Formal relation between Pegg-Barnett and Paul quantum phase frameworks

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The problem of defining a Hermitian quantum phase operator is nearly as old as quantum mechanics itself. Throughout the years, a number of solutions have been proposed, ranging from abstract operator formalisms to phase-space methods. In this work, we make an explicit connection between two of the most prominent approaches by proving that the probability distribution of phase in the Paul formalism follows exactly from the Pegg-Barnett formalism by combining the latter with the quantum-limited amplifier channel. Our findings suggest that the Paul framework may be viewed as a semiclassical limit of the Pegg-Barnett approach.

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I. INTRODUCTION

In the history of quantum mechanics, few problems have received as much attention as the problem of the definition and measurement of the phase of the quantum electromagnetic field. Since the early failed attempt by Dirac [1], a plethora of solutions have been proposed, including phase-space approaches [2,3], as well as operator formalisms by Susskind and Glogower [4], Garrison and Wong [5], Paul [6], Lévy-Leblond [7], Popov and Yarunin [8], and, finally, Pegg and Barnett [9,10].

Significant interest has been especially devoted to the Pegg-Barnett formalism, in which the obstacles standing in the way of a well-defined quantum phase operator are overcome by reducing the problem to a finite dimension. After its discovery, the formalism quickly gave rise to an alternative derivation [11] and extension [12], among other things [13–16]. Nowadays, the Pegg-Barnett formalism is used, e.g., to investigate the phase properties of various nonclassical phenomena, including photon antibunching in the case of photon addition and substraction [17,18], phase-number squeezing in atom-field interactions [19], and nonlinear squeezed states [20].

In this work, we focus on the formal aspects of the Pegg-Barnett framework and relate it to the Paul formalism, a “competing” solution to the quantum phase problem that is especially notable for its close connection to phase-space approaches and a clear experimental interpretation [21,22]. More precisely, we prove that the probability distribution in the Paul formalism can be exactly obtained from its counterpart in the Pegg-Barnett formalism if the Pegg-Barnett phase operator is combined with the quantum-limited amplifier channel. Furthermore, due to the amplifier's association with classicality [23] (discussed in more detail below), we interpret our result as the Paul framework being a semiclassical limit of the Pegg-Barnett formalism. In this way, we bridge the two approaches mathematically and physically.

Interestingly, our findings are not the first to apply the quantum-limited amplifier to the Paul formalism, albeit the context is different. Reference [23] showed that the phase distribution of a quantum state in the Paul framework can be realized experimentally through the amplified state. In Ref. [24], the Paul framework was proved to be the only quantum phase description consistent with the Glauber model of amplification and the natural expectation that large-amplitude coherent states should have a well-defined phase.

This paper is organized as follows. In Sec. II, we briefly summarize the two discussed quantum phase formalisms. In Sec. III, we introduce our main tool: the quantum-limited amplifier channel. In Sec. IV, we state and derive our main result. An in-depth discussion of this result is provided in Sec. V, with explicit examples being given in Sec. VI. We conclude in Sec. VII.

II. PAUL AND PEGG-BARNETT PHASE FORMALISMS

As already stated, our main subject of interest concerns the Paul and Pegg-Barnett formalisms and the connection between them through the quantum-limited amplifier channel. Let us briefly introduce and discuss the two phase formalisms. For a detailed review, see, e.g., [25,26].

A. Paul formalism

Back in 1974, Paul considered the following family of operators [6]:

$$\hat{E}_k := \int \frac{d^2\alpha}{\pi}\left(\frac{\alpha}{|\alpha|}\right)^k |\alpha\rangle\langle\alpha|,$$
$$\hat{E}_{-k} := \int \frac{d^2\alpha}{\pi}\left(\frac{\alpha^*}{|\alpha|}\right)^k |\alpha\rangle\langle\alpha| = \hat{E}_k^\dagger, \tag{1}$$

where $k \in \mathbb{N}$ and

$$|\alpha\rangle = \sum_{n=0}^{\infty}\alpha_n|n\rangle, \quad \alpha_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{2}$$

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is a coherent state with amplitude $\alpha \in \mathbb{C}$. We discuss the case of a single mode, for which $\hat{a}$ is the annihilation operator (so that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$) and $|n\rangle$ for $n = 0, \dots, \infty$ denotes the Fock basis.

The operators (1) correspond to the classical quantities $e^{\pm ik\phi}$. This is most easily seen by setting $\alpha = re^{i\phi}$ and computing the expectation values for an arbitrary state $\hat{\rho}$, which leads to

$$\langle \hat{E}_k \rangle_{\hat{\rho}} = \int_0^{2\pi} d\phi\, e^{ik\phi} \int_0^{\infty} \frac{dr}{\pi}\, r\, Q_{\hat{\rho}}(re^{i\phi}), \tag{3}$$

with analogous notation for $\hat{E}_{-k}$. Here,

$$Q_{\hat{\rho}}(\alpha) := \langle \alpha|\hat{\rho}|\alpha\rangle \tag{4}$$

is the Husimi Q quasiprobability distribution [27]. Due to the properties of the Husimi function, the rightmost integral in Eq. (3) is positive for all ϕ and, when integrated over ϕ from 0 to 2π, yields a value of 1. For this reason, it can be regarded as the probability distribution of ϕ in the formalism:

$$P_{\mathrm{Paul}}(\phi|\hat{\rho}) := \int_0^{\infty} \frac{dr}{\pi}\, r\, Q_{\hat{\rho}}(re^{i\phi}). \tag{5}$$

It is useful to compare the Paul operators with the Glauber-Sudarshan P representation [28] of an arbitrary operator $\hat{X}$, defined through

$$\hat{X} = \int \frac{d^2\alpha}{\pi}\, P_{\hat{X}}(\alpha)\, |\alpha\rangle\langle\alpha|. \tag{6}$$

We can see that the Paul operators (1) are essentially operators whose P distribution is equal to the kth powers of the quantity $e^{i\phi}$.

As a natural generalization, in this paper we consider operators whose P representation is rendered by any complex-valued, bounded function f of $e^{i\phi}$, i.e.,

$$\hat{\phi}_{\mathrm{Paul}}[f] := \int \frac{d^2\alpha}{\pi}\, f\left(\frac{\alpha}{|\alpha|}\right) |\alpha\rangle\langle\alpha|. \tag{7}$$

Clearly, any such operator has properties similar to the original Paul operators. In particular, its expectation value reads

$$\langle \hat{\phi}_{\mathrm{Paul}}[f] \rangle_{\hat{\rho}} = \int_0^{2\pi} d\phi\, f(e^{i\phi})\, P_{\mathrm{Paul}}(\phi|\hat{\rho}), \tag{8}$$

with $P_{\mathrm{Paul}}(\phi|\hat{\rho})$ being the same probability distribution as in Eq. (5).

One of the main strengths of the Paul formalism is its close association with experimental phase detection, such as homodyne measurements using an eight-port interferometer [21,22]. In the strong local oscillatory regime of such an experiment, the phase difference between an arbitrary bosonic state and a reference coherent state is given precisely by the Paul probability distribution (5).

B. Pegg-Barnett formalism

Introduced in 1988 [9] and developed further in subsequent years, the Pegg-Barnett formalism is built upon a family of $s + 1$ "number-phase states"

$$|\theta_{t,s}\rangle := \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_{t,s}} |n\rangle, \tag{9}$$

where

$$\theta_{t,s} := \frac{2\pi t}{s+1}, \qquad t \in \{0, 1, \dots, s\}. \tag{10}$$

Typically, an arbitrary reference phase θ_0 is added to the definition of $\theta_{t,s}$. Indeed, from a practical point of view the phase itself is not well defined, and measurements must be made relative to an auxiliary state. For example, see [12], where an operator measuring the phase difference between two systems was considered. Here, we are concerned with the relation between the Pegg-Barnett and Paul formalisms and not the formalisms themselves. For consistency with the Paul framework, we therefore take the liberty to set the reference phase θ_0 to zero.

For finite s, the number-phase states form an orthonormal basis of the $(s + 1)$-dimensional Hilbert space, which is the s-photon subspace of the single-mode Fock space. Hence, the Pegg-Barnett phase operator

$$\hat{\phi}^{(s)}_{\mathrm{PB}} := \sum_{t=0}^{s} \theta_{t,s} |\theta_{t,s}\rangle\langle\theta_{t,s}| \tag{11}$$

is Hermitian. By considering a formal power series of this operator, we can associate a Pegg-Barnett operator with any complex-valued, bounded function f of the phase exponential:

$$\hat{\phi}^{(s)}_{\mathrm{PB}}[f] := \sum_{t=0}^{s} f(e^{i\theta_{t,s}}) |\theta_{t,s}\rangle\langle\theta_{t,s}|. \tag{12}$$

Computing its expectation value on state $\hat{\rho}$, we find

$$\left\langle \hat{\phi}^{(s)}_{\mathrm{PB}}[f] \right\rangle_{\hat{\rho}} = \sum_{t=0}^{s} f(e^{i\theta_{t,s}}) \langle\theta_{t,s}|\hat{\rho}|\theta_{t,s}\rangle. \tag{13}$$

Since f is arbitrary, we conclude that the probability that the state's phase is equal to $\theta_{t,s}$ is therefore given by

$$\langle\theta_{t,s}|\hat{\rho}|\theta_{t,s}\rangle. \tag{14}$$

Note that the results so far depend on the auxiliary dimension s. This is resolved by taking the limit $s \to \infty$. As the limit is considered, the summation over t in Eq. (13) may be replaced by an integral, so that [29]

$$\left\langle \hat{\phi}^{(s)}_{\mathrm{PB}}[f] \right\rangle_{\hat{\rho}} = \int_0^{2\pi} d\phi\, f(e^{i\phi}) P^{(s)}_{\mathrm{PB}}(\phi|\hat{\rho}). \tag{15}$$

Here,

$$P^{(s)}_{\mathrm{PB}}(\phi|\hat{\rho}) := \frac{s+1}{2\pi} \langle\phi_s|\hat{\rho}|\phi_s\rangle, \tag{16}$$

where

$$|\phi_s\rangle := \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\phi} |n\rangle \tag{17}$$

are the continuous counterparts of the discrete phase states (9). We remark that in the very limit $s = \infty$, the continuous phase states coincide with those in the Susskind-Glogower formalism [4].

For normalizable states, the limit $s \to \infty$ in Eq. (15) can be computed under the integral:

$$\lim_{s\to\infty} \left\langle \hat{\phi}^{(s)}_{\mathrm{PB}}[f] \right\rangle_{\hat{\rho}} = \int_0^{2\pi} d\phi\, f\left(e^{i\phi}\right) \lim_{s\to\infty} P^{(s)}_{\mathrm{PB}}(\phi|\hat{\rho}). \tag{18}$$

In such cases, the formalism has a well-defined probability distribution in the limit of infinite dimension:

$$\lim_{s\to\infty} P_{\mathrm{PB}}^{(s)}(\phi|\hat{\rho}), \tag{19}$$

and the expectation values can be computed after taking the limit.

However, as Barnett himself pointed out [29], states for which the order of integration and limit cannot be exchanged exist, meaning that the formula (18) is not always valid. Then, the probability distribution does not exist in the limit $s \to \infty$, and the expectation values have to be computed through either of the following expressions:

$$\begin{aligned}\lim_{s\to\infty} \langle \hat{\phi}_{\mathrm{PB}}^{(s)}[f]\rangle_{\hat{\rho}} &= \lim_{s\to\infty} \sum_{t=0}^{s} f(e^{i\theta_{t,s}})\langle\theta_{t,s}|\hat{\rho}|\theta_{t,s}\rangle \\ &= \lim_{s\to\infty} \int_0^{2\pi} d\phi\, f(e^{i\phi}) P_{\mathrm{PB}}^{(s)}(\phi|\hat{\rho}),\end{aligned} \tag{20}$$

where we stress that in the discussed singular cases the limit in the bottom line has to be performed after integration. The postulate that, in general, the expectation values should be computed first and only then should the limit $s \to \infty$ be taken is a key feature of the Pegg-Barnett formalism.

III. QUANTUM-LIMITED AMPLIFIER

To make the connection between the Paul and Pegg-Barnett frameworks, we will use the quantum-limited amplifier (QLA) channel.

The action of the (one-mode) QLA channel of arbitrary strength $\kappa \geqslant 1$ on state $\hat{\rho}$ is defined as [30,31]

$$\mathcal{A}_\kappa(\hat{\rho}) := \mathrm{Tr}_B[\hat{U}_\kappa(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}_\kappa^\dagger], \tag{21}$$

where

$$\hat{U}_\kappa := \exp[\operatorname{arcosh}\sqrt{\kappa}(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})] \tag{22}$$

is the two-mode squeezing operator. Here, $\hat{b}$ is the annihilation operator associated with the ancillary system traced out in Eq. (21). The case $\kappa = 1$ corresponds to the identity channel.

From the physical point of view, QLA may be viewed as the process of pumping particles into the system. Because of its properties, it is sometimes viewed as a tool for making a quantum state more "classical" [23]. In particular, it was shown that the Glauber-Sudarshan P quasiprobability distribution of an infinitely amplified state is always non-negative [23], a quality that is associated only with semiclassical states [32].

The action of the amplifier on a state can be calculated explicitly in the number basis. Substituting the convenient decomposition [29] of the squeezing operator (22)

$$\hat{U}_\kappa = \hat{r}_{+,\kappa}^\dagger \exp[-\ln\sqrt{\kappa}(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1)]\hat{r}_{-,\kappa}, \tag{23}$$

where $\hat{r}_{\pm,\kappa} := \exp[\pm\sqrt{\frac{\kappa-1}{\kappa}}\hat{a}\hat{b}]$, into the definition (21), we eventually obtain

$$\begin{aligned}\mathcal{A}_\kappa(\hat{\rho}) = \frac{1}{\kappa}\sum_{j=0}^{\infty}\left(\frac{\kappa-1}{\kappa}\right)^j \sum_{m,n=0}^{\infty} \rho_{mn}\frac{1}{\sqrt{\kappa}^{m+n}} \\ \times\sqrt{\binom{j+m}{j}\binom{j+n}{j}}|j+m\rangle\langle j+n|,\end{aligned} \tag{24}$$

where $\rho_{mn} \equiv \langle m|\hat{\rho}|n\rangle$.

As it will prove important later on, we mention that the QLA channel possesses the semigroup property, according to which

$$\mathcal{A}_x[\mathcal{A}_y(\hat{\rho})] = \mathcal{A}_{xy}(\hat{\rho}) \tag{25}$$

for any $x, y \geqslant 1$. Finally, we remark that, occasionally, for increased readability in subscripts, we will denote QLA by

$$\mathcal{A}(\kappa, \hat{\rho}) \equiv \mathcal{A}_\kappa(\hat{\rho}). \tag{26}$$

IV. MAIN RESULTS

We come back to the Pegg-Barnett formalism. From a practical point of view, to obtain the final, s-independent measurement of phase, one has to perform a number of measurements of finite-dimensional Pegg-Barnett operators given by different values of s. For a large enough number of measurement results obtained for large enough s, one can predict the limiting measurement result for $s \to \infty$.

Let us now imagine that in this setup, before measuring the s-dimensional Pegg-Barnett operator, we first apply to the system s times an infinitesimally weak QLA channel $\mathcal{A}_{1+\epsilon}$, $\epsilon \ll 1$. In other words, before measuring the s-dimensional Pegg-Barnett operator, we prepare the system in state $\mathcal{A}_{1+\epsilon}^s(\hat{\rho})$, which, due to the semigroup property (25), can be rewritten as

$$\mathcal{A}_{1+\epsilon}^s(\hat{\rho}) = \mathcal{A}_{(1+\epsilon)^s}(\hat{\rho}) \approx \mathcal{A}_{1+s\epsilon}(\hat{\rho}). \tag{27}$$

Here, in the second transition we use the fact that $\epsilon \ll 1$. Again, for a large enough number of measurement results obtained for large enough s, one can predict the limiting measurement result $s \to \infty$. Our main claim is that, in the limit of vanishing amplification strength, $\epsilon \to 0$, the results obtained from this procedure are indistinguishable from analogous results obtained from the Paul formalism for an unamplified state.

We are now in the position to state our main result. For clarity, we present it in the form of a proposition.

Proposition 1. The Paul probability distribution (5) can be obtained from the Pegg-Barnett continuous probability distribution (16) through the quantum-limited amplifier as

$$P_{\mathrm{Paul}}(\phi|\hat{\rho}) = \lim_{\epsilon\to 0}\lim_{s\to\infty} P_{\mathrm{PB}}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})]. \tag{28}$$

Before we prove Proposition 1, let us make two important remarks. First, we stress that it is crucial that the order of limits on the right-hand side of Eq. (28) cannot be changed. If we took the limits in the opposite way, we would obtain no amplification at all since $\mathcal{A}_1$ is the identity channel. Consequently, in the process described at the beginning of this section, we would be performing the ordinary Pegg-Barnett measurement, which is obviously different from the Paul measurement.

Second, in Proposition 1, we called the quantity $P_{\mathrm{PB}}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})]$ the continuous probability distribution in the Pegg-Barnett formalism. However, as discussed extensively in Sec. II B, this is true only if, in the formula for the expectation value (20) for the amplified state, one can take the limit

$s \to \infty$ under the integral, i.e.,

$$\lim_{s\to\infty} \langle \hat{\phi}_{\rm PB}^{(s)}[f] \rangle_{\mathcal{A}(1+s\epsilon,\,\hat{\rho})} = \int_0^{2\pi} d\phi\, f(e^{i\phi}) \lim_{s\to\infty} P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})]. \tag{29}$$

Remarkably, we find that Eq. (29) always holds, even if it does not hold for the unamplified state. Thus, $P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})]$ indeed constitutes the continuous probability distribution in the Pegg-Barnett formalism. This technical result is proved in Appendix A.

We now proceed to prove our main result.

Proof of Proposition 1. Setting $\kappa = 1 + s\epsilon$ in Eq. (24) and making use of definitions (16) and (17) leads to

$$P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \frac{1}{2\pi(1+s\epsilon)} \sum_{j=0}^{s} \left(\frac{s\epsilon}{1+s\epsilon}\right)^j \times \sum_{m,n=0}^{s-j} \rho_{mn} \frac{e^{i(n-m)\phi}}{\sqrt{(1+s\epsilon)}^{m+n}} \times \sqrt{\binom{j+m}{j}\binom{j+n}{j}}, \tag{30}$$

where we note that the summation limits on m, n, and j follow from the fact that the Pegg-Barnett operator is limited to dimension s, which means that $j+m, j+n \in \{0, \dots, s\}$. Our goal is to show that after taking the limits $s \to \infty$ and $\epsilon \to 0$, in that order, the above quantity is equal to (5).

To simplify our considerations and shorten the notation, let us observe that Eq. (28) is linear in the density operator. For this reason, it is enough to restrict ourselves to pure states: $\hat{\rho} = |\psi\rangle\langle\psi|$, for which $\rho_{mn} = \psi_m \psi_n^*$. This assumption has no impact on the correctness of the proof. We get

$$P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \frac{1}{2\pi(1+s\epsilon)} \sum_{j=0}^{s} \left(\frac{s\epsilon}{1+s\epsilon}\right)^j \times \left| \sum_{m=0}^{s-j} \psi_m \frac{e^{-im\phi}}{\sqrt{(1+s\epsilon)}^m} \sqrt{\binom{j+m}{j}} \right|^2. \tag{31}$$

In the next step, we rewrite

$$\frac{1}{(1+s\epsilon)^m}\binom{j+m}{j} = \frac{\prod_{k=1}^m (j+k)}{(1+s\epsilon)^m m!} = \frac{1}{m!}\prod_{k=1}^m \frac{j+k}{1+s\epsilon}. \tag{32}$$

Thus, Eq. (31) becomes

$$P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \frac{1}{2\pi(1+s\epsilon)} \sum_{j=0}^{s} \left(\frac{s\epsilon}{1+s\epsilon}\right)^j \times \left| \sum_{m=0}^{s-j} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\prod_{k=1}^m \frac{j+k}{1+s\epsilon}} \right|^2. \tag{33}$$

At this point, it will be beneficial to turn the summation over j into an integral. We do this in complete analogy to the case of particle in a box approaching infinite volume [33]. Instead of summing over j from 0 to s, we sum over $\mu_j := j/s$ from 0 to 1. In the limit of large s, in which we are interested, the sum approaches an integral. As μ_j occupies the volume $1/s$ in the space of indices, we have

$$j \to s\mu, \quad \sum_{j=0}^{s} \to s\int_0^1 d\mu. \tag{34}$$

Therefore, for very large s,

$$P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \frac{s}{2\pi(1+s\epsilon)} \int_0^1 d\mu \left(\frac{s\epsilon}{1+s\epsilon}\right)^{s\mu} \times \left| \sum_{m=0}^{s-s\mu} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\prod_{k=1}^m \frac{s\mu+k}{1+s\epsilon}} \right|^2. \tag{35}$$

We are now ready to take the limit $s \to \infty$. We can do this term by term, which is justified by the fact that each term has a well-defined limit. We have

$$\frac{s}{(1+s\epsilon)} \to \frac{1}{\epsilon}, \quad \left(\frac{s\epsilon}{1+s\epsilon}\right)^{s\mu} \to e^{-\mu/\epsilon}. \tag{36}$$

Finally, the bottom line of Eq. (35) approaches

$$\left| \sum_{m=0}^{s-s\mu} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\prod_{k=1}^m \frac{s\mu+k}{1+s\epsilon}} \right|^2 \to \left| \sum_{m=0}^{\infty} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\frac{\mu}{\epsilon}}^m \right|^2, \tag{37}$$

which we prove in Appendix B.

This altogether yields

$$\lim_{s\to\infty} P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \frac{1}{2\pi\epsilon} \int_0^1 \frac{d\mu}{\epsilon} e^{-\mu/\epsilon} \times \left| \sum_{m=0}^{\infty} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\frac{\mu}{\epsilon}}^m \right|^2, \tag{38}$$

or, upon substituting $r^2 := \mu/\epsilon$ and rearranging,

$$\lim_{s\to\infty} P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \int_0^{\sqrt{1/\epsilon}} \frac{dr}{\pi} r \left| e^{-r^2/2} \sum_{m=0}^{\infty} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} r^m \right|^2. \tag{39}$$

From Eq. (2) for the coherent state in the number basis and the definition (4) of the Husimi distribution one immediately recognizes that the bottom line equals $Q_{\hat{\rho}}(re^{i\phi})$. Thus,

$$\lim_{s\to\infty} P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})] = \int_0^{\sqrt{1/\epsilon}} \frac{dr}{\pi} r\, Q_{\hat{\rho}}(re^{i\phi}), \tag{40}$$

which in the limit $\epsilon \to 0$ becomes the probability distribution (5) in the Paul formalism. This concludes the proof. ■

As an immediate consequence of Proposition 1, the following corollary follows from the definitions of the expectation values in the two formalisms.

Corollary 1. The expectation values (8) in the Paul formalism can be obtained from their Pegg-Barnett counterparts (15) through the quantum-limited amplifier as

$$\langle\hat{\phi}_{\text{Paul}}[f]\rangle_{\hat{\rho}} = \lim_{\epsilon\to 0}\lim_{s\to\infty}\langle\hat{\phi}^{(s)}_{\text{PB}}[f]\rangle_{\mathcal{A}(1+s\epsilon,\,\hat{\rho})}. \quad (41)$$

V. DISCUSSION

Let us discuss our results, beginning with their physical interpretation. As mentioned before, the amplification process is associated with making quantum phenomena more classical. Notably, it is known to transform the Glauber P distribution into the more semiclassical Husimi Q distribution [23] and the von Neumann entropy into the more classical-like Wehrl entropy [31]. In view of our work, this suggests that the Paul formalism may be viewed as a semiclassical limit of the Pegg-Barnett formalism.

This interpretation is strengthened by the fact that, while for a generic quantum state the Pegg-Barnett probability distribution may not exist in the infinite dimension, it does for all amplified states, as if all the quantum "singularities" have been removed. Note also that the Paul formalism, to which the amplification leads from the Pegg-Barnett framework, is itself invariant under state amplification:

$$P_{\text{Paul}}[\phi|\mathcal{A}_\kappa(\hat{\rho})] = P_{\text{Paul}}(\phi|\hat{\rho}) \quad \forall\,\kappa \geqslant 1. \quad (42)$$

To see this, one needs to make use of the known relation [31]

$$Q_{\mathcal{A}(\kappa,\,\hat{\rho})}(\alpha) = \kappa\, Q_{\hat{\rho}}(\sqrt{\kappa}\alpha) \quad (43)$$

in Eq. (5) and change the integration variable to $r' = \sqrt{\kappa}r$. Thus, if we consider the Paul formalism to be the Pegg-Barnett formalism with some of its quantum features suppressed through infinite amplification, it is only natural that further amplification leaves it unaffected.

Why do our results assume the specific amplification choice $\kappa = 1 + s\epsilon$? In particular, why do they connect the amplification rate to the Pegg-Barnett dimension in a linear way? From a mathematical point of view, the necessity of such a connection is clear: if we were to repeat the derivation of our main results with amplification strength κ set to be either independent of s or dependent on it in a nonlinear way (e.g., $\kappa = 1 + s^2\epsilon$), we would quickly find the Pegg-Barnett probability to be vanishing in the limit of infinite amplification. See Appendix C, where we show this explicitly. Thus, setting κ to be linear in s, as we did, is necessary to obtain a nontrivial limit. This also shows that a result similar to ours cannot hold in the Susskind-Glogower formalism since there $s = \infty$ from the beginning, making it impossible to set κ dependent on s.

This necessity of having $\kappa = 1 + s\epsilon$ can also be understood from the physical point of view. In the language of the celebrated Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation [34,35], one of the most prominent master equations for modeling quantum open systems, the GKLS generator of the evolution corresponding to QLA is given by [31].

$$\mathcal{L}(\hat{\rho}) \propto \mathcal{A}_{1+\epsilon}(\hat{\rho}) - \mathcal{A}_1(\hat{\rho}) \propto \hat{a}^\dagger\hat{\rho}\hat{a} - \tfrac{1}{2}\{\hat{a}\hat{a}^\dagger, \hat{\rho}\}. \quad (44)$$

We can see that, due to the action of the creation operator on the state, the infinitesimally weak QLA increases the dimension of the state by one. Combining this with Eq. (27), we conclude that the change in the amplification parameter from s to $s+1$ also increases the dimension of the state by one. At the same time, by construction, the change from s to $s+1$ increases the dimension of the Pegg-Barnett formalism by one. Therefore, setting the amplification parameter $\kappa = 1 + s\epsilon$, i.e., to be linear in s, is special in that it guarantees that the change in dimension of the state induced by amplification is consistent with the limiting procedure in the Pegg-Barnett formalism.

As a final remark, we observe that Corollary 1 can be alternatively formulated as a relation between the phase operators in the two formalisms. Let us deploy the quantum-limited attenuator channel, whose action on arbitrary operator $\hat{O}$ reads [31]

$$\mathcal{E}_\lambda(\hat{O}) := \text{Tr}_B[\hat{V}_\lambda(\hat{O}\otimes|0\rangle\langle 0|)\hat{V}_\lambda^\dagger]. \quad (45)$$

Here, $0 \leqslant \lambda \leqslant 1$ (where $\lambda = 1$ corresponds to the identity channel), and

$$\hat{V}_\lambda := \exp[\arccos\sqrt{\lambda}(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)]. \quad (46)$$

The quantum-limited attenuator is dual to the quantum-limited amplifier, by which we mean that for any state $\hat{\rho}$, operator $\hat{O}$, and $\kappa \geqslant 1$ we have

$$\text{Tr}\mathcal{A}_\kappa(\hat{\rho})\,\hat{O} = \text{Tr}\hat{\rho}\,\frac{1}{\kappa}\mathcal{E}_{1/\kappa}(\hat{O}). \quad (47)$$

Therefore, infinite amplification of the state $\kappa \to \infty$ is equivalent to infinite attenuation $\lambda = 1/\kappa \to 0$ of the operator.

Applying this to Corollary 1, we find that for any $\hat{\rho}$ and f,

$$\text{Tr}\hat{\rho}\left[\hat{\phi}_{\text{Paul}}[f] - \lim_{\epsilon\to 0}\lim_{s\to\infty}\frac{1}{1+s\epsilon}\mathcal{E}_{1/(1+s\epsilon)}\big(\hat{\phi}^{(s)}_{\text{PB}}[f]\big)\right] = 0. \quad (48)$$

Because this equation holds for arbitrary input state $\hat{\rho}$, it is tempting to say that the Paul operator is equal to the infinitely attenuated Pegg-Barnett operator, i.e.,

$$\hat{\phi}_{\text{Paul}}[f] = \lim_{\epsilon\to 0}\lim_{s\to\infty}\frac{1}{1+s\epsilon}\mathcal{E}_{1/(1+s\epsilon)}\big(\hat{\phi}^{(s)}_{\text{PB}}[f]\big). \quad (49)$$

However, we only showed that the two operators coincide when traced with a formal density operator, i.e., a nonnegative, Hermitian operator. Therefore, while the Paul and infinitely attenuated Pegg-Barnett operators are clearly connected, whether they are completely equal, as in Eq. (49), remains to be proved (or disproved).

In any case, because of the general postulate of the Pegg-Barnett formalism to calculate the expectation values first and only then take the limit of infinite dimension, one has to be careful with such an operator interpretation, as an infinitely dimensional phase operator is technically not part of the Pegg-Barnett framework.

VI. EXAMPLES

We illustrate our results with a number of examples. We begin with the simple case of thermal states, for which the convergence of the two phase formalisms is easy to see explicitly.

Example 1. Let us consider the thermal states of the harmonic oscillator:

$$\hat{g}_\beta := \frac{e^{-\beta\hat{a}^\dagger\hat{a}}}{\text{Tr}e^{-\beta\hat{a}^\dagger\hat{a}}}, \quad (50)$$

where $\beta > 0$. For such states, the Paul probability distribution (5) can be calculated analytically, yielding the flat distribution:

$$P_{\mathrm{Paul}}(\phi|\hat{g}_\beta) = \frac{1}{2\pi}. \tag{51}$$

To compare Eq. (51) with the Pegg-Barnett formalism intertwined with QLA, we begin with Eq. (24), finding that

$$\mathcal{A}_\kappa(\hat{g}_\beta) = \hat{g}_{\beta(\kappa)}, \quad \beta(\kappa) = \ln\frac{\kappa}{e^{-\beta}+\kappa-1}. \tag{52}$$

Substituting this into Eq. (31) and simplifying yield

$$P_{\mathrm{PB}}^{(s)}(\phi|\hat{g}_{\beta(1+s\epsilon)}) = \frac{1}{2\pi}\left[1-\left(\frac{e^{-\beta}+s\epsilon}{1+s\epsilon}\right)^{s+1}\right]. \tag{53}$$

Taking the limit $s \to \infty$, we get

$$\lim_{s\to\infty} P_{\mathrm{PB}}^{(s)}(\phi|\hat{g}_{\beta(1+s\epsilon)}) = \frac{1}{2\pi}[1-e^{-(1-e^{-\beta})/\epsilon}], \tag{54}$$

which clearly coincides with the Paul probability distribution (51) after taking the limit $\epsilon \to 0$.

In the remaining examples, we consider states with nontrivial phase dependence. To study the convergence of the two formalisms, we employ numerical methods. To this end, it is useful to define the following object:

$$R_{s,\epsilon}(\phi) := \frac{P_{\mathrm{PB}}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat{\rho})]}{P_{\mathrm{Paul}}(\phi|\hat{\rho})}, \tag{55}$$

which is simply the ratio of the Pegg-Barnett "amplified" probability distribution to the Paul probability distribution. According to Proposition 1, this ratio should approach a value of 1 for large s and small ϵ.

Example 2. As a second example, let us consider a coherent state $\hat{\rho}_\alpha = |\alpha\rangle\langle\alpha|$ with amplitude $\alpha = r'e^{i\psi}$. This example is especially relevant for the study of quantum phase since the phase of such a coherent state is approximately equal to ψ.

In this case, the Husimi Q distribution equals

$$Q_{\hat{\rho}_\alpha}(re^{i\phi}) = e^{-r^2-r'^2+2rr'\cos(\phi-\psi)}, \tag{56}$$

resulting in the following Paul phase distribution:

$$P_{\mathrm{Paul}}(\phi|\hat{\rho}_\alpha) = \frac{e^{-r'^2}}{2\pi}(1+\sqrt{\pi}r'\cos(\phi-\psi)e^{r'^2\cos^2(\phi-\psi)} \times \{\mathrm{erf}[r'\cos(\phi-\psi)\}+1]), \tag{57}$$

where erf stands for the error function. On the other hand, calculating from definition, the corresponding Pegg-Barnett phase distribution reads

$$P_{\mathrm{PB}}(\phi|\hat{\rho}_\alpha) = \frac{e^{-r'^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{e^{in(\psi-\phi)}r'^n}{\sqrt{n!}}\right|^2. \tag{58}$$

Its "amplified" version follows readily by substituting $\psi_m = \alpha_m$, with α_n as in Eq. (2), into Eq. (33).

We compare the two distributions in Fig. 1(a). As expected from coherent states, both phase distributions peak at $\psi = \pi$, with the effect being more pronounced for larger r'. While the Paul and Pegg-Barnett frameworks both give the same qualitative results, the Paul framework yields a noticeably less pronounced peak. Nonetheless, the Paul distribution can be obtained from the Pegg-Barnett distribution by using the quantum-limited amplifier, as seen from Fig. 1(b).

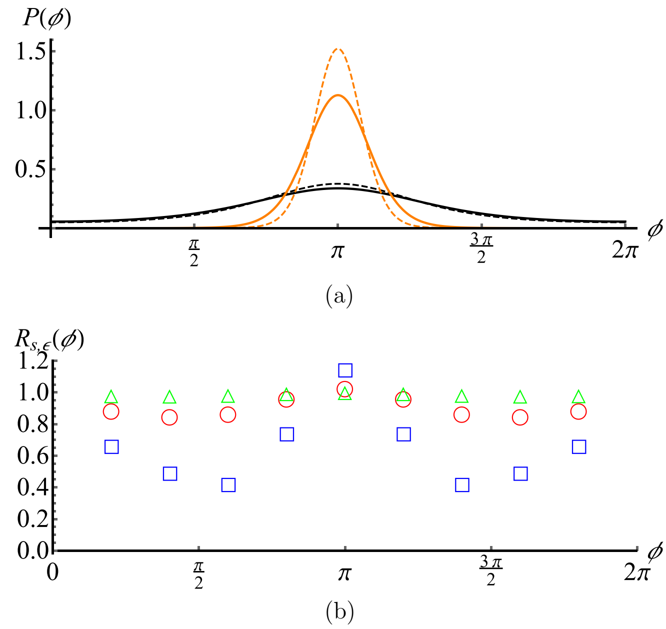

FIG. 1. (a) Comparison between the Paul phase distribution $P_{\mathrm{Paul}}(\phi|\hat{\rho})$ (solid lines) and its Pegg-Barnett counterpart $P_{\mathrm{PB}}(\phi|\hat{\rho})$ (dashed lines) for $\psi = \pi$ and $r' = 0.5$ (black) and $r' = 2$ (orange). To calculate $P_{\mathrm{PB}}(\phi|\hat{\rho})$, the sum in Eq. (58) was approximated by its first 100 terms. (b) Point plot of the ratio $R_{s,\epsilon}(\phi)$ for the coherent state given by $r' = 2$ and $\psi = \pi$ calculated at the points $\phi = 2\pi t/10$ with $t \in \{1, \dots, 9\}$ for $\epsilon = 0.01$. Blue squares, red circles, and green triangles stand for $s+1 \in \{10^2, 10^3, 10^4\}$, respectively. As expected, for large s the ratio approaches the value of 1.

Example 3. The purpose of the final example is to test how the limiting procedure works in practice. In Table I, we provide approximate numerical values of the ratio (55) for various values of s and ϵ numerically averaged over random qubit density matrices, i.e., single-photon states, sampled from the Hilbert-Schmidt ensemble [36]. As can be seen, the ratio approaches a value of 1 in the limit $s \to \infty$, $\epsilon \to 0$, provided s is much bigger than $1/\epsilon$, which we interpret as taking the limit $s \to \infty$ before taking the limit $\epsilon \to 0$.

VII. CONCLUDING REMARKS

To briefly conclude, we successfully demonstrated that the Paul formalism can be obtained from the Pegg-Barnett formalism intertwined with infinite amplification of the state. Since the process we propose has a clear operational interpretation, the "Pegg-Barnett dimension" s acquires, with the help of our procedure, a physical underpinning. Namely, the parameter s determines the number of times we have to apply the infinitesimally weak QLA channel to the system before we measure on it an s-dimensional Pegg-Barnett operator to get the correspondence with the Paul framework.

Our findings suggest a number of closely related directions for future research. First, the Pegg-Barnett framework is known to coincide with the Paul framework for small-amplitude coherent states and with the Paul framework's

TABLE I. Numerical values of the ratio $R_{s,\epsilon}(\phi)$ [defined in Eq. (55)] of the Pegg-Barnett probability distribution calculated for an amplified state to the Paul probability distribution for $\phi = 0.3$. Each entry is an average over 1000 random qubit density matrices (i.e., single-photon states) sampled from the Hilbert-Schmidt ensemble, with terms after $\pm$ standing for the maximum deviation from the mean value. All values are rounded to two significant digits. As can be seen, the ratio approaches the value $R = 1$ in the limit of growing s and vanishing ϵ, as long as s is much bigger than $1/\epsilon$, which we interpret as taking the limit $s \to \infty$ before taking the limit $\epsilon \to 0$.

	$R_{s,\epsilon}(0.3)$				
	$s = 10^0$	$s = 10^1$	$s = 10^2$	$s = 10^3$	$s = 10^4$
ϵ					
1.00	0.61 ± 0.54	0.47 ± 0.37	0.45 ± 0.43	0.45 ± 0.44	0.45 ± 0.44
0.50	1.19 ± 1.01	0.80 ± 0.44	0.73 ± 0.31	0.72 ± 0.30	0.72 ± 0.30
0.10	5.27 ± 4.60	1.79 ± 0.95	1.08 ± 0.14	1.01 ± 0.02	1.00 ± 0.00
0.05	10.17 ± 9.08	2.67 ± 1.80	1.18 ± 0.24	1.02 ± 0.03	1.00 ± 0.00
0.01	49.13 ± 44.92	9.75 ± 8.34	1.95 ± 1.04	1.09 ± 0.14	1.01 ± 0.02

analog based on the Wigner distribution for large-amplitude coherent states [37]. Given the former association's loose resemblance to our main result, perhaps there exists a physical process akin to amplification which connects the Pegg-Barnett operator to the Wigner distribution. Second, the phase-difference operator [12] is based on a construction similar to the Pegg-Barnett operator. Therefore, one may expect that it is also connected to the Paul formalism through some type of system amplification. Finally, our work, as well as the previously mentioned Refs. [23,31], suggests that further "quantum-to-classical" transitions are potentially possible by virtue of the amplification procedure.

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APPENDIX A: PROOF OF EQUATION (29)

In this Appendix, we prove Eq. (29), meaning that in the formula for the expectation value (20) calculated for an amplified state the limiting procedure and the integration can be swapped. In other words, we prove that

$$\lim_{s\to\infty} \int_0^{2\pi} d\phi\, I_s(\phi) = \int_0^{2\pi} d\phi\, \lim_{s\to\infty} I_s(\phi), \tag{A1}$$

where we denote, for short,

$$I_s(\phi) := f\left(e^{i\phi}\right) P_{\rm PB}^{(s)}[\phi|\mathcal{A}_{1+s\epsilon}(\hat\rho)]. \tag{A2}$$

According to the dominated convergence theorem, a sufficient condition for Eq. (A1) to hold is that an s-independent function $J(\phi)$ exists such that

$$\int_0^{2\pi} d\phi\, J(\phi) < \infty \tag{A3}$$

and for all s and ϕ

$$|I_s(\phi)| \leqslant J(\phi). \tag{A4}$$

We make the following guess:

$$J(\phi) = \frac{\max_x |f(x)|}{2\pi\epsilon}\left(\sum_{m=0}^{\infty} \frac{1}{\sqrt{m!\epsilon^m}}\right)^2. \tag{A5}$$

We stress that in Proposition 1, the limit $s \to \infty$ is taken before the limit $\epsilon \to 0$, meaning that $J(\phi)$ is finite. Condition (A3) is thus obviously fulfilled. It remains to show Eq. (A4).

To this end, we start with expression (33). Using the fact that $s/(1+s\epsilon) \leqslant 1/\epsilon$ and $(\frac{s\epsilon}{1+s\epsilon})^{s\mu} \leqslant 1$, as well as basic properties of absolute value, we get

$$|I_s(\phi)| \leqslant \frac{|f(e^{i\phi})|}{2\pi\epsilon}\int_0^1 d\mu \left(\sum_{m=0}^{s-s\mu}\frac{1}{\sqrt{m!}}\sqrt{\prod_{k=1}^{m}\frac{s\mu+k}{1+s\epsilon}}\right)^2. \tag{A6}$$

We then observe that under the product we have $k \leqslant m \leqslant s - s\mu$, which yields

$$|I_s(\phi)| \leqslant \frac{|f(e^{i\phi})|}{2\pi\epsilon}\int_0^1 d\mu \left(\sum_{m=0}^{s-s\mu}\frac{1}{\sqrt{m!}}\sqrt{\frac{s}{1+s\epsilon}}^{m}\right)^2. \tag{A7}$$

In the last step, we once again use $s/(1+s\epsilon) \leqslant 1/\epsilon$. Furthermore, because all the summands are non-negative, we extend the sum to infinity. Finally, we can perform the integral over μ. In the end, we have

$$|I_s(\phi)| \leqslant \frac{|f(e^{i\phi})|}{2\pi\epsilon}\left(\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!\epsilon^m}}\right)^2. \tag{A8}$$

Substituting this into the left-hand side of Eq. (A4) and bounding $|f|$ from above by its largest value finish the proof.

Note that this result also shows that one can interchange the limit $s \to \infty$ with the integral over μ in Eq. (35).

APPENDIX B: PROOF OF EQUATION (37)

In this Appendix, we want to show that the limit $s \to \infty$ in the bottom line of Eq. (35) is given by Eq. (37).

Let us denote

$$S_{x,y} = \sum_{m=x}^{y} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}}\sqrt{\prod_{k=1}^{m}\frac{s\mu+k}{1+s\epsilon}}, \tag{B1}$$

so that the bottom line of Eq. (35) equals $|S_{0,s-s\mu}|^2$. We now introduce the auxiliary parameter $d \in \mathbb{N}$, $d \leqslant s - s\mu$, and split the sum over m into two sums: one from 0 to $d-1$ and one from d to $s - s\mu$. We get

$$|S_{0,s-s\mu}|^2 = |S_{0,d-1} + S_{d,s-s\mu}|^2. \tag{B2}$$

We stress that $S_{x,y}$ is independent of d.

Using basic properties of the absolute value, we get the following bounds:

$$|S_{0,d-1}|^2 - |S_{d,s-s\mu}|^2 \leqslant |S_{0,s-s\mu}|^2 \leqslant |S_{0,d-1}|^2 + |S_{d,s-s\mu}|^2. \tag{B3}$$

Our approach is to calculate the limit $s \to \infty$ separately for the two terms present in the bounds, ultimately showing that both bounds coincide and are therefore equal to the limit of $|S_{0,s-s\mu}|^2$.

Because $|\psi_m| \leqslant 1$ and $|e^{-im\phi}| = 1$, we can see that

$$|S_{d,s-s\mu}| \leqslant \sum_{m=d}^{s-s\mu} \frac{1}{\sqrt{m!}} \sqrt{\prod_{k=1}^{m} \frac{s\mu + k}{1 + s\epsilon}}. \tag{B4}$$

Furthermore, k is bounded from above by m, which is, in turn, bounded by $s - s\mu$. Thus,

$$\begin{aligned} |S_{d,s-s\mu}| &\leqslant \sum_{m=d}^{s-s\mu} \frac{1}{\sqrt{m!}} \sqrt{\prod_{k=1}^{m} \frac{s}{1+s\epsilon}} \\ &\to \sum_{m=d}^{\infty} \frac{1}{\sqrt{m!}} \sqrt{\frac{1}{\epsilon}}^{m} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{(d+m)!}} \sqrt{\frac{1}{\epsilon}}^{d+m}, \end{aligned} \tag{B5}$$

where in the second transition we performed the limit $s \to \infty$ and in the third (final) transition we renumbered the sum.

In the case of $S_{0,d-1}$, m is bounded from above by the finite number d, which means that k yields no contribution to the limit of infinite s and so

$$|S_{0,d-1}|^2 \to \left| \sum_{m=0}^{d-1} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\frac{\mu}{\epsilon}}^{m} \right|^2. \tag{B6}$$

Since d was chosen to be an arbitrary number smaller than $s - s\mu$ and, as already pointed out, $S_{0,s-s\mu}$ is independent of d, after taking the limit $s \to \infty$, we can pick whatever value of $d \in \mathbb{N}$ we want to. In particular, we can now also take the limit $d \to \infty$. Looking at Eq. (B5), we can see that in this limit $S_{d,s-s\mu}$ vanishes—because this series is absolutely convergent, we can interchange the sum with the limit. Thus, due to Eq. (B3), in the limit of infinite s, the bottom line of Eq. (35) coincides with Eq. (B6) with $d \to \infty$. This proves Eq. (37).

APPENDIX C: DISCUSSION OF THE AMPLIFICATION RATE

In this Appendix, we show why Proposition 1 no longer holds if the amplification rate is set to be nonlinear in s, as opposed to the linear dependence $\kappa = 1 + s\epsilon$.

First, let us briefly discuss the case in which κ is independent of s. In this case, we find that the analog of Eq. (30) reads

$$\begin{aligned} P_{\mathrm{PB}}^{(s)}[\phi|\mathcal{A}_\kappa(\hat{\rho})] = &\frac{1}{2\pi\kappa} \sum_{j=0}^{s} \left(\frac{\kappa - 1}{\kappa}\right)^j \\ &\times \sum_{m,n=0}^{s-j} \rho_{mn} \frac{e^{i(n-m)\phi}}{\sqrt{\kappa}^{m+n}} \sqrt{\binom{j+m}{j}\binom{j+n}{j}}. \end{aligned} \tag{C1}$$

Taking the limit of infinite amplification, $\kappa \to \infty$, we get simply zero.

To see what happens for κ depending on s in a nonlinear way, let us consider $\kappa = 1 + w(s, \epsilon)$, where w is a polynomial in s and ϵ of at least the second degree in s. In this case, by following exactly the same steps as in the original derivation, we find that the analog of Eq. (35) reads

$$\begin{aligned} P_{\mathrm{PB}}^{(s)}[\phi|\mathcal{A}_{1+w(s,\epsilon)}(\hat{\rho})] = &\frac{(2\pi)^{-1} s}{[1 + w(s,\epsilon)]} \int_0^1 d\mu \left(\frac{w(s,\epsilon)}{1+w(s,\epsilon)}\right)^{s\mu} \\ &\times \left| \sum_{m=0}^{s-s\mu} \psi_m \frac{e^{-im\phi}}{\sqrt{m!}} \sqrt{\prod_{k=1}^{m} \frac{s\mu + k}{1 + w(s,\epsilon)}} \right|^2. \end{aligned} \tag{C2}$$

By construction, $\lim_{s\to\infty} s/w(s,\epsilon) = 0$. From this, it is easy to see that the whole equation vanishes in the limit $s \to \infty$.

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