


Leggett-Garg violations for continuous-variable systems with Gaussian states

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Macrorealism (MR) is the worldview that certain quantities may take definite values at all times irrespective of past or future measurements and may be experimentally falsified via the Leggett-Garg (LG) inequalities. We put this worldview to the test for systems described by a continuous variable x by seeking LG violations for measurements of a dichotomic variable $Q = \text{sign}(x)$, in the case of Gaussian initial states in a quantum harmonic oscillator. Extending our earlier analysis [C. Mawby and J. J. Halliwell, *Phys. Rev. A* **105**, 022221 (2022)], we find analytic expressions for the temporal correlators. An exploration of parameter space reveals significant regimes in which the two-time LG inequalities are violated, and likewise at three and four times. To obtain a physical picture of the LG violations, we exploit the continuous nature of the underlying position variable and analyze the relevant quantum-mechanical currents, Bohm trajectories, and Wigner function. We also show that larger violations are possible using the Wigner LG inequalities. Further, we extend the analysis to LG tests using coherent state projectors, thermal coherent states, and squeezed states.

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I. INTRODUCTION

The motion of a pendulum has been used by clockmakers and hypnotists alike for centuries, with its regular left, right, left motion. Scaled down enough, the quantum mechanical description becomes necessary, which hints that nonclassical states without a definite left-right property underlie the motion. The existence of these types of states form one of the pillars of many quantum technologies, and hence verification of their existence and an understanding of their persistence to macroscopic scales is of great interest.

The Leggett-Garg (LG) inequalities [1–4] were introduced to provide a quantitative test capable of demonstrating the failure of the precise worldview known as macrorealism (MR). MR is defined as the conjunction of three realist tenets: that a system resides in one observable state only, for all instants of time, which may be measured without influencing future dynamics of the system, and that measurements respect causality. The violation of these inequalities indicates a failure of MR, and hence the presence of nonclassical behavior.

The LG inequalities are typically established for a dichotomic observable Q , which may take value $s_i = \pm 1$, measured in a series of experiments at single times and at pairs of times. This yields a data set consisting of single time averages $\langle Q_i \rangle$, where $Q_i = Q(t_i)$, and the temporal correlators

C_{ij} defined by

$$C_{ij} = \langle Q_i Q_j \rangle = \sum_{i,j} s_i s_j p(s_i, s_j), \quad (1.1)$$

where $p(s_i, s_j)$ is the two time measurement probability, giving the likelihood of measuring s_i and s_j at times t_i, t_j . The temporal correlators must be measured in a noninvasive manner, in keeping with the definition of MR, which is typically done using ideal negative measurements [5–7] (but other approaches exist [8–12]).

For the commonly studied three-time case, the data set consists of three correlators C_{12}, C_{23}, C_{13} and three single-time averages $\langle Q_i \rangle, i = 1, 2, 3$. These quantities then form six puzzle pieces that, should the system obey the assumptions of MR, must be the moments of an underlying joint probability distribution $p(s_1, s_2, s_3)$. In the case where this endeavor is possible, the correlators and averages will satisfy the three-time LG inequalities (LG3):

$$L_1 = 1 + C_{12} + C_{23} + C_{13} \geq 0, \quad (1.2)$$

$$L_2 = 1 - C_{12} - C_{23} + C_{13} \geq 0, \quad (1.3)$$

$$L_3 = 1 + C_{12} - C_{23} - C_{13} \geq 0, \quad (1.4)$$

$$L_4 = 1 - C_{12} + C_{23} - C_{13} \geq 0. \quad (1.5)$$

They will also satisfy a set of 12 two-time LG inequalities (LG2), four of which are of the form

$$1 + s_1 \langle Q_1 \rangle + s_2 \langle Q_2 \rangle + s_1 s_2 C_{12} \geq 0 \quad (1.6)$$

where $s_1, s_2 = \pm 1$, with two more sets of four for the other two-time pairs. This set of 16 inequalities are necessary and sufficient conditions for MR [13–19]. If any one of them is violated, MR fails. From an experimental point of view, the

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LG2 inequalities constitute the simplest place to look first, since only one correlator needs to be measured.

For measurements at four times, there are six correlators, but the natural data set is a cycle of four $C_{12}, C_{23}, C_{34}, C_{14}$, along with the four single-time averages. The LG4 inequalities take the form

$$-2 \leq C_{12} + C_{23} + C_{34} - C_{14} \leq 2, \quad (1.7)$$

together with the six more inequalities permuting the location of the minus sign. The necessary and sufficient conditions for MR at four times consist of these eight LG4 inequalities, together with the set of 16 LG2s for the four two-time pairs [13–16].

This general framework has been put to the test experimentally in many types of systems. See, for example, Refs. [5–7,9,20–22] and also the useful review in Ref. [4]. Most of these experiments are on systems that are essentially microscopic but some come close to macroscopicity [23]. Note also the useful critique of the LG approach in Ref. [24].

In the present paper we investigate LG tests for the quantum harmonic oscillator (QHO). The LG framework is readily adapted to this physical situation using a single dichotomic variable $Q = \text{sgn}(x)$, where x is the particle position, measured at different times. We build on our earlier work [25], which explored LG violations in the QHO for initial states consisting of harmonic oscillator eigenstates and superpositions thereof, finding close to maximal violations in some cases. Here we focus on the case of an initial coherent state and closely related states. Such an initial state is intriguing in this context since it is often regarded as essentially classical, being a phase-space localized state evolving along a classical trajectory. Hence, any LG violations arising from this state constitute particularly striking examples of nonclassical behavior.

LG tests for the QHO with an initial coherent state were first explored by Bose *et al.* [26], who proposed an experiment to measure the temporal correlators and carried out calculations of LG4 violations. A subsequent paper [27] explored both LG2 violations and also violations of the no-signaling in time conditions [28–30]. Our work in part parallels Ref. [27].

The first main aim of this paper is to undertake a thorough analysis of LG2, LG3, and LG4 inequalities for an initial coherent state. This is carried out in Sec. II, where we set out the formalism and, drawing on our earlier work, calculate the temporal correlators [25]. We carry out a detailed parameter search and find the largest LG violations possible for a coherent state.

A second aim is to explore the physical origins of the LG violations. So, in Sec. III we examine the difference between quantum-mechanical currents and their classical counterparts for initial coherent states projected onto the positive or negative x axis. In this section we also provide a second approach to calculating correlators in the small-time limit, which is in fact valid for general states. We also calculate the Bohmian trajectories, to give a further physical portrait of what underlies the observed LG violations. Finally, we examine the measurement process in the Wigner representation, noting that the initially positive Wigner function of the coherent state acquires negativity as a result of the projective measurement process.

Modifications to the above framework are considered in Sec. IV. We investigate what violations are possible using the Wigner LG inequalities [31,32]. We briefly discuss other types of measurements beyond the simple projective position measurements used so far and also consider LG2 violations with projections onto coherent states. We also determine how the LG violations may be modified for squeezed states or thermal states. We summarize in Sec. V. We relegate to a series of appendices the grisly details of the calculations involved in this analysis.

II. CALCULATION OF CORRELATORS AND LG VIOLATIONS

A. Conventions and strategy

For most of this paper, we will work with coherent states of the harmonic potential, which can of course be thought of as the ground state of the QHO, shifted in phase space. The intricacy of calculating temporal correlators within QM stems from the complexity in the time evolution of a post-measurement state. By considering a comoving frame for the postmeasurement state, we develop a time-evolution result which explicitly separates the quantum behavior from the classical trajectories.

We will work with systems defined exactly (or approximately) by the harmonic oscillator Hamiltonian,

$$\hat{H} = \frac{\hat{p}_{\text{phys}}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_{\text{phys}}^2, \quad (2.1)$$

with physical position and momenta x_{phys} and p_{phys} . In calculations we use the standard dimensionless variables $x\sqrt{\hbar/(m\omega)} = x_{\text{phys}}$ and $p\sqrt{\hbar m\omega} = p_{\text{phys}}$. We denote energy eigenstates $|n\rangle$, writing $\psi_n(x)$ in the position basis, with corresponding energies $E_n = \hbar\omega(n + \frac{1}{2}) = \varepsilon_n\hbar\omega$.

We write coherent states as $|\alpha\rangle$, where its eigenvalue α relates to rescaled variables as

$$\langle\hat{x}(t)\rangle = \sqrt{2}\text{Re}\alpha(t), \quad (2.2)$$

$$\langle\hat{p}(t)\rangle = \sqrt{2}\text{Im}\alpha(t). \quad (2.3)$$

These are the classical paths underlying the motion of coherent states, and we adopt the shorthand $x_1 = \langle\hat{x}(t_1)\rangle$ and likewise $p_1 = \langle\hat{p}(t_1)\rangle$. A coherent state may be represented in terms of α and an initial phase; however, in this work we will largely represent them in terms of the initial averages x_0 and p_0 , for clarity of physical understanding. We construct coherent states with the unitary displacement operator $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ operating on the ground state,

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad (2.4)$$

which results in the wave function

$$\psi^\alpha(x, t) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{1}{2}(x - x_t)^2 + i\frac{p_t}{\hbar}x + i\gamma(t)\right]. \quad (2.5)$$

We will calculate the quantities $\langle Q_i \rangle$, C_{ij} appearing in the LG inequalities. A convenient way to proceed is to first note that the combination appearing in the LG2 inequalities is proportional to the quantity

$$q(s_1, s_2) = \frac{1}{4}(1 + s_1\langle\hat{Q}_1\rangle + s_2\langle\hat{Q}_2\rangle + s_1s_2C_{12}). \quad (2.6)$$

Classically, this quantity is nonnegative and is the probability distribution matching the data set with moments of $\langle Q_1 \rangle$, $\langle Q_2 \rangle$, C_{12} . In the quantum-mechanical case, this quantity may be written

$$q(s_1, s_2) = \text{ReTr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho], \quad (2.7)$$

where $s = \pm 1$, and P_s are projection operators corresponding to the measurement made, with

$$P_s = \frac{1}{2}(1 + s\hat{Q}). \quad (2.8)$$

Since Eq. (2.7) can be negative (up to a maximum of $-\frac{1}{8}$ the Lüders bound), it is referred to as a quasi-probability (QP) [13,33]. Purely from a calculational point of view it is a convenient object to work with, as we found in Ref. [25], since it is proportional to the LG2 inequalities in the quantum case, and since the correlators are easily read from its moment expansion, so we make use of it here.

The QP may be rewritten in the form given in Ref. [33],

$$q(s_1, s_2) = \frac{1}{8}(1 + s_1\hat{Q}_1 + s_2\hat{Q}_2)^2 - 1, \quad (2.9)$$

which makes explicit that the maximally violating state satisfies the eigenvalue equation

$$(s_1\hat{Q}_1 + s_2\hat{Q}_2)|\psi\rangle = -|\psi\rangle. \quad (2.10)$$

We have not been able to find the maximally violating state, but Eq. (2.10) suggests it is probably discontinuous at $x = 0$. This in turn suggests that we will not be able to get close to maximal violations with the simple Gaussian states explored here.

B. Calculation of the correlators

We now calculate the temporal correlators for the case $\hat{Q} = \text{sgn}(\hat{x})$. The QP Eq. (2.7) is given by

$$q(s_1, s_2) = \text{Re} \langle \alpha | e^{\frac{iH\tau}{\hbar}} \theta(s_2\hat{x}) e^{-\frac{iH\tau}{\hbar}} \theta(s_1\hat{x}) e^{-\frac{iH\tau}{\hbar}} | \alpha \rangle. \quad (2.11)$$

By considering the displacement operator as acting on the measurements instead of the state, the quasi-probability is shown in Appendix A to be

$$q(s_1, s_2) = \text{Re} e^{\frac{i\omega\tau}{2}} \langle 0 | \theta[s_2(\hat{x} + x_2)] e^{-iH\tau} \theta[s_1(\hat{x} + x_1)] | 0 \rangle, \quad (2.12)$$

which reveals that the quasi-probability for coherent states can be understood as the quasi-probability for the pure ground state, with measurement profiles translated according to the classical paths. This shows that any LG test on any coherent state may be directly mapped to an LG test on the ground state of the QHO, with translated measurements.

Using the surprising result that $\int_a^b \psi_n(x)\psi_m(x)dx$ has an exact and general solution, where $\psi_n(x)$ are energy eigenstates [25,34], we are able to calculate the temporal correlators as an infinite sum,

$$C_{12} = \text{erf}(x_1)\text{erf}(x_2) + 4 \sum_{n=1}^{\infty} \cos(n\omega\tau) J_{0n}(x_1, \infty) J_{0n}(x_2, \infty), \quad (2.13)$$

where

$$J_{0n}(x, \infty) = \langle 0 | \theta(\hat{x} - x) | n \rangle. \quad (2.14)$$

TABLE I. Tabulation of parameter space results.

Inequality	Largest violation	Percent of Lüders bound	Location (x_0 , p_0)
LG2	-0.113	22%	(0.550, 1.925)
LG3	-0.141	28%	(0.859, 3.317)
LG4	2.216	26%	(0.929, 3.666)

The J_{0n} terms are given in terms of the $\psi_n(x)$ and its derivative. The details of this calculation are given in Appendix A. The infinite sum may be evaluated approximately using numerical methods, by summing up to a finite n . This calculation matches the analytically calculated special case of $x_0 = 0$ given in Ref. [35].

C. LG violations

The freely choosable parameters are the initial parameters of the coherent state x_0, p_0 and the time interval between measurements τ . Where there are more than two measurements, we use equal time spacing. In Fig. 1(a), we plot a parameter space exploration of violations for the LG2 inequalities, where for a given x_0, p_0 we have numerically searched for the largest violation for that state. The LG3 and LG4 inequalities have a similar distribution, but with progressive broadening. Figures for the LG3 and LG4 inequalities are included in Appendix B.

The parameters leading to the largest violation and the magnitude of those violations are reported in Table I. In Figs. 1(b) and 2, we plot the temporal behavior of the LG2, LG3, and LG4 violations for the states in Table I leading to the largest violations.

III. PHYSICAL MECHANISMS OF VIOLATION

Given the LG violations exhibited in the previous section, a natural question to ask concerns the underlying physical effects producing the nonclassical behavior responsible for the violations. Since $Q(t) = \text{sgn}[x(t)]$, a classical picture of the system would involve a set of trajectories $x(t)$ and probabilities for those trajectories. It is then natural to look at the parallel structures in quantum theory and compare with the classical analogs. We therefore look at the quantum-mechanical currents associated with the LG inequalities, which correspond to the time evolution of certain probabilities, and also to the Bohm trajectories associated with those currents, in terms of which the probability flow in space-time is easily seen. What we will see is that the departures from classicality are essentially the ‘‘diffraction in time’’ effect first investigated by Moshinsky, who considered the time evolution of an initial plane wave in one dimension restricted to $x < 0$ [36,37]. The key mathematical object is the Moshinsky function

$$M(x, p, t) = \langle x | e^{-iHt} \theta(\hat{x}) | p \rangle \quad (3.1)$$

for an initial momentum state $|p\rangle$, which we will see below appears in the calculation of the quasi-probability.

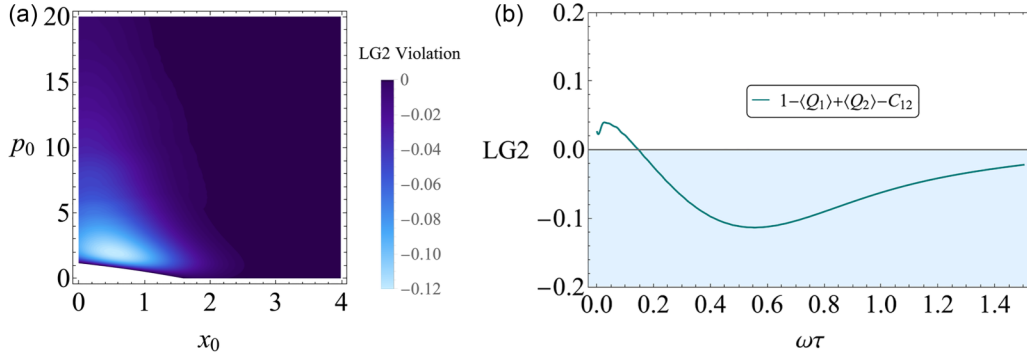


FIG. 1. (a) Parameter space exploration, showing the largest LG2 violation for a given coherent state. (b) Temporal behavior of the LG2s for a state leading to the largest violation of -0.113 .

A. Analysis with currents

As we saw in Sec. II C, the quasi-probability component $q(-, +)$ exhibits a healthy degree of negativity. We start by writing it as

$$q(-, +) = \int_0^\tau dt \frac{dq}{dt}. \quad (3.2)$$

It is then simple to relate $\frac{dq}{dt}$ to a set of quantum mechanical currents, which can be calculated analytically. Overall negativity of the QP can then be spotted by the nonclassicality or negativity of certain combinations of currents. With details in Appendix C 2, we are able to write the quasi-probability as the following combinations of currents at the origin:

$$q(-, +) = \int_{t_1}^{t_2} dt J_-(t) + \frac{1}{2} \int_{t_1}^{t_2} dt [\mathbb{J}_-(t) - J_-(t) + \mathbb{J}_+(t) - J_+(t)], \quad (3.3)$$

where $J_\pm(t)$ is the current following a measurement of $s_1 = \pm$, and $\mathbb{J}_\pm(t)$ are the classical analogs of this. The chopped currents contain the complexity of the influence of the earlier measurement, and are hence quite complicated and given in Appendix C 3.

Note that the first time integral is simply the sequential measurement probability $p_{12}(-, +)$, which is nonnegative. The negativity of the quasi-probability therefore arises as a result of the difference between the classical and quantum chopped currents.

The classical and quantum chopped currents and the current combination appearing in Eq. (C17) are all plotted in Fig. 3 for the initial state giving the LG2 violation described in Sec. II C. The departures from classicality are clearly seen and are consistent with a broadening of the momentum distribution produced by the measurement. Note also that the quantum chopped currents diverge initially due to the sharpness of the measurement.

Most importantly, we see that the combination of currents appearing in the quasi-probability Eq. (C17) will clearly produce an overall negativity when integrated over time, thereby confirming the LG2 violation shown in Fig. 1. Furthermore, we have integrated Eq. (C17) numerically and find an exact agreement with the calculation of Sec. II, Eq. (A14), thereby providing an independent check of this result.

It is also convenient to explore the currents in the small-time limit. This is done in Appendix D and gives a clear analytic picture of the departures from classicality. Since these expressions are valid for any initial state, they could provide a useful starting point in the search for other initial states giving LG2 violations larger than the somewhat modest violations found here.

B. Bohm trajectories

To give a visual demonstration of how the measurements influence motion in a way that leads to LG2 violations, we now calculate and plot the de Broglie–Bohm trajectories [38–40].

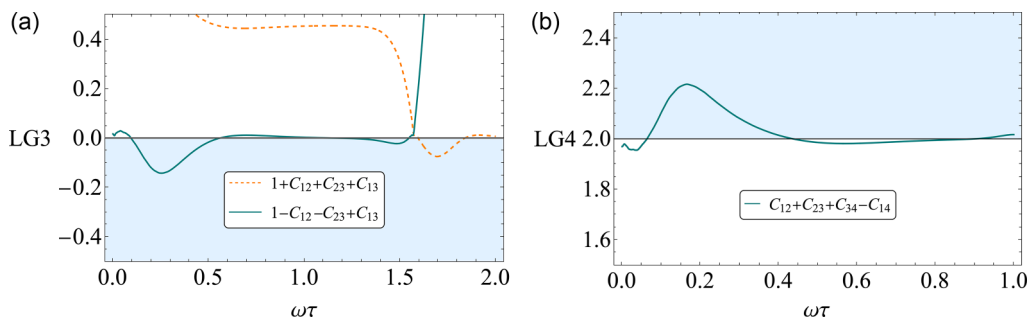


FIG. 2. Temporal behavior of (a) LG3s for a coherent state with the largest violation of -0.141 and (b) LG4s with a coherent state leading to the largest violation of 2.216.

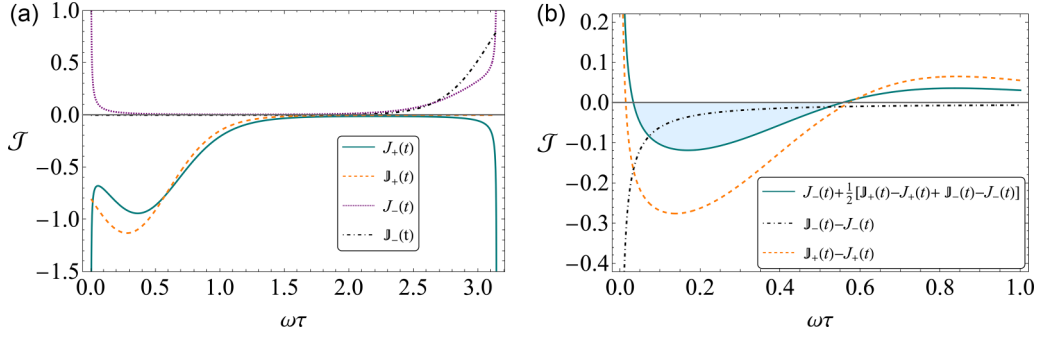


FIG. 3. (a) Postmeasurement quantum currents $J_{\pm}(t)$ and their classical analogs $\mathbb{J}_{\pm}(t)$, normalized to $\mathcal{J} = \frac{J}{\omega}$, for the coherent state with $x_0 = 0.55$, $p_0 = -1.925$, and (b) their combination appearing in the time derivative of $q(-, +)$, Eq. (3.3).

As noted earlier, the Moshinsky function underlies the behavior of the quasi-probability for these measurements, so we initially examine the Bohm trajectories for this scenario. Using Moshinsky's calculation (free-particle dynamics), we calculate the quantum-mechanical current $J_M(x, t)$, which we then use in the guidance equation for Bohm trajectories,

$$\dot{x}(t) = \frac{J_M(x, t)}{|M(x, p, t)|^2}, \quad (3.4)$$

which we proceed to solve numerically.

In Fig. 4, we plot the trajectories for a state initially constrained to the right-hand side of the axis, with a leftward momentum, with classical trajectories shown dotted.

From this we see two distinct phases of deviation from the classical result. Initially the trajectories rapidly exit the right-hand side, with a negative momentum larger than in the classical case, an anti-Zeno effect [41]. After a short while, a Zeno effect [42,43] happens, and the trajectories bend back relative to the classical trajectories, staying in the right-hand side longer than in the classical case. We will see both of these behaviors at play in the case studied in this paper.

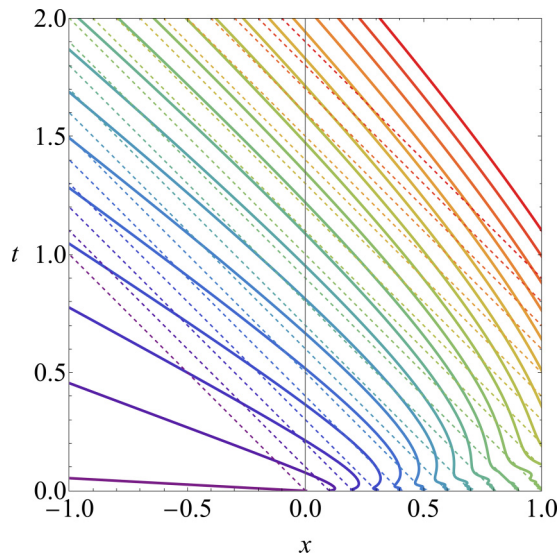


FIG. 4. The Bohm trajectories associated with the Moshinsky function, $\langle x | e^{-iHt} \theta(\hat{x}) | p \rangle$, with $p = -1$. The equivalent classical paths are shown dotted.

Using the expressions for the chopped current Eq. (C37) and chopped wave function Eq. (C33), we can write the guidance equation for the harmonic oscillator case,

$$\dot{x}(t) = \frac{J_{\pm}(x, t)}{|\phi_{\alpha}^{\pm}(x, t)|^2}, \quad (3.5)$$

which we again solve numerically.

In Fig. 5, we show the Bohm trajectories for the state with $x_0 = 0.55$, $p_0 = -1.925$, initially found on the right-hand side of the well. This corresponds to the behavior of the current $J_+(x, t)$ from the previous section. Looking at the zoom of the trajectories in Fig. 5(b), we can observe the same behavior that is seen in the Moshinsky case—initially an anti-Zeno effect, during which the trajectories exit faster than they would classically, followed by a Zeno effect a short while later, where trajectories exit more slowly than in the classical case.

This lines up with the behavior of $\mathbb{J}_+(t) - J_+(t)$ displayed in Fig. 3(b), and is hence a representation on the trajectory level of the source of the LG2 violations.

C. Wigner function approach

Another way to understand the LG2 violations is within the Wigner representation [44–48]. Since coherent states have nonnegative Wigner functions, the source of MR violation lies in the non-Gaussianity of the Wigner transform of the operators describing the measurement procedure. We calculate these transformations in Appendix E, which allows us to write the quasi-probability as a phase-space integral,

$$q(s_1, s_2) = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dp f_{s_1, s_2}(X, p), \quad (3.6)$$

with the phase-space density $f_{s_1, s_2}(X, p)$ given by

$$f_{s_1, s_2}(X, p) = \frac{1}{2} W_{\rho}(X, p) \{1 + \text{Re erf}[i(p - p_0) + s_1 X]\} \theta(s_2 X - \tau), \quad (3.7)$$

where $W_{\rho}(X, p)$ is the Wigner function of the initial state. In Fig. 6, we plot this phase-space density $f_{-+}(X, p)$ for the state $x_0 = 0.55$, $p_0 = -1.925$ and $\omega\tau = 0.55$. To make it clear that this integrates to a negative number, we numerically determine the marginals $f_{-+}(p) = \int_{-\infty}^{\infty} dX f_{-+}(X, p)$ and likewise $f_{-+}(X) = \int_{-\infty}^{\infty} dp f_{-+}(X, p)$, plotting them as

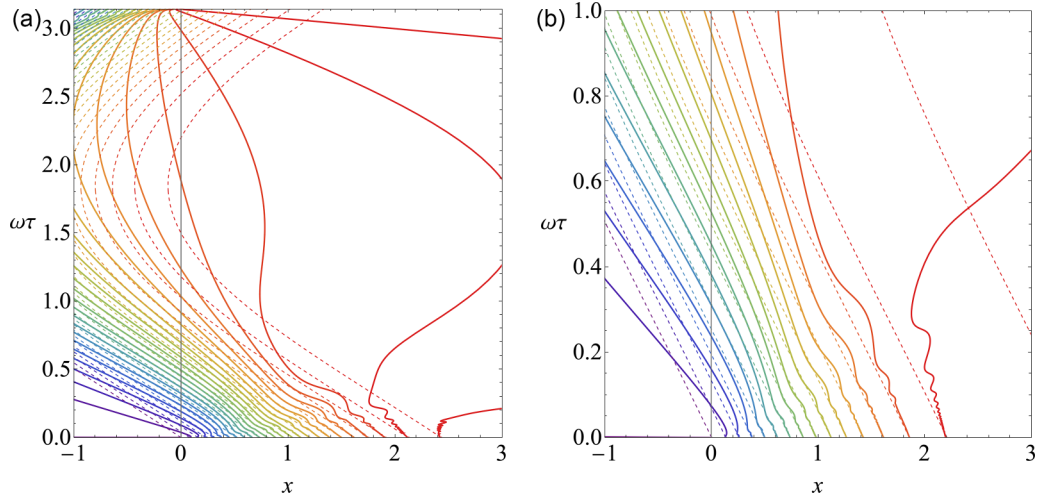


FIG. 5. The Bohm trajectories for the case of the particle being initially found on the right-hand side of the axis. (a) Trajectories are separated such that two adjacent lines bound the evolution of 6.67% of the probability density, and (b) they bound 10% of the probability density.

insets in Fig. 6. It is clear from a simple inspection of these marginals that they will integrate to a negative number.

In Appendix E, we plot the intermediate result Eq. (E8), where it is apparent how the choice of second measurement hones in on the negativity introduced by the initial measurement, ultimately leading to the LG2 violations in Sec. II C.

IV. MODIFIED FRAMEWORKS

In this section, we broadly generalize our analysis, finding larger violations using the Wigner variant of the LG2s, and by analyzing different measurements and extending the analysis to squeezed and thermal coherent states.

A. Achieving larger violations using the Wigner LG2 inequalities

Given the modest size of the LG violations obtained for a single Gaussian, compared to the Lüders bound, it is natural to ask if there are modified situations in which larger violation can be obtained. One way of doing this is to examine slightly different types of inequalities known as the Wigner-Leggett-Garg (Wigner LG) inequalities [31,32]. For the two-time case these arise as follows. The quasi-probability is readily rewritten as

$$\begin{aligned}
 q(s_1, s_2) &= \text{Re}\langle [1 - P_{-s_2}(t_2)][1 - P_{-s_1}(t_1)] \rangle \\
 &= 1 - \langle P_{-s_1}(t_1) \rangle - \langle P_{-s_2}(t_2) \rangle + q(-s_1, -s_2).
 \end{aligned}
 \tag{4.1}$$

However, from a macrorealistic perspective, there is nothing against considering the similar quasi-probability

$$q^W(s_1, s_2) = 1 - \langle P_{-s_1}(t_1) \rangle - \langle P_{-s_2}(t_2) \rangle + p_{12}(-s_1, -s_2),
 \tag{4.2}$$

[where recall p_{12} is the sequential measurement probability Eq. (C13)] since the two are the same classically. The relation $q^W(s_1, s_2) \geq 0$ is a set of Wigner LG2 inequalities (recalling the factor of $\frac{1}{4}$ difference between an LG2 and a QP). It differs from the usual LG2 inequalities by the presence of interference terms, which can be positive or negative, which indicates that violations larger the usual Lüders bound (on the QP) of $-\frac{1}{8}$ might be obtained. The difference between them from an experimental point of view is that the original quasi-probability is measured from three different experiments (determining $\langle Q_1 \rangle$, $\langle Q_2 \rangle$, and C_{12}) but the sequential measurement formula appearing in the Wigner version is measured in a single experiment.

To get a sense of how much larger the maximum violation might be, we take the simple case of one-dimensional projec-

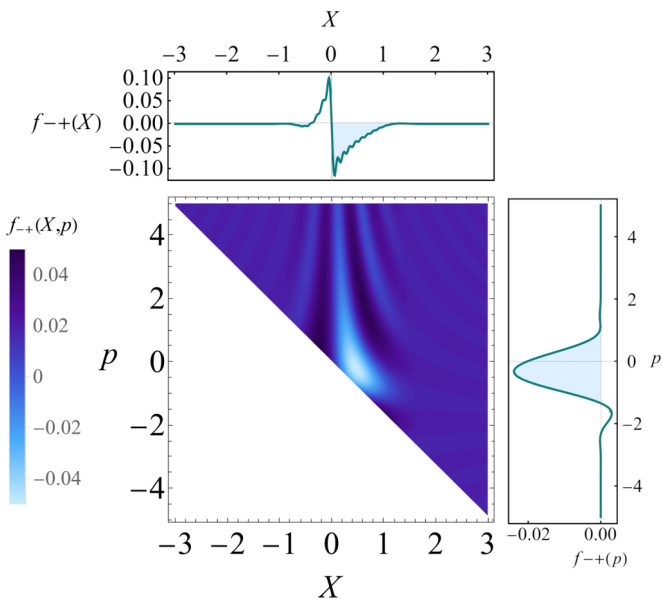


FIG. 6. We plot the phase-space density $f_{-,+}(X, p)$ in the case corresponding to an LG2 violation of -0.113 . Inset are the X and p marginals, shaded over regions of negativity.

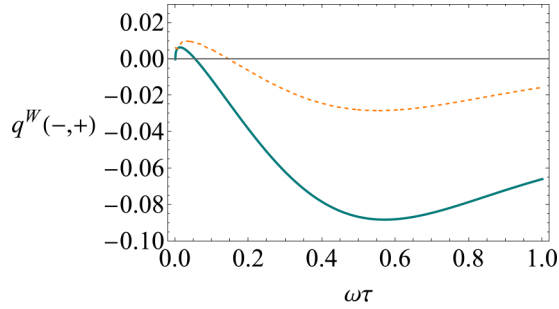


FIG. 7. Plot of $q^W(-, +)$ for the coherent state $x_0 = -0.55$, $p_0 = -1.925$, as well as the standard quasi-probability (dashed), i.e., the LG2 in Fig. 1(b) times a factor of $\frac{1}{4}$.

tors $P_{-s_1}(t_1) = |A\rangle\langle A|$ and $P_{-s_2}(t_2) = |B\rangle\langle B|$ and we find

$$q^W(A, B) = 1 - |\langle\psi|A\rangle|^2 - |\langle\psi|B\rangle|^2 + |\langle\psi|A\rangle|^2|\langle A|B\rangle|^2. \quad (4.3)$$

Simple algebra reveals the lower bound as $-\frac{1}{3}$, which is achieved with $\langle A|B\rangle = 1/\sqrt{3}$ and $|\psi\rangle = (1/\sqrt{6})(|A\rangle + \sqrt{3}|B\rangle)$. (It seems that this is the most negative lower bound for all possible choices of projection but we have not proved this.) This bound is significantly larger than the usual Lüders bound on the QP of $-\frac{1}{8}$.

In Sec. 3 we found that the quasi-probability $q(-, +)$ gives the greatest negativity, so we compare with the corresponding Wigner expression,

$$\begin{aligned} q^W(-, +) &= 1 - \langle P_+(t_1) \rangle - \langle P_-(t_2) \rangle + p_{12}(+, -) \\ &= q(-, +) + [p_{12}(+, -) - q(+, -)], \end{aligned} \quad (4.4)$$

where we have made use of Eq. (4.2) for $q(-, +)$. We first note that Eq. (3.3) may be written $q(-, +) = p_{12}(-, +) + I$, where I denotes the interference terms [i.e., the difference between the classical and quantum currents, the second term in Eq. (3.3)]. The analogous relations for $q(+, -)$ are readily derived and we find $q(+, -) = p_{12}(+, -) - I$. (The difference in sign is expected on general grounds [13]). We thus find

$$q^W(-, +) = p_{12}(-, +) + 2I, \quad (4.5)$$

so the interference term producing the violations is twice as large as the one in $q(-, +)$.

The computation of $p_{12}(-, +)$ can be carried out by integrating the chopped current $J_-(t)$, Eq. (C18) in Appendix C. Using the maximally violating state found in Sec. II C, we find a largest violation of -0.0881 , approximately three times larger than the standard LG2 violation, as well as a larger fraction of the conjectured Wigner LG2 Lüders bound of $-\frac{1}{3}$. This is plotted alongside the standard LG2 in Fig. 7, where the violation is both larger in magnitude and present for a larger range of measurement intervals.

As an aside, we note an interesting aspect of Eq. (4.3), which is that the last term, corresponding to the sequential measurement probability, factors in two parts. This factoring will also hold for more general projections at t_2 as long as the projection at t_1 is one-dimensional. This may have some advantages in terms of meeting the noninvasiveness

requirement on the measurements. It seems plausible that one could find macrorealistic arguments implying the sequential measurement probability factors. Then the first factor is the probability of finding $|A\rangle$ in an initial state $|\psi\rangle$ and the second factor is the probability of finding $|B\rangle$ when the systems is prepared in state $|A\rangle$. These quantities could therefore be obtained in two different experiments with two different preparations with just a single measurement in each, for which there is no issue with invasiveness.

The other obvious way of getting larger violations is to consider von Neumann measurements, which involves making finer-grained measurements than the simple dichotomic ones used here and then coarse graining the probability to compute the correlators [49–53]. For example, one could make measurements onto three regions of the x axis, $x < 0$, $0 \leq x \leq L$, and $x > L$, at the first time and then coarse grain the two-time probabilities into probabilities for the usual coarse graining $x < 0$ and $x > 0$. This produces extra interference terms which can enhance the violations. For the LG3 inequalities, von Neumann measurements can produce violations up to the algebraic maximum of -2 . For the LG2 inequalities, the enhancement is smaller since there is only one correlator and the LG2 violations can be no more than -1 . This corresponds to $-\frac{1}{4}$ in the quasi-probability, which we see is not as big as the violation of $-\frac{1}{3}$ that can be produced by the Wigner LG2.

B. Coherent state projectors

It is useful to know what else may be possible beyond using $\theta(\hat{x})$ projectors. We note investigations [26] into smoothed $\theta(\hat{x})$ measurements, showing LG violations persist under smoothing of measurements up to the characteristic length scale of the oscillator [25]. Modular variables such as $\cos(k\hat{x})$ have also been investigated, and readily produce significant LG violations [54]. These examples show that the LG violations are not due to the sharpness of projective measurements with $\theta(\hat{x})$.

In this section we will look at tests of macrorealism using coherent state projectors [55–57], which are interesting since they leave the postmeasurement state Gaussian and are easily experimentally realized.

We now consider coherent state projectors, where we have $P_+ = |\beta\rangle\langle\beta|$ and $P_- = \mathbb{1} - |\beta\rangle\langle\beta|$ defining a dichotomic variable in the usual way through Eq. (2.8). The quasi-probability is given by

$$q(+, +) = \text{Re} \langle\alpha|e^{iHt_2}|\beta_2\rangle\langle\beta_2|e^{-iHt_1}|\beta_1\rangle\langle\beta_1|e^{-iHt_1}|\alpha\rangle, \quad (4.6)$$

where we make two simplifying observations. Firstly, all the time evolution may be absorbed into the measurement projectors. Secondly, without loss of generality, we work with $\alpha = 0$, where the change in phase-space location may be absorbed into β_1 and β_2 . It is hence entirely equivalent to analyze

$$q(+, +) = \text{Re} \langle 0|\gamma_1\rangle\langle\gamma_1|\gamma_2\rangle\langle\gamma_2|0\rangle, \quad (4.7)$$

with the relation $\gamma_i = e^{-i\omega t_i}\beta_i - \alpha$. The overlap between two coherent states is given by

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha\beta^*)}, \quad (4.8)$$

and we readily find

$$q(+, +) = \exp(-|\gamma_1|^2 - |\gamma_2|^2) \text{Re} \exp(\gamma_1 \gamma_2^*), \quad (4.9)$$

$$q(+, -) = \exp(-|\gamma_1|^2) [1 - \text{Re} \exp(\gamma_1 \gamma_2^* - |\gamma_2|^2)], \quad (4.10)$$

where $q(-, +)$ is found by a relabeling, and $q(-, -)$ does not lead to any violations. To determine the largest violations, it is useful to note that these quasi-probabilities depend only on the magnitude of γ_1 and γ_2 , and the phase difference between them.

In $q(+, +)$, $|\gamma_1|$ and $|\gamma_2|$ appear in the same way, so we set them to be equal, and find a largest violation of -0.0133 at $\gamma_1 = 1.55$, $\gamma_2 = 1.55e^{-1.047i}$, which is about 10% of the maximal violation.

For $q(-, +)$, since the violation is aided by the negative sign on $\text{Re} \exp(\gamma_1 \gamma_2^* - |\gamma_2|^2)$, it is easy to see the largest violation will occur when both γ_1 and γ_2 are purely real. We readily find that the largest violation is approximately -0.1054 with $\gamma_2 = \frac{1}{2}\gamma_1 = 0.536$, which is about 84% of the maximum.

Violations meeting the Lüders bound may be achieved if a superposition state is chosen which satisfies Eq. (2.10). The superposition state $|\psi\rangle = -|\beta_1\rangle - |\beta_2\rangle$ is properly normalized and gives a maximal violation for $q(+, +)$ if the coherent states are chosen so that $\langle\beta_1|\beta_2\rangle = -\frac{1}{2}$. Similarly for $q(+, -)$, the state $|\psi\rangle = |\beta_1\rangle - \sqrt{3}|\beta_2\rangle$ leads to a maximal violation if we choose $\langle\beta_1|\beta_2\rangle = \frac{\sqrt{3}}{2}$.

C. Squeezed states

The squeezed coherent state may be written [58]

$$|\alpha, \zeta\rangle = D(\alpha)S(\zeta)|0\rangle. \quad (4.11)$$

While the squeezing operator $S(\zeta)$ does not commute with the displacement operator $D(\alpha)$, there is a simple braiding relation, allowing us to write

$$|\alpha, \zeta\rangle = S(\zeta)D(\beta)|0\rangle = S(\zeta)|\beta\rangle, \quad (4.12)$$

with β depending on both α and ζ , with the quasi-probability given by

$$q(+, +) = \text{Re} \langle\psi|\theta(\hat{x})\theta[\hat{x}(t)]|\psi\rangle, \quad (4.13)$$

for $|\psi\rangle$ given by a squeezed coherent state. We can consider moving the $S(\zeta)$ in $|\psi\rangle$ onto each $\theta(\hat{x})$ function, resulting in $S^\dagger(\zeta)\theta(\hat{x})S(\zeta)$ twice. Since the squeezing operator has the action of a canonical transform, taking \hat{x} and \hat{p} into a linear combination of themselves, we have that

$$S^\dagger(\zeta)\theta(\hat{x})\theta[\hat{x}(t)]S(\zeta) = \theta(a\hat{x} + b\hat{p})\theta(c\hat{x} + d), \quad (4.14)$$

for some a, b, c, d that may depend on t . We now note that $a\hat{x} + b\hat{p}$ may be written as $\lambda[\hat{x} \cos(t') + \hat{p} \sin(t')]$ for some $\lambda > 0$ and some t' , and since the theta function is invariant under scaling, we see that

$$S^\dagger(\zeta)\theta(\hat{x})\theta[\hat{x}(t)]S(\zeta) = \theta[\hat{x}(t_1')]\theta[\hat{x}(t_2')]. \quad (4.15)$$

This means the QP for a squeezed coherent state is equal to the QP for some other coherent state β , with different measurement times t_1', t_2' . Hence, the operation of squeezing will

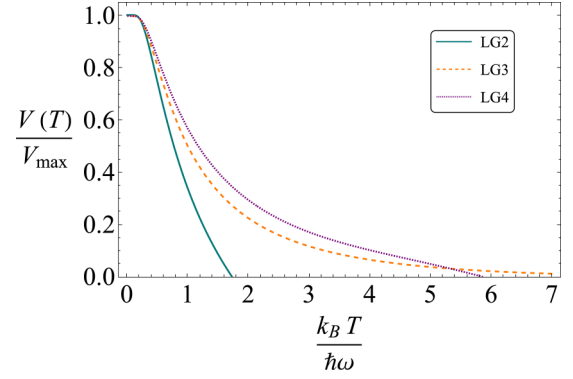


FIG. 8. The LG2, LG3, and LG4 violations are plotted as a fraction of their largest violation, for a state and time which realizes V_{\max} , with varying temperature.

not increase the largest possible violation reported in Sec. II C, although for certain states with suboptimal violation, squeezing can increase the amount of violation.

D. Thermal states

The thermal coherent state at a temperature T is given by

$$\rho_{\text{th}}(\alpha, T) = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} |n, \alpha\rangle\langle n, \alpha|, \quad (4.16)$$

where k_B is the Boltzmann constant, and $|n, \alpha\rangle$ are energy eigenstates displaced by α in phase space [59]. The partition function Z is given by

$$Z = \frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}}. \quad (4.17)$$

Since this state is a mixture, it is simple to update the calculation Eq. (A11) to using this state, leading to

$$q(+, -) = -\frac{1}{Z} \text{Re} \sum_{\ell=0}^{\infty} e^{-\frac{\ell\hbar\omega}{k_B T}} \sum_{n=0}^{\infty} e^{-i(n-\ell)\omega t} \times J_{\ell n}(x_1, \infty) J_{\ell n}(x_2, \infty), \quad (4.18)$$

where the $J_{n\ell}$ matrices are given by Eq. (A12), except for the cases $n = \ell$, where they must be calculated explicitly. A similar result may be calculated for the correlators using Eq. (A15), allowing the analysis of LG3 and LG4 inequalities.

In Fig. 8 using the states found in Sec. II C, we plot the behavior of the largest violation, as temperature is increased. We see the violation persists up to temperatures $k_B T \approx \hbar\omega$, with some preliminary evidence that LG3 and LG4 violations may be more robust against thermal fluctuations in the initial state.

V. SUMMARY

We have undertaken a study of LG violations in the quantum harmonic oscillator for a dichotomic variable $Q = \text{sgn}(x)$ and for an initial state given by a coherent state and closely related states. In Sec. II, building on our earlier work with energy eigenstates of the QHO [25], we showed how the quasi-probability, and hence the temporal correlators, may

be expressed as a discrete infinite sum which is amenable to numerical analysis. We applied this analysis to the LG2, LG3, and LG4 inequalities and carried out parameter space searches. We found LG violations of magnitude 22%, 28%, and 26% of the maximum possible for the LG2, LG3, and LG4 inequalities, respectively, and gave the specific parameters for which these violations are achieved. These violations appear to be robust under small parameter adjustments.

The LG2 violation in the case $x_0 = 0$ agrees with that reported in Ref. [35]. The LG4 violation is significantly smaller than that reported in Ref. [26], which used a coherent state with large momentum, and in fact we found no violations in that regime, although the authors note it involves a very narrow parameter range.

In Sec. III we sought a physical understanding of the mechanism producing the violations. We showed how to relate the quasi-probability (LG2) to a set of currents for projected initial states. We calculated and plotted these currents and also plotted their associated Bohm trajectories along with their classical counterparts. The plots showed the clear departures from classicality and give both a visual understanding and independent check of the LG2 violations described in Sec. II. We also provide a small-time expansion for the LG2s, which is valid for general states. We noted that the quantum effect producing the violations is essentially the diffraction in time effect first noted by Moshinsky [36,37].

We explored the same issues from a different angle using the Wigner representation. The Wigner function of the initial coherent state is everywhere nonnegative. We determined and plotted the Wigner function of the chopped initial state appearing in the quasi-probability. It has significant regions of negativity which are clearly the source of the LG2 violation.

In Sec. IV A, we extended our results to the slightly different Wigner LG inequalities, which are phrased in terms of the sequential measurement probability, and allow for larger violations, where we found a two-time violation three times greater in magnitude than the standard LG2s. We also noted it is likely possible to increase the LG violations through the use of von Neumann measurements. In addition we noted a possible advantage of the Wigner LG2 inequalities, in some cases, in terms of meeting the noninvasiveness requirement.

We briefly noted in Sec. IV B that our work is readily generalized from pure projective measurements to smoothed step function projectors, Gaussian projectors, and modular variables such as $\cos(\hat{x})$. We also examined the LG2 inequality for the case in which both projectors are taken to be projections onto coherent states. We showed that decent violations are possible for an initial coherent state and that a maximal violation arises when the initial state is a superposition of two coherent states.

Finally, we finished Sec. IV briefly discussing how the LG violations may be modified using families of states similar to a coherent state. We showed that the QP for any squeezed state is equal to the QP for some other coherent state, hence squeezing will not increase the largest violation found, however for a state with suboptimal violation it may improve the violation. We also considered a thermal initial state and estimated the degree to which thermal fluctuations may affect the degree of violation.

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APPENDIX A: CALCULATION OF CORRELATORS

The two-time quasi-probability for a coherent state with a generic position basis measurement $m(\hat{x})$ is defined,

$$q(+, +) = \text{Re} \langle \alpha | e^{\frac{iHt_2}{\hbar}} m(\hat{x}) e^{-\frac{iHt_1}{\hbar}} m(\hat{x}) e^{-\frac{iHt_1}{\hbar}} | \alpha \rangle. \quad (\text{A1})$$

We are primarily interested in the case $m(\hat{x}) = \theta(\hat{x})$, but what follows holds for more general $m(\hat{x})$, e.g., Gaussian measurements. Writing this in terms of the displacement operator, we have

$$q(+, +) = \text{Re} \langle 0 | D^\dagger(\alpha) e^{\frac{iHt_2}{\hbar}} m(\hat{x}) e^{-\frac{iHt_1}{\hbar}} m(\hat{x}) e^{-\frac{iHt_1}{\hbar}} D(\alpha) | 0 \rangle. \quad (\text{A2})$$

Since the displacement operator is unitary, we have $D^\dagger(\alpha)D(\alpha) = \mathbb{1}$. Hence, if we may commute the two displacement operators to be neighbors, we will clearly reach a vast simplification of the calculation.

To make the exposition clearer, we consider splitting this expression into two states:

$$|M(\alpha, t_1, \tau)\rangle = e^{-i\hat{H}\tau} m(\hat{x}) e^{-\frac{iHt_1}{\hbar}} D(\alpha) | 0 \rangle, \quad (\text{A3})$$

$$\langle M(\alpha, t_2, 0) | = \langle 0 | D^\dagger(\alpha) e^{\frac{iHt_2}{\hbar}} m(\hat{x}), \quad (\text{A4})$$

where we then have $q(+, +) = \text{Re} \langle M(\alpha, t_2, 0) | M(\alpha, t_1, \tau) \rangle$, where we have introduced the notation $|M(\alpha, t, s)\rangle$ to represent the coherent state measured with $m(\hat{x})$ at time t , then evolved by time s .

Considering now the displacement operator acting to the left, we write $m(\hat{x})D(\alpha) = D(\alpha)D^\dagger(\alpha)m(\hat{x})D(\alpha) = D(\alpha)m(\hat{x} + x_\alpha)$, with $x_\alpha = \sqrt{2}\text{Re}\alpha$. We then have

$$|M(\alpha, t_1, \tau)\rangle = e^{-\frac{i\omega t_1}{2}} e^{-i\hat{H}\tau} D[\alpha(t_1)] m(\hat{x} + x_1) | 0 \rangle. \quad (\text{A5})$$

Using the standard result that $e^{-iHt}D(\alpha)e^{iHt} = D[\alpha(t)]$, we can rewrite this as

$$|M(\alpha, t_1, \tau)\rangle = e^{-\frac{i\omega t_1}{2}} D[\alpha(t_2)] e^{-iH\tau} m(\hat{x} + x_1) | 0 \rangle. \quad (\text{A6})$$

This says that the postmeasurement state is the evolution of the regular ground state undergone a translated measurement, translated by the displacement operator, to a classical trajectory. Proceeding similarly with the other term, we find

$$\langle M(\alpha, t_2, 0) | = e^{\frac{i\omega t_2}{2}} \langle 0 | m(\hat{x} + x_2) D^\dagger[\alpha(t_2)]. \quad (\text{A7})$$

Finally, contracting the two terms, we are able to exploit the unitarity of $D(\alpha)$ to find

$$q(+, +) = \text{Re} e^{\frac{i\omega\tau}{2}} \langle 0 | m(\hat{x} + x_2) e^{-iH\tau} m(\hat{x} + x_1) | 0 \rangle. \quad (\text{A8})$$

A calculation similar to that in our earlier paper [25] shows the quasi-probability is

$$q(+, +) = \text{Re} e^{\frac{i\omega\tau}{2}} \sum_{n=0}^{\infty} e^{-i\omega\tau(n+\frac{1}{2})} \times \langle 0|\theta(\hat{x} + x_2)|n\rangle\langle n|\theta(\hat{x} + x_1)|0\rangle, \quad (\text{A9})$$

and similarly for the other three components. The matrix elements here are given by the J_{mn} matrices from our earlier paper [25,34],

$$J_{mn}(x_1, x_2) = \int_{x_1}^{x_2} dx \langle m|x\rangle\langle x|n\rangle. \quad (\text{A10})$$

The quasi-probability is then

$$q(+, +) = \text{Re} \sum_{n=0}^{\infty} e^{-in\omega\tau} J_{0n}(x_1, \infty) J_{0n}(x_2, \infty). \quad (\text{A11})$$

For $m \neq n$, the J matrices take the value

$$J_{mn}(x_1, x_2) = \frac{1}{2(\varepsilon_n - \varepsilon_m)} [\psi'_m(x_2)\psi_n(x_2) - \psi'_n(x_2)\psi_m(x_2) - \psi'_m(x_1)\psi_n(x_1) + \psi'_n(x_1)\psi_m(x_1)], \quad (\text{A12})$$

where $\psi_n(x) = \langle x|n\rangle$. For the $n = m = 0$ case, the integration is completed manually, yielding

$$J_{00}(x, \infty) = \frac{1}{2}(1 - \text{erf}(x)). \quad (\text{A13})$$

Hence, writing out the quasi-probability with $n = 0$ case of the sum handled, we have

$$q(s_1, s_2) = \frac{1}{4} \left\{ 1 + s_1 \text{erf}(x_1) + s_2 \text{erf}(x_2) + s_1 s_2 \left[\text{erf}(x_1) \text{erf}(x_2) + 4 \sum_{n=1}^{\infty} \cos(n\omega\tau) J_{0n}(x_1, \infty) J_{0n}(x_2, \infty) \right] \right\}. \quad (\text{A14})$$

Comparing to the moment expansion of the quasi-probability, we obtain the correlators

$$C_{12} = \text{erf}(x_1) \text{erf}(x_2) + 4 \sum_{n=1}^{\infty} \cos(n\omega\tau) J_{0n}(x_1, \infty) J_{0n}(x_2, \infty). \quad (\text{A15})$$

The infinite sum may be evaluated approximately using numerical methods, by summing up to a finite n . This calculation matches the analytically calculated special case of $x_0 = 0$ given in Ref. [35], and while it is possible to make an analytic calculation for the more general case, it turned out not to be as useful as the numerical evaluation. The exact result is found in terms of Owen-T functions, but for complex arguments, which rendered the behavior chaotic when computed [60].

The only source of nonclassicality here lies in the infinite sum, and with the J_{0n} matrices expressed in terms of the oscillator eigenstates, this means there is a double exponential suppression $e^{-x_1^2 - x_2^2}$ of this nonclassical term. This corresponds to the requirement that at least two measurements must make a significant chop of the state, which fits the intuition that without significant chopping, there is no mystery attached to which side of the axis the particle may be found on.

APPENDIX B: DETERMINATION OF LG VIOLATIONS

In this Appendix we fill in the details of the LG violations reported in Sec. II C. Recall the variable parameters of the problem are x_0 and p_0 and the equal time spacing parameter τ .

We also note that it is sufficient to explore a single quadrant of the x_0, p_0 parameter space, which we take to be the positive quadrant. For states with $x_0 < 0$, the quasi-probability may be recovered by inverting the sign of s_1 . Likewise for states with $p_0 < 0$, by allowing the interval between measurements to take values $0 < \tau \leq 2\pi$, their behavior is included in the positive quadrant. This same argument applies to the LG inequalities in general, where their different permutations correspond to flips of measurement signs.

To represent the three-dimensional parameter space, for each x_0, p_0 , we use numerical minimization over $0 < \tau \leq 2\pi$ to find the largest possible violation for that coherent state. In this numerical procedure, we take the largest possible violation from all of the inequalities involved.

The results of this parameter space search for the LG3 and LG4 inequalities are shown in Fig. 9, which shows similar behavior to the LG2 inequality parameter space behavior in Fig. 1. As more measurement intervals are included in the LG tests, a broader range of states lead to violation.

LG tests on QHO coherent states are mathematically equivalent to LG tests on the pure ground state $|0\rangle$, which we found to have violations only when at least one of the $\theta(\hat{x})$ measurements involved is displaced from the axis by order of magnitude 1. Hence, at the center of each of these parameter space plots is the region where the coherent state is too similar to the ground state to have any LG violation.

All the violations we have found are in states with initial position and momenta approximately on the length scale of the width of the coherent state, $\sqrt{\hbar/(m\omega)}$. However, by appealing to Eq. (A15), we note that if one were to consider translating the measurement to $\theta(\hat{x} - x_i)$, the classical motion could be subtracted, and at least theoretically, the same magnitude of violations would exist for arbitrarily high x_0 and p_0 .

In Figs. 10 and 11, we plot the temporal behavior of the LG2s, LG3s, and LG4s, respectively, for the case in which the parameters are chosen to give the largest violation.

APPENDIX C: CURRENTS ANALYSIS

1. Classical analogs

To understand the connection between the negativity of the quasi-probability Eq. (C17) and the behavior of the currents, it is very convenient to consider the analogous classical currents, which are in general defined by

$$\mathbb{J}(t) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx p(t) \delta[x(t)] w(x, p), \quad (\text{C1})$$

for a suitably chosen initial phase-space distribution $w(x, p)$. For the unchopped current $\mathbb{J}(t)$ this is taken to be the Wigner function of the coherent state, $W(x, p, x_0, p_0)$, Eq. (C3), which conveniently is nonnegative. For the chopped currents it is taken to be $\theta(\pm x)W(x, p, x_0, p_0)$. We then easily see that

$$J(t) = \mathbb{J}_-(t) + \mathbb{J}_+(t), \quad (\text{C2})$$

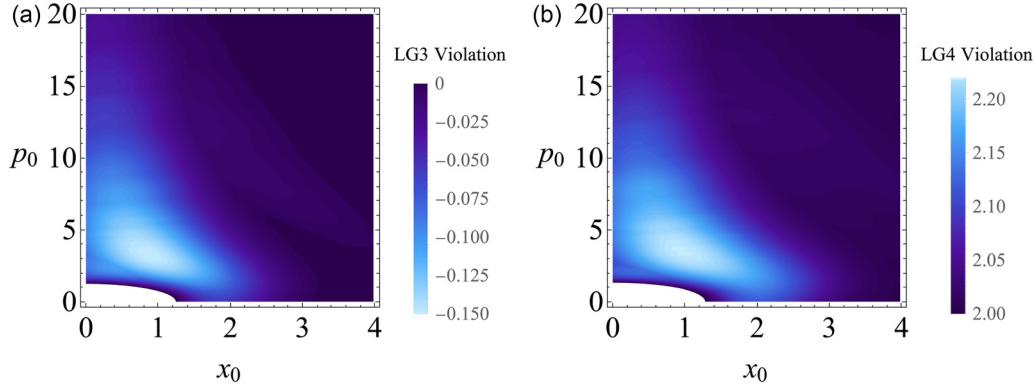


FIG. 9. The greatest possible violations for (a) the LG3 inequalities and (b) the LG4 inequalities.

where $\mathbb{J}_{\pm}(t)$ are the classical analogs to the postmeasurement currents, where since we have used the coherent state Wigner function, we have $\mathbb{J}(t) = J(t)$.

We begin by writing the classical phase-space density for the Gaussian state,

$$\mathbb{W}(X, p, x_0, p_0) = \frac{1}{\pi} \exp[-(X - x_0)^2 - (p - p_0)^2], \quad (\text{C3})$$

where harmonic time evolution leads to rigid rotation in phase space,

$$\mathbb{W}(X, p, x_0, p_0, t) = \mathbb{W}(X \cos \omega t - p \sin \omega t, p \cos \omega t + X \sin \omega t, x_0, p_0). \quad (\text{C4})$$

We have a similar result for the measured classical state, with

$$\mathbb{W}_{\pm}(X, p, x_0, p_0) = \frac{1}{\pi} \theta(\pm X) \exp[-(X - x_0)^2 - (p - p_0)^2], \quad (\text{C5})$$

and

$$\mathbb{W}_{\pm}(X, p, x_0, p_0, t) = \mathbb{W}_{\pm}(X \cos \omega t - p \sin \omega t, p \cos \omega t + X \sin \omega t, x_0, p_0). \quad (\text{C6})$$

The chopped classical current is given by

$$\mathbb{J}_{\pm}(x, t) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dX p \delta(X - x) \mathbb{W}_{\pm}(X, p, x_0, p_0, t). \quad (\text{C7})$$

Completing the X integral trivially, we have

$$\mathbb{J}_{\pm}(x, t) = \int_{-\infty}^{\infty} dp p \mathbb{W}_{\pm}(x, p, x_0, p_0, t). \quad (\text{C8})$$

We are interested in the case of $x = 0$, which we shorthand $\mathbb{J}_{\pm}(0, t) = \mathbb{J}_{\pm}(t)$, and is given by

$$\mathbb{J}_{\pm}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp p \theta(\mp p \sin \omega t) \exp[-(p \cos \omega t - p_0)^2 - (-p \sin \omega t - x_0)^2]. \quad (\text{C9})$$

The step function here just flips the integral between the positive or negative half-plane, dependent on $\text{sgn}(\mp \sin \omega t) = 1$ and $\text{sgn}(\mp \sin \omega t) = -1$, respectively. Computing the integral, this yields the result

$$\mathbb{J}_{\pm}(t) = \frac{1}{2\pi} e^{-p_0^2 - x_0^2} (\mp \text{sgn}(\sin \omega t) + \sqrt{\pi} e^{g(x_0, p_0, t)^2} \times g(x_0, p_0, t) \{1 \mp \text{sgn}(\sin \omega t) \text{erf}[g(x_0, p_0, t)]\}), \quad (\text{C10})$$

with $g(x_0, p_0, t) = p_0 \cos \omega t - x_0 \sin \omega t$.

2. Time derivative of the quasi-probability

To calculate the time derivative of the QP as it appears in Eq. (3.2), we begin by writing the simple projector identity

$$P\rho + \rho P = P\rho P - \bar{P}\rho\bar{P} + \rho, \quad (\text{C11})$$

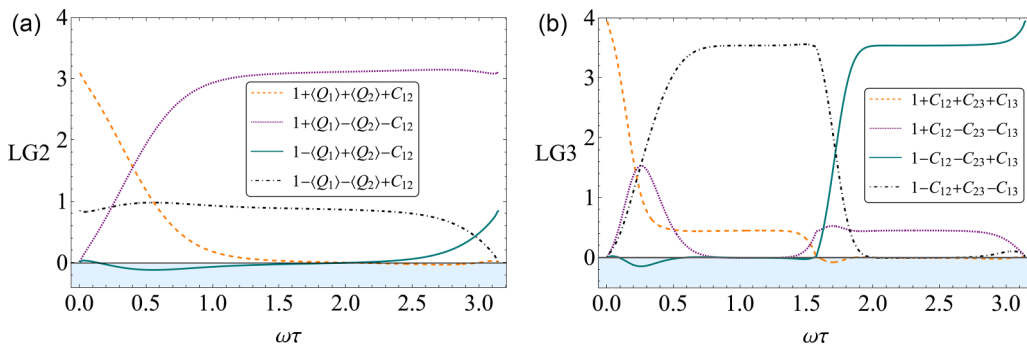


FIG. 10. (a) Four LG2 inequalities for the state with $x_0 = 0.55$, $p_0 = -1.925$, which reaches a largest violation of -0.113 at $\omega\tau = 0.555$. (b) Four LG3 inequalities for the state with $x_0 = 0.859$, $p_0 = -3.317$, reaching a largest violation of -0.141 at $\omega\tau = 0.254$.

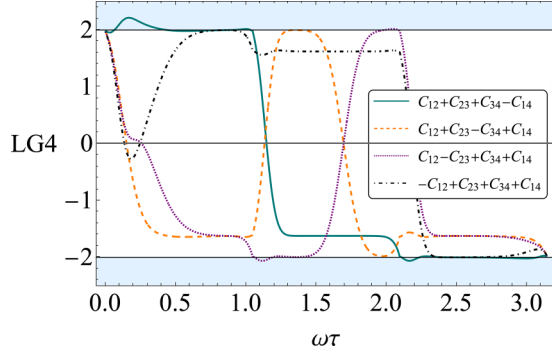


FIG. 11. Four LG4 inequalities, for state $x_0 = 0.929$, $p_0 = -3.666$, reaching a largest violation of 2.216 at $\omega\tau = 0.166$.

where $\bar{P} = 1 - P$. Hence, the quasi-probability, Eq. (2.7) is given by

$$q(-, +) = \frac{1}{2} \text{Tr}\{P_+(t_2)[P_-(t_1)\rho P_-(t_1) - P_+(t_1)\rho P_+(t_1) + \rho]\}. \quad (\text{C12})$$

This may be written in terms of the nonnegative sequential measurement probabilities

$$p_{12}(s_1, s_2) = \text{Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1}(t_1)], \quad (\text{C13})$$

in the form

$$q(-, +) = \frac{1}{2}[p_{12}(-, +) - p_{12}(+, +) + \langle P_+(t_2) \rangle]. \quad (\text{C14})$$

It is now simple to take the derivative with respect to t_2 , noting that

$$\frac{d}{dt}\theta[\hat{x}(t)] = \frac{1}{2m}\{\hat{p}(t)\delta[\hat{x}(t)] + \delta[\hat{x}(t)]\hat{p}(t)\} = \hat{J}(t), \quad (\text{C15})$$

yielding

$$\frac{dq(-, +)}{dt} = \frac{1}{2} \text{Tr}\{\hat{J}(t)[P_-(t_1)\rho P_-(t_1) - P_+(t_1)\rho P_+(t_1) + \rho]\}. \quad (\text{C16})$$

We can hence rewrite $q(-, +)$ as

$$q(-, +) = \frac{1}{2} \int_{t_1}^{t_2} dt [J_-(t) - J_+(t) + J(t)], \quad (\text{C17})$$

where we have introduced the ‘‘chopped current’’

$$J_{\pm}(t) = \langle \psi | \theta(\pm\hat{x}) \hat{J}(t) \theta(\pm\hat{x}) | \psi \rangle, \quad (\text{C18})$$

which corresponds to the current at the origin, after the initial measurement. The chopped currents are therefore the currents of the wave functions $\langle x | e^{-iHt} \theta(\pm\hat{x}) | \psi \rangle$, and note the connection to the Moshinsky function when $|\psi\rangle$ is expanded in the momentum basis.

Using Eq. (C2), we may rewrite Eq. (C17) in terms of the difference between quantum and classical postmeasurement currents, as

$$q(-, +) = \int_{t_1}^{t_2} dt J_-(t) + \frac{1}{2} \int_{t_1}^{t_2} dt [\mathbb{J}_-(t) - J_-(t) + \mathbb{J}_+(t) - J_+(t)]. \quad (\text{C19})$$

3. Chopped currents calculation

We calculate the chopped current first, that is,

$$J_{\pm}(x, t) = \frac{1}{2m} \langle \phi_{\alpha}^{\pm}(t) | \delta(\hat{x} - x) \hat{p} + \hat{p} \delta(\hat{x} - x) | \phi_{\alpha}^{\pm}(t) \rangle, \quad (\text{C20})$$

where $\phi_{\alpha}^{\pm}(t)$ is the time evolution of a coherent state, initially projected on $\theta(\pm\hat{x})$ at t_0 ,

$$|\phi_{\alpha}^{\pm}(t)\rangle = e^{-iHt} \theta(\pm\hat{x}) |\alpha\rangle. \quad (\text{C21})$$

Calculating the current in the position basis, we have

$$J_{\pm}(x, t) = -\frac{i\hbar}{2m} \left[\phi_{\alpha}^{\pm*}(x, t) \frac{\partial \phi_{\alpha}^{\pm}(x, t)}{\partial x} - \frac{\partial \phi_{\alpha}^{\pm*}(x, t)}{\partial x} \phi_{\alpha}^{\pm}(x, t) \right], \quad (\text{C22})$$

equivalent to

$$J_{\pm}(x, t) = \frac{\hbar}{m} \text{Im} \left[\phi_{\alpha}^{\pm*}(x, t) \frac{\partial \phi_{\alpha}^{\pm}(x, t)}{\partial x} \right]. \quad (\text{C23})$$

We calculate the evolved chopped state by

$$\phi_{\alpha}^{\pm}(x, t) = \int_{\Delta(\pm)} dy K(x, y, t) \psi_{\alpha}(y, t_0), \quad (\text{C24})$$

where $\Delta(+)$ = $[0, \infty)$, $\Delta(-)$ = $(-\infty, 0]$,

$$\psi_{\alpha}(x, t_0) = \frac{1}{\pi^{\frac{1}{4}}} \exp \left[-\frac{1}{2}(x - x_0)^2 + ip_0 x \right] \quad (\text{C25})$$

is the nondimensionalized coherent state wave function, with time evolution $\alpha(t) = e^{-i\omega t} \alpha(0)$, and

$$K(x, y, t) = \left(\frac{1}{2\pi i \sin \omega t} \right)^{\frac{1}{2}} \times \exp \left\{ -\frac{1}{2i \sin \omega t} [(x^2 + y^2) \cos \omega t - 2xy] \right\} \quad (\text{C26})$$

is the propagator for the harmonic potential.

Inserting the relevant expressions within Eq. (C24) yields

$$\begin{aligned} \phi_{\alpha}^{\pm}(x, t) &= \left(\frac{1}{2\pi i \sin \omega t} \right)^{\frac{1}{2}} \left(\frac{1}{\pi} \right)^{\frac{1}{4}} \int_{\Delta(\pm)} dy \\ &\times \exp \left[-\frac{1}{2}(y - x_0)^2 + iy p_0 \right] \\ &\times \exp \left(-\frac{x^2 + y^2}{2i \tan \omega t} + \frac{xy}{i \sin \omega t} \right). \end{aligned} \quad (\text{C27})$$

We proceed writing the integral as

$$I_{\pm}(a, b, c) = \int_{\Delta(\pm)} dr \exp \left[-\frac{1}{2}(r - a)^2 + ibr + icr^2 \right], \quad (\text{C28})$$

where

$$a = x_0, \quad (\text{C29})$$

$$b = p_0 - \frac{x}{\sin \omega t}, \tag{C30}$$

$$c = \frac{1}{2 \tan \omega t}. \tag{C31}$$

Completing the integration, we have

$$I_{\pm}(a, b, c) = \frac{\sqrt{\pi} e^{-\frac{2a^2c+2ab+ib^2}{4c+2i}} [1 \pm \operatorname{erf}(\frac{a+ib}{\sqrt{2-4ic}})]}{\sqrt{2-4ic}}. \tag{C32}$$

We hence can write the chopped wave function as

$$\phi_{\alpha}^{\pm}(x, t) = \left(\frac{1}{2\pi i \sin \omega t}\right)^{\frac{1}{2}} \left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2i \tan \omega t}} I_{\pm}\left(x_0, p_0 - \frac{x}{\sin \omega t}, \frac{1}{2 \tan \omega t}\right). \tag{C33}$$

Putting Eq. (C23) into rescaled units as well, we have

$$J_{\pm}(x, t) = \omega \operatorname{Im} \left[\phi_{\alpha}^{\pm*}(x, t) \frac{\partial \phi_{\alpha}^{\pm}(x, t)}{\partial x} \right]. \tag{C34}$$

To take the derivative, we note $I_{\pm}(a, b, c)$ depends on s only in its second argument, and so we define

$$K_{\pm}(a, b, c) = \frac{\partial b}{\partial s} \frac{\partial}{\partial b} I_{\pm}(a, b, c), \tag{C35}$$

which explicitly yields

$$K_{\pm}(a, b, c) = \frac{-1}{\sin \omega t} \frac{e^{-\frac{a^2}{2}} \{ -\sqrt{2\pi}(a+ib)e^{\frac{(a+ib)^2}{2-4ic}} [1 \pm \operatorname{erf}(\frac{a+ib}{\sqrt{2-4ic}})] \mp 2\sqrt{1-2ic} \}}{2\sqrt{1-2ic}(2c+i)}. \tag{C36}$$

Altogether, this yields the current

$$J_{\pm}(x, t) = \frac{\omega}{2\pi^{\frac{3}{2}}} \operatorname{Im} \left\{ \frac{1}{|\sin \omega t|^{\frac{3}{2}}} I_{\pm}\left(x_0, -p_0 + \frac{x}{\sin \omega t}, -\frac{1}{2 \tan \omega t}\right) \times \left[\frac{ix}{\tan \omega t} I_{\pm}\left(x_0, p_0 - \frac{x}{\sin \omega t}, \frac{1}{2 \tan \omega t}\right) + K_{\pm}\left(x_0, p_0 - \frac{x}{\sin \omega t}, \frac{1}{2 \tan \omega t}\right) \right] \right\}. \tag{C37}$$

We also calculate the current of the original unperturbed coherent state, in these same rescaled units. Since coherent states are eigenfunctions of the annihilation operator, with eigenvalue $\alpha(t)$, it follows that

$$\frac{\partial}{\partial x} \psi_{\alpha}(x, t) = \psi_{\alpha}(x, t) [\sqrt{2}\alpha(t) - x]. \tag{C38}$$

Hence, the current is given by

$$J(x, t) = \frac{\omega}{\sqrt{\pi}} \operatorname{Im} \{ e^{-[x - \sqrt{2}\operatorname{Re} \alpha(t)]^2} [\sqrt{2}\alpha(t) - x] \}, \tag{C39}$$

where taking the imaginary part, and using Eq. (2.2), we have

$$J(x, t) = \frac{\omega}{\sqrt{\pi}} p_t e^{-(x-x_t)^2}. \tag{C40}$$

APPENDIX D: SMALL-TIME CURRENT EXPANSIONS

We are interested in a small-time expansion of the chopped current $J_{\pm}(t)$, for any general state $|\psi\rangle$. We start by defining the chopped-evolved state,

$$|\psi^{\pm}(t)\rangle = e^{-iHt} \theta(\pm \hat{x}) |\psi\rangle. \tag{D1}$$

We now follow Appendix C2 up to Eq. (C34) to calculate the current, however this time using the general $|\psi\rangle$ state,

leading to

$$J_{\pm}(t) = \omega \operatorname{Im} \left[\psi^{\pm*}(0, t) \frac{\partial \psi^{\pm}(0, t)}{\partial x} \right], \tag{D2}$$

with $|\psi\rangle$ represented in the nondimensional position basis, i.e., $\langle x|\psi\rangle = (\frac{m\omega}{\hbar})^{\frac{1}{4}} \psi(x)$, with a normalized $\psi(x)$ which is purely a function of x .

Using the QHO propagator as in Appendix C2 up to Eq. (C27), now with the $|\psi\rangle$ state, we have

$$\psi^{\pm}(x, t) = \left(\frac{1}{2\pi i \sin \omega t}\right)^{\frac{1}{2}} \int_{\Delta(\pm)} dy \psi(y, 0) \times \exp\left(-\frac{x^2+y^2}{2i \tan \omega t} + \frac{xy}{i \sin \omega t}\right). \tag{D3}$$

We are interested in the current at $x = 0$, which means we only need $\psi(x)$ and its first derivative evaluated at $x = 0$, leaving

$$\psi^{\pm}(0, t) = \left(\frac{1}{2\pi i \sin \omega t}\right)^{\frac{1}{2}} \int_{\Delta(\pm)} dy \psi(y, 0) \times \exp\left(-\frac{y^2}{2i \tan \omega t}\right). \tag{D4}$$

We now argue that for small t the main contribution to the integral will come from near the boundary of the chop, and hence the Taylor expansion $\psi(y, 0)$ expands around $y = 0$. By using the parametrization $y = z(\tan \omega t)^{\frac{1}{2}}$, we simplify the

exponential part of the integrand.

$$\psi(y, 0) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0, 0)}{n!} z^n (\tan \omega t)^{\frac{n}{2}}. \quad (\text{D5})$$

Using this in Eq. (D4), and using the substitution within the integral, we have

$$\psi^{\pm}(0, t) = \left(\frac{\tan \omega t}{2\pi i \sin \omega t} \right)^{\frac{1}{2}} \int_{\Delta(\pm)} dz \exp\left(-\frac{z^2}{2i}\right) \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0, 0)}{n!} z^n (\tan \omega t)^{\frac{n}{2}}. \quad (\text{D6})$$

Interchanging the order of summation and integration, we have

$$\psi^{\pm}(0, t) = \left(\frac{1}{2\pi i \cos \omega t} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0, 0)}{n!} (\tan \omega t)^{\frac{n}{2}} \int_{\Delta(\pm)} dz e^{-\frac{z^2}{2i}} z^n. \quad (\text{D7})$$

We now define

$$K_{\pm}(n) = \int_{\Delta(\pm)} dz e^{-\frac{z^2}{2i}} z^n, \quad (\text{D8})$$

which can be calculated by taking the Fourier transform of $e^{iz^2} \theta(\pm z)$, taking n derivatives in Fourier space, and then evaluating with the conjugate variable set to 0, to give the result

$$K_{\pm}(n) = (\pm 1)^n 2^{\frac{n-1}{2}} e^{\frac{1}{4}i\pi(n+1)} \Gamma\left(\frac{n+1}{2}\right), \quad (\text{D9})$$

where Γ is the Gamma function. This gives a final result of

$$\psi^{\pm}(0, t) = \left(\frac{1}{2\pi i \cos \omega t} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} K_{\pm}(n) \frac{\psi^{(n)}(0, 0)}{n!} (\tan \omega t)^{\frac{n}{2}}. \quad (\text{D10})$$

We now calculate the derivative of the chopped wave function by taking the derivative of Eq. (D3), evaluated at $x = 0$ to find

$$\left. \frac{\partial \psi^{\pm}(x, t)}{\partial x} \right|_{x=0} = \frac{-i}{\sin \omega t} \left(\frac{1}{2\pi i \sin \omega t} \right)^{\frac{1}{2}} \int_{\Delta(\pm)} dy \psi(y, 0) y \exp\left(-\frac{y^2}{2i \tan \omega t}\right). \quad (\text{D11})$$

We now note that with the exception of the prefactor $\frac{-i}{\sin \omega t}$, this is the same result as before, only with $\psi(y, 0)$ swapped for $y\psi(y, 0)$, leading to the result

$$\frac{\partial \psi^{\pm}(0, t)}{\partial x} = -\left(\frac{-1}{2\pi i \sin^2 \omega t \cos \omega t} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} K_{\pm}(n) \frac{\frac{\partial^n}{\partial y^n} [y\psi(y, 0)]|_{y=0}}{n!} (\tan \omega t)^{\frac{n}{2}}. \quad (\text{D12})$$

Then, since

$$\left. \frac{\partial^n}{\partial y^n} y\psi(y, 0) \right|_{y=0} = n\psi^{(n-1)}(0, 0), \quad (\text{D13})$$

we have as a final result

$$\frac{\partial \psi^{\pm}(0, t)}{\partial x} = -\left(\frac{-1}{2\pi i \sin^2 \omega t \cos \omega t} \right)^{\frac{1}{2}} \sum_{n=1}^{\infty} K_{\pm}(n) \frac{n\psi^{(n-1)}(0, 0)}{n!} (\tan \omega t)^{\frac{n}{2}}, \quad (\text{D14})$$

noting the change on the sum's lower limit.

We now combine Eqs. (D10) and (D14) to yield the small-time expansion for the chopped current

$$J_{\pm}(t) = \frac{-\omega}{2\pi \sin \omega t \cos \omega t} \text{Im} \left(i \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} K_{\pm}(n) K_{\pm}^*(\ell) \frac{n\psi^{(n-1)}(0, 0) \psi^{*(\ell)}(0, 0)}{n! \ell!} (\tan \omega t)^{\frac{n+\ell}{2}} \right). \quad (\text{D15})$$

We note that this result is in fact trivial to integrate over time by noting the derivative of $\tan(\omega t)$, yielding

$$\int_0^{\tau} dt \frac{(\tan \omega t)^{\frac{k}{2}}}{\sin \omega t \cos \omega t} = \frac{2}{k\omega} (\tan \omega \tau)^{\frac{k}{2}}. \quad (\text{D16})$$

The time integral of the chopped current is thus

$$\int_0^{\omega\tau} J_{\pm}(t) dt = -\frac{1}{\pi} \text{Im} \left[i \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} K_{\pm}(n) K_{\pm}^*(\ell) \frac{n\psi^{(n-1)}(0,0)\psi^{*(\ell)}(0,0)}{(n+\ell)n!\ell!} (\tan \omega\tau)^{\frac{n+\ell}{2}} \right]. \quad (\text{D17})$$

We also note that by defining

$$L(n) = K_+(n) + K_-(n), \quad (\text{D18})$$

we may adapt the result to the unchopped current as

$$J(t) = -\frac{\omega}{2\pi \sin \omega t \cos \omega t} \text{Im} \left[i \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} L(n) L^*(\ell) \frac{n\psi^{(n-1)}(0,0)\psi^{*(\ell)}(0,0)}{n!\ell!} (\tan \omega t)^{\frac{n+\ell}{2}} \right], \quad (\text{D19})$$

with time integral

$$\int_0^{\omega\tau} J(t) dt = -\frac{1}{\pi} \text{Im} \left[i \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} L(n) L^*(\ell) \frac{n\psi^{(n-1)}(0,0)\psi^{*(\ell)}(0,0)}{(n+\ell)n!\ell!} (\tan \omega\tau)^{\frac{n+\ell}{2}} \right] \quad (\text{D20})$$

Using Eq. (C17) to express the quasi-probability as the time integral of currents, and noting the similarity of the summands, we can write the quasi-probability as

$$q(-, +) = -\frac{1}{2\pi} \text{Im} \left[i \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} \mathcal{Q}(n, \ell) \frac{n\psi^{(n-1)}(0,0)\psi^{*(\ell)}(0,0)}{(n+\ell)n!\ell!} (\tan \omega\tau)^{\frac{n+\ell}{2}} \right], \quad (\text{D21})$$

where

$$\mathcal{Q}(n, \ell) = K_-(n)K_+^*(\ell) - K_+(n)K_-^*(\ell) + L_n L_{\ell}^*. \quad (\text{D22})$$

By approximating the infinite sums to finite order, we are able to approximate the quasi-probability. To get the first three terms of the approximation, we limit both sums to $\ell_{\max} = 2$ and $n_{\max} = 3$, yielding

$$q(-, +) = \frac{1}{2\sqrt{\pi}} |\psi(0,0)|^2 \tan^{\frac{1}{2}} \omega\tau + \frac{J(0)}{2} \tan \omega\tau + \frac{1}{6\sqrt{\pi}} \left(|\psi'(0,0)|^2 - \left[\frac{1}{4} + \frac{3i}{4} \right] \psi''^*(0,0)\psi(0,0) - \left[\frac{1}{4} - \frac{3i}{4} \right] \psi''(0,0)\psi^*(0,0) \right) \tan^{\frac{3}{2}} \omega\tau + \mathcal{O}(\tan^2 \omega\tau). \quad (\text{D23})$$

Taking the $\omega \rightarrow 0$ expansions of the trigonometric terms recovers the result for the free particle,

$$q(-, +) = \frac{1}{2\sqrt{\pi}} |\psi(0,0)|^2 \tau^{\frac{1}{2}} + \frac{J(0)}{2} \tau + \frac{1}{6\sqrt{\pi}} \left(|\psi'(0,0)|^2 - \left[\frac{1}{4} + \frac{3i}{4} \right] \psi''^*(0,0)\psi(0,0) - \left[\frac{1}{4} - \frac{3i}{4} \right] \psi''(0,0)\psi^*(0,0) \right) \tau^{\frac{3}{2}}, \quad (\text{D24})$$

which we note has a term in $\tau^{\frac{1}{2}}$ which was missing from an earlier calculation of this expansion in Ref. [33], as well as a different coefficient on the $\tau^{\frac{3}{2}}$ term.

The initial divergence of the quantum chopped current is clearly seen. These results also agree with the small-time expansion of chopped currents given by Sokolowski [61], giving another useful check on our calculations. For our Gaussian initial state we find agreement with the results above, and we plot this expansion alongside our original calculation in Fig. 12.

APPENDIX E: WIGNER FUNCTION CALCULATIONAL DETAILS

The Wigner-Weyl transform, which maps Hermitian operators to real phase-space functions [45–48], is defined by

$$W_A(X, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{-ip\xi} \left\langle X + \frac{\xi}{2} | A | X - \frac{\xi}{2} \right\rangle. \quad (\text{E1})$$

Traces of pairs of operators may be expressed in the Wigner representation as

$$\text{Tr}(\hat{A}\hat{B}) = 2\pi \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dp W_A(X, p) W_B(X, p). \quad (\text{E2})$$

To apply this formula to the quasi-probability, there are two natural ways to proceed. First, in Ref. [33], the free-particle quasi-probability was explored in Wigner-Weyl form using $\hat{A} = \frac{1}{2}(P_{s_1}P_{s_2} + P_{s_2}P_{s_1})$ and $\hat{B} = \rho$. However, this was not found to be very useful since $W_A(X, p)$ in this case is highly oscillatory and it was not possible to clearly identify the regions of negativity, hence we proceed with a different approach.

We first write the QP in the form $q(s_1, s_2) = \text{Tr}[\bar{\rho}_{s_1} P_{s_2}(\tau)]$, where $\bar{\rho}_{s_1} = \frac{1}{2}(P_{s_1}\rho + \rho P_{s_1})$, and $t_1 = 0$ without loss of gen-

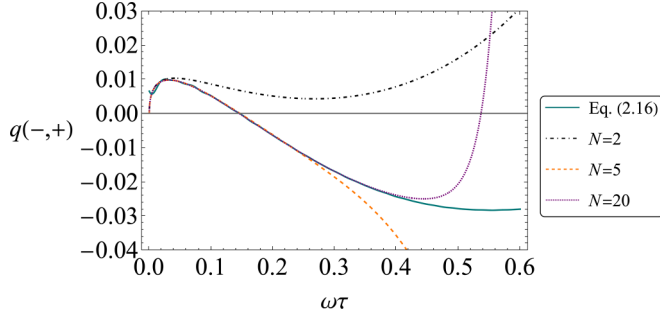


FIG. 12. The small-time expansion of $q(-, +)$ plotted alongside the previous calculation, at varying degrees of truncation, where $N = 2$ corresponds to Eq. (D23).

erality. Hence, by Eq. (E2) we have

$$q(s_1, s_2) = 2\pi \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} W_{\tilde{\rho}_{s_1}}(X, p) W_{P_{s_2}(\tau)}(X, p), \quad (\text{E3})$$

where $W_{P_{s_2}}(X, p) = \theta[s_2(X \cos \omega\tau + p \sin \omega\tau)]$. Using Eq. (E1), the transform of $\tilde{\rho}_{s_1}$ is given by

$$W_{\tilde{\rho}_{s_1}}(X, p) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi e^{-ip\xi} \left\langle X + \frac{\xi}{2} | P_{s_1} \rho + \rho P_{s_1} | X - \frac{\xi}{2} \right\rangle. \quad (\text{E4})$$

Without loss of generality we take $t_1 = 0$, and so $P_{s_1} = \theta(s_1 \hat{x})$, leading to

$$W_{\tilde{\rho}_{s_1}}(X, p) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi e^{-ip\xi} \left\{ \theta \left[s_1 \left(X + \frac{\xi}{2} \right) \right] + \theta \left[s_1 \left(X - \frac{\xi}{2} \right) \right] \right\} \left\langle X + \frac{\xi}{2} | \rho | X - \frac{\xi}{2} \right\rangle. \quad (\text{E5})$$

Since coherent states are pure states, we have simply that $\rho(x, y) = \psi(x)\psi^*(y)$, where we will use natural units with $\psi(x) = \frac{1}{\pi^{1/4}} \exp[-\frac{1}{2}(x - x_0)^2 + ip_0x]$. This yields the integrand as $I(X, \xi) = e^{-ip\xi} \exp(ip_0\xi - X^2 + 2Xx_0 - x_0^2 - \frac{\xi^2}{4})$.

Demonstrating with the $s_1 = +1$ case, the theta functions are handled by splitting the integral into two integrals

over the regions $[-2X, \infty)$ and $(-\infty, 2X]$, and so we have

$$W_{\tilde{\rho}_{s_1}}(X, p) = \frac{1}{4\pi\sqrt{\pi}} \left[\int_{-2X}^{\infty} d\xi I(X, \xi) + \int_{-\infty}^{2X} d\xi I(X, \xi) \right]. \quad (\text{E6})$$

Computing the integral involved, we reach the result

$$W_{\tilde{\rho}_{s_1}}(X, p) = \frac{1}{2\pi} e^{-(p-p_0)^2 - (X-x_0)^2} \times \{1 + \text{Re erf}[i(p - p_0) + s_1X]\}, \quad (\text{E7})$$

which may be written as

$$W_{\tilde{\rho}_{s_1}}(X, p) = \frac{1}{2} W_{\rho}(X, p) \{1 + \text{Re erf}[i(p - p_0) + s_1X]\}. \quad (\text{E8})$$

in terms of $W_{\rho}(X, p)$, the Wigner function of the pure coherent state, given by

$$W_{\rho}(X, p) = \frac{1}{\pi} \exp[-(p - p_0)^2 - (X - x_0)^2]. \quad (\text{E9})$$

The classical equivalent for Eq. (E8) is $\frac{1}{2} W_{\rho}(X, p) [1 + \text{sgn}(X)]$, which is approached for p close to p_0 and for large $|X|$.

The time evolution of the Wigner function in the case of the QHO is given by rigid rotation in accordance with classical paths $X_{\tau} = x_0 \cos \omega\tau - p_0 \sin \omega\tau$. Hence, in Eq. (E3) $W_{P_{s_2}(\tau)}(X, p) = \theta(s_2 X_{-\tau})$, and the final expression for the quasi-probability is

$$q(s_1, s_2) = \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dp f_{s_1, s_2}(X, p), \quad (\text{E10})$$

with the phase-space density $f_{s_1, s_2}(X, p)$ given by

$$f_{s_1, s_2}(X, p) = \frac{1}{2} W_{\rho}(X, p) (1 + \text{Re erf}[i(p - p_0) + s_1X]) \theta(s_2 X_{-\tau}). \quad (\text{E11})$$

Equations (E8) and (E11) are plotted in Fig. 13.

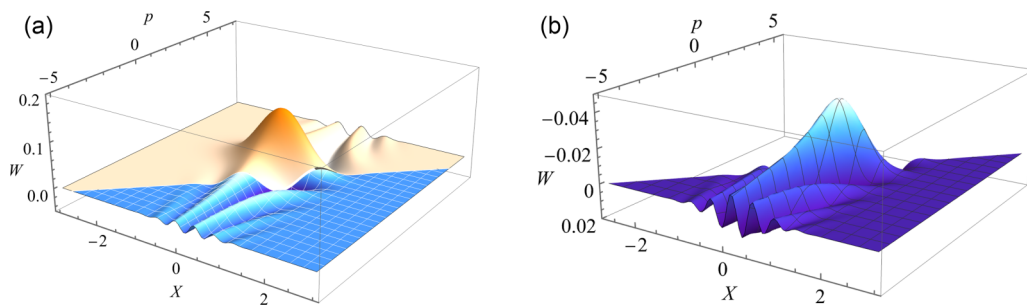


FIG. 13. (a) $W_{\tilde{\rho}_-}(X, p)$, Eq. (E8) with $s_1 = -1$, for an initial coherent state with $x_0 = 0.55$, $p_0 = -1.925$, and $\omega\tau = 0.55$. The orange (smooth) region shows the region removed when we go to the phase-space density Eq. (E11), thereby showing how the most significant positive parts are removed. (b) (flipped axis) Phase-space density Eq. (E11), which will from visual inspection integrate to a negative number overall.

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