Persistency of non-*n*-local correlations in noisy linear networks

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Linear n-local networks are compatible with quantum-repeater-based entanglement distribution protocols. Different sources of imperfections such as error in entanglement generation, communication over noisy quantum channels, and imperfections in measurements result in decay of quantumness across such networks. From practical perspectives it becomes imperative to analyze the nonclassicality of quantum network correlations in the presence of different types of noise. The present discussion provides a formal characterization of non-n-local features of quantum correlations in a noisy network scenario. In this context, persistency of non-n-locality has been introduced. Such a notion helps in analyzing decay of non-n-local features of network correlations with increasing length of the linear network in the presence of one or more causes of imperfections.

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I. INTRODUCTION

Formulation of the Einstein-Podolsky-Rosen (EPR) paradox [1] points out the inexplicability of quantum predictions in terms of only local hidden variable models. Such an impossibility in turn gives rise to the notion of nonlocality [2]. Quantum nonlocality serves as a resource in multifaceted practical tasks [3–10]. Over the past few years study of nonlocality has been extended beyond paradigm of the standard Bell scenario. The manifestation of nonlocal network correlations has been a recent trend of analysis in the field of quantum information theory [11].

Unlike the standard Bell-CHSH scenario, any measurement scenario compatible with network topology involves multiple distant sources. Each of the sources distributes physical systems to a subset of distant observers. In the case where all the sources in the entire network are independent of each other (*n-local assumption*), non-*n-local* correlations may emerge under suitable measurement contexts [11]. The simplest of this type of network, commonly known as a *bilocal* network (see Fig. 1 for $n = 2$), was first introduced in [12] followed by a vivid analysis in [13]. Keeping pace with the utility of quantum networks in various information processing tasks [15–19], the study of n-local networks has witnessed multidirectional development [20–33].

The assumption of source independence adds new physical insights in analyzing nonclassical behavior of quantum network correlations. For example, consider a $(n+1)$-partite entanglement-swapping network (see Fig. 1) involving n independent sources $\mathcal{S}_i (i = 1, 2, \ldots, n)$ arranged in a linear fashion. Each source distributes a two-qubit entangled state between a pair of parties (see a detailed discussion in Sec. II C). All the parties are thus not receiving qubits from a single source. Hence, unlike the standard Bell scenario, initially they do not share any common past. Moreover some of the parties perform a single measurement. This leads to another striking difference with the standard Bell experiment where each party must randomly and independently choose from a collection of two or more inputs [10]. The n-local assumption thus reduces requirements for exploiting nonclassicality in quantum networks [12–14].

Quantum repeaters form building blocks of any network meant for distributing entanglement between distant observers across a large length of quantum channel [15]. Now entanglement swapping forms the basis of designing quantum repeater networks. So any such network structure can be considered as a n-local network [11]. In an ideal scenario, under suitable measurement contexts, nonlocality in terms of non-n-locality is thus generated in the network. However, in practical situations various factors of difficulties such as imperfection in entanglement generation, communication over noisy quantum channels, and many others hinder distribution of entanglement over the entire length of the chain. Consequently, unlike that in the idealistic scenario, simulation of non-n-local correlations in the entire network structure becomes impossible. At this junction it becomes imperative to explore for how long such non classical behavior can be observed. To facilitate the discussion we have introduced the concept of *persistency* in this context.

In the literature, the idea of persistency has been used to characterize different types of multipartite quantum correlations [34–38]. Starting from a given m partite state ρ, for example, exhibiting some form of quantum correlation $\mathcal{C}$, e.g.,

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the number of parties is gradually decreased so as to find the minimum number of parties m', for example, such that none of the possible m' partite reduced state exhibits $\mathcal{C}$. m' is usually referred to as the persistency of ρ with respect to the specified quantum correlation ($\mathcal{C}$). For the current discussion, we have introduced the concept of persistency on a different note. Here it will be used for exploiting sustainability of a non-n-local feature varying with the length of a network in the presence of different types of noise.

We have analyzed generation of non-n-local correlations in the presence of various sources of imperfection. For our purposes we have considered n-local linear [21] networks. There may be error in entanglement generation at the sources. Distribution of qubits may then occur over noisy channels. Also the observers may be using local imperfect measurement devices at their end. We have considered all such potential sources of errors. For the rest of our work, n-local networks affected by at least one such type of imperfection are referred to as *noisy n-local networks*. To characterize non-n-local correlations in such networks we put forward the notion of *persistency of non-n-locality*.

First, we have derived the non-n-locality detection criterion for noisy networks. That criterion is further used to develop the concept of persistency. The first concept of persistency has been introduced in the presence of a single noise factor at a time. Then the notion has been generalized for more practical situations when the network is affected by two or more noise factors simultaneously.

The rest of the work is organized as follows: Some basic preliminaries are discussed in Sec. II. Characterization of the noisy n-local linear is given in Sec. III. Persistency of non-n-local correlations is studied in Sec. IV followed by some concluding remarks in Sec. V.

II. PRELIMINARIES

We first proceed to discuss some basic prerequisites to be used in forthcoming sections.

A. Density matrix representation of arbitrary two-qubit state

Let ϱ denote an arbitrary two-qubit state. The density matrix of ϱ in terms of Bloch parameters is given by

$$\varrho = \frac{1}{4}\Bigg(\mathbb{I}_2 \times \mathbb{I}_2 + \vec{a}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{b}.\vec{\sigma} + \sum_{j_1,j_2=1}^{3} w_{j_1 j_2}\sigma_{j_1} \otimes \sigma_{j_2}\Bigg), \tag{1}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_{j_k} denote Pauli operators along three mutually perpendicular directions ($j_k = 1, 2, 3$). $\vec{a} = (x_1, x_2, x_3)$ and $\vec{b} = (y_1, y_2, y_3)$ denote local Bloch vectors ($\vec{a}, \vec{b} \in \mathbb{R}^3$) corresponding to party $\mathcal{A}$ and $\mathcal{B}$, respectively, with $|\vec{a}|, |\vec{b}| \leqslant 1$ and $(w_{i,j})_{3\times3}$ denotes correlation tensor $\mathcal{W}$(real). Matrix elements $w_{j_1 j_2}$ are given by $w_{j_1 j_2} = \mathrm{Tr}[\rho\, \sigma_{j_1} \otimes \sigma_{j_2}]$. $\mathcal{W}$ can be diagonalized by subjecting it to suitable local unitary operations [39,40]. A simplified expression is then given by

$$\varrho' = \frac{1}{4}(\mathbb{I}_2 \times \mathbb{I}_2 + \mathfrak{a}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathfrak{b}.\vec{\sigma} + \sum_{j=1}^{3} t_{jj}\sigma_j \otimes \sigma_j). \tag{2}$$

$T = \mathrm{diag}(t_{11}, t_{22}, t_{33})$ denotes the correlation matrix in Eq. (2) where t_{11}, t_{22}, t_{33} are the eigenvalues of $\sqrt{\mathcal{W}^T\mathcal{W}}$, i.e., singular values of $\mathcal{W}$.

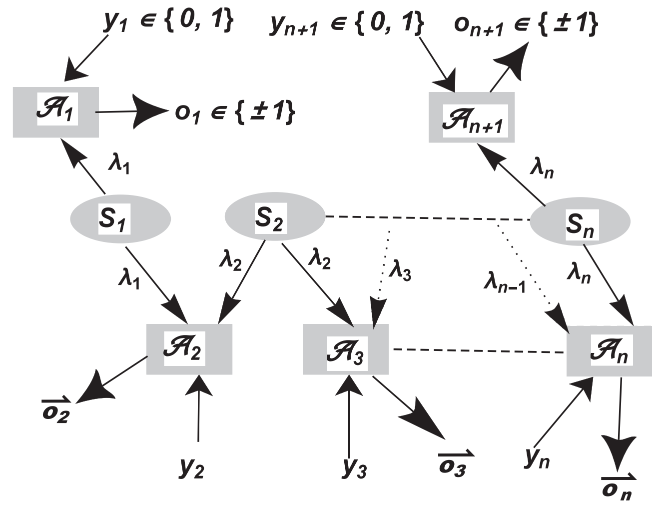

FIG. 1. Schematic diagram of n-local linear network [21].

B. n-local linear networks

Consider a network with n sources $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ and $n+1$ parties $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n+1}$ arranged in a linear pattern (see Fig. 1). $\forall i = 1, 2, \ldots, n$, and source $\mathcal{S}_i$ independently distributes physical systems (characterized by λ_i) to $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$. For each of $i = 2, 3, \ldots, n$, $\mathcal{A}_i$ receives two particles and is referred to as the *central* party. Each of other two parties $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$ receives one particle and is referred to as the *extreme* party. $\mathcal{S}_i$ is characterized by variable λ_i. As sources are independent, joint distribution of $\lambda_1, \ldots, \lambda_n$ is factorizable:

$$\rho(\lambda_1, \ldots, \lambda_n) = \Pi_{i=1}^{n}\rho_i(\lambda_i), \tag{3}$$

where $\forall i$, ρ_i denotes the normalized distribution of λ_i. Equation (3) represents an n-local constraint.

$\forall i = 2, 3, \ldots, n-1$ party $\mathcal{A}_i$ performs a single measurement y_i on the joint state of two subsystems received from $\mathcal{S}_{i-1}$ and $\mathcal{S}_i$. Each of $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$ selects from a collection of two dichotomous inputs. $n+1$ partite network correlations are local if

$$p(o_1, \vec{o}_2, \ldots, \vec{o}_n, o_{n+1}|y_1, y_{n+1}) = \int_{\Lambda_1}\int_{\Lambda_2}\cdots\int_{\Lambda_n} \times d\lambda_1 d\lambda_2 \ldots \lambda_n\, \rho(\lambda_1, \lambda_2, \ldots, \lambda_n)N_1, \text{ where}$$
$$N_1 = p(o_1|y_1, \lambda_1)\Pi_{j=2}^{n}p(\vec{o}_j|\lambda_{j-1}, \lambda_j)p(o_{n+1}|y_{n+1}, \lambda_n). \tag{4}$$

Notations appearing in Eq. (4) are detailed below:

(1) $\forall j$, Λ_j labels the set of all possible values of local hidden variable λ_j.

(2) $y_1, y_{n+1} \in \{0, 1\}$ denotes measurements of $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$, respectively.

(3) $o_1, o_{n+1} \in \{\pm 1\}$ denotes outputs of $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$, respectively.

(4) $\forall j,\ \vec{\mathfrak{o}}_j = (o_{j1}, o_{j2})$ labels four outputs of input y_j for $o_{ji} \in \{0, 1\}$.

Correlations are n-local if those satisfy both Eqs. (3) and (4). So any set of $n+1$ partite correlations that do not satisfy both of these constraints are termed non-n-local.

An n-local inequality [21] corresponding to this network scenario is

$$\sqrt{|I|} + \sqrt{|J|} \leqslant 1, \text{ where}$$
$$I = \frac{1}{4}\sum_{y_1, y_{n+1}} \langle O_{1,y_1} O_2^0 \dots O_n^0 O_{n+1,y_{n+1}} \rangle$$
$$J = \frac{1}{4}\sum_{y_1, y_{n+1}} (-1)^{y_1+y_{n+1}} \langle O_{1,y_1} O_2^1 \dots O_n^1 O_{n+1,y_{n+1}} \rangle \text{ with}$$
$$\times \langle O_{1,y_1} O_2^i \dots O_n^i O_{n+1,y_{n+1}} \rangle = \sum_{\mathcal{D}} (-1)^{\mathfrak{o}_1+\mathfrak{o}_{n+1}+\mathfrak{o}_{2i}+\dots+\mathfrak{o}_{ni}} N_2,$$
$$\text{where } N_2 = p(\mathfrak{o}_1, \vec{\mathfrak{o}}_2, \dots, \vec{\mathfrak{o}}_n, \mathfrak{o}_{n+1} | y_1, y_{n+1}),\ i = 0, 1$$
$$\text{and } \mathcal{D} = \{\mathfrak{o}_1, \mathfrak{o}_{21}, \mathfrak{o}_{22}, \dots, \mathfrak{o}_{n1}, \mathfrak{o}_{n2}, \mathfrak{o}_{n+1}\}. \quad (5)$$

Violation of Eq. (5) ensures the non-n-local nature of corresponding correlations.

C. Quantum linear n-local network scenario

In the n-local network, let $\mathcal{S}_i (i = 1, 2, \dots, n)$ generate an arbitrary two qubit state ϱ_i. Each of the central parties $\mathcal{A}_i (i = 2, 3, \dots, n)$ thus receives two qubits: one of ϱ_{i-1} and another of ϱ_i. $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$ receive a single qubit of ϱ_1 and ϱ_n, respectively. Let each of the central parties perform projection in the Bell basis $\{|\psi^{\pm}\rangle, |\phi^{\pm}\rangle\}$, often referred to as a Bell state measurement (BSM [13]). Let M_i denote BSM of central party $\mathcal{A}_i$. Let each of the extreme parties perform projective measurements in any one of two arbitrary directions: $\{\vec{m}_0.\vec{\sigma}, \vec{m}_1.\vec{\sigma}\}$ for $\mathcal{A}_1$ and $\{\vec{n}_0.\vec{\sigma}, \vec{n}_1.\vec{\sigma}\}$ for $\mathcal{A}_{n+1}$ with $\vec{m}_0, \vec{m}_1, \vec{n}_0, \vec{n}_1 \in \mathcal{R}^3$. Under these measurement settings, non-n-local correlations are detected by violation of Eq. (5) if [30]

$$\sqrt{\Pi_{i=1}^n t_{i11} + \Pi_{i=1}^n t_{i22}} > 1 \quad (6)$$

with t_{i11}, t_{i22} denoting the largest two singular values of correlation tensor (T_i) of ϱ_i $(i = 1, 2, \dots, n)$. In the case where Eq. (6) is violated nothing can be concluded about the n-local behavior of the correlations.

III. NOISY n-LOCAL LINEAR NETWORK

Consider a n-local linear network (Fig. 1). The entire procedure in the network can be divided into two phases: *preparation phase* and *measurement phase*. The former phase further comprises two parts: *generation* and *distribution of entanglement*. For analysis of non-n-locality in the noisy network, errors are considered in all these stages. In the case where the network is used for distribution of entanglement, ideally pure entanglement is to be distributed from each source $S_i (i = 1, 2, \dots, n)$. However, errors in the preparation phase lead to distribution of a mixed two-qubit state ϱ_i among the parties A_i and $A_{i+1} \forall i$.

We first analyze the correlations considering error in the measurement stage. Precisely speaking, under measurement imperfections, a closed form of an upper bound of n-local inequality [Eq. (5)] is derived for arbitrary two-qubit states. This form is further utilized in exploiting non-n-locality under the effect of errors in the preparation phase.

A. Imperfection in measurements

As discussed in Sec. II C, each of $\mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n$ performs BSM. Now let the devices fail to detect particles with some probability. Let $\beta_i \in [0, 1]$ characterize imperfection in measurement operator M_i in the sense that it fails to detect with probability $1 - \beta_i$. Measurement operator M_i^{noisy}, e.g., of $\mathcal{A}_i$, thus turns out to be a positive operator valued measures (POVM) with the elements $\{M_{ij_1j_2}^{\text{noisy}}\}$ given by

$$M_{i,00}^{\text{noisy}} = \beta_i |\phi^+\rangle\langle\phi^+| + \frac{1-\beta_i}{4}\mathbb{I}_{2\times2},$$
$$M_{i,01}^{\text{noisy}} = \beta_i |\phi^-\rangle\langle\phi^-| + \frac{1-\beta_i}{4}\mathbb{I}_{2\times2},$$
$$M_{i,10}^{\text{noisy}} = \beta_i |\psi^+\rangle\langle\psi^+| + \frac{1-\beta_i}{4}\mathbb{I}_{2\times2},$$
$$M_{i,11}^{\text{noisy}} = \beta_i |\psi^-\rangle\langle\psi^-| + \frac{1-\beta_i}{4}\mathbb{I}_{2\times2},\ \forall i = 2, 3, \dots, n. \quad (7)$$

Now, it may be noted that in the case where $\mathcal{A}_i$ performs perfect BSM then $\{|\phi^{\pm}\rangle\langle\phi^{\pm}|, |\psi\pm\rangle\langle\psi^{\pm}|\}$ is the set of possible projectors. Where $\forall i = 2, 3, \dots, n$, denoting $M_{i,00}^{\text{ideal}}, M_{i,01}^{\text{ideal}}, M_{i,10}^{\text{ideal}}, M_{i,11}^{\text{ideal}}$ as the measurement operators corresponding to the BSM projectors, POVM elements of imperfect BSM [Eq. (7)] can be represented as

$$M_{i,j_1j_2}^{\text{noisy}} = \beta_i M_{i,j_1j_2}^{\text{ideal}} + \frac{1-\beta_i}{4}\mathbb{I}_{2\times2}. \quad (8)$$

Similar to imperfection in measurement settings of central parties, let each of the two extreme parties also use imperfect detecting devices. For party $\mathcal{A}_1$, let $\mu \in [0, 1]$ parametrize a faulty measurement device. For single-qubit projection, such a device fails to detect any output with probability $1 - \mu$. POVM resulting due to imperfection in $\vec{m}_k.\vec{\sigma}$ thus has two elements $\{P_{kj}^{\text{noisy}}\}_{j=0,1}$ given by

$$P_{k0}^{\text{noisy}} = \mu\mathcal{O}^+ + \frac{1-\mu}{2}\mathbb{I}_2,$$
$$P_{k1}^{\text{noisy}} = \mu\mathcal{O}^- + \frac{1-\mu}{2}\mathbb{I}_2,\ k = 0, 1, \quad (9)$$

where $\mathcal{O}^+$ ($\mathcal{O}^-$) denotes the projection operator corresponding to the $+1$ (-1) eigenvalue. $\mathcal{O}^{\pm}$ denotes projectors corresponding to perfect projective measurement. Labeling $P_{k0}^{\text{ideal}}, P_{k1}^{\text{ideal}}$ as the projectors corresponding to perfect measurement $\vec{m}_k.\vec{\sigma}$, alternate representation of POVM elements [Eq. (9)] is given by

$$P_{ki}^{\text{noisy}} = \mu P_{ki}^{\text{ideal}} + \frac{1-\mu}{2}\mathbb{I}_2,\ i, k = 0, 1. \quad (10)$$

Similarly for $\mathcal{A}_{n+1}$, let $1-\nu$ denote failure probability in $\vec{n}_k.\vec{\sigma}$. Elements of corresponding POVM are given by

$$Q_{k0}^{\text{noisy}} = \nu \mathcal{Q}^{+} + \frac{1-\nu}{2}\mathbb{I}_2,$$
$$Q_{k1}^{\text{noisy}} = \nu \mathcal{Q}^{-} + \frac{1-\nu}{2}\mathbb{I}_2,\ k=0,1 \tag{11}$$

where $\mathcal{Q}^{+}$ ($\mathcal{Q}^{-}$) denotes a projection operator corresponding to the $+1$ (-1) eigenvalue.

We now put forward the criterion that suffices to detect non-n-locality when all the parties are performing imperfect measurements. For $\mathcal{A}_{n+1}$, the analog of representation given by Eq. (10) is

$$Q_{ki}^{\text{noisy}} = \nu Q_{ki}^{\text{ideal}} + \frac{1-\nu}{2}\mathbb{I}_2,\ i,k=0,1. \tag{12}$$

Theorem 1. With each source $\mathcal{S}_i$ generating an arbitrary two-qubit state and all the parties performing imperfect measurements, a sufficient criterion for detecting non-n-locality in a linear n-local network is given by

$$\sqrt{\Pi_{i=1}^{n} t_{i11} + \Pi_{i=1}^{n} t_{i22}} > \frac{1}{\left(\mu\nu\Pi_{j=2}^{n}\beta_j\right)^{\frac{1}{2}}}. \tag{13}$$

Proof. See the Appendix.

Equation (13) being a sufficient detection criterion, violation of the same gives no definite conclusion regarding simulation of non-n-local correlations in a corresponding noisy network. The above criterion points out the effect of the imperfection parameters over the usual non-n-locality criterion [Eq. (6)]. Comparing the right-hand side of both Eqs. (6) and (13) it is observed that if at least one of the detectors turns out to be imperfect with some nonzero probability, then that reduces the chances for generation of non-n-locality in the noisy network compared to the ideal situation. Moreover, if any of the detectors used in the network always fail to detect, i.e., the corresponding success probability turns out be 0, then the above criterion [Eq. (13)] can never be satisfied.

B. Noisy entanglement generation

Let us first discuss an ideal entanglement generation procedure [15]. Without loss of any generality, we consider the ideal generation of $|\phi^-\rangle\langle\phi^-|$. Let $\varrho = |01\rangle\langle 01|$ be the state at each source $\mathcal{S}_i$. To generate entanglement, the Hadamard gate ($\mathcal{H}$) is applied on the first qubit. Considering the first qubit as the control qubit, the CNOT gate is then applied resulting in generation of the Bell state $|\phi^-\rangle\langle\phi^-|$ [15]. Ideally, each of $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ is supposed to generate $|\phi^-\rangle\langle\phi^-|$.

However, in practical situations imperfections in preparation devices lead to generation of mixed entangled states. Such imperfections result from erroneous applications of Hadamard and/or CNOT gates. At each source $\mathcal{S}_i$, let α_i and δ_i denote the imperfection parameters characterizing $\mathcal{H}$ and CNOT gates, respectively. $\forall i = 1, 2, \ldots, n$, starting from $\varrho_i = |01\rangle\langle 01|$, and the noisy Hadamard gate generates [41]

$$\begin{aligned}\varrho_i' &= \alpha_i(\mathcal{H}\otimes\mathbb{I}_2\varrho\mathcal{H}^{\dagger}\otimes\mathbb{I}_2) + \frac{1-\alpha_i}{2}\mathbb{I}_2\otimes\varrho_{2i},\ \text{with}\,\alpha_i\in[0,1]\\ &\quad\text{and } \varrho_{2i} = \text{Tr}_1(\varrho_i)\\ &= \frac{1}{2}(|00\rangle\langle 00|+|10\rangle\langle 10|) - \frac{\alpha_i}{2}(|00\rangle\langle 10|+|10\rangle\langle 00|).\end{aligned} \tag{14}$$

Subjection of ϱ_i' to noisy CNOT gives [41]

$$\begin{aligned}\varrho_i'' &= \delta_i(\text{CNOT}\varrho_i'(\text{CNOT})^{\dagger}) + \frac{1-\delta_i}{4}\mathbb{I}_2\otimes\mathbb{I}_2\\ &= \frac{1}{4}\Bigg[\sum_{i,j=0}^{1}(1+(-1)^{i+j}\delta_i]|ij\rangle\langle ij|\\ &\quad - 2\alpha_i\delta_i[|11\rangle\langle 00| + |00\rangle\langle 11|)\Bigg].\end{aligned} \tag{15}$$

The correlation tensor of ϱ_i'' is diag$(-\alpha_i\delta_i, \alpha_i\delta_i, \delta_i)$. In the case where $\mathcal{S}_i$ distributes ϱ_i'', and the parties perform imperfect measurements, non-n-locality is observed if

$$\sqrt{\Pi_{i=2}^{n}\delta_i\beta_i\mu\nu\delta_1\left(1+\Pi_{j=1}^{n}\alpha_j\right)} > 1. \tag{16}$$

C. Noisy quantum communication

Let us now consider that communication of ϱ_i'' from $\mathcal{S}_i$ to respective parties is occurring through noisy channels. Such a communication affects generation of non-n-local correlations for obvious reasons. To analyze the effect of such noise parameters over n-locality detection, we are considering a few standard noisy channels [42].

1. Amplitude-damping channel

$\forall i = 1, 2, \ldots, n$, and let $\gamma_i^{\text{amp}}, \xi_i^{\text{amp}}$ characterize channels connecting $\mathcal{S}_i$ with $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$, respectively. Two qubits of ϱ_i'' [Eq. (15)] are thus passed through two different amplitude-damping channels. Let ϱ_i''' denote a corresponding noisy state. Any amplitude-damping channel (e.g., parametrized by γ^{amp}) is represented by Krauss operators $|0\rangle\langle 0| + \sqrt{1-\gamma^{\text{amp}}}|1\rangle\langle 1|$ and $\sqrt{\gamma^{\text{amp}}}|0\rangle\langle 1|$. The correlation tensor of ϱ_i''' is given by diag$(-\alpha_i\delta_i\sqrt{D_i^{\text{amp}}}, \alpha_i\delta_i\sqrt{D_i^{\text{amp}}}, \delta_i D_i^{\text{amp}} + \gamma_i^{\text{amp}}\xi_i^{\text{amp}})$ where $D_i^{\text{amp}} = (1-\gamma_i^{\text{amp}})(1-\xi_i^{\text{amp}})$. Using the closed form of the n-local bound [Eq. (13)] under the imperfect measurement context, non-n-locality is detected if

$$\begin{aligned}&\sqrt{\Pi_{i=2}^{n}\beta_i\mu\nu\text{Max}(2F_1, F_2)} > 1\ ,\ \text{where}\\ &F_1 = \Pi_{j=1}^{n}\alpha_j\delta_j\sqrt{\left(1-\gamma_j^{\text{amp}}\right)\left(1-\xi_j^{\text{amp}}\right)}\ \text{and}\\ &F_2 = \Pi_{j=1}^{n}\alpha_j\delta_j\sqrt{\left(1-\gamma_j^{\text{amp}}\right)\left(1-\xi_j^{\text{amp}}\right)}\\ &\quad + \Pi_{j=1}^{n}\left[\delta_j\left(1-\gamma_j^{\text{amp}}\right)\left(1-\xi_j^{\text{amp}}\right) + \gamma_j^{\text{amp}}\xi_j^{\text{amp}}\right].\end{aligned} \tag{17}$$

2. Phase-damping channel

$\forall i = 1, 2, \ldots, n$, Let $\gamma_i^{ph}, \xi_i^{ph}$ characterize channels connecting $\mathcal{S}_i$ with $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ respectively. Let ϱ_i''' denote the corresponding noisy state. Krauss operators corresponding to the phase-damping channel (e.g., having noise parameter γ^{ph}) are given by $|0\rangle\langle 0| + \sqrt{1-\gamma^{ph}}|1\rangle\langle 1|$ and $\sqrt{\gamma^{ph}}|0\rangle\langle 1|$. The correlation tensor of ϱ_i''' is given by diag$(-\alpha_i\delta_i\sqrt{D_i^{ph}}, \alpha_i\delta_i\sqrt{D_i^{ph}}, \delta_i)$ where $D_i^{ph} = (1-\gamma_i^{ph})$

$(1-\xi_i^{ph})$. In this case non-n-locality is detected if

$$\sqrt{\Pi_{i=2}^{n}\beta_i\mu\nu\text{Max}(G_1,G_2)} > 1 \text{ where}$$
$$G_1 = 2\Pi_{j=1}^{n}\alpha_j\delta_j\sqrt{\left(1-\gamma_j^{ph}\right)\left(1-\xi_j^{ph}\right)} \text{ and}$$
$$G_2 = \Pi_{j=1}^{n}\alpha_j\delta_j\sqrt{\left(1-\gamma_j^{ph}\right)\left(1-\xi_j^{ph}\right)} + \Pi_{j=1}^{n}\delta_j. \quad (18)$$

After analyzing non-n-locality detection in the presence of noise we next introduce the notion of persistency in this context.

IV. PERSISTENCY OF NON-n-LOCALITY

The discussion in Sec. III clearly points out the dependence of the upper bound of n-local inequality over noise parameters. Closer observation of different relations derived therein gives rise to the intuition that increasing the length of a network hinders simulation of non-n-local correlations. Formal characterization of such an interpretation will be provided in this section. In this context, let us now consider that for each of the three categories of errors discussed in Sec. III, noise parameters remain identical. To be precise:

(1) Each of n noisy sources is identical: $(\alpha_i,\delta_i)=(\alpha,\delta)$, e.g., $\forall i=1,2,\dots,n$.

(2) Parties are interconnected via identical noisy quantum channels.

(3) A single parameter characterizes imperfection in measurements of central parties, e.g., $\mathcal{A}_2,\dots,\mathcal{A}_n:\ \beta_2=\dots=\beta_n=\beta$.

Under such assumptions Eq. (16) becomes

$$\sqrt{\mu\nu\delta^n\beta^{n-1}(1+\alpha^n)} > 1. \quad (19)$$

We now define persistency of non-n-locality for each of the three types of errors individually.

A. First type of persistency of non-n-locality

Definition 1. The first type of persistency of non-n-locality $\mathcal{P}_I$(say) may be defined as the maximum number (n) of independent identical sources that can be connected so as to form a linear n-local network where Eq. (5) detects non-n-locality under the assumption that each source distributes a two-qubit mixed entangled state through noiseless quantum channels and all parties perform perfect measurements.

The above definition can be interpreted as a measure of the maximum length of an entanglement distribution network in which non-n-local correlations can be detected when the sources fail to generate pure entanglement. The measure is given in terms of independent sources as the length of any network can be specified by it. If $\mathcal{P}_I=m$ for a noisy network, addition of even a single source (generating mixed entanglement) to the network will result in generation of $(m+1)$ partite correlations whose non-$(m+1)$-local feature cannot be detected by Eq. (5).

Let us now consider a noisy network where error in entanglement generation is the only source of noise. For $\beta=\mu=\nu=1$, Eq. (19) becomes

$$\sqrt{\delta^n(1+\alpha^n)} > 1. \quad (20)$$

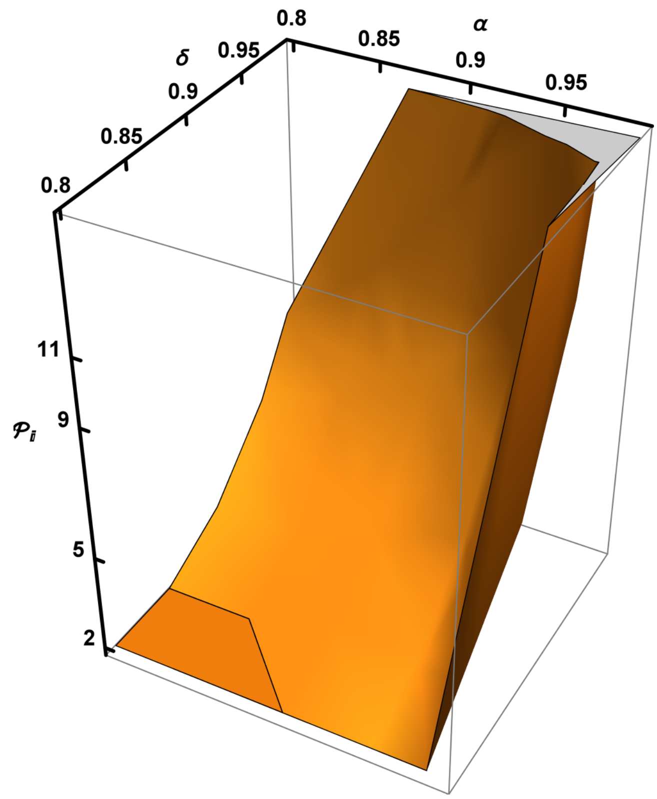

FIG. 2. Variation of the first type of persistency of non-n-locality with that of variables parametrizing error in entanglement generation.

For any specified values of $\alpha,\delta,\ \mathcal{P}_I$ is given by

$$\mathcal{P}_I = \lfloor n_I \rfloor, \quad (21)$$

where n_I denotes the upper bound of n given by Eq. (20). Variation of $\mathcal{P}_I$ with that of (α,δ) is plotted in Fig. 2. For an example, let $\alpha=\delta=0.9$. Equation (20) gives

$$n < n_I = 4.567. \quad (22)$$

So $\mathcal{P}_I=4$ in this case.

B. Second type of persistency of non-n-locality

Definition 2. The second type of persistency of non-n-locality $\mathcal{P}_{II}$, e.g., may be defined as the maximum number (n) of independent identical sources that can be connected so as to form a linear n-local network where non-n-locality is detected by Eq. (5) when communication over identical noisy quantum channels is the only source of error in the network.

Let $\mathcal{N}$ denote a linear n-local network configuration for some fixed value of $n=m$, e.g., such that each of m identical sources distributes pure entanglement through identical noisy channels and all parties perform perfect measurements. Let corresponding correlations turn out to be non m-local. Now let $\mathcal{N}$ be extended to an $(m+1)$-local network $\mathcal{N}'$ under prevalent conditions. $\mathcal{P}_{II}$ turns out to be m if Eq. (5) fails to detect non $(m+1)$-locality (if any) of corresponding correlations.

Let us first consider that the sources and the parties are interconnected via identical amplitude-damping channels: for example, $\gamma_i^{\text{amp}}=\xi_i^{\text{amp}}=\gamma_{\text{amp}},\ \forall i=1,2,\dots,n$.

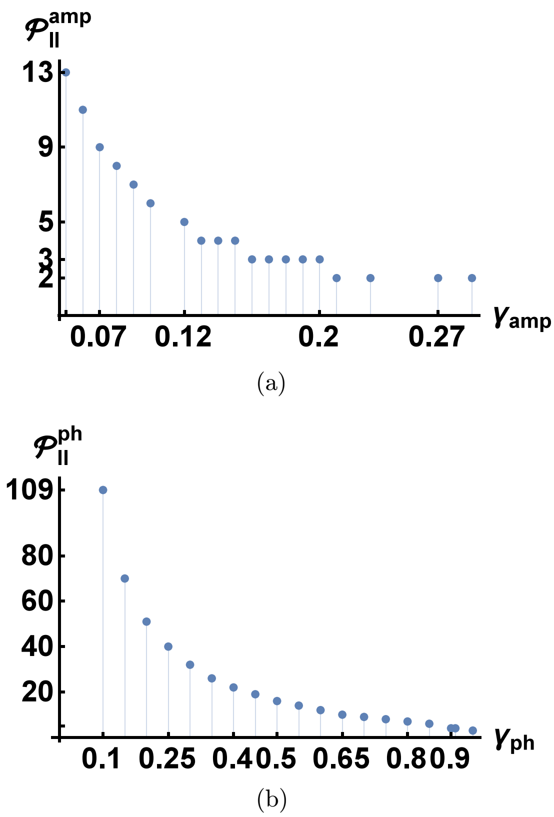

FIG. 3. Decrease in $\mathcal{P}_{II}$, i.e., $\mathcal{P}_{II}^{\text{amp}}$ with increasing noise level (γ_{amp}) in amplitude-damping channels (a). Similarly for the phase-damping channel, $\mathcal{P}_{II}^{ph}$ vs γ_{ph} (b).

Non-n-locality is detected in the network if

$$\sqrt{\text{Max}(2F_1, F_2)} > 1, \text{ where}$$
$$F_1 = (1-\gamma_{\text{amp}})^n \text{ and}$$
$$F_2 = F_1 + [(1-\gamma_{\text{amp}})^2 + (\gamma_{\text{amp}})^2]^n. \tag{23}$$

Equation (23) is obtained from Eq. (17) under the assumption of noisy communication as the only source of error in the n-local network. Equation (23) provides an upper bound on the number of sources n. If n_{II}^{amp} denotes the corresponding upper bound of n, then the second type of persistency of non-n-locality is given by $\mathcal{P}_{II} = \lfloor n_{II}^{\text{amp}} \rfloor$. Dependence of the source count (n) and hence that of $\mathcal{P}_{II}^{\text{amp}}$ on the noise parameter (see Fig. 3) is given by Eq. (23).

Let us now consider the noisy network where communication through identical phase-damping channels is the only source of noise. Setting, e.g., $\gamma_i^{ph} = \xi_i^{ph} = \gamma_{ph}, \forall i = 1, 2, , \dots, n$ and the rest of the parameters to be 1 in Eq. (18), the non-n-locality detection criterion is given by

$$\sqrt{1 + (1-\gamma_{ph})^n} > 1. \tag{24}$$

Equation (24) in turn gives $\mathcal{P}_{II}^{ph} = \lfloor n_{II}^{ph} \rfloor$ with n_{II}^{ph} denoting the upper bound of n given by the detection criterion [Eq. (24]. Comparison of Eqs. (23) and (24) indicates $\mathcal{P}_{II}^{ph} > \mathcal{P}_{II}^{\text{amp}}$ for any fixed value of noise parameter $\gamma_{\text{amp}} = \gamma_{ph} = \gamma$ (see Fig. 3).

C. Third type of persistency of non-*n*-locality

Definition 3. The third type of persistency of non-n-locality may be defined as the maximum number (n) of independent identical sources that can be connected so that non-n-locality is detected by Eq. (5) in the corresponding network under the assumption that each source distributes pure entanglement over noiseless channels and all parties perform imperfect measurements.

Let $\mathcal{P}_{III}$ denote the third type of persistency of non-n-locality. Consider an n-local network $\mathcal{N}$ for some fixed value of, e.g., $n = m$ such that each of m identical sources distributes pure entanglement through noiseless channels. Let each of the extreme parties($\mathcal{A}_1, \mathcal{A}_{m+1}$) perform imperfect projective measurements, whereas each of central parties($\mathcal{A}_2, \dots, \mathcal{A}_m$) performs imperfect Bell basis measurements. Under such measurement contexts, let corresponding correlations turn out to be non-m-local. Now let $\mathcal{N}$ be extended to an $(m+1)$-local network $\mathcal{N}'$ by adding another identical source $\mathcal{S}_{m+1}$. In $\mathcal{N}'$, there are m central parties. With all the parties performing imperfect measurements, if Eq. (5) fails to detect non-$(m+1)$-locality (if any) in $\mathcal{N}'$, then $\mathcal{P}_{III} = m$.

With imperfection in measurements considered as the only source of error, the non-n-locality detection criterion (16) becomes

$$\sqrt{2\mu\nu\beta^{n-1}} > 1. \tag{25}$$

If n_{III} denotes the upper bound of n in Eq. (25), then $\mathcal{P}_{III} = \lfloor n_{III} \rfloor$. With an increase in imperfection, $\mathcal{P}_{III}$ decreases (see Fig. 4). For example, in the case $\mu = \nu = \beta = 0.9$, non-n-locality is detected up to $n = 5$. Hence, $P_{III} = 5$. Until now persistency of non-n-locality has been analyzed under the presence of only one type of noise at a time. However, for practical purposes, it is important to study the same when at least two of the three possible factors of noise are present in the network. So generalization of the notion follows below.

D. Persistency of non-*n*-locality

Definition 4. The persistency of non-n-locality (e.g., $\mathcal{P}$) may be defined as the maximum number (n) of independent identical sources that can be connected to form a network such that Eq. (5) detects non-n-local correlations when at least two of the three noise factors are present in the network.

The above definition corresponds to the most general notion of persistency. Consider an n-local network $\mathcal{N}$ for some fixed value of $n = m$, e.g., under the assumption that at least two of the three categories of noise are present in the network. For better understanding, without loss of any generality, let sources distribute mixed entanglement, whereas parties perform imperfect measurements. Let non-m-locality be observed in the network. On extension of $\mathcal{N}$ to an $(m+1)$-local network in the presence of existing noise factors only, if Eq. (5) fails to detect non-n-locality for $n = m+1$, then $\mathcal{P} = m$ for $\mathcal{N}$. At this point it must be noted that in order to measure $\mathcal{P}$, extension of any network $\mathcal{N}$ to another one $\mathcal{N}'$, e.g., by adding identical sources, must be considered under the assumption that noise factors of $\mathcal{N}$ and $\mathcal{N}'$ remain invariant.

Clearly $\mathcal{P} < \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. Let us now provide an example for further illustration.

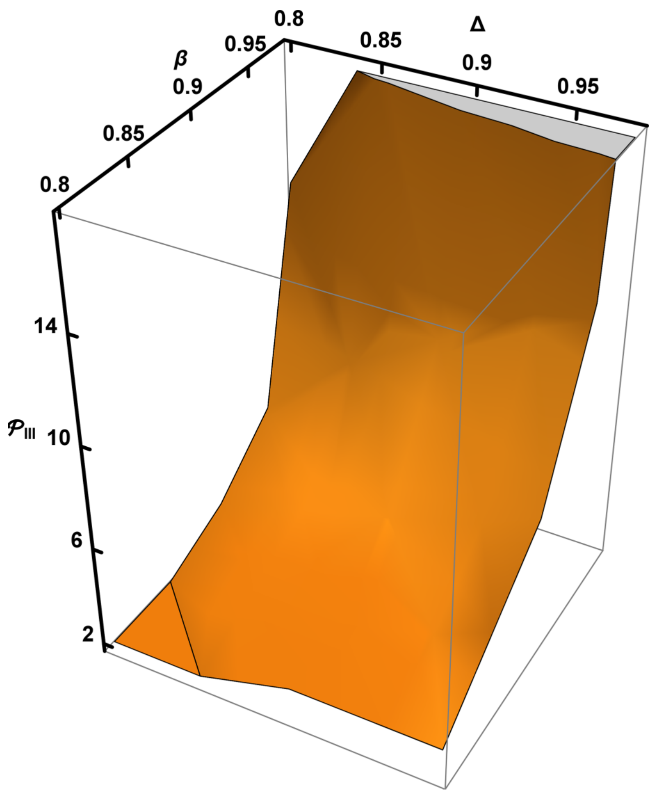

FIG. 4. Variation of the third type of persistency of non-n-locality with that of imperfection in measurements. Imperfections in single-qubit projective measurements of the extreme parties are parametrized by a single variable, e.g., $\mu = \nu = \Delta$.

Let us consider a noisy network where noise due to quantum communication may occur due to use of phase damp channels. The non-n-locality detection criterion [Eq. (18)] is given by

$$\sqrt{\beta^{n-1}\mu\nu \text{Max}(2G_1, G_2)} > 1, \text{ where}$$
$$G_1 = \alpha\delta(1-\gamma_{ph})^n \text{ and}$$
$$G_2 = [\alpha\delta(1-\gamma_{ph})]^n + \delta^n. \tag{26}$$

Persistency of non-n-locality for some specific values of noise parameters is given in Table I.

V. DISCUSSION

Characterization of non-n-local correlations in the presence of three different noise factors is provided in the present work. For our purpose the existing upper bound [Eq. (6)] of n-local inequality [Eq. (5)] has been used as the detection criterion of non-n-locality. Persistency of non-n-locality has been introduced for analyzing decay of non-n-local correlations with increasing length of a noisy network. Considering persistency of n-locality for each of three types of noise individually, the notion has been generalized to the more practical situation when at least two of the error factors exist in the network.

To this end, it must be pointed out that from experimental perspectives, the current study turns out to be a simple form of error analysis for exploiting non-n-locality. Though we have considered three broad categories of errors that usually occur in any entanglement-swapping-based network scenario, yet the discussion is oversimplified as any discussion of the technical difficulties, associated with experimental realization of quantum networks [43], lies beyond the scope of this paper.

In [44] the authors have pointed out multiple problems associated with physical implementation of any network configuration based on entanglement swapping. For instance, one of the most significant problem is the exponential decrease in coherence of quantum states. Such decoherence of a quantum system occurs due to long-range distribution of entanglement over noisy channels and on being subjected to operations over a considerably long span of time [43]. Methods such as entanglement purification [45–48], concentration [45,48, 52–55], and distillation [49–51] have been developed to distribute entanglement along a long chain of networks. With technological advancements in the field of quantum information science, practical implementations of these procedures with tolerable error rates have become possible [47–55]. Apart from loss in coherence, long-range fiber-based quantum communications become challenging due to photon loss, noise in photon detection, and many other factors [43]. The exponential decrease of the signal-to-noise ratio with increasing length of the fiber in quantum key distribution protocols is one of the consequences of limitations over long-range quantum communication. Though remarkable technological progress has been attained over the years [56–61], yet, to date, there exist several limitations constraining quantum communication over large distances.

Apart from the issues mentioned above, experimentalists face several other challenges while implementing a network configuration [43]. Hence, from experimental perspectives, it becomes important to consider at least some of these crucial factors while making any form of error analysis in a network scenario. However, our analysis has not included any such practical problem. At this junction, it is needed to be

TABLE I. Persistency of non-n-locality in networks under variation of noise factors. Error in communication of qubits is considered to be due to the use of phase-damping channels. The first row gives persistency of non-n-locality($\mathcal{P}$) when there is no error in entanglement generation. The second row gives $\mathcal{P}$ when parties perform perfect measurements, and the third row gives $\mathcal{P}$ when qubits pass through a noiseless channel. The last row gives $\mathcal{P}$ when all three forms of errors exist in the network.

Error in entanglement generation	Noisy communication	Imperfect measurements	$\mathcal{P}$
$(\alpha, \delta) = (1, 1)$	$\gamma = 0.1$	$(\mu, \nu, \beta) = (0.94, 0.93, 0.92)$	4
$(\alpha, \delta) = (0.94, 0.93)$	$\gamma = 0.1$	$(\mu, \nu, \beta) = (1, 1, 1)$	7
$(\alpha, \delta) = (0.92, 0.95)$	$\gamma = 0$	$(\mu, \nu, \beta) = (0.92, 0.94, 0.95)$	9
$(\alpha, \delta) = (0.92, 0.95)$	$\gamma = 0.12$	$(\mu, \nu, \beta) = (0.94, 0.93, 0.95)$	4

mentioned that the sole aim of the present study is to introduce the notion of persistency of non-n-locality in the context of exploiting nonclassicality in a linear network configuration. So the study, in its present form, can be considered to be in a nascent stage which needs gradual upgrading keeping pace with present day technology. Consequently, it will be interesting to extend the study taking into consideration the practical problems.

In [62] an entanglement-swapping network has been used as a Bell nonlocality activation protocol. The key role of any such protocol is to generate Bell-CHSH nonlocal quantum states starting from two or more Bell-CHSH local states. Over the years such a type of protocol has been generalized so as to activate different other notions of nonlocality [63–65]. Now, as already discussed in Sec. I, an entanglement-swapping network is a n-local network where each independent source distributes an entangled state. The present study on persistency of non-n-local correlations can thus be considered as one way of analyzing errors in exploiting a particular form of nonclassicality (non-n-locality) of $n+1$-partite correlations generated across any such network. In place of considering correlations across the entire network configuration, it will also be interesting to study the persistency of Bell nonlocality or any other notion of nonclassicality of the conditional states generated at the end of any such activation network [62,63].

Our entire analysis is limited to noisy linear n-local networks only. It will be interesting to exploit the same for any nonlinear configuration. Also in the network scenarios considered here, each source distributes two-qubit entangled states. Analyzing the decay of nonclassicality with growing imperfections in network when each of the sources generate multipartite and/or higher dimensional entangled states is a potential direction of future research.

APPENDIX

Proof of Theorem 1. Let us first consider the n-local inequality [Eq. (5)] for the noisy network:

$$\sqrt{|I_{\text{noisy}}|}+\sqrt{|J_{\text{noisy}}|}\leqslant 1, \text{ where}$$
$$I_{\text{noisy}}=\frac{1}{4}\sum_{y_1,y_{n+1}}\langle O_{1,y_1}O_{2n}^{00}O_{n+1,y_{n+1}}\rangle_{\text{noisy}}$$
$$J_{\text{noisy}}=\frac{1}{4}\sum_{y_1,y_{n+1}}(-1)^{y_1+y_{n+1}}\langle O_{1,y_1}O_2^1\dots O_n^1O_{n+1,y_{n+1}}\rangle_{\text{noisy}} \text{ with}$$
$$\langle O_{1,y_1}O_2^i\dots O_n^iO_{n+1,y_{n+1}}\rangle_{\text{noisy}}=\sum_{\mathcal{D}}(-1)^{\mathfrak{o}_1+\mathfrak{o}_{n+1}+\mathfrak{o}_{2i}+\cdots\mathfrak{o}_{ni}}N_{\text{noisy}},$$
$$\text{where } N_{\text{noisy}}=p'(\mathfrak{o}_1,\vec{\mathfrak{o}}_2,\dots,\vec{\mathfrak{o}}_n,\mathfrak{o}_{n+1}|y_1,y_{n+1}),\ i=0,1$$
$$\text{and } \mathcal{D}=\{\mathfrak{o}_1,\mathfrak{o}_{21},\mathfrak{o}_{22},\dots,\mathfrak{o}_{n1},\mathfrak{o}_{n2},\mathfrak{o}_{n+1}\}. \tag{A1}$$

A different symbol $p'()$ has been used for probability terms so as to discriminate those arising in a noisy scenario from that in an ideal scenario. Let us denote the overall state in the network as $\varrho=\otimes_{l=1}^n\varrho_l$. Next we consider the expectation terms given by Eq. (5). Without loss of any generality let us fix $i=0$ and fix the labeling of (y_1,y_{n+1}) as (0,0) and consider the corresponding expectation term $\langle O_{1,0}O_2^0\dots O_n^0O_{n+1,0}\rangle$:

$$\begin{aligned}\langle O_{1,0}O_2^0\dots O_n^0O_{n+1,0}\rangle_{\text{noisy}} &= \sum_{\mathfrak{o}_1,\mathfrak{o}_{21},\mathfrak{o}_{22},\dots,\mathfrak{o}_{n1},\mathfrak{o}_{n2},\mathfrak{o}_{n+1}}(-1)^{\mathfrak{o}_1+\mathfrak{o}_{n+1}+\mathfrak{o}_{20}+\cdots+\mathfrak{o}_{n0}}p(\mathfrak{o}_1,\vec{\mathfrak{o}}_2,\dots,\vec{\mathfrak{o}}_n,\mathfrak{o}_{n+1}|0,0)\\ &=\sum_{i,j=0}^1\sum_{g_2,h_2=0}^1\sum_{g_3,h_3=0}^1\cdots\sum_{g_n,h_n=0}^1(-1)^{i+j+g_2+g_3+\cdots+g_n}\text{Tr}\big[P_{0i}^{\text{noisy}}\otimes_{k=2}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]\\ \text{by Eqs. (7), (9), and (11)} &=\sum_{i,j=0}^1\sum_{g_3,h_3=0}^1\cdots\sum_{g_n,h_n=0}^1R_2\end{aligned}$$
$$\begin{aligned}\text{where } R_2=(-1)^{i+j+g_3+\cdots+g_n}\Big(&\text{Tr}\big[P_{0i}^{\text{noisy}}M_{2,00}^{\text{noisy}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]+\text{Tr}\big[P_{0i}^{\text{noisy}}M_{2,01}^{\text{noisy}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]\\ &-\text{Tr}\big[P_{0i}^{\text{noisy}}M_{2,10}^{\text{noisy}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]-\text{Tr}\big[P_{0i}^{\text{noisy}}M_{2,11}^{\text{noisy}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]\Big).\end{aligned}\tag{A2}$$

Further simplifying R_2, we get

$$R_2=(-1)^{i+j+g_3+\cdots+g_n}\text{Tr}\big[P_{0i}^{\text{noisy}}\big(M_{2,00}^{\text{noisy}}+M_{2,01}^{\text{noisy}}-M_{2,10}^{\text{noisy}}-M_{2,11}^{\text{noisy}}\big)\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big] \tag{A3}$$
$$=\beta_2(-1)^{i+j+g_2+g_3+\cdots+g_n}\text{Tr}\big[P_{0i}^{\text{noisy}}\otimes M_{2,g_2h_2}^{\text{ideal}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]\ \text{[using Eq. (8)]}. \tag{A4}$$

Using Eq. (A3) in Eq. (A2), we get

$$\langle O_{1,0}O_{2n}^{00}O_{n+1,0}\rangle_{\text{noisy}}=\beta_2\sum_{i,j=0}^1\sum_{g_2,h_2=0}^1\sum_{g_3,h_3=0}^1\cdots\sum_{g_n,h_n=0}^1(-1)^{i+j+g_2+g_3+\cdots+g_n}\text{Tr}\big[P_{0i}^{\text{noisy}}\otimes M_{2,g_2h_2}^{\text{ideal}}\otimes_{k=3}^nM_{k,g_kh_k}^{\text{noisy}}\,Q_{0j}^{\text{noisy}}\varrho\big]. \tag{A5}$$

Following a similar approach of breaking sums over the rest of the indices appearing in Eq. (A5), we have

$$\langle O_{1,0}O_2^0\dots O_n^0O_{n+1,0}\rangle_{\text{noisy}}=\Pi_{i=2}^n\beta_i\mu\nu\sum_{i,j=0}^1\sum_{g_2,h_2=0}^1\sum_{g_3,h_3=0}^1\dots\sum_{g_n,h_n=0}^1(-1)^{i+j+g_2+g_3+\cdots+g_n}\text{Tr}\big[P_{0i}^{\text{ideal}}\otimes_{k=2}^nM_{k,g_kh_k}^{\text{ideal}}\,Q_{0j}^{\text{ideal}}\varrho\big]. \tag{A6}$$

Following the same procedure for each expectation term appearing in Eq. (A1), we get

$$I_{\text{noisy}} = \mu\nu\Pi_{i=2}^{n}\beta_i I, \quad (A7)$$

$$J_{\text{noisy}} = \mu\nu\Pi_{i=2}^{n}\beta_i J. \quad (A8)$$

Equation (A1) thus gives

$$\sqrt{\mu\nu\Pi_{i=2}^{n}\beta_i}(\sqrt{|I|} + \sqrt{|J|}) = 1. \quad (A9)$$

Equation (A9) is the n-local inequality for a linear n-local network where the parties perform imperfect measurements and each of the sources generates an arbitrary two-qubit state. As upper bound of n-local inequality [Eq. (5)] in an ideal linear n-local network is given by Eq. (6), and clearly an upper bound of Eq. (A9) is given by $\sqrt{\mu\nu\Pi_{j=2}^{n}\beta_j}\sqrt{\Pi_{i=1}^{n}t_{i11} + \Pi_{i=1}^{n}t_{i22}}$. A non-$n$-locality detection criterion is thus given by Eq. (13). Proved.

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