Canonical steering ellipsoids of pure symmetric multiqubit states with two distinct spinors and volume monogamy of steering

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(Received 15 January 2023; accepted 28 March 2023; published 11 April 2023)

Quantum steering ellipsoid formalism provides a faithful representation of all two-qubit states and is useful for obtaining their correlation properties. The steering ellipsoids of two-qubit states that have undergone local operations on both the qubits, in order to bring the state to its canonical form, are the so-called *canonical steering ellipsoids*. The steering ellipsoids corresponding to the two-qubit subsystems of permutation-symmetric N-qubit states are considered here. We construct and analyze the geometric features of the canonical steering ellipsoids corresponding to pure permutation-symmetric N-qubit states with two distinct spinors. Depending on the degeneracy of the two spinors in the pure symmetric N-qubit state, several families arise which cannot be converted into one another through stochastic local operations and classical communication (SLOCC). The canonical steering ellipsoids of the two-qubit states drawn from the pure symmetric N-qubit states with two distinct spinors allow for a geometric visualization of the SLOCC-equivalent class of states. We show that the states belonging to the W class correspond to oblate spheroids centered at $(0, 0, 1/(N-1))$ with fixed semiaxis lengths $1/\sqrt{N-1}$ and $1/(N-1)$. The states belonging to all other SLOCC-inequivalent families correspond to ellipsoids centered at the origin of the Bloch sphere. We also explore volume monogamy relations of states belonging to these families, mainly the W class of states.

DOI: 10.1103/PhysRevA.107.042207

I. INTRODUCTION

The Bloch sphere representation of a single qubit contains valuable geometric information needed for quantum information processing tasks. A natural generalization and an analogous picture for a two-qubit system are provided by the *quantum steering ellipsoid* [1–3] and are helpful for understanding correlation properties such as quantum discord [4,5], volume monogamy of steering [2,3], and so on. A quantum steering ellipsoid is the set of all Bloch vectors to which one party's qubit could be "steered" when all possible measurements are carried out on the qubit belonging to other party. The volume of the steering ellipsoids [1] corresponding to the two-qubit subsystems of an N-qubit state, $N > 3$, captures monogamy properties of the state effectively [2,3] and provides insightful information about two-qubit entanglement.

While the quantum steering ellipsoid [1–3] is the set of all Bloch vectors of the first qubit steered by local operations on the second qubit, the so-called *canonical steering ellipsoid* [6–8] is the steering ellipsoid of a two-qubit state that has attained a canonical form under suitable stochastic local operations and classical communication (SLOCC) operations on *both the qubits*. It has been shown that the SLOCC canonical forms of a two-qubit state can be either a Bell diagonal form or a nondiagonal one (when the two-qubit state is rank deficient) [6,8]. The canonical steering ellipsoids corresponding to the two-qubit states can thus have only two distinct forms [6,8] and provide a much simpler geometric picture representing the set of all SLOCC-equivalent two-qubit states.

The canonical steering ellipsoids corresponding to the two-qubit subsystems of pure three-qubit permutation-symmetric states are analyzed in Ref. [9]. It has been shown that [9] the two SLOCC-inequivalent families of pure three-qubit permutation-symmetric states, the W class of states (with two distinct spinors) and the Greenberger-Horne-Zeilinger (GHZ) class of states (with three distinct spinors), correspond to distinct canonical steering ellipsoids. While an ellipsoid centered at the origin of the Bloch sphere is the canonical steering ellipsoid for the GHZ class of states, an oblate spheroid with its center shifted along the polar axis is the one for the W class of states. Using these, the volume monogamy relations are established, and the obesity of the steering ellipsoids is used to obtain expressions for the concurrence of states belonging to these two SLOCC-inequivalent families in Ref. [9].

In this paper, we extend the analysis to a class of N-qubit pure states which are symmetric under the exchange of qubits. Through the SLOCC canonical forms of the two-qubit reduced state, extracted from pure symmetric *multiqubit* states with two distinct spinors and the Lorentz canonical forms of their real-matrix representation, we examine the features of the *canonical steering ellipsoids* associated with them. We

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identify the special features of the canonical steering ellipsoid representing N-qubit states of the W class, and these features distinguish this class from all other SLOCC-inequivalent families of pure symmetric N-qubit states. We discuss the volume monogamy of steering for pure permutation-symmetric N-qubit states and obtain the volume monogamy relation satisfied by the W class of states. An expression for the obesity of the steering ellipsoid, and thereby an expression for the concurrence of two-qubit subsystems of N-qubit states belonging to the W class, is obtained.

The contents of this paper are organized as follows: In Sec. II, we give a brief review of the SLOCC classification of pure permutation-symmetric multiqubit states based on the Majorana representation [10–13] and obtain the two-qubit subsystems of the states belonging to SLOCC-inequivalent families of pure symmetric multiqubit states with two distinct spinors. Section III provides an outline of the real-matrix representation of a two-qubit density matrix and their Lorentz canonical forms under SLOCC transformation of the two-qubit density matrix. We also obtain the Lorentz canonical forms of two-qubit subsystems corresponding to SLOCC-inequivalent families in Sec. III. In Sec. IV, we analyze the nature of steering ellipsoids associated with the distinct Lorentz canonical forms obtained in Sec. III. The volume monogamy of the steering for pure symmetric multiqubit states with two distinct spinors is discussed, along with illustration of the W class of states, in Sec. V. A summary of our results is presented in Sec. VI.

II. MAJORANA GEOMETRIC REPRESENTATION OF PURE SYMMETRIC N-QUBIT STATES WITH TWO DISTINCT SPINORS

Ettore Majorana, in his 1932 paper [10], proposed that a pure spin $j = \frac{N}{2}$ quantum state can be represented as a *symmetrized* combination of N constituent spinors as follows:

$$|\Psi_{\rm sym}\rangle = \mathcal{N} \sum_{P} \hat{P}\,\{|\epsilon_1, \epsilon_2, \ldots \epsilon_N\rangle\}, \tag{1}$$

where

$$|\epsilon_l\rangle = e^{-i\beta_l/2} \cos\frac{\alpha_l}{2}\,|0\rangle + e^{i\beta_l/2} \sin\frac{\alpha_l}{2}\,|1\rangle, \quad l = 1, 2, \ldots, N. \tag{2}$$

The symbol $\hat{P}$ in (1) corresponds to the set of all $N!$ permutations of the spinors (qubits), and $\mathcal{N}$ corresponds to an overall normalization factor. The name Majorana *geometric* representation is due to the fact that it leads to an intrinsic geometric picture of the state in terms of N points on the unit sphere. In fact, the spinors $|\epsilon_l\rangle$, $l = 1, 2, \ldots, N$, in (2) correspond geometrically to N points on the unit sphere S^2, with the pair of angles (α_l, β_l) determining the orientation of each point on the sphere.

The pure symmetric N-qubit states characterized by *two* distinct qubits are given by [11–13]

$$|D_{N-k,k}\rangle = \mathcal{N} \sum_{P} \hat{P}\,\{|\underbrace{\epsilon_1, \epsilon_1, \ldots, \epsilon_1}_{N-k};\ \underbrace{\epsilon_2, \epsilon_2, \ldots, \epsilon_2}_{k}\rangle\}. \tag{3}$$

Here, one of the spinors, say, $|\epsilon_1\rangle$, occurs $N-k$ times, whereas the other spinor, $|\epsilon_2\rangle$, occurs k times in each term of the symmetrized combination.

Under identical local unitary transformations, the pure symmetric N-qubit states with two distinct spinors can be brought to the following canonical form [13]:

$$|D_{N-k,k}\rangle \equiv \sum_{r=0}^{k} \beta_r^{(k)} \left|\frac{N}{2}, \frac{N}{2} - r\right\rangle, \tag{4}$$

$$\beta_r^{(k)} = \mathcal{N} \sqrt{\frac{N!(N-r)!}{r!}}\,\frac{a^{k-r}\, b^r}{(N-k)!(k-r)!}, \tag{5}$$

where $k = 1, 2, 3, \ldots, [N/2] = N/2$ if N is even and $k = 1, 2, 3, \ldots, [N/2] = (N-1)/2$ when N is odd. Here, a and b are two real parameters constrained by the normalization condition $a^2 + b^2 = 1$. Thus, the class of pure symmetric N-qubit states constituted by two distinct spinors is characterized by a single real parameter $0 \leqslant a < 1$. We denote this one-parameter family of states by $\{\mathcal{D}_{N-k,k}\}$ [13,14]. Notice that $|\frac{N}{2}, \frac{N}{2} - r\rangle$, $r = 0, 1, 2\ldots$, are the Dicke states, which are common eigenstates of collective angular momentum operators J^2 and J_z. They correspond to the basis states of the $(N+1)$-dimensional symmetric subspace indexed by the maximum angular momentum $N/2$.

When $a = 0$, the states $|D_{N-k,k}\rangle$ reduce to the Dicke states $|N/2, N/2 - k\rangle$ [13,14] in which $|\epsilon_1\rangle = |0\rangle$ and $|\epsilon_2\rangle = |1\rangle$ [see (3)]. When a approaches the value 1, the state $|D_{N-k,k}\rangle$ turns out to be a product state of N qubits consisting of only one spinor, $|\epsilon_1\rangle$ or $|\epsilon_2\rangle$.

It is important to note that in the family $\{\mathcal{D}_{N-k,k}\}$, different values of k ($k = 1, 2, 3, \ldots, [\frac{N}{2}]$) correspond to different SLOCC-inequivalent classes [13]. That is, a state $|D_{N-k,k}\rangle$ cannot be converted into $|D_{N-k',k'}\rangle$, $k \neq k'$, through any choice of local unitary (identical) transformations. In fact, different values of k lead to different *degeneracy configurations* [13] of the two spinors, $|\epsilon_1\rangle$ and $|\epsilon_2\rangle$, in the state $|D_{N-k,k}\rangle$. When $k = 1$, one gets the W class of states $\{\mathcal{D}_{N-1,1}\}$ in which one of the qubits, say, $|\epsilon_1\rangle$, repeats only once in each term of the symmetrized combination [see (3)] and the other qubit, $|\epsilon_2\rangle$, repeats $N-1$ times. The N-qubit W state

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|000\cdots 1\rangle + |000\cdots 10\rangle + \cdots + |100\cdots 00\rangle) \equiv \left|\frac{N}{2}, \frac{N}{2} - 1\right\rangle$$

belongs to the family $\{\mathcal{D}_{N-1,1}\}$, hence the name *W class of states*. The Dicke state

$$\left|\frac{N}{2}, \frac{N}{2} - 2\right\rangle = \sqrt{\frac{2}{N(N-1)}}(|110\cdots 000\rangle + |011\cdots 000\rangle + \cdots + |000\cdots 011\rangle)$$

is a typical state of the family $\{\mathcal{D}_{N-2,2}\}$. In all, there are $[\frac{N}{2}]$ SLOCC-inequivalent families in the set of all pure permutation-symmetric N-qubit states with two-distinct spinors [15].

A. Two-qubit reduced density matrices of the states $|D_{N-k,k}\rangle$

The two-qubit marginal $\rho^{(k)}$ corresponding to any random pair of qubits in the pure symmetric N-qubit state $|D_{N-k,\,k}\rangle \in \{\mathcal{D}_{N-k,k}\}$ is obtained by tracing over the remaining $N-2$ qubits in it. In Ref. [16], it has been shown, using the algebra of addition of angular momenta, $j_1 = 1$ (corresponding to the two-qubit marginal) and $j_2 = (N-2)/2$ (corresponding to the remaining $N-2$ qubits), that the two-qubit reduced density matrix $\rho^{(k)}$ has the form

$$\rho^{(k)} = \begin{pmatrix} A^{(k)} & B^{(k)} & B^{(k)} & C^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ C^{(k)} & E^{(k)} & E^{(k)} & F^{(k)} \end{pmatrix}. \tag{6}$$

The elements $A^{(k)}$, $B^{(k)}$, $C^{(k)}$, $D^{(k)}$, $E^{(k)}$, and $F^{(k)}$ are real and are explicitly given by [16]

$$A^{(k)} = \sum_{r=0}^{k} \left(\beta_r^{k}\right)^2 \left(c_1^{(r)}\right)^2, \quad B^{(k)} = \frac{1}{\sqrt{2}} \sum_{r=0}^{k-1} \beta_r^{(k)} \beta_{r+1}^{(k)}\, c_1^{(r)} c_0^{(r+1)}, \quad C^{(k)} = \sum_{r=0}^{k-2} \beta_r^{(k)} \beta_{r+2}^{(k)}\, c_1^{(r)} c_{-1}^{(r+2)},$$

$$D^{(k)} = \frac{1}{2} \sum_{r=1}^{k} \left(\beta_r^{(k)}\right)^2 \left(c_0^{(r)}\right)^2, \quad E^{(k)} = \frac{1}{\sqrt{2}} \sum_{r=0}^{k-1} \beta_r^{(k)} \beta_{r+1}^{(k)}\, c_0^{(r)} c_{-1}^{(r+1)}, \quad F^{(k)} = \sum_{r=0}^{k} \left(\beta_r^{(k)}\right)^2 \left(c_{-1}^{(r)}\right)^2, \tag{7}$$

where $\beta_r^{(k)}$ are given as functions of the real parameter a in (5) and

$$c_1^{(r)} = \sqrt{\frac{(N-r)(N-r-1)}{N(N-1)}}, \quad c_{-1}^{(r)} = \sqrt{\frac{r\,(r-1)}{N(N-1)}},$$

$$c_0^{(r)} = \sqrt{\frac{2r\,(N-r)}{N(N-1)}} \tag{8}$$

are the Clebsch-Gordan coefficients $c_{m_2}^{(r)} = C(\frac{N}{2}-1,\, 1,\, \frac{N}{2}; m-m_2,\, m_2, m)$, with $m = \frac{N}{2} - r$ and $m_2 = 1,\, 0,\, -1$ [17]. In particular, for the W class of states, i.e., when $k=1$, we have

$$\begin{aligned}\rho^{(1)} &= \mathrm{Tr}_{N-2}(|D_{N-1,\,1}\rangle\langle D_{N-1,\,1}|) \\ &= \left[\left(\beta_0^{(1)}\right)^2 + \left(\beta_1^{(1)} c_1^{(1)}\right)^2\right]|1,\,1\rangle\langle 1,\,1| \\ &\quad + \left(\beta_1^{(1)} c_0^{(1)}\right)^2 |1,\,0\rangle\langle 1,\,0| + \beta_0^{(1)}\beta_1^{(1)} c_0^{(1)} |1,\,1\rangle\langle 1,\,0| \\ &\quad + \beta_0^{(1)}\beta_1^{(1)} c_0^{(1)} |1,\,0\rangle\langle 1,\,1|.\end{aligned} \tag{9}$$

Here [see (5)], we have $\beta_0^{(1)} = \mathcal{N} N a$ and $\beta_1^{(1)} = \mathcal{N}\sqrt{N(1-a^2)}$, with $\mathcal{N} = \frac{1}{\sqrt{N^2a^2+N(1-a^2)}}$, and the associated nonzero Clebsch-Gordan coefficients [see (8)] are given by

$$c_1^{(1)} = \sqrt{\frac{N-2}{N}}, \quad c_0^{(1)} = \sqrt{\frac{2}{N}}. \tag{10}$$

In the standard two-qubit basis $\{|0_A, 0_B\rangle, |0_A, 1_B\rangle, |1_A, 0_B\rangle, |1_A, 1_B\rangle\}$, the two-qubit density matrix $\rho^{(1)}$ drawn from the states $|D_{N-1,1}\rangle$ takes the form

$$\rho^{(1)} = \begin{pmatrix} A^{(1)} & B^{(1)} & B^{(1)} & 0 \\ B^{(1)} & D^{(1)} & D^{(1)} & 0 \\ B^{(1)} & D^{(1)} & D^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

where

$$A^{(1)} = \frac{N^2a^2 + (N-2)(1-a^2)}{N^2a^2 + N(1-a^2)}, \quad B^{(1)} = \frac{a\sqrt{1-a^2}}{1 + a^2(N-1)},$$

$$D^{(1)} = \frac{1-a^2}{N^2a^2 + N(1-a^2)}. \tag{12}$$

In a similar manner, the two-qubit subsystems of pure symmetric N-qubit states $|D_{N-k,k}\rangle$ belonging to each SLOCC-inequivalent family $\{\mathcal{D}_{N-k,\,k}\}$, $k = 2,\, 3, \ldots, [\frac{N}{2}]$, can be obtained as a function of N and the parameter a characterizing the family of states using Eqs. (6), (7), and (8). As shown in Refs. [8,9], the real-matrix representation $\Lambda^{(k)}$ of the two-qubit subsystem $\rho^{(k)}$ and its Lorentz canonical form $\widetilde{\Lambda}^{(k)}$ are essential for obtaining the geometric representation of states $|D_{N-k,k}\rangle$ for all k. We thus proceed to obtain $\Lambda^{(k)}$ and its Lorentz canonical form $\widetilde{\Lambda}^{(k)}$ in the following.

III. THE REAL-MATRIX REPRESENTATION OF $\rho^{(k)}$ AND ITS LORENTZ CANONICAL FORMS

The real-matrix representation $\Lambda^{(k)}$ of the two-qubit state $\rho^{(k)}$ is a 4×4 real matrix, whose elements are given by [8]

$$\Lambda_{\mu\,\nu}^{(k)} = \mathrm{Tr}\,[\rho^{(k)}\,(\sigma_\mu \otimes \sigma_\nu)\,]. \tag{13}$$

More specifically, $\Lambda_{\mu\,\nu}^{(k)}$, $\mu,\, \nu = 0,\, 1,\, 2,\, 3$, are the coefficients of expansion in the Hilbert-Schmidt basis $\{\sigma_\mu \otimes \sigma_\nu\}$ of the density matrix $\rho^{(k)}$:

$$\rho^{(k)} = \frac{1}{4} \sum_{\mu,\,\nu=0}^{3} \Lambda_{\mu\,\nu}^{(k)}\,(\sigma_\mu \otimes \sigma_\nu). \tag{14}$$

Here, σ_i, $i = 1,\, 2,\, 3$, are the Pauli spin matrices, and σ_0 is the 2×2 identity matrix:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{15}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be readily seen that [see (13) and (14)] the real 4×4 matrix $\Lambda^{(k)}$ has the form

$$\Lambda^{(k)} = \begin{pmatrix} 1 & r_1 & r_2 & r_3 \\ s_1 & t_{11} & t_{12} & t_{13} \\ s_2 & t_{21} & t_{22} & t_{23} \\ s_3 & t_{31} & t_{32} & t_{33} \end{pmatrix}, \tag{16}$$

where $\mathbf{r} = (r_1, r_2, r_3)^T$ and $\mathbf{s} = (s_1, s_2, s_3)^T$ are Bloch vectors of the individual qubits and $T = (t_{ij})$ is the correlation matrix:

$$r_i = \Lambda^{(k)}_{i0} = \mathrm{Tr}\,[\rho^{(k)}\,(\sigma_i \otimes \sigma_0)\,], \tag{17}$$

$$s_j = \Lambda^{(k)}_{0j} = \mathrm{Tr}\,[\rho^{(k)}\,(\sigma_0 \otimes \sigma_j)\,], \tag{18}$$

$$t_{ij} = \Lambda^{(k)}_{ij} = \mathrm{Tr}\,[\rho^{(k)}\,(\sigma_i \otimes \sigma_j)\,],\; i,\; j = 1,\,2,\,3. \tag{19}$$

For a symmetric two-qubit density matrix, the Bloch vectors $\mathbf{r}$ and $\mathbf{s}$ are identical, and hence, $r_i = s_i$, $i = 1, 2, 3$. From the structure of $\rho^{(k)}$ in (6) and using (17), (18), and (19), we obtain the general form of the real matrix $\Lambda^{(k)}$ as follows:

$$\Lambda^{(k)} = \begin{pmatrix} 1 & \frac{2(B^{(k)}+E^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} & 0 & \frac{A^{(k)}-F^{(k)}}{A^{(k)}+2D^{(k)}+F^{(k)}} \\ \frac{2(B^{(k)}+E^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} & \frac{2(C^{(k)}+D^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} & 0 & \frac{2(B^{(k)}-E^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} \\ 0 & 0 & \frac{2(D^{(k)}-C^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} & 0 \\ \frac{A^{(k)}-F^{(k)}}{A^{(k)}+2D^{(k)}+F^{(k)}} & \frac{2(B^{(k)}-E^{(k)})}{A^{(k)}+2D^{(k)}+F^{(k)}} & 0 & 1-\frac{4D^{(k)}}{A^{(k)}+2D^{(k)}+F^{(k)}} \end{pmatrix}. \tag{20}$$

The elements of $\Lambda^{(k)}$ for different k can be evaluated using (7) and (8).

A. Lorentz canonical forms of $\Lambda^{(k)}$

Under SLOCC transformation, the two-qubit density matrix $\rho^{(k)}$ transforms to $\widetilde{\rho}^{(k)}$ as

$$\rho^{(k)} \longrightarrow \widetilde{\rho}^{(k)} = \frac{(A\otimes B)\,\rho^{(k)}\,(A^\dagger \otimes B^\dagger)}{\mathrm{Tr}[\rho^{(k)}\,(A^\dagger A \otimes B^\dagger B)]}. \tag{21}$$

Here, $A, B \in \mathrm{SL}(2, \mathrm{C})$ denote 2×2 complex matrices with unit determinant. A suitable choice of A and B takes the two-qubit density matrix $\rho^{(k)}$ to its canonical form $\widetilde{\rho}^{(k)}$.

The transformation of $\rho^{(k)}$ in (21) leads to the transformation [8,9]

$$\Lambda^{(k)} \longrightarrow \widetilde{\Lambda}^{(k)} = \frac{L_A\,\Lambda^{(k)}\,L_B^T}{\left(L_A\,\Lambda^{(k)}\,L_B^T\right)_{00}} \tag{22}$$

of its real-matrix representation $\Lambda^{(k)}$. In (22), $L_A, L_B \in \mathrm{SO}(3, 1)$ are 4×4 proper orthochronous Lorentz transformation matrices [18] corresponding, respectively, to $A, B \in \mathrm{SL}(2, C)$, and the superscript T denotes transpose operation. The Lorentz canonical form $\widetilde{\Lambda}^{(k)}$ of $\Lambda^{(k)}$ and thus the SLOCC canonical form of the two-qubit density matrix $\rho^{(k)}$ [see (21)] can be obtained by constructing the 4×4 real symmetric matrix $\Omega^{(k)} = \Lambda^{(k)}\,G\,(\Lambda^{(k)})^T$, where $G = \mathrm{diag}\,(1, -1, -1, -1)$ denotes the Lorentz metric. Using the defining property [18] $L^T\,G\,L = G$ of Lorentz transformation L, it can be seen that $\Omega^{(k)}$ undergoes a *Lorentz-congruent transformation* under SLOCC (up to an overall factor) [8] as

$$\begin{aligned}\Omega^{(k)} \to \widetilde{\Omega}^{(k)}_A &= \widetilde{\Lambda}^{(k)}\,G\,(\widetilde{\Lambda}^{(k)})^T \\ &= L_A\,\Lambda^{(k)}\,L_B^T\,G\,L_B\,\Lambda^{(k)^T}\,L_A^T \\ &= L_A\,\Omega^{(k)}\,L_A^T.\end{aligned} \tag{23}$$

In Ref. [8], it has been shown that $\widetilde{\Lambda}^{(k)}$ can be either a real 4×4 diagonal matrix or a nondiagonal matrix with only one off-diagonal element, depending on the eigenvalues and eigenvectors of $G\,\Omega^{(k)} = G(\Lambda^{(k)}\,G\,(\Lambda^{(k)})^T)$.

(i) The diagonal canonical form $\widetilde{\Lambda}^{(k)}_{I_c}$ results when the eigenvector X_0 associated with the highest eigenvalue λ_0 of $G\,\Omega^{(k)}$ obeys the Lorentz-invariant condition $X_0^T\,G X_0 > 0$. The diagonal canonical form $\widetilde{\Lambda}^{(k)}_{I_c}$ is explicitly given by

$$\begin{aligned}\Lambda^{(k)} \longrightarrow \widetilde{\Lambda}^{(k)}_{I_c} &= \frac{L_{A_1}\,\Lambda^{(k)}\,L_{B_1}^T}{\left(L_{A_1}\,\Lambda^{(k)}\,L_{B_1}^T\right)_{00}} \\ &= \mathrm{diag}\left(1,\, \sqrt{\frac{\lambda_1}{\lambda_0}},\, \sqrt{\frac{\lambda_2}{\lambda_0}},\, \pm\sqrt{\frac{\lambda_3}{\lambda_0}}\right),\end{aligned} \tag{24}$$

where $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0$ are the *non-negative* eigenvalues of $G\,\Omega^{(k)}$. The Lorentz transformations $L_{A_1}, L_{B_1} \in \mathrm{SO}(3, 1)$ in (24) respectively correspond to $\mathrm{SL}(2, C)$ transformation matrices A_1 and B_1, which take the two-qubit density matrix $\rho^{(k)}$ to its SLOCC canonical form $\widetilde{\rho}^{(k)}_{I_c}$ through the transformation (21). The diagonal form of $\widetilde{\Lambda}^{(k)}_{I_c}$ readily leads, on using (14), to the Bell-diagonal form

$$\widetilde{\rho}^{(k)}_{I_c} = \frac{1}{4}\left(\sigma_0\otimes\sigma_0 + \sum_{i=1,2}\sqrt{\frac{\lambda_i}{\lambda_0}}\,(\sigma_i\otimes\sigma_i) \pm \sqrt{\frac{\lambda_3}{\lambda_0}}\,(\sigma_3\otimes\sigma_3)\right) \tag{25}$$

as the canonical form of the two-qubit state $\rho^{(k)}$.

(ii) The Lorentz canonical form of $\Lambda^{(k)}$ turns out to be a nondiagonal matrix (with only one nondiagonal element) given by

$$\begin{aligned}\Lambda^{(k)} \longrightarrow \widetilde{\Lambda}^{(k)}_{II_c} &= \frac{L_{A_2}\,\Lambda^{(k)}\,L_{B_2}^T}{\left(L_{A_2}\,\Lambda^{(k)}\,L_{B_2}^T\right)_{00}} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 1-a_0 & 0 & 0 & a_0 \end{pmatrix}\end{aligned} \tag{26}$$

when the non-negative eigenvalues of $G\Omega^{(k)}$ are doubly degenerate with $\lambda_0 \geqslant \lambda_1$ and the eigenvector X_0 belonging to the highest eigenvalue λ_0 satisfies the Lorentz-invariant condition $X_0^T\,G X_0 = 0$. In Ref. [16], it has been shown that when the maximum of the doubly degenerate eigenvalues of $G\Omega^{(k)}$

possesses an eigenvector X_0 satisfying the condition $X_0^T\,G X_0 = 0$, the real symmetric matrix $\Omega^{(k)} = \Lambda^{(k)} G(\Lambda^{(k)})^T$ attains the nondiagonal Lorentz canonical form given by

$$\Omega^{(k)}_{II_c} = \widetilde{\Lambda}^{(k)}_{II_c}\, G\left(\widetilde{\Lambda}^{(k)}_{II_c}\right)^T = L_{A_2}\,\Omega^{(k)}\,L^T_{A_2} = \begin{pmatrix} \phi_0 & 0 & 0 & \phi_0-\lambda_0 \\ 0 & -\lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ \phi_0-\lambda_0 & 0 & 0 & \phi_0-2\lambda_0 \end{pmatrix}. \quad (27)$$

The parameters a_0 and a_1 in (26) are related to the eigenvalues λ_0 and λ_1 of $G\Omega^{(k)}$ and the 00th element of $\widetilde{\Omega}^{(k)}_{II_c}$ [see (27)]. It can be seen that [8]

$$a_0 = \frac{\lambda_0}{\phi_0}, \quad a_1 = \sqrt{\frac{\lambda_1}{\phi_0}},$$

where

$$\phi_0 = \left(\Omega^{(k)}_{II_c}\right)_{00} = \left[\left(L_{A_2}\,\Lambda^{(k)}\,L^T_{B_2}\right)_{00}\right]^2. \quad (28)$$

The Lorentz matrices $L_{A_2}, L_{B_2} \in SO(3,1)$ correspond to the $SL(2,C)$ transformations A_2 and B_2 that transform $\rho^{(k)}$ to its SLOCC canonical form $\rho^{(k)}_{II_c}$ [see (21)]. The nondiagonal canonical form $\widetilde{\Lambda}^{(k)}_{II_c}$ leads to the SLOCC canonical form $\widetilde{\rho}^{(k)}_{II_c}$ of the two-qubit density matrix $\rho^{(k)}$ on using (14):

$$\widetilde{\rho}^{(k)}_{II_c} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1-a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & a_0 \end{pmatrix}, \quad 0 \leqslant a_1^2 \leqslant a_0 \leqslant 1. \quad (29)$$

B. Lorentz canonical form of $\Lambda^{(1)}$ corresponding to the W class of states $\{\mathcal{D}_{N-1,1}\}$:

Using the explicit structure of the two-qubit state $\rho^{(1)}$ given in (11) and (12), its real-matrix representation $\Lambda^{(1)}$ is obtained as [see (13)]

$$\Lambda^{(1)} = \begin{pmatrix} 1 & \frac{2a\sqrt{1-a^2}}{1+a^2(N-1)} & 0 & 1+\frac{2a^2}{1+a^2(N-1)}-\frac{2}{N} \\ \frac{2a\sqrt{1-a^2}}{1+a^2(N-1)} & \frac{2(1-a^2)}{N(1+a^2(N-1))} & 0 & \frac{2a\sqrt{1-a^2}}{1+a^2(N-1)} \\ 0 & 0 & \frac{2(1-a^2)}{N(1+a^2(N-1))} & 0 \\ 1+\frac{2a^2}{1+a^2(N-1)}-\frac{2}{N} & \frac{2a\sqrt{1-a^2}}{1+a^2(N-1)} & 0 & 1+\frac{4a^2}{1+a^2(N-1)}-\frac{4}{N} \end{pmatrix} = (\Lambda^{(1)})^T. \quad (30)$$

We now construct the 4×4 symmetric matrix $\Omega^{(1)}$ and obtain

$$\Omega^{(1)} = \Lambda^{(1)}\,G\,(\Lambda^{(1)})^T = \Lambda^{(1)}\,G\,\Lambda^{(1)} = \chi\begin{pmatrix} N-1 & 0 & 0 & N-2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ N-2 & 0 & 0 & N-3 \end{pmatrix}, \quad (31)$$

where $\chi = [\frac{2(1-a^2)}{N[1+a^2(N-1)]}]^2$. The eigenvalues of the matrix $G\,\Omega^{(1)}$, $G = \text{diag}\,(1,\,-1,\,-1,\,-1)$, are readily seen to be fourfold degenerate and are given by

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \chi = \left[\frac{2(1-a^2)}{N[1+a^2(N-1)]}\right]^2. \quad (32)$$

It can be seen that $X_0 = (1,\,0,\,0,\,-1)$ is an eigenvector of $G\,\Omega^{(1)}$ belonging to the fourfold-degenerate eigenvalue λ_0 and obeys the Lorentz-invariant condition $X_0^T\,G X_0 = 0$. We note here that $\Omega^{(1)}$ is already in the canonical form (27). On comparing (31) with (27), we get

$$\phi_0 = (\Omega^{(1)})_{00} = (N-1)\chi. \quad (33)$$

On substituting the parameters a_0 and a_1 [see (28), (32), and (33)] in (26), we arrive at the Lorentz canonical form of the real matrix $\Lambda^{(1)}$:

$$\widetilde{\Lambda}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{N-1}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{N-1}} & 0 \\ \frac{N-2}{N-1} & 0 & 0 & \frac{1}{N-1} \end{pmatrix}. \quad (34)$$

It can be readily seen that $\widetilde{\Lambda}^{(1)}$, the Lorentz canonical form corresponding to the W class of states, is independent of the parameter a.

C. Lorentz canonical form of $\Lambda^{(k)}, k = 2,\ 3, \ldots, [\frac{N}{2}]$

Here, we evaluate the real-matrix representation $\Lambda^{(k)}$ of $\rho^{(k)}$ for different values of k ($k = 2,\ 3, \ldots, [\frac{N}{2}]$) making use of Eqs. (7), (8), and (20). We then construct the real symmetric matrix $\Omega^{(k)} = \Lambda^{(k)}\,G(\Lambda^{(k)})^T$ for $k = 2,\ 3, \ldots, [\frac{N}{2}]$ and observe that $G\Omega^{(k)} = G\Lambda^{(k)}\,G\,(\Lambda^{(k)})^T$ has *nondegenerate eigenvalues* $\lambda_0 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ when $k = 2, 3,\ \ldots, [\frac{N}{2}]$ and the highest eigenvalue λ_0 possesses an eigenvector X_0 satisfying the relation $X_0^T\,G X_0 > 0$. The Lorentz canonical form $\widetilde{\Lambda}^{(k)}$, $k = 2,\ 3, \ldots, [\frac{N}{2}]$, is thus given by the diagonal matrix [see (24)]:

$$\widetilde{\Lambda}^{(k)} = \text{diag}\,(1,\ \sqrt{\lambda_1/\lambda_0},\ \sqrt{\lambda_2/\lambda_0},\ \pm\sqrt{\lambda_3/\lambda_0}).$$

The eigenvalues λ_μ ($\mu = 0,\ 1,\ 2,\ 3$) of $G\Omega^{(k)}$ are dependent on the parameters a, k, and N characterizing the state $|D_{N-k,\,k}\rangle$ when k takes any of the integral values greater than 1 and less than $[\frac{N}{2}]$. Hence, the canonical form $\widetilde{\Lambda}^{(k)}$, $k = 2, 3,\ \ldots, [\frac{N}{2}]$, is different for different states $|D_{N-k,\,k}\rangle$, unlike in the case of $\widetilde{\Lambda}^{(1)}$ [see (34)], the canonical form of the W class of states, which depends only on the number of qubits N.

IV. GEOMETRIC REPRESENTATION OF THE STATES $|D_{N-k,k}\rangle$

In this section, based on the two different canonical forms of $\Lambda^{(k)}$ obtained in Sec. III, we find the nature of canonical steering ellipsoids associated with the pure symmetric multiqubit states $|D_{N-k,k}\rangle$ belonging to SLOCC-inequivalent families $\{\mathcal{D}_{N-k,\,k}\}$. To begin with, we give a brief outline [8,9] of obtaining the steering ellipsoids of a two-qubit density matrix $\rho^{(k)}$ based on the form of its real-matrix representation $\Lambda^{(k)}$.

In the two-qubit state $\rho^{(k)}$, local projection-valued measurements (PVMs) $Q > 0,\ Q = \sum_{\mu=0}^{3} q_\mu\,\sigma_\mu,\ q_0 = 1,\ \sum_{i=1}^{3} q_i^2 = 1$ on Bob's qubit lead to a collapsed state of Alice's qubit characterized by its Bloch vector $\mathbf{p}_A = (p_1,\ p_2,\ p_3)^T$ through the transformation [8]

$$(1, p_1,\ p_2,\ p_3)^T = \Lambda^{(k)}\,(1, q_1,\ q_2,\ q_3)^T. \tag{35}$$

Note that the vector $\mathbf{q}_B = (q_1,\ q_2,\ q_3)^T,\ q_1^2 + q_2^2 + q_3^2 = 1$, represents the entire Bloch sphere and the steered Bloch vectors $\mathbf{p}_A$ of Alice's qubit constitute an ellipsoidal surface $\mathcal{E}_{A|B}$ enclosed within the Bloch sphere. When Bob employs convex combinations of PVMs, i.e., positive operator-valued measures, to steer Alice's qubit, he can access the points inside the steering ellipsoid. The case will be similar when Bob's qubit is steered by Alice through local operations on her qubit.

For the Lorentz canonical form $\widetilde{\Lambda}^{(k)}_{I_c}$ [see (24)] of the two-qubit state $\widetilde{\rho}^{(k)}_{I_c}$, it follows from (35) that

$$p_1 = \sqrt{\frac{\lambda_1}{\lambda_0}}\,q_1,\quad p_2 = \sqrt{\frac{\lambda_2}{\lambda_0}}\,q_2,\quad p_3 = \pm\sqrt{\frac{\lambda_3}{\lambda_0}}q_3 \tag{36}$$

are steered Bloch points $\mathbf{p}_A$ of Alice's qubit. They are seen to obey the equation

$$\frac{\lambda_0\,p_1^2}{\lambda_1} + \frac{\lambda_0\,p_2^2}{\lambda_2} + \frac{\lambda_0\,p_3^2}{\lambda_3} = 1 \tag{37}$$

for an ellipsoid with semiaxes $(\sqrt{\lambda_1/\lambda_0},\ \sqrt{\lambda_2/\lambda_0},\ \sqrt{\lambda_3/\lambda_0})$ and center (0,0,0) inside the Bloch sphere $q_1^2 + q_2^2 + q_3^2 = 1$. We refer to this as the *canonical steering ellipsoid* representing the set of all two-qubit density matrices which are on the SLOCC orbit of the state $\widetilde{\rho}^{(k)}_{I_c}$ [see (21)].

For the second Lorentz canonical form $\widetilde{\Lambda}_{II_c}$ [see (26)], we get the coordinates of Alice's steered Bloch vector $\mathbf{p}_A$ on using (35):

$$p_1 = a_1 q_1,\quad p_2 = -a_1 q_2,\quad p_3 = (1 - a_0) + a_0 q_3,\qquad q_1^2 + q_2^2 + q_3^2 = 1, \tag{38}$$

and they satisfy the equation

$$\frac{p_1^2}{a_1^2} + \frac{p_2^2}{a_1^2} + \frac{[p_3 - (1 - a_0)]^2}{a_0^2} = 1. \tag{39}$$

Equation (39) represents the canonical steering spheroid (traced by Alice's Bloch vector $\mathbf{p}_A$) inside the Bloch sphere with its center at $(0,\ 0,\ 1 - a_0)$ and lengths of the semiaxes given by $a_0 = \lambda_0/\phi_0$ and $a_1 = \sqrt{\lambda_1/\phi_0}$ [see (28)]. In other words, a shifted spheroid inscribed within the Bloch sphere represents two-qubit states that are SLOCC equivalent to $\widetilde{\rho}^{(k)}_{II_c}$ [see (29)].

A. Canonical steering ellipsoids of the *W* class of states

We saw in Sec. III B that the Lorentz canonical form of $\Lambda^{(1)}$, the real-matrix representation of the symmetric two-qubit state $\rho^{(1)}$ drawn from the *W* class of states $|D_{N-1,1}\rangle$, has a *nondiagonal* form [see (34)]. On comparing (34) with the canonical form in (26), we get

$$a_1 = \frac{1}{\sqrt{N-1}},\quad a_0 = \frac{1}{N-1}. \tag{40}$$

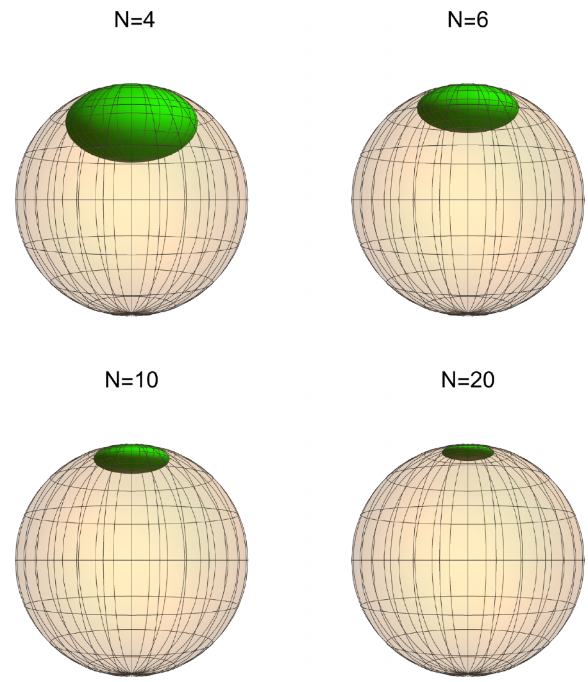

FIG. 1. Steering spheroids inscribed within the Bloch sphere representing the Lorentz canonical form $\widetilde{\Lambda}^{(1)}$ [see (34)] of the *W* class of states $\{\mathcal{D}_{N-1,\,1}\}$. The spheroids are centered at $(0, 0, \frac{N-2}{N-1})$, and the length of the semiaxes is given by $(\frac{1}{\sqrt{N-1}},\ \frac{1}{\sqrt{N-1}},\ \frac{1}{N-1})$. The reduction in the volume of the spheroids with the increase in N is a clear indication of the monogamous nature of the states $|D_{N-1,\,1}\rangle$. (All quantities plotted are dimensionless.)

From (39) and the discussions prior to it, it can readily be seen that the quantum steering ellipsoid associated with $\widetilde{\Lambda}^{(1)}$ in (34) is a spheroid centered at $(0, 0, \frac{N-2}{N-1})$ inside the Bloch sphere, with fixed semiaxes lengths $(\frac{1}{\sqrt{N-1}},\ \frac{1}{\sqrt{N-1}},\ \frac{1}{N-1})$ (see Fig. 1). It is interesting to note that the Lorentz canonical form $\widetilde{\Lambda}^{(1)}$ is not dependent on the state parameter a and hence all states $|D_{N-1,\,1}\rangle$ in the family $\{\mathcal{D}_{N-1,\,1}\}$ are represented by a spheroid; all its parameters, such as center, semiaxes, and volume, depend only on the number of qubits N. Another feature worth observing is the decrease in the volume of the spheroid (depicting two-qubit subsystems of $|D_{N-1,\,1}\rangle$) with the increase in N (see Fig. 1). The inherent monogamous nature [19] of the states $|D_{N-1,\,1}\rangle$ readily follows from this observation.

B. Canonical steering ellipsoids of the states $|D_{N-k,k}\rangle$, $k = 2,\ 3, \ldots, [\frac{N}{2}]$

As seen in Sec. III C, the Lorentz canonical form of $\Lambda^{(k)}$, $k = 2,\ 3, \ldots, [\frac{N}{2}]$, the real-matrix representation of the two-qubit states $\rho^{(k)}$ drawn from the pure symmetric N-qubit states $|D_{N-k,k}\rangle$, has a diagonal form [see (24)]. The values of λ_0,

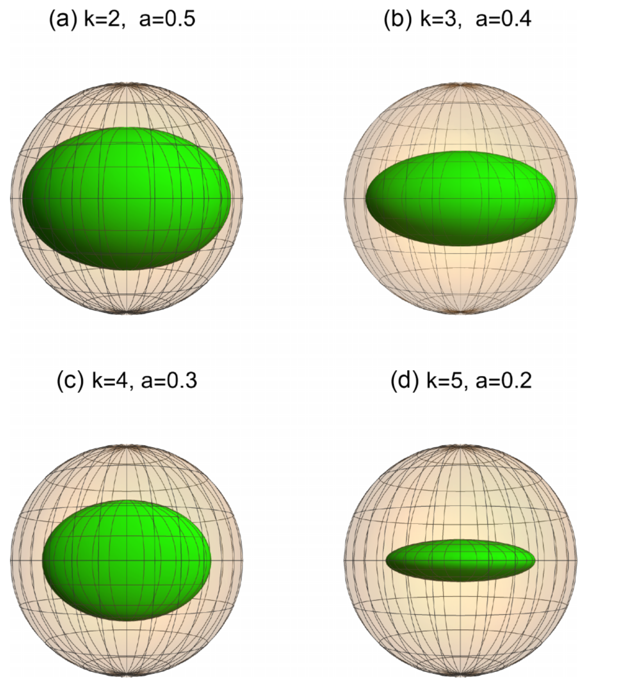

FIG. 2. Steering ellipsoids centered at the origin of the Bloch sphere representing the Lorentz canonical form of pure symmetric 10-qubit states $|D_{10-k,k}\rangle$ for $k = 2$ to $k = 5$. The lengths of the semiaxes of the ellipsoids depicted above are given by (a) (0.91, 0.71, 0, 62), (b) (0.83, 0.59, 0.42), (c) (0.74, 0.53, 0.28), and (d) (0.66, 0.53, 0.18). (All quantities plotted are dimensionless.)

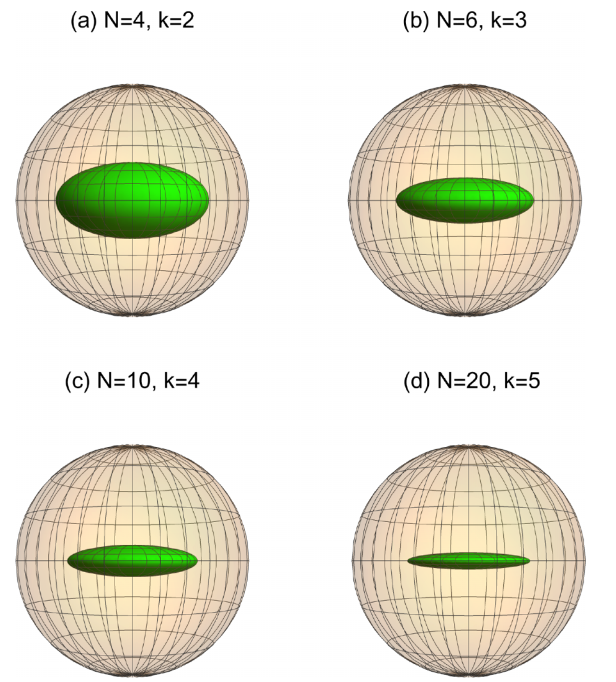

FIG. 3. Oblate spheroids centered at the origin representing the Lorentz canonical form of the N-qubit Dicke states $|N/2, N/2-k\rangle$ (equivalently, the states $|D_{N-k,k}\rangle$, with $a = 0$). (All quantities plotted are dimensionless.)

λ_1, λ_2, and λ_3, the eigenvalues of the matrix $G\,\Omega^k$, can be evaluated for each value of k, $k = 2, 3, \ldots, [\frac{N}{2}]$, for a chosen N. From (37) and the corresponding discussion, it follows that the canonical steering ellipsoids of the states $|D_{N-k,k}\rangle$, $k = 2, 3, \ldots, [\frac{N}{2}]$, are ellipsoids centered at the origin of the Bloch sphere with the lengths of the semiaxes given by $\sqrt{\lambda_1/\lambda_0}$, $\sqrt{\lambda_2/\lambda_0}$, and $\sqrt{\lambda_3/\lambda_0}$. In this case the eigenvalues λ_μ, $\mu = 0, 1, 2, 3$, of $G\Omega^{(k)}$ depend on the state parameter a (unlike in the case of the W class of states, where they depended only on N, the number of qubits). Thus, each state $|D_{N-k,k}\rangle$ belonging to the family $\{\mathcal{D}_{N-k,\,k}\}$, $k = 2, 3, \ldots, [\frac{N}{2}]$, is represented by an ellipsoid whose semiaxes depend on the values of k, N, and a. The canonical steering ellipsoids corresponding to the 10-qubit pure symmetric states $|D_{10-k,k}\rangle$ with chosen values of k and a are shown in Fig. 2.

In particular, the canonical steering ellipsoids corresponding to Dicke states $|N/2, N/2-k\rangle$ are *oblate spheroids* centered at the *origin* (see Fig. 3).

V. VOLUME MONOGAMY RELATIONS FOR PURE SYMMETRIC MULTIQUBIT STATES $|D_{N-k,k}\rangle$

Monogamy relations restrict the shareability of quantum correlations in a multipartite state. They have potential applications in ensuring security in quantum key distribution [20,21]. Milne *et al.* [2,3] introduced a geometrically intuitive monogamy relation for the volumes of the steering ellipsoids representing the two-qubit subsystems of multiqubit pure states which is stronger than the well-known Coffman-Kundu-Wootters monogamy relation [19]. In this section we explore how the volume monogamy relation [2] imposes limits on volumes of the quantum steering ellipsoids representing the two-qubit subsystems $\rho^{(k)} = \mathrm{Tr}_{N-2}\,[|D_{N-k,k}\rangle\langle D_{N-k,k}|]$ of pure symmetric multiqubit states $|D_{N-k,k}\rangle$.

For the two-qubit state $\rho_{AB}(=\rho^{(k)})$ [see (14)], we denote by $\mathcal{E}_{A|B}$ the quantum steering ellipsoid containing all steered Bloch vectors belonging to Alice when Bob carries out local operations on his qubit. The volume of $\mathcal{E}_{A|B}$ is given by [1]

$$V_{A|B} = \left(\frac{4\pi}{3}\right)\frac{|\det\Lambda|}{(1-r^2)^2}, \tag{41}$$

where $r^2 = \mathbf{r}\cdot\mathbf{r} = r_1^2 + r_2^2 + r_3^2$ [see (17)]. As the steering ellipsoid is constrained to lie within the Bloch sphere, i.e., $V_{A|B} \leqslant V_{\text{unit}} = (4\pi/3)$, one can choose to work with the *normalized volumes* $v_{A|B} = \frac{V_{A|B}}{4\pi/3}$, the ratio of the volume of the steering ellipsoid to the volume of a unit sphere [3].

The volume monogamy relation satisfied by a *pure* three-qubit state shared by Alice, Bob, and Charlie is given by [1–3]

$$\sqrt{V_{A|B}} + \sqrt{V_{C|B}} \leqslant \sqrt{\frac{4\pi}{3}}. \tag{42}$$

where $V_{A|B}$, $V_{C|B}$ are respectively the volumes of the ellipsoids corresponding to steered states of Alice and Charlie when Bob performs all possible local measurements on his qubit. The *normalized* form of the volume monogamy relation (42) turns out to be

$$\sqrt{v_{A|B}} + \sqrt{v_{C|B}} \leqslant 1, \tag{43}$$

where $v_{A|B} = \frac{V_{A|B}}{4\pi/3}$ are the *normalized volumes*.

The monogamy relation (43) is not, in general, satisfied by mixed three-qubit states [3], and it has been shown that

$$(v_{A|B})^{\frac{2}{3}} + (v_{C|B})^{\frac{2}{3}} \leqslant 1 \tag{44}$$

is the volume monogamy relation for pure as well as mixed three-qubit states [3].

As there are $\frac{1}{2}(N-2)(N-1)$ three-qubit subsystems in an N-qubit state, each of which obeys monogamy relation (44), after adding these relations and simplifying, one gets [3]

$$(v_{A|B})^{\frac{2}{3}} + (v_{C|B})^{\frac{2}{3}} + (v_{D|B})^{\frac{2}{3}} + \cdots \leqslant \frac{N-1}{2}. \tag{45}$$

Relation (45) is the volume monogamy relation satisfied by pure as well as mixed N-qubit states [3]. For $N = 3$, it reduces to (44).

For multiqubit states that are invariant under the exchange of qubits, $v_{A|B} = v_{C|B} = v_{D|B} = \cdots = v_N$, where v_N denotes the normalized volume of the steering ellipsoid corresponding to any of the $N-1$ qubits, the steering performed by, say, the Nth qubit. Equation (45) thus reduces to

$$(N-1)(v_N)^{\frac{2}{3}} \leqslant \frac{N-1}{2} \Longrightarrow (v_N)^{\frac{2}{3}} \leqslant \frac{1}{2}, \tag{46}$$

implying that $(v_N)^{\frac{2}{3}} \leqslant \frac{1}{2}$ is the volume monogamy relation for permutation-symmetric multiqubit states.

A. Volume monogamy relations governing the *W* class of states $\{\mathcal{D}_{N-1,1}\}$

On denoting the normalized volume of a steering ellipsoid corresponding to the states $|D_{N-1,1}\rangle$ by $v_N^{(1)}$, we have [see (41)]

$$v_N^{(1)} = \frac{|\det \Lambda^{(1)}|}{(1-r^2)^2}, \tag{47}$$

where $\Lambda^{(1)}$ is given in (30) and

$$r_1 = \frac{2a\sqrt{1-a^2}}{1+a^2(N-1)}, \quad r_2 = 0, \quad r_3 = 1 + \frac{2a^2}{1+a^2(N-1)} - \frac{2}{N}. \tag{48}$$

Under suitable Lorentz transformations, the real matrix $\Lambda^{(1)}$ [see (30)] associated with the state $\rho^{(1)}$ [see (11)] gets transformed to its Lorentz canonical form $\widetilde{\Lambda}^{(1)}$ [see (34)]. It follows that [see (28), (31), and (32)]

$$\left(L_A\, \Lambda^{(1)}\, L_B^T\right)_{00} = \sqrt{\phi_0} = 2\sqrt{N-1}\left[\frac{1-a^2}{N(1+(N-1)\,a^2)}\right]. \tag{49}$$

Using the property $\det L_A = \det L_B = 1$ of orthochronous proper Lorentz transformations [18] and substituting $|\det \widetilde{\Lambda}^{(1)}| = \frac{1}{(N-1)^2}$ in (22), we obtain

$$\begin{aligned} |\det \widetilde{\Lambda}^{(1)}| &= \frac{1}{(N-1)^2} \\ &= |\det L_A|\,|\det L_B| \left|\det\left(\frac{\Lambda^{(1)}}{\sqrt{\phi_0}}\right)\right| \\ &= \frac{|\det \Lambda^{(1)}|}{\phi_0^2}. \end{aligned} \tag{50}$$

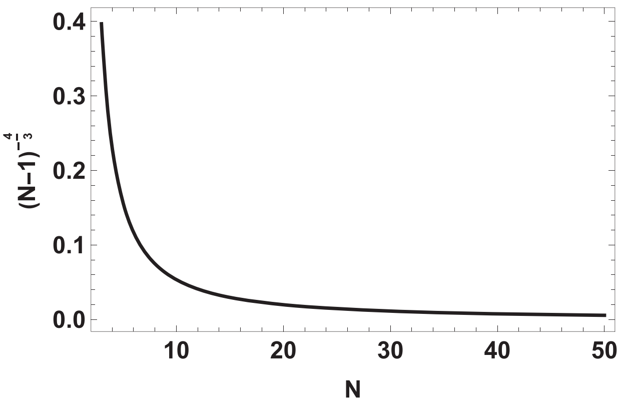

FIG. 4. The left-hand side of the monogamy relation $(N-1)^{-\frac{4}{3}} \leqslant \frac{1}{2}$ is seen to be less than $\frac{1}{2}$ for the W class of states $|D_{N-1,1}\rangle$ for any $N \geqslant 3$. (All quantities plotted are dimensionless.)

Equation (50) leads to $|\det \Lambda^{(1)}| = \phi_0^2\, |\det \widetilde{\Lambda}^{(1)}|$. The normalized volume $v_N^{(1)}$ of the quantum steering ellipsoid corresponding to the W class of states thus becomes [see (47)]

$$v_N^{(1)} = |\det \widetilde{\Lambda}^{(1)}| \frac{\phi_0^2}{(1-r^2)^2}. \tag{51}$$

From (48) and (49) it readily follows that $\phi_0^2 = (1-r^2)^2$, and hence [see (51)], the simple form for the normalized volume of the corresponding steering ellipsoid associated with the two-qubit state $\rho^{(1)}$ turns out to be

$$v_N^{(1)} = \frac{\phi_0^2}{(N-1)^2\,(1-r^2)^2} = \frac{1}{(N-1)^2}. \tag{52}$$

The volume monogamy relation $(v_N^{(1)})^{\frac{2}{3}} \leqslant \frac{1}{2}$ [see (46)] takes the form

$$\left(\frac{1}{(N-1)^2}\right)^{2/3} \leqslant \frac{1}{2} \Longrightarrow (N-1)^{-\frac{4}{3}} \leqslant \frac{1}{2} \tag{53}$$

and is readily satisfied for any $N \geqslant 3$, as can be seen in Fig. 4.

B. Relation between obesity of steering ellipsoids and concurrence

We recall here that the *obesity* $\mathcal{O}(\rho_{AB}) = |\det \Lambda|^{1/4}$ of the quantum steering ellipsoid [2] depicting a two-qubit state ρ_{AB} is an upper bound for the concurrence $C(\rho_{AB})$:

$$C(\rho_{AB}) \leqslant \mathcal{O}(\rho_{AB}) = |\det \Lambda|^{1/4}. \tag{54}$$

Furthermore, if $\rho_{AB} \longrightarrow \widetilde{\rho}_{AB} = (A \otimes B)\rho_{AB}\,(A^\dagger \otimes B^\dagger)/[\mathrm{Tr}(A^\dagger A \otimes B^\dagger B)\rho_{AB}]$, $A, B \in \mathrm{SL}(2, C)$, it follows that [2]

$$\frac{\mathcal{O}(\rho_{AB})}{C(\rho_{AB})} = \frac{\mathcal{O}(\widetilde{\rho}_{AB})}{C(\widetilde{\rho}_{AB})}. \tag{55}$$

Relation (55) can be used to obtain a relation for concurrence [22] of a pair of qubits in the symmetric N-qubit pure states $|D_{N-k,k}\rangle$, $k = 1, 2, \ldots, [\frac{N}{2}]$. For the states $|D_{N-1,1}\rangle$ belonging to the W class, we readily get [see (30) and (34)]

$$\det \Lambda^{(1)} = \left(\frac{2(1-a^2)}{N[1+a^2(N-1)]}\right)^4, \quad \det \widetilde{\Lambda}^{(1)} = \left(\frac{1}{N-1}\right)^2 \tag{56}$$

and therefore the obesities $\mathcal{O}(\rho^{(1)})$ and $\mathcal{O}(\widetilde{\rho}^{(1)})$:

$$\mathcal{O}(\rho^{(1)}) = \frac{2(1-a^2)}{N[1+a^2(N-1)]}, \quad \mathcal{O}(\widetilde{\rho}^{(1)}) = \frac{1}{\sqrt{N-1}}. \tag{57}$$

It is not difficult to evaluate the concurrence of the canonical state $\widetilde{\rho}^{(1)}$, and it is seen that

$$C(\widetilde{\rho}^{(1)}) = \mathcal{O}(\widetilde{\rho}^{(1)}) = \frac{1}{\sqrt{N-1}}. \tag{58}$$

We thus obtain [see (55) and (58)]

$$C(\rho^{(1)}) = \mathcal{O}(\rho^{(1)}) = \frac{2(1-a^2)}{N[1+a^2(N-1)]}. \tag{59}$$

The value of concurrence in (59) matches exactly with that obtained using $C(\rho^{(1)}) = \max(0,\ \mu_1 - \mu_2 - \mu_3 - \mu_4)$, where $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \mu_4$ are square roots of the eigenvalues of the matrix $R = \rho^{(1)}\,(\sigma_2 \otimes \sigma_2)\,\rho^{(1)*}\,(\sigma_2 \otimes \sigma_2)$ [22]. We have seen that the state $|D_{N-1,\,1}\rangle$ reduces to the W state when $a = 0$, and hence for the N-qubit W state, concurrence of any pair of qubits is given by $C(\rho_W^{(1)}) = \frac{2}{N}$ [see (59)]. The decrease in concurrence with the increase in the number of qubits in the W class of states $\{\mathcal{D}_{N-1,1}\}$ is pictorially indicated in Fig. 1 (the reduction in size of the canonical steering spheroids with an increase in N).

VI. SUMMARY

In this work, we have analyzed the canonical steering ellipsoids and volume monogamy relations of the pure symmetric N-qubit states characterized by two distinct Majorana spinors. We have shown that the entire W class of states has a geometric representation in terms of a *shifted oblate spheroid* inscribed within the Bloch sphere. The center of the spheroid, the length of its semiaxes, and its volume are shown to be dependent only on the number of qubits N, implying that all states in the N-qubit W class are characterized by a *single* spheroid, shifted along the polar axis of the Bloch sphere. All other SLOCC-inequivalent families of pure symmetric N-qubit states with two distinct spinors are shown to be geometrically represented by *ellipsoids centered at the origin*. Except for the W state (and its obverse counterpart), which is represented by a *shifted spheroid*, all other N-qubit Dicke states are represented by an *oblate spheroid centered at the origin*. A discussion of the volume monogamy relations applicable to the identical subsystems of a pure symmetric N-qubit state is given here, and a volume monogamy relation applicable for the W class of states is obtained. A relation connecting the concurrence of the two-qubit state and obesity of the associated quantum steering ellipsoid with its canonical counterparts is used to obtain the concurrence of the states belonging to the W class. It would be interesting to examine the features of canonical steering ellipsoids and volume monogamy relations for the SLOCC-inequivalent families of pure symmetric multiqubit states with more than two distinct spinors, in particular, the class of pure symmetric N-qubit states belonging to the GHZ class (with three distinct spinors).

ACKNOWLEDGMENTS

B.G.D. thanks IASC-INSA-NASI for the award of Summer Research Fellowship-2022 during this work. Sudha, A.R.U.D., and I.R. are supported by the Department of Science and Technology (DST), India, through Project No. DST/ICPS/QuST/Theme-2/2019 (Proposal Ref. No. 107). P.K.P. acknowledges the financial support from DST, India through Grant No. DST/ICPS/QuST/Theme-1/2019/2020-21/01.

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