous, then (we take $n = 3$ for example)

$$J^\alpha(A : B : C) \geqslant J^\alpha(A : B) + J^\alpha(A : C) + J^\alpha(B : C)$$

holds for any state in $\mathcal{S}^{ABC}$ with α as above. (iii) If J is tightly complete monogamous, then

$$J^\alpha(A : B : C) \geqslant J^\alpha(A : BC) + J^\alpha(B : C)$$

holds for any state in $\mathcal{S}^{ABC}$ for some $\alpha > 0$ as above. We are now ready to present the first main result of this article, which shows that the MQMI I is a nice measure of quantumness, but another alternative I' is not, since it is neither completely monogamous nor tightly complete monogamous. For the reader's convenience, we put the proof in Appendix B ■.

Theorem 1. (i) I is monogamous only on pure states. (ii) I is not only completely monogamous but also tightly complete monogamous. (iii) I' is neither completely monogamous nor tightly complete monogamous.

Theorem 1 indicates that complete monogamy does not imply monogamy in general, although the measure is a complete one.

In Ref. [42], we showed that the Markov quantum state satisfies the disentangling condition

$$E(A|BC) = E(AB) \tag{20}$$

for any bipartite entanglement monotone E. Hereafter, we always assume that E is a bipartite entanglement monotone. Thus, from the proof of Theorem 1, it turns out that

(i) $I(A:BC) = I(A:B)$ implies $E(A|BC) = E(AB)$ and $E(C|AB) = E(CA)$;

(ii) $I'(A:B:C) = I(A:B)$ implies $E(A|BC) = E(B|AC) = E(AB)$ and $E(C|AB) = E(CA) = E(CB)$.

In general, we can prove that

$$I'(A_1 : A_2 : \cdots : A_n) = I'(A_1 : A_2 : \cdots : A_k)$$

implies

$$E(A_i|\overline{A_i}) = E(A_i|\overline{A_iA_n}) = E(A_i|\overline{A_iA_{i_1}A_{i_2}\cdots A_{i_l}A_n})$$

and

$$E(A_n|\overline{A_n}) = E(A_n|\overline{A_iA_n}) = E(A_n|\overline{A_iA_{i_1}A_{i_2}\cdots A_{i_l}A_n})$$

for any $i_s \neq i$, $1 \leqslant i \leqslant k$, and $l < n-2$. That is, the discorrelated condition and the complete discorrelated condition of MQMI are closely related to the disentangling condition of entanglement.

V. COMPLETE MONOGAMY OF THE MQMI VIA THE TSALLIS ENTOPY

In this section, we explore the mutual information deduced by the Tsallis entopy. The Tsallis q-entropy S_q is defined by [51]

$$S_q(\rho) = \frac{1 - \mathrm{Tr}\rho^q}{q-1}, \quad q > 0, \quad q \neq 1.$$

S_q is subadditive when $q > 1$ [52]; i.e.,

$$S_q(AB) \leqslant S_q(A) + S_q(B), \quad q > 1, \tag{21}$$

for any $\rho^{AB} \in \mathcal{S}^{AB}$, where $\rho^{A,B} = \mathrm{Tr}_{B,A}\rho^{AB}$. If we replace the von Neumann entropy with S_q in Eqs. (4) and (5), we get

$$I_q(A_1 : A_2 : \cdots : A_n) := \sum_{k=1}^{n} S_q(A_k) - S_q(A_1A_2\cdots A_n) \tag{22}$$

and

$$I'_q(A_1 : A_2 : \cdots : A_n) := \sum_{k=1}^{n} S_q(\overline{A_k}) - (n-1)S_q(A_1A_2\cdots A_n), \tag{23}$$

respectively. It is straightforward that

$$I_q(A_1 : A_2 : \cdots : A_n) \geqslant 0, \quad q > 1. \tag{24}$$

In Ref. [45], we proved that, for any bipartite state $\rho_{AB} \in \mathcal{S}^{AB}$, $1 + \mathrm{Tr}(\rho_{AB}^2) = \mathrm{Tr}(\rho_A^2) + \mathrm{Tr}(\rho_B^2)$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$ with $\min\{\mathrm{Rank}(\rho_A), \mathrm{Rank}(\rho_B)\} = 1$. Thus, the equality for $q = 2$ in Eq. (24) holds if and only if $\rho^{A_1A_2\cdots A_n}$ is a product state where, at most, one of the reduced states ρ^{A_i} has a rank greater than 1. $I'_q(A_1 : A_2) = I_q(A_1 : A_2) \geqslant 0$ for any bipartite state. However,

$$I'_q(A_1 : A_2 : \cdots : A_n) \ngeqslant 0 \tag{25}$$

in general when $n > 2$ due to the fact that S_q is not strongly subadditive [53], i.e.,

$$S_q(AB) + S_q(BC) \ngeqslant S_q(ABC) + S_q(B), \ q > 0, \ q \neq 1,$$

in general.

By definition, I_q and I'_q are symmetric under permutation of the subsystems. We next show that I_q is additive while I'_q is not. If $I_q(A_1A_2\cdots A_k : A_{k+1}\cdots A_n) = 0$, then $S_q(A_1A_2\cdots A_k) + S_q(A_{k+1}\cdots A_n) = S_q(A_1A_2\cdots A_n)$, which yields

$$I_q(A_1 : A_2 : \cdots : A_n) = I_q(A_1 : A_2 : \cdots : A_k) + I_q(A_{k+1} : \cdots : A_n). \tag{26}$$

But I'_q does not obey such a equality. In order to see this, we take $\rho^{ABCD} = \rho^{ABC} \otimes \rho^D$, with

$$\rho^{ABC} = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&\frac14&0&\frac14&0&0&0\\ 0&0&0&\frac14&0&\frac14&0&0\\ 0&0&\frac14&0&\frac14&0&0&0\\ 0&0&0&\frac14&0&\frac14&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\rho^D = \begin{pmatrix} \frac12 & 0\\ 0 & \frac12 \end{pmatrix}. \tag{27}$$

It follows that

$$\rho^{AB} = \begin{pmatrix} 0&0&0&0\\ 0&\frac12&\frac12&0\\ 0&\frac12&\frac12&0\\ 0&0&0&0 \end{pmatrix},$$

$$\rho^{AC} = \rho^{BC} = \rho^{CD} = \begin{pmatrix} \frac14&0&0&0\\ 0&\frac14&0&0\\ 0&0&\frac14&0\\ 0&0&0&\frac14 \end{pmatrix}.$$

Short computation gives

$$I'_q(AB:CD) = S_q(AB) + S_q(CD) - S_q(ABCD) = S_q(CD) - S_q(ABCD)$$
$$= \frac{1}{q-1}(1-4^{1-q}) - \frac{1}{q-1}(1-4^{1-q}) = 0,$$

but

$$\frac{1}{q-1}\{I'_q(A:B:C:D) - [I_q(A:B) + I_q(C:D)]\}$$
$$= \frac{1}{q-1}\big\{[(1-2^{1-q}) + (1-8^{1-q}) + (1-2^{1-q}) + (1-8^{1-q}) - 3(1-4^{1-q})] - [4(1-2^{1-q}) - (1-4^{1-q})]\big\}$$
$$= \frac{1}{q-1}(4^{1-q} + 2^{1-q} - 1 - 8^{1-q}) = \frac{1}{q-1}(2^{1-q}-1)(1-4^{1-q})$$
$$> 0$$

for any $q > 1$. From the subadditivity of S_q, the following is straightforward.

Proposition 2. Let $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ be two partitions of $A_1A_2\cdots A_n$ or subsystems of $A_1A_2\cdots A_n$. If $X_1|X_2|\cdots|X_k \succ^{a,b} Y_1|Y_2|\cdots|Y_l$, then

$$I_q(X_1:X_2:\cdots:X_k) \geqslant I_q(Y_1:Y_2:\cdots:Y_l) \tag{28}$$

holds for any $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$.

However, one can readily check that I_q may increase under the coarsening relation of type (c). That is, I_q displays some degree of "completeness" as a multipartite measure but I'_q fails.

Notice that there is another approach of MQMI, which is defined by (we take $n = 3$ and 4 for example) [23,54,55]

$$I''(A:B:C) = S_A + S_B + S_C - (S_{AB} + S_{AC} + S_{BC}) + S_{ABC} \tag{29}$$

and

$$I''(A:B:C:D) = S_A) + S_B + S_C + S_D - [S_{AB} + S_{AC} + S_{BC} + S_{AD} + S_{BD} + S_{CD}] + [S_{ABC} + S_{BCD} + S_{ACD} + S_{ABD}] - S_{ABCD}.$$

$I''(A:B:C)$ can be negative [54,55] and thus it is not a good alternative of MQMI. We now consider this quantity by replacing S with S_q (for example, the case of $n = 3$), i.e.,

$$I''_q(A:B:C) = S_q(A) + S_q(B) + S_q(C) - [S_q(AB) + S_q(AC) + S_q(BC)] + S_q(ABC). \tag{30}$$

Take the three-qubit state $\rho = \frac{1}{2}|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1}{16}I$, one can easily get $I''_q(A:B:C) < 0$ whenever $q = \frac{1}{2}$. In addition, for the state $\rho^{ABCD} = \rho^{ABC} \otimes \rho^D$ in Eq. (27), we have $I''_q(A:B:C:D) < 0$ whenever $q = 2$. Namely, this approach is not valid for the Tsallis q-entropy MQMI, either.

We next explore the complete monogamy and monogamy of I_q. Let $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ be two partitions of $A_1A_2\cdots A_n$ or subsystems of $A_1A_2\cdots A_n$. If $X_1|X_2|\cdots|X_k \succ^{a,b} Y_1|Y_2|\cdots|Y_l$ and

$$I_q(X_1:X_2:\cdots:X_k) = I_q(Y_1:Y_2:\cdots:Y_l),$$

we can easily get that

$$I_q(\Gamma) = 0 \tag{31}$$

holds for all $\Gamma \in \Xi(X_1|X_2|\cdots|X_k - Y_1|Y_2|\cdots|Y_l)$. In particular, for $q = 2$, Γ is a product state where at most one of the reduced states has a rank greater than 1. We thus get the following result.

Theorem 2. (i) I_q is monogamous on pure states. (ii) I_q is not only completely monogamous but also tightly complete monogamous under the coarsening relation of types (a) and (b).

Proof. We only need to check item (i). For pure state, I_q reduces to the Tsallis q-entropy of entanglement, where the Tsallis q-entropy of entanglement is defined by [45]

$$E_q^{(n)}(|\psi\rangle) = \tfrac{1}{2}\big[S_q(A_1) + S_q(A_2) + \cdots + S_q(A_n)\big],\ q > 1,$$

for the pure state $|\psi\rangle \in \mathcal{H}^{ABC}$, and then is defined by the convex-roof extension for mixed states. Thus I_q is monogamous on pure states since the Tsallis q-entropy of entanglement is monogamous [43]. ■

It is worth mentioning that I_q is monogamous iff $S_q(AB) + S_q(BC) = S_q(ABC) + S_q(B)$ implies $S_q(AC) = S_q(A) + S_q(C)$. We remark here that this is not true. Taking [56]

$$\rho^{ABC} = p|000\rangle\langle000| + (1-p)|111\rangle\langle111|, \tag{32}$$

we get $S_q(AB) + S_q(BC) = S_q(ABC) + S_q(B)$, but $S_q(AC) < S_q(A) + S_q(C)$. Comparing with I and I', as a measure of mutual information, I_q is nicer than I'_q, but worse than I. For $0 < q < 1$, S_q is neither subadditive nor superadditive [51] [superadditive refers to $S_q(AB) \geqslant S_q(A) + S_q(B)$], so we cannot define the associated mutual information whenever $0 < q < 1$. The Rényi α-entropy, i.e.,

$$R_\alpha(\rho) := (1-\alpha)^{-1}\ln(\mathrm{Tr}\rho^\alpha), \quad 0 < \alpha < 1,$$

is the same since it is not subadditive [57] either. We call I and I_q the type-1 MQMIs, I' and I'_q the type-2 MQMIs, and I'' and I''_q the type-3 MQMI. Together with Proposition 1 and Theorem 1, we find out that the type-1 MQMI is nicer than the type-2 MQMI for characterizing the mutual information as a measure of multipartite correlation, and the type-3 MQMI cannot be an alternative indeed.

TABLE I. Comparing of I, I', I'', I_q, I'_q, and I''_q. M, CM, TCM, and TI signify the measure is monogamous, completely monogamous, tightly complete monogamous, and satisfies the triangle inequality, respectively. "$\succ^{a,b,c}$" denotes the MQMI is nonincreasing under the coarsening relation "$\succ^{a,b,c}$". "$--$" means the item is senseless or unknown.

MQMI	Entropy	Non-negative	Symmetric	Additivity	$\succ^a$	$\succ^b$	$\succ^c$	M	CM	TCM	TI
I	S	✓	✓	✓	✓	✓	✓	Pure states	✓	✓	✓
I'	S	✓	✓	✓	✓	✓	✓	Pure states	×	×	×
I''	S	×	✓	—	—	—	—	—	—	—	—
I_q	$S_q, q>1$	✓	✓	✓	✓	✓	×	Pure states	✓[a]	✓[b]	×
I'_q	$S_q, q>1$	×	✓	×	×	×	×	Pure states	×	×	×
I''_q	$S_q, q>1$	×	✓	—	—	—	—	—	—	—	—

[a]It is completely monogamous under the coarser relation $\succ^{a,b}$.
[b]It is tightly complete monogamous under the coarser relation $\succ^{a,b}$.

Let us further remark that, for $q=2$, S_q is the linear entropy, which can be regarded as a measure of purity [58]. In fact, S_q reflects the degree of purity for any $q>0$. Hence, MQMI is, indeed, a measure of the multipartite "mutual purity."

VI. TRIANGLE RELATION OF QMI

The first triangle relation for entanglement is the concurrence triangle for the three-qubit pure state [59,60]:

$$C^2_{A|BC} \leqslant C^2_{AB|C} + C^2_{B|AC}. \tag{33}$$

Very recently, we have shown that such a triangle relation is generally true [61]. Let E be a continuous bipartite entanglement measure. Then there exists $0<\alpha<\infty$ such that [61]

$$E^\alpha(A|BC) \leqslant E^\alpha(B|AC) + E^\alpha(AB|C) \tag{34}$$

for all pure states $|\psi\rangle^{ABC} \in \mathcal{H}^{ABC}$ with fixed $\dim \mathcal{H}^{ABC} = d < \infty$. Let $E^{(3)}$ be a continuous unified tripartite entanglement measure. Then [61]

$$E^\alpha(A|B|CD) \leqslant E^\alpha(A|BD|C) + E^\alpha(AD|B|C) \tag{35}$$

for all $|\psi\rangle^{ABCD} \in \mathcal{H}^{ABCD}$ with α as above. Here we omit the superscript $^{(3)}$ of $E^{(3)}$ for brevity. Let E be a continuous bipartite entanglement measure that is determined by the eigenvalues of the reduced state. Then [61]

$$E^\alpha(AB|CD) \leqslant E^\alpha(AC|BD) + E^\alpha(AD|BC) \tag{36}$$

for all $|\psi\rangle^{ABCD} \in \mathcal{H}^{ABCD}$ with α as above. For mutual information I and I', we have the triangle relation below analogously (the proof is given in Appendix C).

Proposition 3. The MQMIs I and I' admit the following triangle relations:

$$I(A:BC) \leqslant I(B:AC) + I(AB:C) \tag{37}$$

for any state in $\mathcal{S}^{ABC}$, and

$$I(AB:CD) \leqslant I(AC:BD) + I(AD:BC), \tag{38}$$

$$I(A:B:CD) \leqslant I(A:BD:C) + I(AD:B:C), \tag{39}$$

$$I'(A:B:CD) \leqslant I'(A:BD:C) + I'(AD:B:C) \tag{40}$$

hold for any state in $\mathcal{S}^{ABCD}$, but I_q fails.

That is, the von Neumann entropy MQMI reflects the same triangle relation as that of entanglement. We now also conclude that the von Neumann entropy MQMI sounds nicer than that of Tsallis entropy. For more clarity, we list all the properties of these measures so far in Table I. We close this section with the following inequalities which reveal the relation between entanglement and the mutual information.

Proposition 4. Let ρ be any state in $\mathcal{S}^{A_1A_2\cdots A_n}$. Then

$$I(\rho) + S(\rho) \geqslant 2E_f^{(n)}(\rho) \tag{41}$$

and

$$I_q(\rho) + S_q(\rho) \geqslant 2E_q^{(n)}(\rho), \tag{42}$$

and the equality holds iff ρ is a pure state.

Proof. We assume with no loss of generality that $n=3$. For any given $\rho \in \mathcal{S}^{ABC}$, let

$$\begin{aligned} E_f^{(3)}(\rho) &= \sum_i p_i\big[E_f^{(3)}(|\psi_i\rangle\langle\psi_i|)\big] \\ &= \frac{1}{2}\sum_i p_i\big[S(\rho_i^A) + S(\rho_i^B) + S(\rho_i^C)\big], \end{aligned}$$

where $\rho_i^X = \mathrm{Tr}_{\bar{X}}|\psi_i\rangle\langle\psi_i|$. It follows that

$$\begin{aligned} I(\rho) + S(\rho) - 2E_f^{(n)}(\rho) &= S(\rho^A) + S(\rho^B) + S(\rho^C) - \sum_i p_i\big[S(\rho_i^A) + S(\rho_i^B) + S(\rho_i^C)\big] \\ &= S(\rho^A) - \sum_i p_i S(\rho_i^A) + S(\rho^B) - \sum_i p_i S(\rho_i^B) + S(\rho^C) - \sum_i p_i S(\rho_i^C) \\ &\geqslant 0 \end{aligned}$$

since S is concave. Notice additionally that S is strictly concave [62], and the equality is immediate. Applying the same strategy for the Tsallis q-entropy, we get the second inequality and the equality makes sense for pure states in light of the strict concavity of the Tsallis q-entropy. ■

That is, the sum of the mutual information and the total entropy acts as an upper bound of entanglement. It can be interpreted as the QMI referring more quantum correlation than entanglement. We also need to note that $E_f^{(n)}(\rho) \geqslant S(\rho)$ and $E_q^{(n)}(\rho) \geqslant S(\rho)$ for pure states but $E_f^{(n)}(\rho) < S(\rho)$ and $E_q^{(n)}(\rho) < S(\rho)$ for any separable mixed state, namely, entanglement and the global entropy are incomparable.

VII. CONCLUSIONS AND DISCUSSIONS

The completeness is a basic requirement for any multipartite measure of correlation. We have shown that the two types of MQMI via the standard von Neumann entropy are complete measures, and the type-1 MQMI via the Tsallis q-entropy demonstrates some weak completeness while the type-2 MQMI via the Tsallis q-entropy is not complete any more. Moreover, we have proven that the type-1 MQMI is not only completely monogamous but also tightly complete monogamous, but the type-2 MQMI fails. We have also found that the von Neumann entropy MQMI obeys the triangle relation which is the same as that of the entanglement measure.

We thus conclude that the von Neumann entropy is better than all the other versions of entropy from such a point of view, and the type-1 von Neumann entropy MQMI represents the same quality as that of the complete measure of multipartite entanglement since both of them are complete and completely monogamous. Despite the fact the type-1 MQMI we proposed is completely monogamous and tightly complete monogamous, it is monogamous only on pure states. This indicates that monogamy and complete monogamy may be independent of each other. However, it remains an open problem whether there exists a MQMI which is not only completely monogamous and tightly complete monogamous but also monogamous. We conjecture that the answer is no.

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APPENDIX A: PROOF OF PROPOSITION 1

Proof. It is clear that

$$I(A:B:C) - I(A:B) = (S_A + S_B + S_C - S_{ABC}) - (S_A + S_B - S_{AB}) = S_{AB} + S_C - S_{ABC} \geqslant 0$$

since S is subadditive, and

$$\begin{aligned} I'(A:B:C) - I'(A:B) &= (S_{AB} + S_{BC} + S_{AC} - 2S_{ABC}) - (S_A + S_B - S_{AB}) \\ &= (S_{AB} + S_{BC} - S_{ABC} - S_B) + (S_{AB} + S_{AC} - S_{ABC} - S_A) \geqslant 0 \end{aligned}$$

since S is strongly subadditive (i.e., $S_{AB} + S_{BC} \geqslant S_{ABC} + S_B$ for any state). In general,

$$I(A_1 : A_2 : \cdots A_n) - I(A_1 : A_2 : \cdots A_{n-1}) = S_{A_1A_2\cdots A_{n-1}} + S_{A_n} - S_{A_1A_2\cdots A_n} \geqslant 0,$$

and

$$I'(A_1 : A_2 : \cdots A_n) - I'(A_1 : A_2 : \cdots A_{n-1}) = \sum_{i=1}^{n-1} \left(S_{\overline{A_i}} + S_{\overline{A_n}} - S_{A_1A_2\cdots A_n} - S_{\overline{A_iA_n}}\right) \geqslant 0.$$

That is, both I and I' are nonincreasing under the coarsening relation of type (a).

Similarly, in light of the subadditivity and the strong subadditivity of the von Neumann entropy, we can obtain Eqs. (14) and (15) under the coarsening relation of types (b) and (c). For example, we can get $I'(AB : CD : EF) \geqslant I'(AB : C : E)$ due to the subadditivity and the strong subadditivity of the von Neumann entropy. In fact,

$$\begin{aligned} I'(AB:CD:EF) - I'(AB:C:E) &= (S_{ABCD} + S_{ABEF} + S_{CDEF} - 2S_{ABCDEF}) - (S_{ABC} + S_{ABE} + S_{CE} - 2S_{ABCE}) \\ &= (S_{ABCD} + S_{ABEF} + S_{CDEF} + 2S_{ABCE}) - (S_{ABC} + S_{ABE} + S_{CE} + 2S_{ABCDEF}) \\ &= [(S_{ABCD} + S_{ABCE}) + (S_{ABEF} + S_{ABCE}) + S_{CDEF}] - (S_{ABC} + S_{ABE} + S_{CE} + 2S_{ABCDEF}) \\ &\geqslant [(S_{ABC} + S_{ABCDE}) + (S_{ABE} + S_{ABCEF}) + S_{CDEF}] - (S_{ABC} + S_{ABE} + S_{CE} + 2S_{ABCDEF}) \\ &= [S_{ABCDE} + (S_{ABCEF} + S_{CDEF})] - (S_{CE} + 2S_{ABCDEF}) \\ &\geqslant [S_{ABCDE} + (S_{CEF} + S_{ABCDEF})] - (S_{CE} + 2S_{ABCDEF}) \\ &= (S_{ABCDE} + S_{CEF}) - (S_{CE} + S_{ABCDEF}) \geqslant 0. \end{aligned}$$

This completes the proof. ■

APPENDIX B: PROOF OF THEOREM 1

Proof. (i) If $I(A:BC) = I(A:B)$, then $S_{AB} + S_{BC} - S_B - S_{ABC} = 0$. According to Theorem 6 in Ref. [63], such a state ρ^{ABC} is precisely the state that saturates the strong subadditivity of the von-Neumann entropy (i.e., the Markov state). For such states, the state space of system B, $\mathcal{H}^B$, must have a decomposition into a direct sum of tensor products $\mathcal{H}^B = \bigoplus_j \mathcal{H}^{B_j^L} \otimes \mathcal{H}^{B_j^R}$, such that ρ^{ABC} admits the form

$$\rho^{ABC} = \bigoplus_j q_j \rho^{AB_j^L} \otimes \rho^{B_j^R C}, \tag{B1}$$

where q_j is a probability distribution [63]. However, $\rho^{ABC} \neq \rho^{AB} \otimes \rho^C$ whenever $\rho^{AB_j^L} \neq \rho^A \otimes \rho^{B_j^L}$ or $\rho^{B_j^R C} \neq \rho^{B_j^R} \otimes \rho^C$. That is, $I(A:BC) = I(A:B)$ cannot guarantee $I(A:C) = 0$ if ρ^{ABC} is a mixed state, but it is true for pure states since it is reduced to $2E_f$ for pure states and E_f is monogamous [43].

(ii) If $I(A:B:C) = I(A:B)$, then $S_{AB} + S_C = S_{ABC}$, which implies that $\rho^{ABC} = \rho^{AB} \otimes \rho^C$. Hence, $I(A:C) = I(B:C) = 0$. If $I(A:B:C:D) = I(A:B:C)$, then $S_{ABC} + S_D = S_{ABCD}$, which leads to $\rho^{ABCD} = \rho^{ABC} \otimes \rho^D$. Thus, $I(A:D) = I(B:D) = I(C:D) = 0$. Similarly, $I(A:B:C:D) = I(A:B)$ implies $\rho^{ABCD} = \rho^{AB} \otimes \rho^C \otimes \rho^D$. In general,

$$I(A_1 : A_2 : \cdots : A_n) = I\big(A_{k_1} : A_{k_2} : \cdots A_{k_m}\big)(m < n, k_i \neq k_j \text{ whenever } i \neq j, 1 \leqslant k_i \leqslant n)$$

implies

$$\rho^{A_1A_2\cdots A_n} = \rho^{A_{k_1}A_{k_2}\cdots A_{k_m}} \otimes \rho^{A_{k_{m+1}}} \otimes \cdots \otimes \rho^{A_{k_n}},$$

and therefore

$$I(A_{k_1}A_{k_2}\cdots A_{k_m} : A_{k_{m+1}} : \cdots : A_{k_n}) = 0.$$

This yields $I(\Gamma) = 0$ for any $\Gamma \in \Xi(A_1|A_2|\cdots|A_n - A_{k_1}|A_{k_2}|\cdots|A_{k_m})$. Namely, I is completely monogamous.

One can readily check that $I(A:B:C:D) = I(A:BCD)$ implies ρ^{BCD} is a product state [i.e., $I(B:C:D) = 0$] and $I(A:B:C:D) = I(AB:CD)$ implies ρ^{AB} and ρ^{CD} are product states [i.e., $I(A:B) = I(C:D) = 0$]. In general,

$$I(A_1 : A_2 : \cdots : A_n) = I\left(A_{k_1^{(1)}}A_{k_2^{(1)}}\cdots A_{k_s^{(1)}} : A_{k_1^{(2)}}A_{k_2^{(2)}}\cdots A_{k_t^{(2)}} : \cdots : A_{k_1^{(l)}}A_{k_2^{(l)}}\cdots A_{k_r^{(l)}}\right)$$

implies

$$I\big(A_{k_1^{(p)}} : A_{k_2^{(p)}} : \cdots : A_{k_q^{(p)}}\big) = 0$$

for any $1 \leqslant p \leqslant l$ and $q \in \{s, t, \ldots, r\}$, where $A_{k_1^{(1)}}\cdots A_{k_s^{(1)}}|A_{k_1^{(2)}}\cdots A_{k_t^{(2)}}|\cdots|A_{k_1^{(l)}}\cdots A_{k_r^{(l)}}$ is an l-partition of $A_1A_2\cdots A_n$ up to some permutation of the subsystems. That is, I is tightly complete monogamous.

(iii) We assume that $I'(A:B:C) = I(A:B)$, i.e., $S(AB) + S(AC) + S(BC) - 2S(ABC) = S(A) + S(B) - S(AB)$. Since $S(AB) + S(AC) \geqslant S(ABC) + S(A)$ and $S(AB) + S(BC) \geqslant S(ABC) + S(B)$, we get $S(AB) + S(AC) = S(ABC) + S(A)$ and $S(AB) + S(BC) = S(ABC) + S(B)$. If the state Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ have decompositions into a direct sum of tensor products as

$$\mathcal{H}^A = \bigoplus_j \mathcal{H}^{A_j^L} \otimes \mathcal{H}^{A_j^R}, \quad \mathcal{H}^B = \bigoplus_j \mathcal{H}^{B_j^L} \otimes \mathcal{H}^{B_j^R},$$

such that

$$\rho^{ABC} = \bigoplus_j q_j \rho_{a_j^L} \otimes \rho_{a_j^R b_j^L} \otimes \rho_{b_j^R} \otimes \rho_{c_j}, \tag{B2}$$

it is easily checked that $S(AB) + S(AC) = S(ABC) + S(A)$ and $S(AB) + S(BC) = S(ABC) + S(B)$. However, $I(B:C) > 0$ and $I(A:C) > 0$ provided that $\rho_{c_i} \neq \rho_{c_j}$ whenever $i \neq j$. Thus I' is not completely monogamous.

If $I'(A:B:C) = I(A:BC)$, we get $S(AB) + S(AC) = S(ABC) + S(A)$. That is, ρ^{BAC} admits the form as Eq. (B1), which reveals that ρ^{BC} is not necessarily a product state. Therefore, I' is not tightly complete monogamous. Together with Proposition 1, the proof is completed. ■

APPENDIX C: PROOF OF PROPOSITION 3

Proof. It is easy to derive from the subadditivity and the strong subadditivity of the von Neumann entropy that

$$\begin{aligned}
&I(B:AC) + I(AB:C) - I(A:BC)\\
&\quad = (S_B + S_{AC} - S_{ABC}) + (S_{AB} + S_C - S_{ABC})\\
&\qquad - (S_A + S_{BC} - S_{ABC})\\
&\quad = (S_{AB} + S_{AC} - S_{ABC} - S_A) + (S_B + S_C - S_{BC})\\
&\quad \geqslant 0,\\
&I(AC:BD) + I(AD:BC) - I(AB:CD)\\
&\quad = (S_{AC} + S_{BC} + S_{AD} + S_{BD}) - (S_{AB} + S_{CD} + S_{ABCD})\\
&\quad \geqslant (S_{ABC} + S_C + S_{ABD} + S_D) - (S_{AB} + S_{CD} + S_{ABCD})\\
&\quad = (S_{ABC} + S_{ABD} - S_{ABCD} - S_{AB}) + (S_C + S_D - S_{CD})\\
&\quad \geqslant 0,\\
&I(A:BD:C) + I(AD:B:C) - I(A:B:CD)\\
&\quad = (S_{AD} + S_{BD} + 2S_C) - (S_{ABCD} + S_{CD})\\
&\quad \geqslant (S_{ABD} + S_D + 2S_C) - (S_{ABCD} + S_{CD})\\
&\quad \geqslant (S_{ABD} + S_{CD} + S_C) - (S_{ABCD} + S_{CD})\\
&\quad = S_{ABD} + S_C - S_{ABCD}\\
&\quad \geqslant 0,
\end{aligned}$$

and

$$\begin{aligned} I'(A:BD:C)+I'(AD:B:C)-I'(A:B:CD) &= (2S_{ABD}+S_{AC}+S_{BC})-(2S_{ABCD}+S_{AB}) \\ &\geqslant (S_{ABCD}+S_A+S_{ABD}+S_{BC})-(2S_{ABCD}+S_{AB}) \\ &\geqslant S_{ABC}+S_{ABD}-S_{ABCD}-S_{AB} \\ &\geqslant 0. \end{aligned}$$

For I_q, by the invalidation of the strong subadditivity of the Tsallis q-entropy, the proof is completed. ■

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