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No-go result for quantum postselection measurements of a rank-*m* degenerate subspace

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We present a no-go result for postselection measurements where the conditional expectation value of a joint system-device observable under postselection is nothing else than the conventional expectation value. Such a no-go result relies on the rank-m degenerate of the joint observable, where m is the dimension of the device subspace. Remarkably, we show that the error and disturbance in quantum measurements obey the no-go result, which implies that the error-disturbance uncertainty is unaffected under postselection measurements.

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I. INTRODUCTION

The theoretical description of quantum measurements with a postselection protocol is fundamental and of practical interest [1–5]. It is a sequential measurements of a generalized (positive operator-valued measure) measurement followed by another projective measurement [1]. The postselection process alters the statistical results of the measured observable and leads to an extraordinary amplification effect: the expectation value obtained by the postselection can go far beyond the conventional eigenvalues of the measured observable [1,6,7]. Beyond the fundamental interest [8–15], postselection measurements have attracted tremendous research interest in multiple fields, including testing of quantum paradoxes and nonlocality [16–25], measurement uncertainty [26], weak value amplification [27–34], quantum-enhanced metrology [35–41], and direct quantum state measurement [42–51], among others.

Consider a prepared state ρ and a measured observable A represented by a self-adjoint operator $\boldsymbol{A}$. Assume the spectral decomposition of $\boldsymbol{A}$ has a purely discrete spectrum as $\boldsymbol{A} = \sum_k r_k \boldsymbol{P}_k$, where r_k is the eigenvalue, and projection operators $\boldsymbol{P}_k = |r_k\rangle\langle r_k|$ satisfy $\boldsymbol{P}_j\boldsymbol{P}_k = \delta_{jk}\boldsymbol{P}_k$; $\sum_k \boldsymbol{P}_k = \boldsymbol{I}$. Following the projection postulate [52,53], the expectation value gives $\langle \boldsymbol{A}\rangle_\rho = \sum_k r_k P(r_k|\rho)$, where $P(r_k|\rho) = \mathrm{Tr}[\boldsymbol{P}_k\rho]$ is the probability upon obtaining outcome r_k. The state transforms to, following the Lüders rule [52,54], the state transforms to $\rho' = \boldsymbol{P}_k\rho\boldsymbol{P}_k/\mathrm{Tr}[\boldsymbol{P}_k\rho]$. After the $\boldsymbol{A}$ measurement, a subsequent projection measurement is carried out using $\boldsymbol{\Pi}_\phi = |\phi\rangle\langle\phi|$, such that we postselect the system onto a final state $|\phi\rangle$. The expectation value of $\boldsymbol{A}$ now conditions on the postselected state $|\phi\rangle$ and reads ${}_\phi\langle \boldsymbol{A}\rangle_\rho = \sum_k r_k P(r_k|\phi,\rho)$, where $P(r_k|\phi,\rho) = \mathrm{Tr}[\boldsymbol{\Pi}_\phi\boldsymbol{P}_k\rho\boldsymbol{P}_k]/\sum_{k'}\mathrm{Tr}[\boldsymbol{\Pi}_\phi\boldsymbol{P}_{k'}\rho\boldsymbol{P}_{k'}]$ is the conditional probability following the Aharonov-Bergmann-Lebowitz (ABL) rule [55]. The conditional expectation value becomes the weak value when the system weakly couples to the device.

Even though the effect of postselections on measurement results is significant, it is not always so. For example, with a full degenerated spectrum, i.e., $r_k = r\ \forall k$, or with a projective observable $\boldsymbol{A} = |r\rangle\langle r|$, we have $\langle \boldsymbol{A}\rangle_\rho = {}_\phi\langle \boldsymbol{A}\rangle_\rho$. Previously, Vaidman *et al.* [56] have claimed that the nature of weak values is the same as the eigenvalues for an infinitesimally small interaction strength. Besides, weak values become conventional expectation values in the enlarged Hilbert space [13].

In this paper, we generalize these intuitive claims by presenting a "no-go" theorem, where the postselection does not affect the measurement results. We first extend the projection postulate to a composite system, such as a measured system and its apparatus (device). (A composite system also induces subsystem-subsystem and system-environment interaction models.) Whenever a joint system-device observable has rank-m degenerate subspace, where m is the device's dimension, the measured observable's results will not be affected by the postselection measurement. This is the main statement of the theorem. Afterward, we illustrate the no-go theorem in the error and disturbance of quantum measurements. Following Ozawa's interpretation [57,58], the error is a root-mean-square of the noise operator formed by the device's operator after the interaction and the system operator before the interaction, and the disturbance in a root-mean-square of the disturbance operator formed by the system's observables after and before the interaction.

II. CONDITIONAL EXPECTATION VALUES

In the von Neumann mechanism [59], we consider a measured system $\mathcal{S}$ and a device $\mathcal{M}$, initially prepared in uncorrelated state $|\Psi\rangle = |\psi\rangle\otimes|\xi\rangle$. The interaction is given by a unitary $\mathcal{U} = \exp(-it\boldsymbol{H}_\mathcal{S}\otimes\boldsymbol{H}_\mathcal{M})$, where $\boldsymbol{H}_\mathcal{S}$ and $\boldsymbol{H}_\mathcal{M}$ are Hamiltonians over the system's and device's complex Hilbert spaces $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{M}$, respectively. Given any initial joint observable $\mathcal{O}_0 = \boldsymbol{S}_0\otimes\boldsymbol{M}_0$ in the joint $\mathcal{SM}$ Hilbert space, following the Heisenberg picture, it evolves to $\mathcal{O}_t = \mathcal{U}^\dagger\mathcal{O}_0\mathcal{U}$ after interaction. Let $\mathcal{O}$ be a joint measured operator after interaction, which is a function of $\mathcal{O}_t$ and satisfies $\mathcal{O} \equiv f(\mathcal{O}_t) = \sum_k \boldsymbol{S}_k\otimes\boldsymbol{M}_k$ [60] (e.g., $\mathcal{O} = \mathcal{O}_t - \mathcal{O}_0$, which pertains to the error and disturbance operators discussed later). To calculate the expectation value of $\mathcal{O}$, we start with an element $\mathcal{O}_k \equiv \boldsymbol{S}_k\otimes\boldsymbol{M}_k$ which is an $(nm\times nm)$ matrix, where n (m) is the system (device) dimension. Let $|u_i\rangle$, $|v_j\rangle$ where $i \leqslant n$ and

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$j \leqslant m$ are eigenvectors associated to the eigenvalues u_i and v_j of $\boldsymbol{S}_k$ and $\boldsymbol{M}_k$, respectively, then $\mathcal{O}_k$ can be expressed in the spectral representation as $\mathcal{O}_k = \sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}^{(k)}\boldsymbol{P}_{ij}$, where $r_{ij}^{(k)} = u_i v_j$ are eigenvalues, and $\boldsymbol{P}_{ij} = |u_i v_j\rangle\langle u_i v_j|$ satisfies the orthonormality relations $\boldsymbol{P}_{ij}\boldsymbol{P}_{i'j'} = \delta_{i,i'}\delta_{j,j'}\boldsymbol{P}_{i'j'}$ and a completeness relation $\sum \boldsymbol{P}_{ij} = \boldsymbol{I}$. Following von Neumann, the probability to obtain the outcome $r_{ij}^{(k)}$ is given by [54]

$$P\big(r_{ij}^{(k)}\big|\rho\big) = \mathrm{Tr}[\boldsymbol{P}_{ij}\rho], \tag{1}$$

where we set $\rho = |\Psi\rangle\langle\Psi|$. The expectation value of $\mathcal{O}_k$ is given by

$$\langle \mathcal{O}_k\rangle_\rho = \sum_{i,j} r_{ij}^{(k)} P\big(r_{ij}^{(k)}\big|\rho\big) = \sum_{i,j} r_{ij}^{(k)}\mathrm{Tr}[\boldsymbol{P}_{ij}\rho]. \tag{2}$$

After the projection measurement $\boldsymbol{P}_{ij}$, the joint state transforms to a conditional (not normalized) $\rho'_{ij} = \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}$. We then postselect system $\mathcal{S}$ onto a final state $|\phi\rangle$, represented by a projection operator $\boldsymbol{\Pi}_\phi = |\phi\rangle\langle\phi| \otimes \boldsymbol{I}$. The joint probability to obtain $r_{ij}^{(k)}$ and postselection is

$$P\big(r_{ij}^{(k)}, \phi\big|\rho\big) = \mathrm{Tr}[\boldsymbol{\Pi}_\phi \rho'_{ij}] = \mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}]. \tag{3}$$

Using the Bayesian theorem, the conditional probability to obtain $r_{ij}^{(k)}$ for given pre- and postselected states is

$$\begin{aligned} P\big(r_{ij}^{(k)}\big|\phi,\rho\big) &= \frac{P\big(r_{ij}^{(k)},\phi\big|\rho\big)}{\sum_{i',j'} P\big(r_{i'j'}^{(k)},\phi\big|\rho\big)} \\ &= \frac{\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}]}{\sum_{i',j'}\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{i'j'}\rho\boldsymbol{P}_{i'j'}]}. \end{aligned} \tag{4}$$

Then, the conditional expectation value of $\mathcal{O}_k$ yields

$${}_\phi\langle \mathcal{O}_k\rangle_\rho = \sum_{i,j} r_{ij}^{(k)} P(r_{ij}|\phi,\rho) = \frac{\sum_{i,j} r_{ij}^{(k)}\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}]}{\sum_{i',j'}\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{i'j'}\rho\boldsymbol{P}_{i'j'}]}. \tag{5}$$

We present the following theorem:

Theorem (no-go postselection theorem). For any given joint state $\rho = |\Psi\rangle\langle\Psi|$ and postselected state $|\phi\rangle$, the following rank-m degenerate for every joint operator $\mathcal{O}_k$,

$$r_{ij}^{(k)} = r_{i'j}^{(k)} \equiv \widetilde{r}_j^{(k)}, \forall\ 1 \leqslant i, i' \leqslant n,\ \text{and}\ 1 \leqslant j \leqslant m, \tag{6}$$

must lead to a no-go for postselection measurement

$${}_\phi\langle \mathcal{O}_k\rangle_\rho = \langle \mathcal{O}_k\rangle_\rho \quad \text{and} \quad {}_\phi\langle \mathcal{O}\rangle_\rho = \langle \mathcal{O}\rangle_\rho, \tag{7}$$

where $\langle\mathcal{O}\rangle = \sum_k \langle\mathcal{O}_k\rangle$

Proof of Theorem. Let $\{|e_i\rangle \otimes |g_j\rangle\}$ be the canonical basis of the joint space, in which the joint state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = \begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{pmatrix} \otimes \begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_m\end{pmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{m}\psi_i\xi_j|e_i g_j\rangle, \tag{8}$$

where $\psi_i = \langle e_i|\psi\rangle$ and $\xi_j = \langle g_j|\xi\rangle$. Let the eigenvalues $|r_{ij}^{(k)}\rangle \equiv |u_i v_j\rangle = \sum_{i',j'} a_{i'j',ij}|e_{i'}g_{j'}\rangle$ be the eigenbasis, and $\boldsymbol{T} = (|r_{11}^{(k)}\rangle, \dots, |r_{nm}^{(k)}\rangle)$ is the transformation matrix that formed by the ket vector of all eigenvalues. In the eigenbases, the joint state is expressed as $|\Psi\rangle = \sum_{i,j}\psi'_i\xi'_j|u_i v_j\rangle$, where $\psi'_i = \langle u_i|\psi\rangle$ and $\xi'_j = \langle v_j|\xi\rangle$, who obey $\sum_i |\psi'_i|^2 = 1$ and $\sum_j |\xi'_j|^2 = 1$. We then obtain $\langle r_{ij}^{(k)}|\Psi\rangle = \psi'_i\xi'_j$.

In the postselected projection operator $\boldsymbol{\Pi}_\phi = |\phi\rangle\langle\phi| \otimes \boldsymbol{I}$, let $|\phi\rangle = \sum_i \phi_i|e_i\rangle$, where $\phi_i = \langle e_i|\phi\rangle$ a complex amplitude, then we obtain the $(nm\times nm)$ matrix:

$$\boldsymbol{\Pi}_\phi = \begin{pmatrix} \boxed{\begin{matrix}|\phi_1|^2 & & \\ & \ddots & \\ & & |\phi_1|^2\end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix}|\phi_n|^2 & & \\ & \ddots & \\ & & |\phi_n|^2\end{matrix}} \end{pmatrix}, \tag{9}$$

where the off-diagonal elements are omitted since they will vanish in the canonical basis; see more details in Appendix A. Here, each box is an $(m\times m)$-matrix, with totally n boxes. In the eigenbasis, the postselected state is expressed as $\boldsymbol{\Pi}'_\phi = \boldsymbol{T}^\dagger\boldsymbol{\Pi}_\phi\boldsymbol{T}$. There is an extra requirement (for the transformation matrix) that the diagonal elements of $\boldsymbol{\Pi}'_\phi$ admit the rank-m degenerated subspace similar to $\boldsymbol{\Pi}_\phi$, i.e., $\mathrm{diag}(\boldsymbol{\Pi}'_\phi) = (|\phi'_1|^2, \dots, |\phi'_1|^2, \dots, |\phi'_n|^2, \dots, |\phi'_n|^2)$. We emphasize that this requirement is always satisfied when the eigenbasis is the canonical basis as given in Eq. (9).

Now, from Eq. (5), we have the denominator

$$\sum_{i,j=1}^{n,m}\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}] = \sum_{i=1}^{n}|\phi'_i|^2\psi'_i|^2 \tag{10}$$

and the numerator

$$\sum_{i,j} r_{ij}^{(k)}\mathrm{Tr}[\boldsymbol{\Pi}_\phi \boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}] = \sum_{j=1}^{m}\widetilde{r}_j^{(k)}|\xi'_j|^2 \cdot \sum_{i=1}^{n}|\phi'_i|^2\psi'_i|^2, \tag{11}$$

where we applied condition (6). Equation (5) is recast as

$${}_\phi\langle \mathcal{O}_k\rangle_\rho = \sum_{j=1}^{m}\widetilde{r}_j^{(k)}|\xi'_j|^2. \tag{12}$$

Similarly, we have $\langle\mathcal{O}_k\rangle_\rho = \sum_{j=1}^{m}\widetilde{r}_j^{(k)}|\xi'_j|^2$, then ${}_\phi\langle\mathcal{O}_k\rangle_\rho = \langle\mathcal{O}_k\rangle_\rho$ (see detailed proof in Appendix A). The proof for the second term in (7) is straightforward since all $\mathcal{O}_k$ satisfy condition (6). ■

Corollary 1. For any eigenbasis where $\{|r_{ij}\rangle\}$ is the canonical basis, the no-go theorem states

$${}_\phi\langle\mathcal{O}\rangle_\rho = \langle\mathcal{O}\rangle_\rho = \sum_{j=1}^{m}\widetilde{r}_j|\xi_j|^2. \tag{13}$$

The proof for this Corollary is the same as above.

III. REMARKS

Different from the conditional expectation values we are considering here, the weak value of an observable $\boldsymbol{A}$ in system $\mathcal{S}$ generally depends on the postselected state, as it is $\langle \boldsymbol{A}\rangle_w = \langle\phi|\boldsymbol{A}|\psi\rangle/\langle\phi|\psi\rangle$, except for some certain conditions wherein it can be an expectation value [13] or an eigenvalue [56]. However, this is not a consequential result from our theorem here. Instead, a consequence from the no-go theorem can be stated as follows:

Corollary 2. Weak values can reduce to eigenvalues if the measured observable $\boldsymbol{A}$ has a full-rank degenerate subspace. To proof this Corollary, let us say that $\boldsymbol{A} = \sum_k a_k|k\rangle\langle k|$ with $a_k = a$ for all $k = 1,\ldots,n$, then $\langle \boldsymbol{A}\rangle_w = \langle \boldsymbol{A}\rangle = a$.

IV. OBSERVATION

Any joint observable, i.e., $\mathcal{O} = \boldsymbol{S}\otimes\boldsymbol{M}$ with $\boldsymbol{S} = \boldsymbol{I}$, always satisfies the no-go theorem. The proof for this observation is given directly by noting that the eigenvalues of $\boldsymbol{I}$ are all one. Thus, an eigenvalue of $\mathcal{O}$ satisfies (6), and thus satisfies the theorem.

V. NO-GO THEOREM IN THE ERROR AND DISTURBANCE

Error and disturbance are essential quantities for determining measurements' uncertainties [57,58,61]. We consider the error of an $\boldsymbol{A}$ measurement in system $\mathcal{S}$ through an $\boldsymbol{M}$ measurement in device $\mathcal{M}$ and the disturbance of a $\boldsymbol{B}$ measurement in system $\mathcal{S}$. In the joint $\mathcal{SM}$ system, we denote $\boldsymbol{\mathcal{A}}_0 = \boldsymbol{A}\otimes\boldsymbol{I}$, and $\boldsymbol{\mathcal{B}}_0 = \boldsymbol{B}\otimes\boldsymbol{I}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are the observables to be measured in system $\mathcal{S}$. We also define a device observable $\boldsymbol{M}$ in the device space, such that it becomes $\boldsymbol{\mathcal{M}}_0 = \boldsymbol{I}\otimes\boldsymbol{M}$ in the joint $\mathcal{SM}$ space [57,58,61]. The interaction is switched on during a short time t, where the joint $\mathcal{SM}$ system evolves under the unitary transformation $\boldsymbol{\mathcal{U}}$. These operators transform to $\boldsymbol{\mathcal{M}}_t = \boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\mathcal{M}}_0\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{B}}_t = \boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\mathcal{B}}_0\boldsymbol{\mathcal{U}}$. According to Ozawa [57,58,61], the error and disturbance operators are defined by $\boldsymbol{\mathcal{N}}_{\boldsymbol{A}} = \boldsymbol{\mathcal{M}}_t - \boldsymbol{\mathcal{A}}_0$ and $\boldsymbol{\mathcal{D}}_{\boldsymbol{B}} = \boldsymbol{\mathcal{B}}_t - \boldsymbol{\mathcal{B}}_0$, respectively. Then, the mean square error and the disturbance are given by

$$\epsilon_{\boldsymbol{A}}^2 = \left\langle \boldsymbol{\mathcal{N}}_{\boldsymbol{A}}^2\right\rangle_\Psi \quad\text{and}\quad \eta_{\boldsymbol{B}}^2 = \left\langle \boldsymbol{\mathcal{D}}_{\boldsymbol{B}}^2\right\rangle_\Psi, \tag{14}$$

where the bra-ket symbol $\langle\ldots\rangle_\Psi$ means $\langle\Psi|\ldots|\Psi\rangle$ throughout this paper.

In the following, we illustrate that such an error and disturbance satisfy condition (6) in the theorem, and thus it makes no sense for postselection measurements of the error and disturbance. In other words, the error and disturbance will not be affected under the postselection.

Concretely, let us consider a CNOT-type measurement, where both system $\mathcal{S}$ and device $\mathcal{M}$ are qubits initially prepared in $|\psi\rangle = |i^+\rangle$ and $|\xi\rangle = \sqrt{\frac{1+s}{2}}|0\rangle + \sqrt{\frac{1-s}{2}}|1\rangle$, where $|0\rangle$ and $|1\rangle$ are two eigenstates of Pauli matrix $\boldsymbol{Z}$, and $|i^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$; s is the measurement strength ranging from 0 (weak measurement) to 1 (strong measurement). The measured observables $\boldsymbol{A}$ and $\boldsymbol{B}$ in system $\mathcal{S}$ are chosen to be Pauli matrices $\boldsymbol{Z}$ and $\boldsymbol{X}$, respectively, and the device observable $\boldsymbol{M}$ is also $\boldsymbol{Z}$. The interaction is CNOT gate, i.e., $\boldsymbol{\mathcal{U}} = |0\rangle\langle 0|\otimes\boldsymbol{I} + |1\rangle\langle 1|\otimes\boldsymbol{X}$. The square error and disturbance operators give

$$\boldsymbol{\mathcal{N}}_{\boldsymbol{Z}}^2 = 4\boldsymbol{I}\otimes|1\rangle\langle 1| \quad\text{and}\quad \boldsymbol{\mathcal{D}}_{\boldsymbol{X}}^2 = 2\boldsymbol{I}\otimes(\boldsymbol{I}-\boldsymbol{X}). \tag{15}$$

Fortunately, CNOT is a typical interaction, which leads to simplifying measured operators (square error and disturbance). The eigenvalues of the square error are (0, 4, 0, 4), and the same for the square disturbance, which all satisfy the rank-2 degenerate. As a result, ${}_\phi\langle\boldsymbol{\mathcal{N}}_{\boldsymbol{Z}}^2\rangle_\Psi = \langle\boldsymbol{\mathcal{N}}_{\boldsymbol{Z}}^2\rangle_\Psi = 2(1-s)$, and ${}_\phi\langle\boldsymbol{\mathcal{D}}_{\boldsymbol{X}}^2\rangle_\Psi = \langle\boldsymbol{\mathcal{D}}_{\boldsymbol{X}}^2\rangle_\Psi = 2(1-\sqrt{1-s^2})$, for any postselected state $|\phi\rangle = \cos\theta|0\rangle + e^{-i\varphi}\sin\theta|1\rangle$. These results imply that postselection measurements affect neither the error nor the disturbance. (See detailed calculation in Appendix B.)

The observations in the example can be explained as follows. The error is determined via the device after the first measurement, and thus it is not affected by postselection measurements as long as the rank-m degenerate in the device holds. For the same reason, the disturbance is affected by the backaction caused by the device while unpolluted with postselection measurements.

VI. CONCLUSION

We introduced a no-go theorem for postselection measurements, where the obtained conditional expectation value is not affected by postselection measurements and is equal to a conventional expectation value. This can happen when the joint observable has a rank-m degenerate subspace, where m is the dimension of the device space. As a consequence, the error and disturbance in quantum measurements are immune with postselection measurements.

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APPENDIX A: DETAILED PROOF FOR THE NO-GO THEOREM

In this proof, we omit the indicator k for short. We first derive $\mathrm{Tr}[\boldsymbol{\Pi}_\phi\boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}]$ in Eq. (10) in the main text. We have

$$\begin{aligned}\mathrm{Tr}[\boldsymbol{\Pi}_\phi\boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}] &= \langle\Psi|\boldsymbol{P}_{ij}\boldsymbol{\Pi}_\phi\boldsymbol{P}_{ij}|\Psi\rangle\\ &= \langle\Psi|r_{ij}\rangle\langle r_{ij}|\boldsymbol{\Pi}_\phi|r_{ij}\rangle\langle r_{ij}|\Psi\rangle,\end{aligned} \tag{A1}$$

where we used $\boldsymbol{P}_{ij} = |r_{ij}\rangle\langle r_{ij}|$. Concretely, we derive

$$\langle r_{ij}|\Psi\rangle = \langle u_i v_j|\psi\xi\rangle = \psi_i'\xi_j', \tag{A2}$$

where we set $\psi_i' = \langle u_i|\psi\rangle$, $\xi_j' = \langle v_j|\xi\rangle$, and

$$\langle r_{ij}|\boldsymbol{\Pi}_\phi|r_{ij}\rangle = \langle e_i g_j|\boldsymbol{T}^\dagger\boldsymbol{\Pi}_\phi\boldsymbol{T}|e_i g_j\rangle = (\boldsymbol{\Pi}_\phi')_{ij\times ij}, \tag{A3}$$

where we applied the bases transformation rule $|r_{ij}\rangle = \boldsymbol{T}|e_i g_j\rangle$, and set $\boldsymbol{\Pi}_\phi' = \boldsymbol{T}^\dagger\boldsymbol{\Pi}_\phi\boldsymbol{T}$. We also consider the case $\mathrm{diag}(\boldsymbol{\Pi}_\phi') = (|\phi_1'|^2,\ldots,|\phi_1'|^2,\ldots,|\phi_n'|^2,\ldots,|\phi_n'|^2)$. Then, substituting Eqs. (A2) and (A3) into Eq. (A1), we obtain Eq. (10),

$$\begin{aligned}\sum_{i,j=1}^{n,m}\mathrm{Tr}[\boldsymbol{\Pi}_\phi\boldsymbol{P}_{ij}\rho\boldsymbol{P}_{ij}] &= \sum_{i,j=1}^{n,m}\langle\Psi|r_{ij}\rangle\langle r_{ij}|\boldsymbol{\Pi}_\phi|r_{ij}\rangle\langle r_{ij}|\Psi\rangle\\ &= \sum_{i,j=1}^{n,m}|\psi_i'|^2\,|\xi_j'|^2\,|\phi_i'|^2\\ &= \sum_{i=1}^{n}|\psi_i'|^2\,|\phi_i'|^2.\end{aligned} \tag{A4}$$

Next, we derive Eq. (11) in the main text

$$\sum_{i,j} r_{ij}\mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{ij}\rho \boldsymbol{P}_{ij}] = \sum_{i,j=1}^{n,m} r_{ij}\ |\psi_i'|^2\ |\xi_j'|^2\ |\phi_i'|^2 = \sum_{i=1}^{n} |\psi_i'|^2\ |\phi_i'|^2 \cdot \sum_{j=1}^{m} \widetilde{r}_j\ |\xi_j'|^2. \quad (A5)$$

Finally, the conditional expectation value of the observable $\mathcal{O}$ is given by

$$_\phi\langle\mathcal{O}\rangle_\rho = \frac{\sum_{i,j} r_{ij}\mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{ij}\rho \boldsymbol{P}_{ij}]}{\sum_{i',j'} \mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{i'j'}\rho \boldsymbol{P}_{i'j'}]} = \sum_{j=1}^{m} \widetilde{r}_j\ |\xi_j'|^2. \quad (A6)$$

We compare the conditional expectation value with the conventional expectation value in Eq. (A7) in the main text,

$$\langle\mathcal{O}\rangle_\rho = \sum_{i,j} r_{ij}\mathrm{Tr}[\boldsymbol{P}_{ij}\rho] = \sum_{i,j} \widetilde{r}_j\langle\Psi|r_{ij}\rangle\langle r_{ij}|\Psi\rangle = \sum_{i,j} \widetilde{r}_j\ |\psi_i'|^2\ |\xi_j'|^2 = \sum_{j} \widetilde{r}_j\ |\xi_j'|^2, \quad (A7)$$

and obtain $_\phi\langle\mathcal{O}\rangle_\rho = \langle\mathcal{O}\rangle_\rho$, which completes the proof.

APPENDIX B: ERROR AND DISTURBANCE IN CNOT-TYPE MEASUREMENT

In this section, we give a detailed calculation of square error and square disturbance. First, we explicitly derive the initial joint state as

$$|\Psi\rangle = |\psi\rangle \otimes |\xi\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{1+s} \\ \sqrt{1-s} \\ i\sqrt{1+s} \\ i\sqrt{1-s} \end{pmatrix}, \quad (B1)$$

where $\psi_1 = \psi_2 = 1/\sqrt{2}$, and $\xi_1 = \sqrt{\frac{1+s}{2}}, \xi_2 = \sqrt{\frac{1-s}{2}}$. The square error operator in Eq. (15) is decomposed into its eigenvalue and eigenstate as

$$\mathcal{N}_Z^2 = 4\boldsymbol{I} \otimes |1\rangle\langle 1| = 0|00\rangle\langle 00| + 4|01\rangle\langle 01| + 0|10\rangle\langle 10| + 4|11\rangle\langle 11| \equiv \sum_{i,j} r_{ij}\boldsymbol{P}_{ij}. \quad (B2)$$

Here, obviously, we have $r_{11} = r_{21} = 0$ $(\equiv \widetilde{r}_1)$ and $r_{12} = r_{22} = 4$ $(\equiv \widetilde{r}_2)$, which satisfy condition (6). The projectors are $\boldsymbol{P}_{11} = \mathrm{diag}(1, 0, 0, 0); \boldsymbol{P}_{12} = \mathrm{diag}(0, 1, 0, 0); \boldsymbol{P}_{21} = \mathrm{diag}(0, 0, 1, 0)$; and $\boldsymbol{P}_{22} = \mathrm{diag}(0, 0, 0, 1)$. The postselected state explicitly reads

$$\mathbf{\Pi}_\phi = \begin{pmatrix} \cos^2\theta & & & \\ & \cos^2\theta & & \\ & & \sin^2\theta & \\ & & & \sin^2\theta \end{pmatrix}, \quad (B3)$$

while we ignore the off-diagonal terms, hence $\phi_1 = \cos\theta$, $\phi_2 = e^{-i\varphi}\sin\theta$. We emphasize that in this case, the eigenbasis is also the canonical basis, such that $\boldsymbol{T} = \boldsymbol{I}$. Now, we calculate (10)

$$\sum_{i,j=1}^{2,2} \mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{ij}\rho \boldsymbol{P}_{ij}] = \sum_{i=1}^{2} |\phi_i|^2\ |\psi_i|^2 = \frac{1}{2}. \quad (B4)$$

Note that direct calculating the left-hand-side gives us exactly the same result. Next, we evaluate (11)

$$\sum_{i,j} r_{ij}\mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{ij}\rho \boldsymbol{P}_{ij}] = \sum_{j=1}^{2} \widetilde{r}_j\ |\xi_j|^2\ \sum_{i=1}^{2} |\phi_i|^2\ |\psi_i|^2 = 1 - s. \quad (B5)$$

Again, direct calculating the left-hand side gives us exactly the same result. Now, Eq. (5) explicitly reads

$$_\phi\langle\mathcal{N}_Z^2\rangle_\rho = 2(1 - s), \quad (B6)$$

which is the square error under the pre- and postselection measurement scheme. Furthermore, in the usual case without postselection, the square error also reads $\langle\mathcal{N}_Z^2\rangle_\rho = 2(1 - s)$, which implies no-go for postselection as shown in Eq. (13) of Corollary 1.

Now, we evaluate the square disturbance. The square disturbance operator stated in Eq. (15) is decomposed into its eigenvalue and eigenstate as

$$\mathcal{D}_X^2 = 2\boldsymbol{I} \otimes (\boldsymbol{I} - \boldsymbol{X}) = 0|0+\rangle\langle 0+| + 4|0-\rangle\langle 0-| + 0|1+\rangle\langle 1+| + 4|1-\rangle\langle 1-|. \quad (B7)$$

Here, $|0+\rangle = |0\rangle \otimes |+\rangle$ and likewise for the others, and $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Similar to the square-error case, we have $r_{11} = r_{21} = 0$ $(\equiv \widetilde{r}_1)$ and $r_{12} = r_{22} = 4$ $(\equiv \widetilde{r}_2)$, which satisfy condition (6). The projectors in this case are $\boldsymbol{P}_{11} = |0+\rangle\langle 0+|$; $\boldsymbol{P}_{12} = |0-\rangle\langle 0-|$; $\boldsymbol{P}_{21} = |1+\rangle\langle 1+|$; and $\boldsymbol{P}_{22} = |1-\rangle\langle 1-|$.

We next introduce the transformation matrix

$$\boldsymbol{T} = (|0+\rangle, \quad |0-\rangle, \quad |1+\rangle, \quad |1-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (B8)$$

The postselected state in the eigenbasis reads

$$\mathbf{\Pi}_\phi' = \boldsymbol{T}^\dagger \mathbf{\Pi}_\phi \boldsymbol{T} = \mathrm{diag}(\cos^2\theta, \cos^2\theta, \sin^2\theta, \sin^2\theta), \quad (B9)$$

where we have ignored the off-diagonal terms as it will not affect under the eigenbasis. Now, we calculate (10) by inserting $\boldsymbol{T}\boldsymbol{T}^\dagger$ between operators inside the trade:

$$\sum_{i,j=1}^{2,2} \mathrm{Tr}[\mathbf{\Pi}_\phi' \boldsymbol{P}_{ij}'\rho' \boldsymbol{P}_{ij}'] = \sum_{i=1}^{2} |\phi_i'|^2\ |\psi_i'|^2 = \frac{1}{2}. \quad (B10)$$

Note that direct calculating the left-hand side gives us exactly the same result. Next, we evaluate (11)

$$\sum_{i,j} r_{ij}\mathrm{Tr}[\mathbf{\Pi}_\phi \boldsymbol{P}_{ij}\rho \boldsymbol{P}_{ij}] = \sum_{j=1}^{2} \widetilde{r}_j\ |\xi_j'|^2\ \sum_{i=1}^{2} |\phi_i'|^2\ |\psi_i'|^2 = 1 - \sqrt{1 - s^2}. \quad (B11)$$

Now, Eq. (5) explicitly reads

$$_\phi\langle\mathcal{D}_X^2\rangle_\rho = 2(1-\sqrt{1-s^2}), \tag{B12}$$

which is the square disturbance under the pre- and postselection measurement scheme. Furthermore, in the usual case without postselection, the square error also reads $\langle\mathcal{D}_X^2\rangle_\rho = 2(1-\sqrt{1-s^2})$, which again implies no-go for postselection.

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[60] Using the Baker-Campbell-Hausdorff formula [62], we expand $\mathcal{U}^\dagger\mathcal{O}_0\mathcal{U}=\mathcal{O}_0+it[\boldsymbol{H}_\mathcal{S}\otimes\boldsymbol{H}_\mathcal{M},\mathcal{O}_0]+\frac{(it)^2}{2!}[\boldsymbol{H}_\mathcal{S}\otimes\boldsymbol{H}_\mathcal{M},[\boldsymbol{H}_\mathcal{S}\otimes\boldsymbol{H}_\mathcal{M},\mathcal{O}_0]]+\cdots$. For small t (or weak interaction), it yields $\mathcal{U}^\dagger\mathcal{O}_0\mathcal{U}\approx\mathcal{O}_0+it[\boldsymbol{H}_\mathcal{S}\otimes\boldsymbol{H}_\mathcal{M},\mathcal{O}_0]]=\sum_k\boldsymbol{S}_k\otimes\boldsymbol{M}_k$.

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