


Certification of the maximally entangled state using nonprojective measurements

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In recent times, device-independent certification of quantum states has been one of the intensively studied areas in quantum information. However, all such schemes utilize projective measurements which are practically difficult to generate. In this paper, we consider the one-sided device-independent scenario and propose a self-testing scheme for the two-qubit maximally entangled state using nonprojective measurements, in particular, three three-outcome extremal positive operator-valued measures. We also analyze the robustness of our scheme against white noise.

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I. INTRODUCTION

The existence of nonlocal correlations, as was first realized by Einstein, Podolski, and Rosen in 1935 [1] as a paradox and then subsequently by Schrödinger in the same year [2], is one of the most intriguing features of quantum theory. Consequently, Bell [3,4] proposed a mathematical formulation to detect whether or not quantum theory is inherently nonlocal, and thus it is commonly referred to as Bell nonlocality. Apart from its relevance in the foundations of physics, Bell nonlocality has given rise to an enormous number of applications in computation, communication, and information theory [5].

A recent application of nonlocality is a device-independent (DI) certification where, assuming quantum theory and some other physically well-motivated assumptions, the statistics obtained from a black box are enough to validate the underlying mechanism inside it. The strongest DI certification is referred to as self-testing. First introduced in Ref. [6], self-testing allows one to certify the underlying quantum states and the measurements, up to some freedom based on the maximal violation of a Bell inequality [7]. In recent times, there has been increased interest to find protocols to self-test various quantum systems due to their applicability in various quantum information tasks. Despite the progress in designing schemes to self-test various quantum states using projective measurements, for instance, Refs. [8–18], a scheme that utilizes nonprojective measurements is lacking.

The precision to experimentally generate sets of projective measurements, which are a prerequisite to observe any form of nonlocality, reduces as the dimension of the system grows (see, for instance, Ref. [19]). Thus, a natural question arises whether noisy measurements or nonprojective measurements can also be used to self-test quantum states. Further on, to self-test any state or measurement, one needs to observe the maximal violation of an inequality which touches the set of quantum correlations, or simply the quantum set, at a particular point. It is also an open question in quantum foundations

whether a point on the boundary of the quantum set in some scenarios can be saturated by only nonprojective measurements.

Self-testing quantum states using nonprojective measurements is not possible in the standard Bell scenario, the reason being that the maximal violation of Bell inequalities can always be achieved by projective measurements. Consequently, we consider another form of nonlocality, known as quantum steering [20–22]. To witness quantum steering, one needs to consider the Bell scenario with an additional assumption that one of the parties is trusted. In the DI regime, this is referred to as the one-sided device-independent (1SDI) scenario. The certification of quantum states and measurements in the 1SDI scenario has gained recent interest [23–29] as they are more robust to noise and require detectors with lower efficiencies when compared to fully DI scenarios [30,31].

In this paper, we provide a scheme to certify the two-qubit maximally entangled state

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad (1)$$

using three three-outcome nonprojective extremal measurements in the 1SDI scenario. For this purpose, we first construct a steering inequality with two parties such that each of them chooses three inputs and gets three outputs. We then use the maximal violation of this steering inequality to obtain the self-testing result. We finally show that our scheme is highly robust when the states and measurements are mixed with white noise.

II. PRELIMINARIES

Before proceeding to the results, let us first describe the scenario and notions used throughout this work.

A. Extremal positive operator-valued measure (POVM)

Any measurement in quantum theory, usually referred to as a POVM, is represented as $M = \{M^a\}$, where M^a are the measurement elements corresponding to the a th outcome of

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M . These elements are positive semi-definite operators and $\sum_a M^a = \mathbb{1}$. Now, a POVM that cannot be expressed as a convex combination of other POVMs is defined as an extremal POVM. As shown in Ref. [32], the elements M^a of any rank-one extremal POVM can be expressed as $M^a = \lambda_a |v_a\rangle\langle v_a|$, where $\lambda_a \geq 0$ and the elements are linearly independent.

B. Quantum steering scenario

In this paper, we consider a simple scenario to witness quantum steering, consisting of two spatially separated parties, namely, Alice and Bob. They locally perform measurements on their respective subsystems which they receive from a preparation device. Bob can choose among three measurements denoted by B_y such that $y = 0, 1, 2$, each of which results in three outcomes labeled by $b = 0, 1, 2$. The measurement performed by Bob might affect the received subsystem with Alice which is denoted as $\sigma_b^y \in \mathcal{H}_A$, where σ_b^y are positive semidefinite operators. The collection of these operators $\sigma = \{\sigma_b^y$ such that $b = 0, 1, 2, y = 0, 1, 2\}$ is called an assemblage.

In quantum theory the operators σ_b^y are expressed for any y, b as

$$\sigma_b^y = \text{Tr}_B[(\mathbb{1}_A \otimes N_y^b)\rho_{AB}], \quad (2)$$

where $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is the state shared between Alice and Bob and $B_y = \{N_y^b\}$ denote Bob's measurements. Alice is trusted here, which means that her measurements are known or she can perform tomography on her subsystem. If the shared state is not steerable, then the assemblage has a local hidden state (LHS) model [20] defined as

$$\sigma_b^y = \sum_{\lambda} p(\lambda) \rho_{\lambda}(b|y) \rho_{\lambda}, \quad (3)$$

where $\sum_{\lambda} p(\lambda) = 1$, $p_{\lambda}(b|y)$ are the probability distributions over λ , and ρ_{λ} are density matrices over \mathcal{H}_A . As Alice can perform topographically complete measurements on σ_b^y , in general, quantum steering is witnessed by the so-called "steering functional," a map from the assemblage $\{\sigma_b^y\}$ to a real number [33].

Instead of checking the steerability of the assemblage, quantum steering can be equivalently witnessed similar to a Bell scenario where trusted Alice and untrusted Bob performs the measurements $A_x = \{M_x^a\}$ and $B_y = \{N_y^b\}$, respectively, and obtain the joint probability distribution $\vec{p} = \{p(a, b|x, y)\}$, where $a, b, x, y = 0, 1, 2$. Here, a, x denotes the output and input of Alice, respectively. The probabilities can be computed in quantum theory as

$$p(a, b|x, y) = \text{Tr}[(M_x^a \otimes N_y^b)\rho_{AB}] = \text{Tr}(M_x^a \sigma_b^y). \quad (4)$$

To witness quantum steering, a steering inequality \mathcal{B} can now be constructed from \vec{p} as

$$\mathcal{B}(\vec{p}) = \sum_{a,b,x,y} c_{a,b|x,y} p(a, b|x, y) \leq \beta_L, \quad (5)$$

where $c_{a,b|x,y}$ are real coefficients and β_L denotes the maximum value attainable using assemblages admitting an LHS model (3). The probabilities one obtains from such

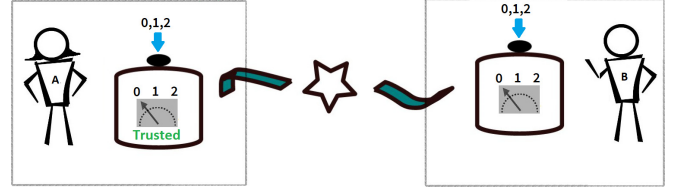


FIG. 1. Quantum steering scenario. Alice and Bob are spatially separated and each of them receive a subsystem on which they perform three three-outcome measurements. They are not allowed to communicate during the experiment. Once it is complete, they construct the joint probability distribution $\{p(a, b|x, y)\}$.

assemblages are expressed using (3) as

$$p(a, b|x, y) = \sum_{\lambda} p(\lambda) p(a|x, \rho_{\lambda}) p(a|x, \lambda). \quad (6)$$

The above representation (6) will be particularly useful to find the LHS bound of the steering inequality proposed in this work. In the DI framework, the above-presented scenario is also referred to as the ISDI scenario (see Fig. 1).

C. Self-testing

Inspired by [28], we now define self-testing in the ISDI scenario.

Definition 1. Consider the above ISDI scenario with the preparation device creating a state $|\psi\rangle_{AB}$. Alice and Bob perform measurements on this state and observe the joint probability distribution $\{p(a, b|x, y)\}$. Alice is trusted and her measurements A_x are fixed and Bob's measurements represented as $B_y = \{N_y^b\}$ are arbitrary. Let us now consider that the distribution $\{p(a, b|x, y)\}$ is generated by an ideal experiment with a state $|\tilde{\psi}\rangle_{AB}$ and Bob's measurements $\tilde{B}_y = \{\tilde{N}_y^b\}$. Then, the state $|\psi\rangle_{AB}$ and measurements B_y are certified from $\{p(a, b|x, y)\}$ if there exists a unitary $U_B : \mathcal{H}_B \rightarrow \mathcal{H}_B$ such that

$$(\mathbb{1}_A \otimes U_B)|\psi\rangle_{AB} = |\tilde{\psi}\rangle_{AB}, \quad (7)$$

and

$$U_B \Pi_B N_y^b \Pi_B U_B^\dagger = \tilde{N}_y^b, \quad (8)$$

where Π_B is the projection onto the support of the local support $\rho_B = \text{Tr}_A(|\psi\rangle\langle\psi|_{AB})$.

Let us now proceed towards the results of this work.

III. RESULTS

We begin by constructing a steering inequality stated using the joint probability distribution \vec{p} as

$$W = \sum_{a,b,x=0}^2 p(a, b \neq a|x, y = x) \leq \beta_L. \quad (9)$$

Using the fact that $\sum_{a,b} p(a, b|x, y) = 1$ for all x, y , we can simplify the above steering inequality as

$$W = 3 - \sum_{a,x=0}^2 p(a, a|x, x) \leq \beta_L. \quad (10)$$

Alice is trusted and performs the measurements $A_x = \{M_x^a\}_{a=0,1,2}$, where $x = 0, 1, 2$ and the measurement elements

are given as $M_x^a = \frac{2}{3}|e_{a,x}\rangle\langle e_{a,x}|$. Here, the vectors $|e_{a,x}\rangle \in \mathbb{C}^2$ are given by

$$\begin{aligned} |e_{0,0}\rangle &= |0\rangle, & |e_{0,1}\rangle &= \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \\ |e_{0,2}\rangle &= \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, & |e_{1,0}\rangle &= |1\rangle, \\ |e_{1,1}\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle, & |e_{1,2}\rangle &= \frac{\sqrt{3}}{2}|0\rangle - \frac{i}{2}|1\rangle, \\ |e_{2,0}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), & |e_{2,1}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{7\pi i}{6}}|1\rangle), \\ |e_{2,2}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{-\frac{\pi i}{6}}|1\rangle). \end{aligned} \tag{11}$$

Notice that Alice’s measurement is extremal.

Let us now compute the LHS bound β_L of the steering inequality (9). Using (6), we rewrite the steering functional W from Eq. (10) as

$$W = 3 - \sum_{a,x=0}^2 \sum_{\lambda} p(\lambda)p(a|x, \rho_{\lambda})p(a|x, \lambda). \tag{12}$$

We focus on the last term in Eq. (12) for each x and notice that they can be bounded from below in the following way,

$$\begin{aligned} \sum_{a=0}^2 \sum_{\lambda} p(\lambda)p(a|x, \rho_{\lambda})p(a|x, \lambda) \\ \geq \sum_{\lambda} p(\lambda) \min_a \{p(a|x, \rho_{\lambda})\}, \end{aligned} \tag{13}$$

where $x = 0, 1, 2$ and we used the fact that $\sum_a p(a|x, \lambda) = 1$ for any x and λ . Now, minimizing over ρ_{λ} , we obtain

$$\sum_{\lambda} p(\lambda) \min_a \{p(a|x, \rho_{\lambda})\} \geq \sum_{\lambda} p(\lambda) \min_{\rho_{\lambda}} \min_a \{p(a|x, \rho_{\lambda})\}. \tag{14}$$

Using the fact $\sum_{\lambda} p(\lambda) = 1$, we get that the LHS bound is upper bounded by

$$\beta_L \leq 3 - \sum_{x=0}^2 \min_{|\psi\rangle \in \mathbb{C}^2} \min_a \{p(a|x, |\psi\rangle)\}. \tag{15}$$

Notice that since the steering functional W is linear, the minimization can be carried out over pure states. Numerically evaluating the above quantity by putting in Alice’s measurements (11), we find that

$$\beta_L \leq 2.673. \tag{16}$$

For this purpose, we choose a state $|\psi\rangle \in \mathbb{C}^2$ parametrized using the Bloch representation as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle, \tag{17}$$

where $0 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq \pi$. Now, the probabilities $p(a|x, |\psi\rangle)$ when Alice performs the measurement M_x^a are given by $p(a|x, |\psi\rangle) = |\langle e_{a,x} | \psi \rangle|^2$ which are a function of θ, ϕ . Then, a simple optimization over the parameters θ, ϕ gives us the local bound (16).

Let us now evaluate the quantum bound β_Q , the maximal value achievable using quantum states and measurements, of the steering functional W (9). The quantum bound is in fact the same as the algebraic bound of W (9), that is, 3 and for instance can be achieved by $|\psi\rangle_{AB} = |\phi^+\rangle_{AB}$ and Bob’s measurements $B_x = \{N_x^a\}_{a=0,1,2}$, such that $x = 0, 1, 2$. Here, the measurement elements $N_x^a = \frac{2}{3}|f_{a,x}\rangle\langle f_{a,x}|$ such that $|f_{a,x}\rangle = |e_{a,x}^{\perp}\rangle \in \mathbb{C}^2$, that is, $\langle e_{a,x}^* | f_{a,x} \rangle = 0$ where $|e_{a,x}\rangle$ are specified in Eq. (11) and $*$ denotes their conjugate.

It is important to note here that to achieve the maximal violation of the steering inequality (9), all the probabilities $p(a, a|x, x)$ in (10) have to be 0, that is,

$$p(a, a|x, x) = 0, \quad a, x = 0, 1, 2, \tag{18}$$

along with the condition that $\sum_{a,b} p(a, b|x, y) = 1$ for all x, y . This simple relation (18) is in fact sufficient to self-test the unknown state and measurements that result in the maximal violation of the steering inequality (9). Let us now proceed to the main result of this work.

Theorem 1. Consider that the steering inequality (10) is maximally violated by a state $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathcal{H}_B$ and three-outcome measurements $B_y = \{N_y^b\}$ ($y = 0, 1, 2$) acting on \mathcal{H}_B . Alice is trusted and her measurements A_x are given in (11). Then, there exists a local unitary transformation on Bob’s side, U_B such that

$$(\mathbb{1}_A \otimes U_B)|\psi\rangle_{AB} = |\phi^+\rangle_{AB}, \tag{19}$$

and

$$U_B \Pi_B N_y^b \Pi_B U_B^\dagger = \frac{2}{3}|e_{b,y}^{\perp}\rangle\langle e_{b,y}^{\perp}|, \tag{20}$$

where $|e_{b,y}\rangle$ are given in (11) and Π_B is the projector onto the support of $\rho_B = \text{Tr}_A(|\psi\rangle\langle\psi|_{AB})$.

Proof. We begin by considering a state ρ_{AB} that maximally violates the steering inequality (9). However, Bob’s dimension is unrestricted and thus we can always purify this state by adding an auxiliary system to Bob. Thus without loss of generality, we consider that the state that results in the maximal violation is given by $|\psi\rangle_{AB}$.

Now, as the dimension of Alice’s Hilbert space is 2, as suggested in Ref. [28] let us consider the Schmidt decomposition of the state $|\psi\rangle_{AB}$ as

$$|\psi\rangle_{AB} = \sum_{i=0,1} \lambda_i |s_i\rangle_A |t_i\rangle_B, \tag{21}$$

where the coefficients $\lambda_i > 0$ and satisfy the condition $\sum_i \lambda_i^2 = 1$. The local vectors $|s_i\rangle \in \mathbb{C}^2$ and $|t_i\rangle \in \mathcal{H}_B$ are orthonormal. Notice that the coefficients $\lambda_i \neq 0$ as any violation of the steering inequality (9) imposes that the state $|\psi\rangle_{AB}$ is entangled.

Let us now observe that there exists a unitary U_B such that $U_B |t_i\rangle = |s_i^*\rangle$ for every i . Thus, the state (21) can be expressed as

$$(\mathbb{1}_A \otimes U_B)|\psi\rangle_{AB} = (\mathbb{1}_A \otimes P_B) \frac{1}{\sqrt{2}} \sum_{i=0,1} |s_i\rangle |s_i^*\rangle, \tag{22}$$

where

$$P_B = \sqrt{2} \sum_{i=0,1} \lambda_i |s_i^*\rangle\langle s_i^*|. \tag{23}$$

Notice that P_B is full rank as $\lambda_i \neq 0$ in the state (21). The state on the right-hand side of Eq. (22) is the two-qubit maximally entangled state. Thus,

$$(\mathbb{1}_A \otimes U_B)|\psi\rangle_{AB} = |\tilde{\psi}\rangle_{AB} = (\mathbb{1}_A \otimes P_B)|\phi^+\rangle_{AB}. \quad (24)$$

Let us now consider that Bob's measurements are POVMs given by $B_y = \{N_y^b\}$ such that $b, y = 0, 1, 2$. We can characterize these measurements only on the support of Bob's reduced state ρ_B . Thus, we project these measurements onto the support of ρ_B to get

$$\Pi_B N_y^b \Pi_B = \bar{N}_y^b, \quad (25)$$

where $\Pi_B = |t_0\rangle\langle t_0| + |t_1\rangle\langle t_1|$ such that $|t_i\rangle$ are specified in Eq. (21). Now, as shown in Ref. [34], a product of two positive semidefinite matrices is also positive semidefinite. Thus, \bar{N}_y^b is also positive semidefinite as Π_B, N_y^b are both Hermitian and positive semidefinite matrices. The condition $\sum_{a,b} p(a, b|x, y) = \sum_b p(b|y) = 1$ imposes that $\sum_b \bar{N}_y^b = \mathbb{1}_B$ for all y . Applying the unitary U_B , we arrive at

$$U_B \bar{N}_y^b U_B^\dagger = \tilde{N}_y^b \quad \forall b, y. \quad (26)$$

Notice from the above formula (26) that \tilde{N}_y^b acts on the Hilbert space \mathbb{C}^2 . Now, evaluating the joint probability $p(a, a|x, x)$ using the state (24) and the measurements (26), we obtain

$$p(a, a|x, x) = \langle \phi^+ | (\mathbb{1}_A \otimes P_B) [M_x^a \otimes \tilde{N}_x^a] (\mathbb{1}_A \otimes P_B) | \phi^+ \rangle, \quad (27)$$

where M_x^a denote Alice's measurement elements and are given in Eq. (11). Now using the condition (18), we arrive at

$$p(a, a|x, x) = \langle \phi^+ | M_x^a \otimes P_B \tilde{N}_x^a P_B | \phi^+ \rangle = 0. \quad (28)$$

Using the fact that $R \otimes Q | \phi^+ \rangle = \mathbb{1} \otimes QR^T | \phi^+ \rangle$, where R^T denotes the transpose of R in the standard basis, we get from Eq. (28) that

$$\text{Tr}[P_B \tilde{N}_x^a P_B M_x^{aT}] = 0. \quad (29)$$

Now, notice from (23) that P_B and \tilde{N}_x^a are positive semidefinite. Thus $P_B \tilde{N}_x^a P_B$ is also positive semidefinite [34]. Now, we take the eigendecomposition of $P_B \tilde{N}_x^a P_B$ as

$$P_B \tilde{N}_x^a P_B = \sum_{i=0,1} \alpha_{i,a,x} |k_{i,a,x}\rangle \langle k_{i,a,x}|, \quad (30)$$

such that $\alpha_{i,a,x} \geq 0$. Expanding M_x^a using (11), we obtain from Eq. (29) that

$$\sum_{i=0,1} \alpha_{i,a,x} |\langle e_{a,x}^* | k_{i,a,x} \rangle|^2 = 0, \quad (31)$$

where we used the fact that for any projector $\Pi^T = \Pi^*$. As \tilde{N}_x^a acts on \mathbb{C}^2 along with the fact that $|e_{a,x}\rangle \in \mathbb{C}^2$ for any a, x , we expand $|k_{i,a,x}\rangle$ in the basis $\{|e_{a,x}^*\rangle, |e_{a,x}^{\perp}\rangle\}$ to obtain from (31) that

$$P_B \tilde{N}_x^a P_B = \beta_{a,x} |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}|, \quad (32)$$

where $\beta_{a,x} > 0$. Then, using the fact that $\sum_a \tilde{N}_x^a = \mathbb{1}$ for any x , we get

$$P_B^2 = \sum_a \beta_{a,x} |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}| \quad \forall x. \quad (33)$$

Thus, $\beta_{a,x}$ must satisfy the following condition,

$$\sum_a \beta_{a,x} |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}| = \sum_a \beta_{a,x'} |e_{a,x'}^{\perp}\rangle \langle e_{a,x'}^{\perp}| \quad (34)$$

for any $x, x' = 0, 1, 2$. Solving the above conditions by putting in the explicit form of $|e_{a,x}\rangle$ (11), we get that $\beta_{a,x} = \beta_{a',x'}$ for any a, x, a', x' . Thus, from Eq. (33) we arrive at

$$P_B^2 = \frac{3\beta_{0,0}}{2} \mathbb{1}_B, \quad (35)$$

where we used the fact that $\sum_a |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}| = \frac{3}{2} \mathbb{1}$ for any x . This implies from (24) that

$$|\tilde{\psi}\rangle_{AB} = \sqrt{\frac{3\beta_{0,0}}{2}} |\phi^+\rangle_{AB}. \quad (36)$$

Normalizing the above state, we get that $\beta_{0,0} = 2/3$. Thus, we have that the state up to some local unitary U_B is the two-qubit maximally entangled state while the measurements from Eq. (32) are

$$\tilde{N}_x^a = \frac{2}{3} |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}|. \quad (37)$$

This completes the proof. \blacksquare

Robustness against white noise

From an experimental perspective, it is important to find the robustness of our certification scheme against noise that might be present in the sources or the detectors. However, to perform an experiment it is not always necessary to find the full robustness, which captures the fidelity between the real and ideal state with respect to the violation of the steering inequality.

Here, inspired by practical experiments, we find the robustness of our scheme with respect to a specific noise model, that is, when the ideal states and measurements, that result in the maximal violation of the steering inequality (9), are mixed with white noise. For this purpose, let us consider ideal Bob's POVMs $B_x = \{\frac{2}{3} |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}| \}_{a=0,1,2}$ where $x = 0, 1, 2$, and $|e_{a,x}\rangle$ given in Eq. (11). We add white noise to every measurement element and define the new measurement as $\bar{B}_x = \{N_x^a\}_{a=0,1,2}$ such that

$$N_x^a = \frac{2}{3} \left((1 - \varepsilon_{a,x}) |e_{a,x}^{\perp}\rangle \langle e_{a,x}^{\perp}| + \frac{\varepsilon_{a,x}}{2} \mathbb{1} \right). \quad (38)$$

Similarly, adding white noise to the maximally entangled state, we obtain that the noisy state shared between Alice and Bob is

$$\rho_{AB} = (1 - 2\varepsilon_s) |\phi^+\rangle \langle \phi^+|_{AB} + \frac{\varepsilon_s}{2} \mathbb{1}. \quad (39)$$

It is worth noting here that the term $1 - \varepsilon_i$ for any index i is usually referred to as the visibility parameter. Notice that the measurement elements and state being positive semidefinite imposes that noise parameters $\varepsilon_{a,x} \geq 0$ for any a, x along with $\varepsilon_s \geq 0$. Let us denote $\varepsilon = \max\{\max_{a,x} \{\varepsilon_{a,x}\}, \varepsilon_s\}$. Without loss of generality, we can replace all the noise parameters $\varepsilon_{a,x}, \varepsilon_s$ in Eqs. (38) and (39) with ε .

Let us now evaluate the steering functional (10), with Alice being trusted and her measurements are given in (11), using

the noisy state (39) and noisy Bob’s measurements (38). For this purpose, let us first compute $p(0, 0|0, 0)$ as

$$p(0, 0|0, 0) = \frac{2}{3} \text{Tr}(|e_{0,0}\rangle\langle e_{0,0}| \otimes N_0^0 \rho_{AB}). \quad (40)$$

Substituting $|e_{0,0}\rangle$ from (11), N_0^0 from (38) and ρ_{AB} from (39), we obtain that

$$p(0, 0|0, 0) = \frac{\varepsilon}{9} (3 - 2\varepsilon). \quad (41)$$

Proceeding in a similar manner, we obtain for any $a, x = 0, 1, 2$ that

$$p(a, a|x, x) = \frac{\varepsilon}{9} (3 - 2\varepsilon). \quad (42)$$

Thus, the value of the steering functional (10) when the ideal states and measurements are mixed with white noise is given by

$$W = 3 + 2\varepsilon^2 - 3\varepsilon \geq 3(1 - \varepsilon). \quad (43)$$

Therefore, the proposed self-testing scheme is highly robust against white noise as the value of the steering functional changes linearly with respect to the noise parameter ε .

Let us also analyze the robustness of our protocol when the state shared between Alice and Bob has a noise model of the form

$$\rho_{AB} = (1 - 2\varepsilon_s) |\phi^{+, \delta}\rangle\langle \phi^{+, \delta}|_{AB} + \frac{\varepsilon_s}{2} \mathbb{1}, \quad (44)$$

where

$$|\phi^{+, \delta}\rangle = \frac{1}{\sqrt{2[1 + (1 - \delta)^2]}} [|00\rangle + (1 - \delta)|11\rangle]. \quad (45)$$

Evaluating the steering functional W (10) using the above state (44) and noisy Bob’s measurements (38), we get that

$$W = 3 - f(\delta, \varepsilon) \quad (46)$$

such that

$$f(\delta, \varepsilon) = \frac{3\sqrt{2}\delta(3 - 2\varepsilon)\varepsilon + 3\varepsilon(-3 + 2\varepsilon) + \delta^2(-2 + \varepsilon)(1 + 2\varepsilon)}{-3 + 3(\sqrt{2} - \delta)\delta}. \quad (47)$$

Thus, even when the state consists of noise along with an imbalance in the coefficient of the maximally entangled state, our scheme is highly robust as $f(\delta, \varepsilon) \sim O(\delta, \varepsilon)$ when $\varepsilon, |\delta| \ll 1$.

IV. CONCLUSIONS

There are a few certification schemes in the prepare-and-measure scenarios, for instance [35,36], that utilize nonprojective measurements. However, none of these schemes can certify entangled states. In this paper, utilizing the quantum steering scenario, we propose a scheme for the certification of the two-qubit maximally entangled state using nonprojective measurements. Along with it, we also certified three three-outcome extremal POVMs on the untrusted Bob’s side. We then show that our scheme is highly robust against the presence of white noise in the experimental devices. It is worth noting here that the certification of states using nonprojective measurements cannot be implemented in the standard Bell scenario. Thus, we identify a task that can be done using quantum steering but not using Bell nonlocality. This work also suggests that quantum steering might be useful towards designing highly noise-tolerant self-testing schemes, that is, quantum states might be certifiable using noisy projective measurements in the quantum steering scenario.

Some follow-up questions arise from our work. First, it will be interesting to find 1SDI certification of any pure two-qubit entangled state using only nonprojective measurements. A challenging problem in this regard would be to find a 1SDI scheme that can certify states of arbitrary dimension using only POVMs. In this work, we utilized extremal measurements, however, it will be interesting if one can find similar certification schemes using nonextremal measurements.

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