Characterizing and quantifying the incompatibility of quantum instruments

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Incompatibility of quantum devices is one of the cornerstones of quantum theory, and the incompatibility of quantum measurements and channels has been linked to quantum advantage in certain information-theoretic tasks. In this work, we focus on the less well explored question of the incompatibility of quantum instruments, that is, devices that describe the measurement process in its entirety, accounting for both the classical measurement outcome and the quantum postmeasurement state. In particular, we focus on the recently introduced notion of parallel compatibility of instruments, which has been argued to be a natural notion of instrument compatibility. We introduce, in a manner similar to the case of measurements and channels, the incompatibility robustness of quantum instruments and derive universal bounds on it. We then prove that postprocessing of quantum instruments is a free operation for parallel compatibility. Last, we provide families of instruments for which our bounds are tight and families of compatible indecomposable instruments.

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I. INTRODUCTION

The incompatibility of devices is a key feature of quantum theory that sets it apart from classical physics [1]. The notion of incompatibility captures the fact that some devices (such as measurements or state transformations) cannot be performed simultaneously. While at first incompatibility may sound like a drawback, it turns out that incompatibility is an essential resource in various information processing tasks with quantum advantage [2–5]. Furthermore, incompatibility is linked to fundamental concepts such as Bell nonlocality [6], Einstein-Podolsky-Rosen steering [3,7], and contextuality [8]. For this reason, characterizing incompatible devices and quantifying incompatibility are fundamental research questions relevant to quantum information processing.

Arguably, the type of quantum devices whose incompatibility is the most studied is that of quantum measurements, although various results regarding the incompatibility of quantum channels (i.e., state transformations) also exist. The third commonly studied family of quantum devices generalizes measurements and channels and is called quantum instruments. Instruments take a quantum state as an input and produce a classical output and a quantum output. That is, quantum instruments describe the full measurement process, including the measurement outcome (the classical output) and the postmeasurement state (the quantum output).

While the compatibility of quantum instruments has been defined in the literature [9–11], its systematic study started only recently [12,13]. It was recently pointed out that the "traditional" notion of instrument compatibility might not be satisfactory in some cases [12]. The authors of Ref. [12] introduced the notion of *parallel* compatibility and argued that it is a more natural notion of instrument compatibility than traditional compatibility. In this work we initiate the characterization of parallel compatible instruments and the quantification of parallel incompatibility.

We start with a few technical preliminaries in Sec. II on quantum devices, their compatibility, and quantifying incompatibility. Then in Sec. III we introduce the incompatibility robustness of quantum instruments, prove bounds on this quantity, prove that postprocessing of instruments is a free operation for parallel compatibility, and provide families of instruments for which our bounds are tight, as well as families of indecomposable compatible instruments. Finally, in Sec. IV, we summarize our results and discuss future directions.

II. PRELIMINARIES

Every quantum system has an associated Hilbert space $\mathcal{H}$ that (although most definitions and statements in this paper generalize to the infinite-dimensional case) we will assume to be finite dimensional. We denote the set of linear operators on the Hilbert space $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and the set of positive semidefinite linear operators on the Hilbert space $\mathcal{H}$ by $\mathcal{L}^+(\mathcal{H})$. We denote the set of quantum states (i.e., density matrices, or positive semidefinite operators with trace 1) on $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$.

A. Quantum measurements

A quantum measurement A with a finite outcome set Ω_A is described by a set of $|\Omega_A|$ positive-semidefinite operators on $\mathcal{H}$, i.e., $A = \{A(x)\}_{x\in\Omega_A}$ such that $\sum_{x\in\Omega_A} A(x) = \mathbb{I}_{\mathcal{H}}$, where $\mathbb{I}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$ and $|\Omega_A|$ is the cardinality of

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Ω_A [14]. If the measurement A is performed on a quantum state $\rho \in \mathcal{S}(\mathcal{H})$, the probability to obtain the outcome x is given by $\mathrm{tr}[\rho A(x)]$. We denote the set of measurements with outcome set Ω acting on the Hilbert space $\mathcal{H}$ by $\mathbb{M}(\Omega, \mathcal{H})$. A measurement A is said to be a trivial measurement if $A(x) = p_x\mathbb{I}$ for all $x \in \Omega_A$ for some probability distribution p_x. In general, we will use the term positive operator-valued measure (POVM) interchangeably with the term quantum measurement. Furthermore, if $A^2(x) = A(x)$ for all $x \in \Omega_A$, then the measurement is called *projective* or a *projection-valued measure*.

A pair of measurements (A, B) on $\mathcal{H}$ is said to be *compatible* [1] if there exists a *joint measurement* G on $\mathcal{H}$ with outcome set $\Omega_A \times \Omega_B$ such that

$$A(x) = \sum_{y\in\Omega_B} G(x, y) \quad \forall x \in \Omega_A, \tag{1}$$

$$B(y) = \sum_{x\in\Omega_A} G(x, y) \quad \forall y \in \Omega_B. \tag{2}$$

The outcome (x, y) of the measurement G on a given quantum state ρ is distributed according to the joint distribution of the outcome of A and B on the same quantum state. Therefore, implementing G amounts to simultaneously implementing A and B. Measurement pairs that are not compatible are called *incompatible*, and the definition of (in)compatibility naturally generalizes to sets of more than two measurements. Measurement incompatibility has been linked to quantum advantage in various information processing tasks [2–5]. Therefore, measurement incompatibility can be thought of as a resource in these tasks [15].

B. Quantum operations and quantum channels

A quantum operation maps every quantum state on a Hilbert space $\mathcal{H}$ to a subnormalized quantum state on another Hilbert space $\mathcal{K}$. Formally, $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})$ is a completely positive (CP) trace-nonincreasing map such that for any set of states $\{\rho_i \in \mathcal{S}(\mathcal{H})\}$ and any probability distribution $\{p_i\}$ we have that $\Phi(\sum_i p_i\rho_i) = \sum_i p_i\Phi_i(\rho_i)$. It is known that every quantum operation has a unique linear extension $\tilde{\Phi} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, i.e., a unique linear map $\tilde{\Phi}$ such that $\Phi = \tilde{\Phi}|_{\mathcal{S}(\mathcal{H})}$, where $\tilde{\Phi}|_{\mathcal{S}(\mathcal{H})}$ is the restriction of $\tilde{\Phi}$ to $\mathcal{S}(\mathcal{H})$ [14].

Quantum operations that map quantum states to normalized quantum states (and hence can be thought of as state transformations) are called *quantum channels*. Formally, a quantum channel $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ is a trace-preserving (TP) quantum operation, also called a CPTP map. We denote the set of channels with input space $\mathcal{H}$ and output space $\mathcal{K}$ by $\mathbb{Ch}(\mathcal{H}, \mathcal{K})$.

Two quantum channels $\Lambda_1 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1)$ and $\Lambda_2 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_2)$ are said to be *compatible* [1] if there exists a *joint channel* $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ such that for all $\rho \in \mathcal{S}(\mathcal{H})$ we have

$$\Lambda_1(\rho) = \mathrm{tr}_{\mathcal{K}_2}\Lambda(\rho), \tag{3}$$

$$\Lambda_2(\rho) = \mathrm{tr}_{\mathcal{K}_1}\Lambda(\rho). \tag{4}$$

The output of the joint channel Λ is a joint (composite) state of the outputs of Λ_1 and Λ_2 on a tensor-product Hilbert space. Channels that are not compatible are called *incompatible*, and the definition of (in)compatibility naturally generalizes to sets of more than two channels. The incompatibility of quantum channels has been studied in various works [16–18], and it has been shown to provide an advantage in quantum state discrimination [19].

C. Quantum instruments

A quantum instrument $\mathcal{I}$ is described by a set of quantum operations, in particular, $\mathcal{I} = \{\Phi_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}_{x\in\Omega_\mathcal{I}}$ such that $\Phi = \sum_{x\in\Omega_\mathcal{I}} \Phi_x$ is a quantum channel [14,20–22]. We denote the set of quantum instruments with outcome set $\Omega_\mathcal{I}$, input space $\mathcal{H}$, and output space $\mathcal{K}$ by $\mathbb{In}(\Omega_\mathcal{I}, \mathcal{H}, \mathcal{K})$. We recall that every quantum instrument induces a unique quantum measurement (a POVM) A [22], that is, a unique set of positive-semidefinite operators $A(x)$ on $\mathcal{H}$ such that $\mathrm{tr}[\rho A(x)] = \mathrm{tr}[\Phi_x(\rho)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and all $x \in \Omega_\mathcal{I}$. It is, in fact, straightforward to see that these operators are defined by $A(x) = \Phi_x^*(\mathbb{I})$, where $\Phi_x^* : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ is the dual of Φ_x defined via $\mathrm{tr}[\Phi_x(\rho)M] = \mathrm{tr}[\rho\Phi_x^*(M)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and for all $M \in \mathcal{L}(\mathcal{K})$. The probability of observing outcome x upon measuring A on ρ is given by $\mathrm{tr}[\Phi_x(\rho)]$, and the (un-normalized) postmeasurement state is given by $\Phi_x(\rho)$. An instrument inducing the POVM A is sometimes called an A-compatible instrument. While every instrument induces a unique POVM, for a given POVM many different instruments implementing it exist. Therefore, one can think of instruments as various ways of implementing a POVM, with various postmeasurement states.

Like in the case of quantum measurements and quantum channels, one might call two quantum instruments *compatible* if they can be performed simultaneously. The following definition of *parallel compatibility* of quantum instruments is one way to make this intuition rigorous [12].

Definition 1. Parallel compatibility. A pair of quantum instruments $\mathcal{I}_1 = \{\Phi_x^1 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}_{x\in\Omega_{\mathcal{I}_1}}$ and $\mathcal{I}_2 = \{\Phi_y^2 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}_{y\in\Omega_{\mathcal{I}_2}}$ is (parallel) compatible if there exists a joint quantum instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}_{x\in\Omega_{\mathcal{I}_1}, y\in\Omega_{\mathcal{I}_2}}$ such that

$$\sum_y \mathrm{tr}_{\mathcal{K}_2}\Phi_{xy} = \Phi_x^1 \quad \forall x, \tag{5}$$

$$\sum_x \mathrm{tr}_{\mathcal{K}_1}\Phi_{xy} = \Phi_y^2 \quad \forall y. \tag{6}$$

In other words, the joint instrument $\mathcal{I}$ simultaneously reproduces both the classical (x and y) and the quantum $[\Phi_x^1(\rho)$ and $\Phi_y^2(\rho)]$ outputs of $\mathcal{I}_1$ and $\mathcal{I}_2$. Reference [12] argued that parallel compatibility is, in some cases, a favorable definition of instrument compatibility compared to the "traditional" definition [9–11], which requires the existence of a joint instrument such that $\sum_y \Phi_{xy} = \Phi_x^1$ and $\sum_x \Phi_{xy} = \Phi_y^2$ for all x and y. Therefore, in the following by "compatible instruments" we mean parallel compatible instruments, unless otherwise specified. It is also clear that this definition naturally generalizes to more than two instruments.

We now recall the notion of *postprocessing* of quantum instruments, which will be useful for characterizing compatible instruments [23].

Definition 2. Consider the instruments $\mathcal{I}_1 = \{\Phi^1_x\}_{x\in\Omega_{\mathcal{I}_1}} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{I}_2 = \{\Phi^2_y\}_{y\in\Omega_{\mathcal{I}_2}} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$. Then, $\mathcal{I}_2$ is said to be the postprocessing of $\mathcal{I}_2$ if there exists a set of instruments $\{\mathcal{R}^x = \{R^x_y\}_{y\in\Omega_{\mathcal{I}_2}} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{K}_1, \mathcal{K}_2)\}_{x\in\Omega_{\mathcal{I}_1}}$ such that

$$\Phi^2_y = \sum_{x\in\Omega_{\mathcal{I}_1}} R^x_y \circ \Phi^1_x \quad \forall\, y \in \Omega_{\mathcal{I}_2}. \tag{7}$$

We denote this relation as $\mathcal{I}_2 \lesssim \mathcal{I}_1$, reflecting the fact that this relation induces a preorder on the set of instruments.

The postprocessing preorder induces an equivalence relation: $\mathcal{I}_1 \sim \mathcal{I}_2$ if both $\mathcal{I}_1 \lesssim \mathcal{I}_2$ and $\mathcal{I}_2 \lesssim \mathcal{I}_1$ hold. Postprocessing defines a partial order on the equivalence classes [23].

Two simple classes of instruments, defined directly through their induced POVMs, are so-called measure-and-prepare instruments [14]. One can intuitively think of them as performing a measurement and then preparing a state depending on the outcome of the measurement.

Definition 3. For a measurement $A \in \mathbb{M}(\Omega_A, \mathcal{H})$, a special measure-and-prepare instrument is defined as $\mathfrak{J}_A = \{J^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}_{x\in\Omega_A}$, with $\dim(\mathcal{K}) = |\Omega_A|$, where

$$J^A_x(\rho) := \mathrm{tr}[\rho A(x)]\, |x\rangle\langle x| \tag{8}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and $x \in \Omega_A$ and $\{|x\rangle\}_{x\in\Omega_A}$ is an orthonormal basis on $\mathcal{K}$. The corresponding (special measure-and-prepare) quantum channel Γ_A acting on a quantum state $\rho \in \mathcal{S}(\mathcal{H})$ is given by $\Gamma_A(\rho) = \sum_x J^A_x(\rho) = \sum_x \mathrm{tr}[\rho A(x)]\, |x\rangle\langle x|$.

Definition 4. For a measurement $A \in \mathbb{M}(\Omega_A, \mathcal{H})$, a measure-and-prepare instrument is defined as $\mathfrak{J}'_A = \{J'^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}_{x\in\Omega_A}$ such that for all $\rho \in \mathcal{S}(\mathcal{H})$ and $x \in \Omega_A$

$$J'^A_x(\rho) := \mathrm{tr}[\rho A(x)]\rho'_x, \tag{9}$$

where $\rho'_x \in \mathcal{S}(\mathcal{K})$ are some fixed quantum states. The corresponding (measure-and-prepare) quantum channel Γ'_A acting on a quantum state $\rho \in \mathcal{S}(\mathcal{H})$ is given by $\Gamma'_A(\rho) = \sum_x J'^A_x(\rho) = \sum_x \mathrm{tr}[\rho A(x)]\rho'_x$.

Last, measure-and-prepare instruments corresponding to trivial measurements are called *trash-and-prepare* instruments since their outcome does not depend on the input state.

Definition 5. A measure-and-prepare instrument $\mathfrak{J}'_T$ is called as a trash-and-prepare instrument if T is a trivial measurement.

D. Incompatibility robustnesses

A common way of quantifying the incompatibility of a set of quantum devices is to ask how much noise needs to be added to the set in order to make it compatible [1]. While this quantifier depends on what is considered to be "noise" [24], in what follows we adopt the definition usually referred to as *generalized robustness*.

For a pair of measurements $\{A_1(x)\}_{x\in\Omega_{A_1}}$ and $\{A_2(y)\}_{y\in\Omega_{A_2}}$ on $\mathcal{H}$, the generalized incompatibility robustness is defined as [25]

$$\begin{aligned} R_M(A_1, A_2) = \min\ & r \\ \text{such that}\ & \frac{A_1(x) + r\tilde{A}_1(x)}{1+r} = \sum_y G(x,y) \quad \forall\, x, \\ & \frac{A_2(y) + r\tilde{A}_2(y)}{1+r} = \sum_x G(x,y) \quad \forall\, y, \\ & \tilde{A}_1(x), \tilde{A}_2(y) \geqslant 0 \quad \forall\, x, y, \\ & \sum_x \tilde{A}_1(x) = \sum_y \tilde{A}_2(y) = \mathbb{I}, \\ & G(x,y) \geqslant 0 \quad \forall\, x, y. \end{aligned} \tag{10}$$

In other words, the generalized incompatibility robustness quantifies how much noise $(\tilde{A}_1, \tilde{A}_2)$ can be added to the pair (A_1, A_2) before the noisy POVM pair $(A_1 + r\tilde{A}_1)/(1+r)$ and $(A_2 + r\tilde{A}_2)/(1+r)$ becomes compatible (with some joint measurement G). The term "generalized" refers to the fact that $\tilde{A}_1$ and $\tilde{A}_2$ can be chosen to be arbitrary POVMs on $\mathcal{H}$ with outcome numbers matching with those of A_1 and A_2, respectively. Note that the incompatibility robustness can be cast as an efficiently computable semidefinite program (SDP) [26] and that this definition naturally generalizes to more than two measurements.

The incompatibility robustness of two quantum channels $\Phi_1 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)$ and $\Phi_2 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)$ can be defined similarly [25]:

$$\begin{aligned} R_C(\Phi_1, \Phi_2) = \min\ & r \\ \text{such that}\ & \frac{\Phi_1 + r\tilde{\Phi}_1}{1+r} = \mathrm{tr}_{\mathcal{K}_2}\Psi, \\ & \frac{\Phi_2 + r\tilde{\Phi}_2}{1+r} = \mathrm{tr}_{\mathcal{K}_1}\Psi, \\ & \Psi \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2), \\ & \tilde{\Phi}_i \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_i) \quad i = 1, 2. \end{aligned} \tag{11}$$

Note that via the Choi representation, the incompatibility robustness of quantum channels can be cast as an SDP as well and that this definition also naturally generalizes to larger sets of channels.

III. CHARACTERIZING THE COMPATIBILITY OF QUANTUM INSTRUMENTS

In this section, we initiate the quantitative characterization of parallel incompatibility of instruments, along lines similar to the characterization of measurement and channel incompatibility through incompatibility robustness. First, we define the generalized incompatibility robustness of quantum instruments in a way analogous to measurements and channels.

Definition 6. The generalized incompatibility robustness of two quantum instruments $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}_{x\in\Omega_1}$

and $\mathcal{I}_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}_{y\in\Omega_2}$ is given by

$$\begin{aligned} R_I(\mathcal{I}_1, \mathcal{I}_2) = \min\ & r \\ \text{such that } & \frac{\Phi^1_x + r\tilde{\Phi}^1_x}{1+r} = \sum_y \mathrm{tr}_{\mathcal{K}_2}\Psi_{xy}, \\ & \frac{\Phi^2_y + r\tilde{\Phi}^2_y}{1+r} = \sum_x \mathrm{tr}_{\mathcal{K}_1}\Psi_{xy}, \\ & \mathcal{I} = \{\Psi_{xy}\} \in \mathbb{In}(\Omega_1\times\Omega_2, \mathcal{H}, \mathcal{K}_1\otimes\mathcal{K}_2), \\ & \tilde{\mathcal{I}}_1 = \{\tilde{\Phi}^1_x\} \in \mathbb{In}(\Omega_1, \mathcal{H}, \mathcal{K}_1), \\ & \tilde{\mathcal{I}}_2 = \{\tilde{\Phi}^2_y\} \in \mathbb{In}(\Omega_2, \mathcal{H}, \mathcal{K}_2). \end{aligned} \tag{12}$$

Similar to the generalized incompatibility robustness of measurements and channels, the generalized incompatibility robustness of instruments can be cast as an SDP (through the Choi states of the individual CP maps), and the definition naturally generalizes to more than two instruments.

In the remainder of this section, we prove some basic characteristics of the incompatibility robustness of quantum instruments. Note that in the following, by "incompatibility robustness" we mean generalized incompatibility robustness, unless otherwise specified.

A. Bounds on the incompatibility robustness

First, we show that the incompatibility robustness of two quantum instruments is lower bounded by the incompatibility robustness of their induced POVMs and their induced channels.

Theorem 1. Consider two instruments $\mathcal{I}_1 = \{\Phi^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{I}_2 = \{\Phi^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$. Let us denote the POVM induced by $\mathcal{I}_i$ by A_i. Furthermore, consider the induced channels, $\Phi^1 = \sum_{x\in\Omega_{\mathcal{I}_1}} \Phi^2_x$ and $\Phi^2 = \sum_{y\in\Omega_{\mathcal{I}_2}} \Phi^2_y$. Then we have that $R_I(\mathcal{I}_1, \mathcal{I}_2) \geqslant \max\{R_M(A_1, A_2), R_C(\Phi_1, \Phi_2)\}$.

Proof. Suppose that $R_I(\mathcal{I}_1, \mathcal{I}_2) = r$. Then, following from the definition of the incompatibility robustness of instruments in Definition 6, there exist quantum instruments $\mathcal{I} = \{\Psi_{xy}\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}\times\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_1\otimes\mathcal{K}_2)$, $\tilde{\mathcal{I}}_1 = \{\tilde{\Phi}^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$, and $\tilde{\mathcal{I}}_2 = \{\tilde{\Phi}^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$ such that

$$\frac{\Phi^1_x + r\tilde{\Phi}^1_x}{1+r} = \sum_y \mathrm{tr}_{\mathcal{K}_2}\Psi_{xy} \quad \forall x \tag{13}$$

$$\frac{\Phi^2_y + r\tilde{\Phi}^2_y}{1+r} = \sum_x \mathrm{tr}_{\mathcal{K}_1}\Psi_{xy} \quad \forall y. \tag{14}$$

Summing up the first equation over x and the second one over y, we see that there exist quantum channels $\tilde{\Phi}^1$ and $\tilde{\Phi}^2$ such that $(\Phi^1 + r\tilde{\Phi}^1)/(1+r)$ and $(\Phi^2 + t\tilde{\Phi}^2)/(1+r)$ are compatible. Therefore, by the definition of the incompatibility robustness of channels in Eq. (11), we have $r \geqslant R_C(\Phi_1, \Phi_2)$.

In a similar way, one can show that $r \geqslant R_M(A_1, A_2)$. Let us take the dual of the operations in Eqs. (13) and (14), and notice that $(\mathrm{tr}_{\mathcal{K}_2}\Psi_{xy})^*(X) = \Psi^*_{xy}(X\otimes\mathbb{I}_{\mathcal{K}_2})$ for all $X\in\mathcal{L}(\mathcal{K}_1)$ and, similarly, $(\mathrm{tr}_{\mathcal{K}_1}\Psi_{xy})^*(X) = \Psi^*_{xy}(\mathbb{I}_{\mathcal{K}_1}\otimes X)$ for all $X\in\mathcal{L}(\mathcal{K}_2)$. We then apply the dual of Eq. (13) to $\mathbb{I}_{\mathcal{K}_1}$ and the dual of Eq. (14) to $\mathbb{I}_{\mathcal{K}_2}$, which yields

$$\frac{A_1(x) + r\tilde{A}_1(x)}{1+r} = \sum_y G(x,y), \tag{15}$$

$$\frac{A_2(y) + r\tilde{A}_2(y)}{1+r} = \sum_x G(x,y), \tag{16}$$

where we have defined the POVMs $\tilde{A}_1(x) = (\tilde{\Phi}^1_x)^*(\mathbb{I}_{\mathcal{K}_1})$, $\tilde{A}_2(y) = (\tilde{\Phi}^2)_y)^*(\mathbb{I}_{\mathcal{K}_2})$, and $G(x,y) = \Psi^*_{xy}(\mathbb{I}_{\mathcal{K}_1}\otimes\mathbb{I}_{\mathcal{K}_2})$. From the definition of the incompatibility robustness of measurements in Eq. (10) it is then clear that $r \geqslant R_M(A_1, A_2)$.

In summary, we have $R_I(\{\mathcal{I}_1, \mathcal{I}_2\}) \geqslant \max\{R_M(A_1, A_2), R_C(\Phi_1, \Phi_2)\}$, which is exactly the statement of the theorem. ∎

The following theorem establishes a universal upper bound on the incompatibility robustness of two instruments.

Theorem 2. Consider two arbitrary instruments $\mathcal{I}_1 = \{\Phi^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{I}_2 = \{\Phi^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$. Then $R_I(\mathcal{I}_1, \mathcal{I}_2) \leqslant 1$.

Proof. Consider two quantum states $\eta_1 = |\eta_1\rangle\langle\eta_1| \in \mathcal{S}(\mathcal{K}_1)$ and $\eta_2 = |\eta_2\rangle\langle\eta_2| \in \mathcal{S}(\mathcal{K}_2)$. Let us define a quantum instrument $\mathcal{I} = \{\Psi_{xy}\}$, where $x\in\Omega_{\mathcal{I}_1}$ and $y\in\Omega_{\mathcal{I}_2}$, given by its action

$$\Psi_{xy}(\rho) = \frac{1}{2}\left(\Phi^1_x(\rho)\otimes\frac{1}{|\Omega_{\mathcal{I}_2}|}\eta_2 + \frac{1}{|\Omega_{\mathcal{I}_1}|}\eta_1\otimes\Phi^2_y(\rho)\right), \tag{17}$$

which clearly defines a quantum instrument.

Now, consider the quantum instruments $\tilde{\mathcal{I}}^1 = \{\tilde{\Phi}^1_x = \sum_{y\in\Omega_{\mathcal{I}_2}} \mathrm{tr}_{\mathcal{K}_2}\Psi_{xy} = \frac{\Phi^1_x + \Phi^{\eta_1}_x}{2}\}_{x\in\Omega_{\mathcal{I}_1}}$ and $\tilde{\mathcal{I}}^2 = \{\tilde{\Phi}^2_y = \sum_{x\in\Omega_{\mathcal{I}_1}} \mathrm{tr}_{\mathcal{K}_1}\Psi_{xy} = \frac{\Phi^2_y + \Phi^{\eta_2}_y}{2}\}_{y\in\Omega_{\mathcal{I}_2}}$, where $\mathfrak{T}_{\eta_1} = \{\Phi^{\eta_1}_x\}_{x\in\Omega_{\mathcal{I}_1}}$ and $\mathfrak{T}_{\eta_2} = \{\Phi^{\eta_2}_y\}_{y\in\Omega_{\mathcal{I}_2}}$ are the trash-and-prepare instruments defined via $\Phi^{\eta_1}_x(\rho) = \frac{\mathrm{tr}(\rho)}{|\Omega_{\mathcal{I}_1}|}\eta_1$ and $\Phi^{\eta_2}_y(\rho) = \frac{\mathrm{tr}(\rho)}{|\Omega_{\mathcal{I}_2}|}\eta_2$ for all $\rho\in\mathcal{S}(\mathcal{H})$, $x\in\Omega_{\mathcal{I}_1}$, and $y\in\Omega_{\mathcal{I}_2}$. By definition, $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are compatible via the joint instrument $\mathcal{I}$. Furthermore, they are noisy versions of the instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ with noise parameter $r = 1$ and noise instruments $\mathfrak{T}_{\eta_1}$ and $\mathfrak{T}_{\eta_2}$. Therefore, from the definition of incompatibility robustness in Definition 6, we find that $R_I(\mathcal{I}_1, \mathcal{I}_2) \leqslant 1$. ∎

B. Postprocessing as a free operation in a resource theory

Resource theories [27] are a systematic way of quantifying the resourcefulness of certain quantum systems or devices. In order to define a resource theory, one needs to define the resource itself, which is usually done by defining the states and devices without the resource (called *free states or devices*). The next step is to define *free operations*, that is, operations that map states to states (devices to devices) in a way that they do not create a resourceful state or device from one without a resource. There is certainly some freedom in defining these operations, and they are often taken to be operations with physical meaning (such as local operations and classical communication in the resource theory of entanglement [28]). Once the resource and free operations are defined, *resource monotones* can be defined to quantify the resourcefulness of states and devices. A resource monotone assigns a real number to states or devices such that the value does not increase under any free operation, and therefore, in this sense it is a faithful representation of the resourcefulness. We refer the interested reader to Ref. [27] for a review on quantum resource theories.

As an example, incompatible measurements are a necessary resource in certain quantum information processing tasks. One might therefore define a resource theory of incompatibility of measurements (defining sets of compatible measurements as free sets of measurements), taking pre- and postprocessing as free operations and the incompatibility robustness as a resource monotone [24] (note that a resource theory of incompatibility of measurements with a larger set of free operations was already developed in Ref. [15]).

Since incompatible measurements and channels provide quantum advantage, it is natural to assume that incompatible instruments are also advantageous in some information processing tasks. As such, characterizing the incompatibility of quantum instruments via resource-theoretic tools is an adequate method for studying the potential quantum advantage of incompatible instruments. In the next theorem we prove that postprocessing preserves the compatibility of quantum instruments (maps free devices to free devices). That is, one may view postprocessing as a free operation in a potential resource theory of incompatibility.

Theorem 3. Consider two arbitrary instruments $\mathcal{I}_1 = \{\Phi^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{I}_2 = \{\Phi^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$ and another two instruments $\tilde{\mathcal{I}}_1 = \{\tilde{\Phi}^1_{\tilde{x}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_1}, \mathcal{H}, \tilde{\mathcal{K}}_1)$ and $\tilde{\mathcal{I}}_2 = \{\tilde{\Phi}^2_{\tilde{y}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_2}, \mathcal{H}, \tilde{\mathcal{K}}_2)$ such that $\tilde{\mathcal{I}}_1 \lesssim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \lesssim \mathcal{I}_2$. Then, if $\mathcal{I}_1$ and $\mathcal{I}_2$ are compatible, then $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are also compatible.

Proof. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are compatible, there exists an instrument $\mathcal{I} = \{\Phi_{xy}\}$ such that Eqs. (5) and (6) hold. Also, since $\tilde{\mathcal{I}}_1 \lesssim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \lesssim \mathcal{I}_2$, there exist two sets of instruments $\{\mathcal{R}^{1,x} = \{R^{1,x}_{\tilde{x}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_1}, \mathcal{K}_1, \tilde{\mathcal{K}}_1)\}_{x\in\Omega_{\mathcal{I}_1}}$ and $\{\mathcal{R}^{2,y} = \{R^{2,y}_{\tilde{y}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_2}, \mathcal{K}_2, \tilde{\mathcal{K}}_2)\}_{y\in\Omega_{\mathcal{I}_2}}$ such that

$$\tilde{\Phi}^1_{\tilde{x}} = \sum_{x\in\Omega_{\mathcal{I}_1}} R^{1,x}_{\tilde{x}} \circ \Phi^1_x \quad \forall\, \tilde{x} \in \Omega_{\tilde{\mathcal{I}}_1}, \tag{18}$$

$$\tilde{\Phi}^2_{\tilde{y}} = \sum_{y\in\Omega_{\mathcal{I}_2}} R^{2,y}_{\tilde{y}} \circ \Phi^2_y \quad \forall\, \tilde{y} \in \Omega_{\tilde{\mathcal{I}}_2}. \tag{19}$$

Let us define the instrument $\tilde{\mathcal{I}} = \{\tilde{\Phi}_{\tilde{x}\tilde{y}}\}_{\tilde{x}\in\Omega_{\tilde{\mathcal{I}}_1}, \tilde{y}\in\Omega_{\tilde{\mathcal{I}}_2}}$, given by its action

$$\tilde{\Phi}_{\tilde{x}\tilde{y}} = \sum_{x\in\Omega_{\mathcal{I}_1}} \sum_{y\in\Omega_{\mathcal{I}_2}} \left(R^{1,x}_{\tilde{x}} \otimes R^{2,y}_{\tilde{y}}\right) \circ \Phi_{xy}. \tag{20}$$

Since $\sum_{\tilde{x}\in\Omega_{\tilde{\mathcal{I}}_1}} R^{1,x}_{\tilde{x}} =: R^{1,x}$ and $\sum_{\tilde{y}\in\Omega_{\tilde{\mathcal{I}}_2}} R^{2,y}_{\tilde{y}} =: R^{2,y}$ are both quantum channels and therefore trace preserving, we have

$$\sum_{\tilde{y}\in\Omega_{\tilde{\mathcal{I}}_2}} \mathrm{tr}_{\tilde{\mathcal{K}}_2}\left(R^{1,x}_{\tilde{x}} \otimes R^{2,y}_{\tilde{y}}\right)(\cdot) = R^{1,x}_{\tilde{x}} \circ \mathrm{tr}_{\mathcal{K}_2}(\cdot), \tag{21}$$

$$\sum_{\tilde{x}\in\Omega_{\tilde{\mathcal{I}}_1}} \mathrm{tr}_{\tilde{\mathcal{K}}_1}\left(R^{1,x}_{\tilde{x}} \otimes R^{2,y}_{\tilde{y}}\right)(\cdot) = R^{2,y}_{\tilde{y}} \circ \mathrm{tr}_{\mathcal{K}_1}(\cdot). \tag{22}$$

Using these relations, it is easy to verify that

$$\sum_{\tilde{y}\in\Omega_{\tilde{\mathcal{I}}_2}} \mathrm{tr}_{\tilde{\mathcal{K}}_2} \tilde{\Phi}_{\tilde{x}\tilde{y}} = \tilde{\Phi}^1_{\tilde{x}}, \tag{23}$$

$$\sum_{\tilde{x}\in\Omega_{\tilde{\mathcal{I}}_1}} \mathrm{tr}_{\tilde{\mathcal{K}}_1} \tilde{\Phi}_{\tilde{x}\tilde{y}} = \tilde{\Phi}^2_{\tilde{y}}. \tag{24}$$

Therefore, $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are compatible. ■

As a consequence of the above theorem, given a pair of compatible instruments, every pair of instruments that is postprocessing equivalent to the given pair is also compatible.

Corollary 1. Consider two arbitrary instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ and another two instruments $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ such that $\tilde{\mathcal{I}}_1 \sim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \sim \mathcal{I}_2$. Then, $\mathcal{I}_1$ and $\mathcal{I}_2$ are compatible if and only if $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are compatible.

In the following, we will show that trash-and-prepare instruments are compatible with any other instrument with the same input space. As a preparation, we prove the following lemma, noting that a quantum channel can be considered a one-outcome instrument.

Lemma 1. The identity channel $\mathrm{id}_{\mathcal{H}} \in \mathbb{Ch}(\mathcal{H}, \mathcal{H})$, viewed as a one-outcome instrument $\mathrm{id}_{\mathcal{H}} \in \mathbb{In}(\{1\}, \mathcal{H}, \mathcal{H})$, is compatible with any trash-and-prepare instrument $\mathfrak{J}'_T \in \mathbb{In}(\Omega_{\mathfrak{J}'_T}, \mathcal{H}, \mathcal{K})$.

Proof. Let us denote the fixed outcome of the identity channel $\mathrm{id}_{\mathcal{H}}$ by 1. Consider the trivial measurement $T = \{T(x) = p_x\mathbb{I}\}$, where $\{p_x\}$ is a probability distribution. Then, $\mathfrak{J}'_T$ has the form $\mathfrak{J}'_T = \{\Phi_x\}$, where, for any quantum state $\rho \in \mathcal{S}(\mathcal{H})$, $\Phi_x(\rho) = p_x\eta_x$, where $\eta_x \in \mathcal{S}(\mathcal{K})$ for all $x \in \Omega_{\mathfrak{J}'_T}$. Now consider the quantum instrument $\mathcal{I} = \{\Phi'_{x1}\}$, where, for any quantum state $\rho \in \mathcal{S}(\mathcal{H})$,

$$\Phi'_{x1}(\rho) = p_x\eta_x \otimes \rho \tag{25}$$

for all $x \in \Omega_{\mathfrak{J}'_T}$. Then, from Eqs. (5) and (6), we find that $\mathcal{I}$ is a joint instrument for $\mathrm{id}_{\mathcal{H}}$ and $\mathfrak{J}'_T$, and hence, they are compatible. ■

From the fact that any instrument can be postprocessed from the identity channel [23], Theorem 3 and Lemma 1, the following proposition follows directly.

Proposition 1. Any instrument $\mathcal{I} \in \mathbb{In}(\Omega_{\mathcal{I}}, \mathcal{H}, \mathcal{K})$ is compatible with any trash-and-prepare instrument $\mathfrak{J}'_T \in \mathbb{In}(\Omega_{\mathfrak{J}'_T}, \mathcal{H}, \mathcal{K}')$.

The following theorem establishes an even stronger relation between incompatibility robustness and the postprocessing preorder: the incompatibility robustness of quantum instruments is monotonic under postprocessing. That is, one may view incompatibility robustness as a resource monotone in a potential resource theory of incompatibility.

Theorem 4. Consider two arbitrary instruments $\mathcal{I}_1 = \{\Phi^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{I}_2 = \{\Phi^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$ and another two instruments $\tilde{\mathcal{I}}_1 = \{\tilde{\Phi}^1_{\tilde{x}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_1}, \mathcal{H}, \tilde{\mathcal{K}}_1)$ and $\tilde{\mathcal{I}}_2 = \{\tilde{\Phi}^2_{\tilde{y}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_2}, \mathcal{H}, \tilde{\mathcal{K}}_2)$ such that $\tilde{\mathcal{I}}_1 \lesssim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \lesssim \mathcal{I}_2$. Then, $R_I(\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2) \leqslant R_I(\mathcal{I}_1, \mathcal{I}_2)$.

Proof. Since $\tilde{\mathcal{I}}_1 \lesssim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \lesssim \mathcal{I}_2$, there exist two sets of instruments $\{\mathcal{R}^{1,x} = \{R^{1,x}_{\tilde{x}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_1}, \mathcal{K}_1, \tilde{\mathcal{K}}_1)\}_{x\in\Omega_{\mathcal{I}_1}}$ and $\{\mathcal{R}^{2,y} = \{R^{2,y}_{\tilde{y}}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_2}, \mathcal{K}_2, \tilde{\mathcal{K}}_2)\}_{y\in\Omega_{\mathcal{I}_2}}$ such that Eqs. (18) and (19) hold.

Suppose that $R_I(\mathcal{I}_1, \mathcal{I}_2) = r$. Then there exist two instruments $\{J^1_x\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\{J^2_y\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$ such that the instruments $\mathcal{J}_1 := \{\frac{\Phi^1_x + rJ^1_x}{1+r}\} \in \mathbb{In}(\Omega_{\mathcal{I}_1}, \mathcal{H}, \mathcal{K}_1)$ and $\mathcal{J}_2 := \{\frac{\Phi^2_x + rJ^2_x}{1+r}\} \in \mathbb{In}(\Omega_{\mathcal{I}_2}, \mathcal{H}, \mathcal{K}_2)$ are compatible. Consider the postprocessing of $\mathcal{J}_1$ and $\mathcal{J}_2$, given by $\tilde{\mathcal{J}}_1 =$

$\{\frac{\tilde{\Phi}^1_{\tilde{x}}+r\tilde{J}^1_{\tilde{x}}}{1+r}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_1}, \mathcal{H}, \tilde{\mathcal{K}}_1)$ and $\tilde{\mathcal{J}}_2 = \{\frac{\tilde{\Phi}^2_{\tilde{y}}+r\tilde{J}^2_{\tilde{y}}}{1+r}\} \in \mathbb{In}(\Omega_{\tilde{\mathcal{I}}_2}, \mathcal{H}, \tilde{\mathcal{K}}_2)$, where

$$\tilde{J}^1_{\tilde{x}} = \sum_{x\in\Omega_{\mathcal{I}_1}} R^{1,x}_{\tilde{x}} \circ J^1_x, \tag{26}$$

$$\tilde{J}^2_{\tilde{y}} = \sum_{y\in\Omega_{\mathcal{I}_2}} R^{2,y}_{\tilde{y}} \circ J^2_y. \tag{27}$$

Then from Theorem 3, we find that $\tilde{\mathcal{J}}_1$ and $\tilde{\mathcal{J}}_2$ are compatible since $\tilde{\mathcal{J}}_1 \lesssim \mathcal{J}_1$ and $\tilde{\mathcal{J}}_2 \lesssim \mathcal{J}_2$. Thus, from the definition of the incompatibility robustness of instruments, we find that $R_I(\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2) \leqslant r$. ■

As a consequence of the above theorem, every postprocessing equivalent pair of instruments has the same incompatibility robustness.

Corollary 2. Consider two arbitrary instruments $\mathcal{I}_1$ and $\mathcal{I}_2$ and another two instruments $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ such that $\tilde{\mathcal{I}}_1 \sim \mathcal{I}_1$ and $\tilde{\mathcal{I}}_2 \sim \mathcal{I}_2$. Then, $R_I(\mathcal{I}_1, \mathcal{I}_2) = R_I(\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2)$.

One might also consider other operations that preserve compatibility in a potential resource theory of incompatibility. Here we show that the operation that was shown to be a free operation for traditional compatibility [29] is not a free operation for parallel compatibility.

Reference [29] uses the terminology *programmable instrument devices* (PIDs) for a set of instruments that induce the same quantum channel. Furthermore, a PID is called *simple* if the instruments that comprise the PID are traditionally compatible. Hence, since traditionally compatible instruments induce the same quantum channel [12], simple PIDs correspond exactly to traditionally compatible instruments. The authors of Ref. [29] introduced a family of maps that map PIDs to PIDs and, moreover, map simple PIDs to simple PIDs, and they called this family *free PID supermaps*. For the sake of clarity, we recall their definition here.

Definition 7. A free PID supermap maps a set of instruments $\{\mathcal{I}_i = \{\Phi^i_x\}_{x\in\Omega_{\mathcal{I}_i}} \in \mathbb{In}(\Omega_{\mathcal{I}_i}, \mathcal{H}, \mathcal{K})\}_{i=1}^{n_\mathcal{I}}$ to another set of instruments $\{\mathcal{J}_j = \{\Psi^j_y\}_{y\in\Omega_{\mathcal{J}_j}} \in \mathbb{In}(\Omega_{\mathcal{J}_j}, \tilde{\mathcal{H}}, \tilde{\mathcal{K}})\}_{j=1}^{n_\mathcal{J}}$. The map is given by

$$\Psi^j_y = \sum_{s,i,x} q_{y|x,s}\Gamma^j_{is} \circ \left(\Phi^i_x \otimes \mathrm{id}\right) \circ \Lambda, \tag{28}$$

where $\Lambda : \mathcal{S}(\tilde{\mathcal{H}}) \to \mathcal{S}(\mathcal{H}\otimes\mathcal{Q})$ is a quantum channel from $\tilde{\mathcal{H}}$ to $\mathcal{H}\otimes\mathcal{Q}$, where $\mathcal{Q}$ is an arbitrary auxiliary Hilbert space. Here id is the identity channel on $\mathcal{Q}$, and s is an element of the set $[n_S] := \{1, \ldots, n_S\}$, where n_S is some integer. $\{\Gamma_j := \{\Gamma^j_{is}\}_{i,s} \in \mathbb{In}([n_S]\otimes[n_\mathcal{I}], \mathcal{K}\otimes\mathcal{Q}, \tilde{\mathcal{K}})\}_{j=1}^{n_\mathcal{J}}$ is an arbitrary simple PID (set of traditionally compatible instruments), and $q_{y|x,s}$ is a probability distribution for every x and s.

The following theorem was proven in Theorem 1 of Ref. [29].

Theorem 5. Free PID supermaps map every set of traditionally compatible instruments to a set of traditionally compatible instruments. Moreover, every set of traditionally compatible instruments can be mapped to any other set of traditionally compatible instruments using a free PID supermap.

The second statement is relatively easy to see by noticing that the definition of a free PID supermap includes applying a set of traditionally compatible instruments. Hence, one can just throw away the output of the original set and apply the new set.

The above theorem essentially states that free PID supermaps are free operations for traditional compatibility. Below, we show that this is not the case for parallel compatibility.

Theorem 6. Free PID supermaps are not free operations for parallel compatibility. In particular, free PID supermaps might map parallel compatible instruments to parallel incompatible instruments.

Proof. Consider the following two instruments:

$$\mathcal{I} = \left\{\Phi_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}) \mid \Phi_x(\rho) = \frac{\mathrm{tr}(\rho)}{n}\eta\right\}_{x=1}^n, \tag{29}$$

$$\mathcal{J} = \left\{\Psi_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}) \mid \Psi_x(\rho) = \frac{1}{n}\rho\right\}_{x=1}^n, \tag{30}$$

where η is some fixed state on $\mathcal{H}$. It is straightforward to verify that two copies of $\mathcal{I}$ are traditionally compatible via the joint instrument

$$\mathcal{I}_t = \left\{\Phi^t_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}) \mid \Phi^t_{xy}(\rho) = \frac{\mathrm{tr}(\rho)}{n^2}\eta\right\}_{x,y=1}^n. \tag{31}$$

Similarly, two copies of $\mathcal{J}$ are also traditionally compatible via the joint instrument

$$\mathcal{J}_t = \left\{\Psi^t_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}) \mid \Psi^t_{xy}(\rho) = \frac{1}{n^2}\rho\right\}_{x,y=1}^n. \tag{32}$$

Therefore, following Theorem 5, there exists a free PID supermap that maps two copies of $\mathcal{I}$ to two copies of $\mathcal{J}$.

One can also show that two copies of $\mathcal{I}$ are parallel compatible via the joint instrument

$$\mathcal{I}_p = \left\{\Phi^p_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}\otimes\mathcal{H}) \mid \Phi^p_{xy}(\rho) = \frac{\mathrm{tr}(\rho)}{n^2}\eta\otimes\eta\right\}_{x,y=1}^n. \tag{33}$$

On the other hand, two copies of $\mathcal{J}$ are not parallel compatible, which is a direct consequence of the no-cloning theorem or, alternatively, the fact that the identity channel is not compatible with itself [12]. Therefore, the free PID supermap mapping two copies of $\mathcal{I}$ to two copies of $\mathcal{J}$ maps a pair of parallel compatible instruments to a pair of parallel incompatible instruments. As such, free PID supermaps are, in general, not free operations for parallel compatibility. ■

Remark 1. In general, the above theorem is a consequence of Theorem 5, the fact that there exist instruments that are both traditionally compatible and parallel compatible, and the fact that there exist instruments that are traditionally compatible but not parallel compatible [12].

Concluding this section, we have identified a natural class of free operations for parallel compatibility, namely, the class of postprocessing of quantum instruments from Ref. [23]. At the same time, we showed that another class of operations—the free PID supermaps of Ref. [29]—that are natural free operations for traditional compatibility are not free operations for parallel compatibility. It could be an interesting research direction to try to identify more physically motivated classes of free operations for parallel compatibility and to develop a resource theory of instrument incompatibility based on these free operations.

C. Cases for which the bounds are tight

The following theorem establishes that the lower bound on the incompatibility robustness of instruments in terms of the incompatibility robustness of the induced POVMs and channels from Theorem 1 is tight for special measure-and-prepare instruments.

Theorem 7. The lower bound in Theorem 1 is tight for any pair of special measure-and-prepare instruments $\mathfrak{I}_{A_1}$ and $\mathfrak{I}_{A_2}$, where $A_1 \in \mathbb{M}(\Omega_{A_1}, \mathcal{H})$ and $A_2 \in \mathbb{M}(\Omega_{A_2}, \mathcal{H})$.

Proof. Suppose that $R_M(A_1, A_2) = r$. Then there exist $A_1' \in \mathbb{M}(\Omega_{A_1}, \mathcal{H})$ and $A_2' \in \mathbb{M}(\Omega_{A_2}, \mathcal{H})$ such that $\bar{A}_1 = \{\bar{A}_1(x) = \frac{A_1(x)+rA_1'(x)}{1+r}\}_{x\in\Omega_{A_1}}$ and $\bar{A}_2 = \{\bar{A}_2(y) = \frac{A_2(y)+rA_2'(y)}{1+r}\}_{y\in\Omega_{A_2}}$ are compatible. That is, there exists a joint measurement $G \in \mathbb{M}(\Omega_{A_1} \times \Omega_{A_2}, \mathcal{H})$ such that

$$\bar{A}_1(x) = \sum_y G(x, y) \quad \forall\, x \in \Omega_{A_1}, \tag{34}$$

$$\bar{A}_2(y) = \sum_x G(x, y) \quad \forall\, y \in \Omega_{A_2}. \tag{35}$$

Now consider the special measure-and-prepare quantum instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}_{x\in\Omega_{A_1}, y\in\Omega_{A_2}}$ given by

$$\Phi_{xy}(\rho) = \mathrm{tr}[\rho G(x, y)] |x\rangle\langle x| \otimes |y\rangle\langle y|. \tag{36}$$

Then, for all $x \in \Omega_{A_1}$ we have

$$\begin{aligned}\sum_y \mathrm{tr}_{\mathcal{K}_2} \Phi_{xy}(\rho) &= \sum_y \mathrm{tr}[\rho G(x, y)] |x\rangle\langle x| \\ &= \mathrm{tr}[\rho \bar{A}_1(x)] |x\rangle\langle x| \\ &= \mathrm{tr}\left[\rho \frac{A_1(x) + rA_1'(x)}{1 + r}\right] |x\rangle\langle x| \\ &= \frac{\mathrm{tr}[\rho A_1(x)] |x\rangle\langle x| + r\,\mathrm{tr}[\rho A_1'(x)] |x\rangle\langle x|}{1 + r},\end{aligned} \tag{37}$$

and for all $y \in \Omega_{A_2}$ we have

$$\begin{aligned}\sum_x \mathrm{tr}_{\mathcal{K}_1} \Phi_{xy}(\rho) &= \sum_x \mathrm{tr}[\rho G(x, y)] |y\rangle\langle y| \\ &= \mathrm{tr}[\rho \bar{A}_2(y)] |y\rangle\langle y| \\ &= \mathrm{tr}\left[\rho \frac{A_2(y) + rA_2'(y)}{1 + r}\right] |y\rangle\langle y| \\ &= \frac{\mathrm{tr}[\rho A_2(y)] |y\rangle\langle y| + r\,\mathrm{tr}[\rho A_2'(y)] |y\rangle\langle y|}{1 + r}.\end{aligned} \tag{38}$$

Hence, from Definition 3 and Definition 1, we have $r = R_M(A_1, A_2) \geqslant R_I(\mathfrak{I}_{A_1}, \mathfrak{I}_{A_2})$. Since from Theorem 1 we have $R_M(A_1, A_2) \leqslant R_I(\mathfrak{I}_{A_1}, \mathfrak{I}_{A_2})$, we establish that $R_M(A_1, A_2) = R_I(\mathfrak{I}_{A_1}, \mathfrak{I}_{A_2})$. Also, from Theorem 1, we know that $R_I(\mathfrak{I}_{A_1}, \mathfrak{I}_{A_2}) \geqslant R_C(\Gamma_{A_1}, \Gamma_{A_2})$. Therefore, we conclude that $R_M(A_1, A_2) \geqslant R_C(\Gamma_{A_1}, \Gamma_{A_2})$, and in summary, the lower bound in Theorem 1 is tight. ■

The following lemma from Ref. [23] allows us to generalize the previous result to arbitrary measure-and-prepare instruments.

Lemma 2. For any measurement $A \in \mathbb{M}(\Omega_A, \mathcal{H})$, a generic measure-and-prepare instrument $\mathfrak{J}_A'$ is a postprocessing of the special measure-and-prepare instrument $\mathfrak{J}_A$, that is, $\mathfrak{J}_A' \lesssim \mathfrak{J}_A$ ([23], Proposition 10).

Corollary 3. The lower bound in Theorem 1 is tight for any pair of generic measure-and-prepare instruments $\mathfrak{J}_{A_1}'$ and $\mathfrak{J}_{A_2}'$ for any $A_1 \in \mathbb{M}(\Omega_{A_1}, \mathcal{H})$ and $A_2 \in \mathbb{M}(\Omega_{A_2}, \mathcal{H})$.

Proof. From Lemma 2, Theorem 4, and Theorem 7 we have $R_I(\mathfrak{J}_{A_1}', \mathfrak{J}_{A_2}') \leqslant R_I(\mathfrak{J}_{A_1}, \mathfrak{J}_{A_2}) = R_M(A_1, A_2)$, and from Theorem 1, we have $R_I(\mathfrak{J}_{A_1}', \mathfrak{J}_{A_2}') \geqslant R_M(A_1, A_2)$. Therefore, we conclude that $R_I(\mathfrak{J}_{A_1}', \mathfrak{J}_{A_2}') = R_M(A_1, A_2)$. ■

D. Indecomposable instruments

Indecomposable instruments were introduced in Ref. [23]. They play a fundamental role in the postprocessing preorder since (although these instruments are not, in general, maximal in the preorder) every instrument can be obtained from an indecomposable instrument via postprocessing. This property is analogous to rank-1 POVMs in the sense that every POVM can be obtained from a rank-1 POVM via classical postprocessing. For the sake of clarity, we recall the definition and a basic characterization of indecomposable instruments from Ref. [23].

Definition 8. A quantum operation Φ is called indecomposable if $\Phi = \Psi + \Psi'$ for some quantum operations Ψ and Ψ' implies $\Psi = \mu\Phi$ and $\Psi' = \mu'\Phi$ for some $\mu, \mu' > 0$. A quantum instrument is called indecomposable if all of its nonzero operations are indecomposable.

Proposition 2. A quantum operation is indecomposable if and only if it has Kraus rank 1.

To show that every instrument is a postprocessing of an indecomposable instrument, one needs to introduce the concept of a *detailed instrument* [23].

Definition 9. Consider a quantum instrument $\mathcal{I} = \{\Phi_x\}$ with Kraus decomposition $\Phi_x(\rho) = \sum_{i=1}^{n_x} K_{ix}\rho K_{ix}^\dagger$. The detailed instrument of $\mathcal{I}$ is defined as $\hat{\mathcal{I}} = \{\hat{\Phi}_{ix}(\rho) = K_{ix}\rho K_{ix}^\dagger\}$.

It is straightforward to see that for every instrument $\mathcal{I}$ we have $\mathcal{I} \lesssim \hat{\mathcal{I}}$ [23], and clearly, every detailed instrument is indecomposable. In Ref. [23], the authors provided a class of instruments for which the converse is true as well, and hence, the instrument is postprocessing equivalent to its detailed instrument.

Proposition 3. An instrument $\mathcal{I} = \{\Phi_x\} \in \mathbb{In}(\Omega, \mathcal{H}, \mathcal{K})$ with Kraus decomposition $\Phi_x(\rho) = \sum_{i=1}^{n_x} K_{ix}\rho K_{ix}^\dagger$ is postprocessing equivalent to its detailed instrument if $K_{ix}^\dagger K_{jx} = 0$ for all $i \neq j$ and $x \in \Omega$.

Since indecomposable instruments are fundamental in the postprocessing preorder, it is useful to characterize compatible sets of indecomposable instruments. Here we provide a family of compatible pairs of indecomposable instruments constructed from compatible measurements.

Theorem 8. For every pair of compatible measurements $A \in \mathbb{M}(\Omega_A, \mathcal{H})$ and $B \in \mathbb{M}(\Omega_B, \mathcal{H})$ there exists a pair of compatible indecomposable instruments.

Proof. We base our construction on the pair of compatible instruments for a given pair (A, B) of compatible measurements from Ref. [12]. While these instruments are, in general, not indecomposable, we show using Proposition 3 that they are postprocessing equivalent to their detailed instruments. Therefore, following Corollary 1, the detailed instruments (which are indecomposable) are also compatible.

Let us recall the construction from Ref. [12]. Consider two POVMs, $\{A(x)\}_{x\in\Omega_A}$ and $\{B(y)\}_{y\in\Omega_B}$, on $\mathcal{H}$ such that they are compatible, that is, there exists a POVM $\{G(x,y)\}_{x\in\Omega_A,y\in\Omega_B}$ on $\mathcal{H}$ such that

$$A(x) = \sum_y G(x,y) \quad \forall\, x, \tag{39}$$

$$B(y) = \sum_x G(x,y) \quad \forall\, y. \tag{40}$$

Now consider a Naimark dilation of the joint measurement G, that is, a projective measurement $\Pi(x,y)$ on a Hilbert space $\mathcal{H}\otimes\mathcal{K}$ such that

$$\mathrm{tr}[(\rho\otimes|0\rangle\langle 0|)\Pi(x,y)] = \mathrm{tr}[\rho G(x,y)] \quad \forall\, \rho\in\mathcal{S}(\mathcal{H}) \tag{41}$$

for some $|0\rangle\in\mathcal{K}$. Furthermore, consider a rank-1 fine graining of $\Pi(x,y)$, given by

$$\Pi(x,y) = \sum_{z_{xy}\in P(x,y)} |\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}|, \tag{42}$$

where $P(x,y)$ is a set indexing the rank-1 projections that comprise $\Pi(x,y)$. It is clear that $\langle\phi_{z_{xy}}|\phi_{z'_{x'y'}}\rangle = \delta_{x,x'}\delta_{y,y'}\delta_{z_{xy},z'_{x'y'}}$.

Take the A-compatible instrument

$$\begin{aligned}\mathcal{I}_A = \Bigg\{ &\Phi^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{H}\otimes\mathcal{K}) \mid \\ \Phi^A_x(\rho) &= \sum_y \sum_{z_{xy}\in P(x,y)} |\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}|(\rho\otimes|0\rangle\langle 0|)|\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}| \\ &= \sum_y \sum_{z_{xy}\in P(x,y)} K^A_{(y,z_{xy}),x}\,\rho\,(K^A_{(y,z_{xy}),x})^\dagger \Bigg\},\end{aligned} \tag{43}$$

where $K^A_{(y,z_{xy}),x} = |\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}|(\mathbb{I}_\mathcal{H}\otimes|0\rangle)$, and the B-compatible instrument

$$\begin{aligned}\mathcal{I}_B = \Bigg\{ &\Phi^B_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+_{\mathcal{H}\otimes\mathcal{K}} \mid \\ \Phi^B_y(\rho) &= \sum_x \sum_{z_{xy}\in P(x,y)} |\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}|(\rho\otimes|0\rangle\langle 0|)|\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}| \\ &= \sum_x \sum_{z_{xy}\in P(x,y)} K^B_{(x,z_{xy}),y}\,\rho\,(K^B_{(x,z_{xy}),y})^\dagger \Bigg\},\end{aligned} \tag{44}$$

where $K^B_{(x,z_{xy}),y} = |\phi_{z_{xy}}\rangle\langle\phi_{z_{xy}}|(\mathbb{I}_\mathcal{H}\otimes|0\rangle)$. Reference [12] showed that $\mathcal{I}_A$ and $\mathcal{I}_B$ are compatible.

Using Proposition 3, we show that $\mathcal{I}_A$ and $\mathcal{I}_B$ are postprocessing equivalent to their detailed instruments, $\hat{\mathcal{I}}_A$ and $\hat{\mathcal{I}}_B$, respectively. Indeed, we have

$$\begin{aligned}&\big(K^A_{(y,_{xy}z),x}\big)^\dagger K^A_{(y',z'_{xy'}),x} \\ &\quad = (\mathbb{I}_\mathcal{H}\otimes\langle 0|)\,|\phi_{z_{xy}}\rangle\,\langle\phi_{z_{xy}}|\phi_{z'_{xy'}}\rangle\,\langle\phi_{z'_{xy'}}|\,(\mathbb{I}_\mathcal{H}\otimes|0\rangle) = 0\end{aligned} \tag{45}$$

for all x whenever $(y, z_{xy}) \neq (y', z'_{xy'})$, which implies that $\mathcal{I}_A \sim \hat{\mathcal{I}}_A$ by Proposition 3. Similarly, we can show that $\mathcal{I}_B \sim \hat{\mathcal{I}}_B$. Therefore, following Corollary 1, we find that the indecomposable instruments $\hat{\mathcal{I}}_A$ and $\hat{\mathcal{I}}_B$ are compatible. ■

Remark 2. While for every pair of compatible measurements there exists a pair of compatible instruments that induce the measurements, it is not true that all such instruments are compatible. Indeed, consider the Lüders instrument $\Phi_x(\rho) = \sqrt{A(x)}\rho\sqrt{A(x)}$ of a measurement A. Given two trivial measurements A and B (which are always compatible), the corresponding Lüders instruments are not compatible since they both induce the identity channel [12]. However, following Ref. [12], there exists a pair of instruments compatible with A and B such that these instruments are compatible. Moreover, Ref. [30] proved that every A-compatible instrument is a postprocessing of the Lüders instrument of A. Thus, it would be interesting to characterize, starting from the (potentially incompatible) Lüders instruments of a pair of compatible measurements A and B, at what point in the postprocessing preorder we obtain a compatible pair of A- and B-compatible instruments. Another consequence of the fact that every A-compatible instrument is a postprocessing of the Lüders instrument of A is that every instrument that is compatible with the Lüders instrument of A is compatible with all A-compatible instruments.

IV. CONCLUSION

In this paper, we have studied the quantification and characterization of the recently introduced concept of parallel compatibility of quantum instruments, which is a natural generalization of more established notions of measurement and channel compatibility. We have defined the incompatibility robustness of quantum instruments to quantify the incompatibility of instruments and have proved universal bounds on this quantity. Furthermore, we have proved that postprocessing of quantum instruments is a free operation in a potential resource theory of quantum instruments. We also have proved that free PID supermaps, which are free operations for traditional compatibility, are not free operations for parallel compatibility. In addition, we have provided families of instruments for which our bounds are tight. Finally, we have proved that for every pair of compatible measurements there exists a pair of compatible indecomposable instruments.

Given this well-defined way of quantifying the incompatibility of quantum instruments, it is a natural further research direction to see how the incompatibility robustness relates to quantitative measures of the performance of instruments in quantum information processing tasks. Since quantum instruments appear in the description of sequential tasks, such as sequential prepare-and-measure protocols [31,32] and sequential Bell scenarios [33–35], it would be an interesting research direction to investigate the quantitative relationship between the incompatibility robustness of instruments and the success probability or Bell violation in these information processing tasks. Furthermore, while we provided some examples and counterexamples for free operations of parallel compatibility, a potential further research direction could be to systematically study such free operations and develop a resource theory of instrument incompatibility.

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