Correspondence between Dicke-model semiclasscial dynamics in the superradiant dipolar phase and the Euler heavy top

S. I. Mukhin ,[1,*] A. Mukherjee ,[1,2,†] and S. S. Seidov [1]
[1]*Theoretical Physics and Quantum Technologies Department, NUST "MISIS," Moscow 119049, Russia*
[2]*Department of Physics, School of Basic Sciences, Manipal University Jaipur, Rajasthan 303007, India*

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Analytic expression is found for the frequency dependence of the transmission coefficient of a transmission line inductively coupled to the microwave cavity with the superradiant condensate. Sharp transmission drops reflect the condensate's frequencies spectrum. These results pave the way to direct detection of emergence of the superradiant condensates in quantum metamaterials. Results are based on the analytic solutions of the nonlinear semiclassical dynamics of the superradiant photonic condensate in the Dicke model of an ensemble of two-level atoms dipolar coupled to the electromagnetic field in the microwave cavity. In the adiabatic limit with respect to the photon degree of freedom, the system is approximately integrable with evolution being expressed via Jacobi elliptic functions of real time. Depending on the coupling strength, the semiclassical coordinate of the superradiant condensate in the ground state either oscillates in one of the two degenerate minima of the condensate's potential energy or traverses between them over the saddle point. An experimental setup for measuring of the breakdown of the normal phase of the Dicke model via coupling to the transmission line is proposed. A one-to-one mapping of the semiclassical motion of the superradiant condensate on the nodding of unstable Lagrange "sleeping top" also turns the Dicke model into an analog device for modeling the dynamics of mechanical systems.

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I. INTRODUCTION

Prediction of superradiant quantum phase transition [1,2], that breaks parity symmetry of the system consisting of $N \gg 1$ two-level (TL) atoms coupled to a single bosonic mode in the resonant cavity, poses an interesting problem concerning observable fingerprints of superradiant condensates emerging in the quantum metamaterials [3–8]. This knowledge is also important for the quantum computation perspectives [9–12]. Recently, an approximate integrability of the Dicke model [13] was established [14] in the adiabatic limit with respect to photon condensate degree of freedom. We show below that, in the vicinity of the quantum phase transition into the superradiant state, the condensate characteristic frequencies obey the adiabaticity condition: $\Omega_n \ll \omega_0$, where $\hbar\omega_0$ is the bare TL splitting. This allows an analytic solution of the semiclassical dynamics equation for the superradiant condensate. This solution is based on the two integrals of motion possessed by the coupled photonic condensate and the TL system in the adiabatic limit. We found solutions that bear unexpected parallelism with the evolution of the polar angle made by the pivoted axis of the Euler symmetric spinning top with direction of the external gravitational field, i.e., nodding of Lagrange "sleeping top" [15–19]. We demonstrate that semiclassical dynamics of the superradiant condensate in the resonant cavity, and, hence, of the Lagrange sleeping top nodding, could be studied by measuring frequency dependence of the transmission coefficient of a transmission line inductively coupled to the cavity. This follows from the derived below analytic expression featuring sharp drops of transmission coefficient at the characteristic frequencies of the superradiant condensate's spectrum. Our analytic solution indicates that patterns of the transmission coefficient drops along the frequencies axis depend on the coupling strength of the TL system to the microwave cavity photons.

II. DICKE HAMILTONIAN IN ADIABATIC APPROXIMATION

In this article, we consider the Dicke model Hamiltonian expressed in terms of the operators of collective variables,

$$\hat{H} = \frac{\omega}{2}(\hat{p}^2 + \hat{q}^2) + \frac{2\gamma}{\sqrt{S}}\hat{q}\hat{S}^x + \omega_0\,\hat{S}^z, \tag{1}$$

where $\hat{S}^\alpha = \sum_i \hat{s}_i^\alpha$ are Cartesian components of the total pseudospin of the TL system, spin-1/2 Pauli operators $\hat{s}_i^\alpha$ characterize states of the ith TL, and the Planck constant $\hbar$ is taken for unity. We use here and below notations introduced in Ref. [14], where $\hbar = 1$. The photon field second quantized operators are as follows:

$$\hat{p} = i\sqrt{\frac{1}{2}}(\hat{a}^\dagger - \hat{a}) \quad \text{and} \quad \hat{q} = \sqrt{\frac{1}{2}}(\hat{a}^\dagger + \hat{a}), \tag{2}$$

where $[\hat{a}, \hat{a}^\dagger] = 1$. The superradiant regime is achieved for $f^2 = \gamma^2/\gamma_c^2 > 1$, $\gamma_c = \sqrt{\omega\omega_0}/2$ [1]. Using the Born-Oppenheimer approximation for a slow semiclassical motion of the photonic condensate on the background of fast coherent TL systems [14] one may substitute operators $\hat{p}$ and $\hat{q}$ with c numbers considered as fixed parameters with respect to

*i.m.sergei.m@gmail.com
†abhikmukherjeesinp15@gmail.com

the fast superspin degrees of freedom. Then, the spin part in Eq. (1) is diagonalized using rotation angle θ around the y axis (compare [14,20]),

$$\hat{S}^z \cos\theta + \hat{S}^x \sin\theta = \hat{J}^z, \tag{3}$$

$$\cos\theta = \frac{\omega_0}{\omega_P(q)}, \qquad \sin\theta = \frac{2q\gamma}{\omega_P(q)\sqrt{S}}, \tag{4}$$

$$\omega_P(q) = \omega_0\sqrt{1 + f^2\frac{q^2\omega}{S\omega_0}}, \tag{5}$$

and, consequently, one obtains the Hamiltonian with adiabatic invariant $\hat{J}^z$,

$$\hat{H}_a = \frac{\omega}{2}(p^2+q^2) + \omega_P(q)\hat{J}^z. \tag{6}$$

Since $\hat{H}_a$ commutes with $\hat{J}^z$, the lowest-energy band is reached in state $|S,-S\rangle$ with $\hat{J}^z|S,-S\rangle = -S|S,-S\rangle$. Substituting this spin projection into Eq. (6), one finds effective Hamiltonian of the condensate,

$$H_a(S,-S) = \frac{\omega}{2}(p^2+q^2) - \omega_P(q)S. \tag{7}$$

Hence, Eq. (7) describes a "particle" moving in the potential,

$$U(q) = \frac{\omega}{2}q^2 - \omega_0 S\sqrt{1 + f^2\frac{\omega q^2}{\omega_0 S}}. \tag{8}$$

Considering the first integral of motion of the Hamiltonian (7) of a particle with effective mass $1/\omega$, one finds the differential equation for the dynamic variable q,

$$H_a(S,-S) = \frac{\dot{q}^2}{2\omega} + U(q) = E. \tag{9}$$

Now we consider the case when the square root in Eq. (8) can be expanded in powers of coordinate q, i.e., the following condition holds:

$$f^2\frac{\omega q^2}{\omega_0 S} \ll 1. \tag{10}$$

Then, expanding the root in Eq. (8) up to fourth order in q and substituting it into Eq. (9), one obtains the equation of motion of a particle in the double-well potential $U_{\rm dw}$ (the constant term $-\omega_0 S$ is absorbed by constant E),

$$E = \frac{\dot{q}^2}{2\omega} + \frac{\omega_0}{8S}\left(f^2\frac{\omega}{\omega_0}\right)^2 q^4 + \frac{\omega}{2}(1-f^2)q^2 \equiv \frac{\dot{q}^2}{2\omega} + U_{\rm dw}(q). \tag{11}$$

The condition of superradiance $f^2 > 1$ makes the last term negative, thus, forming a double-well potential.

A. Applicability of the series expansion

The minima $\pm q_{\rm min}$ of $U_{\rm dw}(q)$ are found readily from the condition $\partial_q U_{\rm dw}(q)|_{q_{\rm min}} = 0$. After substitution $q_{\rm min}$ into applicability condition Eq. (10), one obtains the following inequality condition:

$$f^2\frac{\omega q_{\rm min}^2}{\omega_0 S} \equiv \frac{2(f^2-1)}{f^2} \ll 1. \tag{12}$$

Substituting $q_{\rm min}$ into $U_{\rm dw}(q)$ and using the condition in Eq. (12), one finds

$$|U_{\rm dw}(q_{\rm min})| \equiv |U_{\rm dw}|_{\rm min} = \frac{(f^2-1)^2\omega_0 S}{2f^4} \ll \frac{\omega_0 S}{8}. \tag{13}$$

Hence, the approximate polynomial expression in Eq. (11) for potential energy is safe to use in the close enough vicinity of the phase transition $f^2 \to 1+0$. Therefore, allowing for the limitation Eq. (12), it seems, at first glance, that the adiabaticity condition for the Hamiltonian Eq. (6) would be $\omega \ll \omega_P$, which in our case of $f \approx 1$ could be achieved via inequality $\omega \ll \omega_0$, i.e., far from the resonance: $\omega = \omega_0$, compare Ref. [14]. We will see below that this is not the case in the vicinity of the saddle-point energy $|E| \leqslant |U_{\rm dw}|_{\rm min}$, when adiabaticity is granted already by $f^2 \to 1+0$ itself, even though the resonant condition $\omega_0 = \omega$ holds.

Also one can calculate value of $\sin\theta$ in (4) at $q = q_{\rm min}$, the result is

$$\sin\theta(q_{\rm min}) = \sqrt{1 - \frac{\gamma_c^4}{\gamma^4}} \equiv \sqrt{1 - f^{-4}}. \tag{14}$$

Exactly this result was obtained for the superradiant phase in the Dicke model using the rotated Holstein-Primakoff transformation [20]. Thus, θ is the superradiant angle, which describes (pseudo)spin rotation from the z axis to the x axis under the superradiant phase transition.

B. Applicability of the adiabatic approximation

As the system evolves and the photonic coordinate $q(t)$ changes with time, there exists some probability of tunneling to the upper energy band at $q = 0$ where the gap between the $|S,-S\rangle$ and $|S,-S+1\rangle$ bands is the smallest due to Landau-Zener tunneling. One can think about the problem in a following way: The spin subsystem is controlled by an external field $q(t)$, and its spectrum is changing in time as $q(t)$ changes in time. The original Landau-Zener problem considers the time-dependent Hamiltonian in form [21]

$$H_{\rm LZ} = \begin{pmatrix} \alpha t & \Delta \\ \Delta & -\alpha t \end{pmatrix}. \tag{15}$$

Then if, for example, the particle was initially in the ground-state (GS) $|g\rangle$ with energy $-\sqrt{\alpha^2t^2+\Delta^2}$ at $t = -\infty$, the probability of it ending up in an exited-state $|e\rangle$ with energy $\sqrt{\alpha^2t^2+\Delta^2}$ at $t = \infty$ (Landau-Zener tunneling) is given by

$$P_{\rm LZ} = e^{-\pi\Delta^2/2\hbar|\alpha|}. \tag{16}$$

From this expression follows that the bigger is the gap Δ or smaller is the rate of change of energy α, the smaller is the tunneling probability.

In our case, when considering tunneling from spin state $|J_z = -S\rangle$ to state $|J_z = -S+1\rangle$ due to a change in time of $q(t)$, we restrict the whole phase space to only the considered subspace, and the spin part of Hamiltonian (1) is written as a 2×2 matrix,

$$H_S = \begin{pmatrix} \omega_0(-S+1) & \sqrt{2}\gamma q(t) \\ \sqrt{2}\gamma q(t) & -\omega_0 S \end{pmatrix}. \tag{17}$$

In order to bring it in form (15), we perform a unitary transformation,

$$
\begin{aligned}
&H_S \to O^\dagger H_S O \\
&= \begin{pmatrix} \sqrt{2}\gamma q(t) + \omega_0/2 & -\omega_0/2 \\ -\omega_0/2 & -\sqrt{2}\gamma q(t) + \omega_0/2 \end{pmatrix} - \omega_0 S, \\
&O = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\end{aligned} \tag{18}
$$

The rate of change α is obtained by linearization,

$$
2\gamma q(t) \approx 2\gamma \dot{q}(t_0)(t - t_0). \tag{19}
$$

Thus, the rate α is $\alpha = 2\gamma\dot{q}(t_0)$. Finally, the transition probability is

$$
P_{\rm LZ} = \exp\left\{-\frac{\pi\hbar^2\omega_0^2}{|4\hbar\gamma\dot{q}(t_0)|}\right\}. \tag{20}
$$

The separation between nonadiabatic energy levels $\pm 2\gamma q(t) + \omega_0/2$ is large far from the point $q = 0$, thus, the speed $\dot{q}(t)$ might not be small in this region. However, near $q = 0$ the gap reduces and in order to suppress the tunneling of the rate of change of $q(t)$ should be slow. The velocity $\dot{q}(t)$ at $q = 0$ can be easily found using the energy conservation law (11),

$$
\dot{q}|_{q=0} = \sqrt{\frac{2\omega}{\hbar}[E - U_{\rm dw}(0)]} = \sqrt{\frac{2\omega}{\hbar}E}. \tag{21}
$$

The closer is the total energy to the maximum of the potential (11) at $E = 0$ from above, the less is the particle's speed when passing the maximum. And if the energy is below the maximum, i.e., $E < 0$, the particle does not even reach the point $q = 0$ at the maximum of the potential, so the tunneling to the upper band is suppressed by a large energy gap. Finally, we express the Landau-Zener tunneling probability via energy,

$$
P_{\rm LZ}|_{E>0} = \exp\left\{-\frac{\pi\hbar^2\omega_0^2}{4\gamma\sqrt{2\hbar\omega E}}\right\}. \tag{22}
$$

Later, in Sec. IV, we will calculate the Landau–Zener tunneling probability for energy E equal to the ground-state energy of the Dicke model.

III. SOLUTION IN JACOBI FUNCTIONS

Analytic solutions of equation of motion Eq. (11) in the quartic double-well potential are well known [22] and found below in the form of the different Jacobi elliptic functions, depending on the value of E of the total energy of the system. Using conservation law (11), the motion of the photonic subsystem can be described by the differential equation,

$$
\frac{\dot{q}^2}{2\omega} = E - \frac{\omega^2 f^4}{8\omega_0 S}q^4 - \frac{\omega}{2}(1 - f^2)q^2. \tag{23}
$$

If the total energy of the system is positive: $E > 0$, i.e., the particle has enough momentum to surpass the potential barrier at $q = 0$, the solution is as follows:

$$
\begin{aligned}
&q(t)_{E>0} = A\ {\rm cn}(\Omega t, k), \\
&A^2 = \frac{2S(f^2-1)}{f^4}\frac{\omega_0}{\omega}\left(1 + \sqrt{1 + \frac{2f^4 E}{S\omega_0(1-f^2)^2}}\right), \\
&\Omega^2 = \omega^2(f^2-1)\sqrt{1 + \frac{2f^4E}{S\omega_0(1-f^2)^2}}, \\
&k^2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \frac{2f^4E}{S\omega_0(1-f^2)^2}}}\right).
\end{aligned} \tag{24}
$$

One should note that $q(t)$ is a sign changing function, meaning that the particle travels between the two wells $q = \pm q_0$.

For negative total energy $E < 0$, the solution is as follows:

$$
\begin{aligned}
&q(t)_{E<0} = \pm A\, dn(\Omega t, k), \quad A^2 = \frac{\omega_0}{\omega}\frac{4S(1-f^2)2z}{f^4(1-\sqrt{4z+1})}, \\
&\Omega^2 = \frac{\omega^2(1-f^2)2z}{1-\sqrt{4z+1}}, \quad k^2 = \frac{\sqrt{4z+1}}{2z}(\sqrt{4z+1}-1), \\
&z = \frac{Ef^4}{2\omega_0 S(1-f^2)^2}.
\end{aligned} \tag{25}
$$

In this case $q(t)$ does not change sign, so that particle remains in one of the two potential wells.

If the total energy is zero $E = 0$, the period of motion becomes infinite and the solution is as follows:

$$
q(t)_{E=0} = \pm\frac{2\sqrt{S}}{f^2}\sqrt{\frac{\omega_0}{\omega}}(f^2-1){\rm sech}(\omega t\sqrt{f^2-1}). \tag{26}
$$

This solution describes particle moving infinitely long from the "turning point" at $q = q(t = 0)$ to the saddle point at $q(t = \infty) = 0$. In all the solutions above, the inequality in Eq. (12) ensures adiabaticity condition $\Omega \ll \omega_0$ provided that $\omega \leqslant \omega_0$, and,

$$
|E| \leqslant |U_{\rm dw}|_{\rm min} = \frac{(f^2-1)^2\omega_0 S}{2f^4}. \tag{27}
$$

Simultaneously, the semiclassical approximation is justified by the proportionality of the photonic condensate coordinate $q(t) \propto \sqrt{S} = \sqrt{N/2}$, where N is the macroscopic number of the TLs.

IV. GROUND-STATE ENERGY OF THE DICKE MODEL

Our solutions above describe quasiclassical motion with arbitrary energy. However, in the quantum case, the energy of course should be quantized, and its arbitrary values are not allowed. In analogy with the Bohr model of an electron in an atom, we consider discrete energy levels of the Dicke model as allowed values of energy of quasiclassical motion. In particular, the analytical expression for the ground-state energy $E_{\rm GS}$ of the Dicke model in the superradiant phase can be obtained in the thermodynamic limit of large S [1,20], e.g., see Eq. (B15) in Ref. [20]. In the notation, used in the current

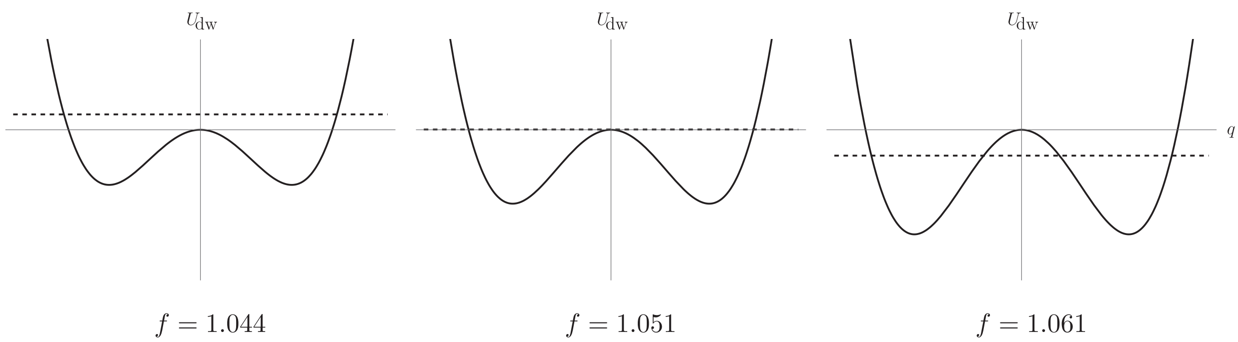

FIG. 1. Illustration of the ground-state energy level (dashed line) of the Dicke model with respect to the maximum of the effective potential in (11) (solid line). The parameters are chosen as $\omega = 1,\ \omega_0 = 5,\ S = 10$. As parameter f grows, the ground-state energy level drops below the potential-energy maximum.

paper, the expression becomes

$$E_{\rm GS} = \omega_0 S - \frac{\omega_0 f^2}{2}(S+1) - \frac{\omega_0}{2f^2}S + \frac{1}{2}(\varepsilon_1 + \varepsilon_2),$$

$$\varepsilon_{1,2}^2 = \frac{1}{2}\Big[\omega_0^2 f^4 + \omega^2 \pm \sqrt{\left(\omega_0^2 f^4 - \omega^2\right)^2 + 4\omega^2\omega_0^2}\Big]. \quad (28)$$

This expression is valid for $f \geqslant 1$. Simultaneously, the value of the potential-energy U in (11) at its maximum equals zero. In particular, in the case of $f = 1$,

$$E_{\rm GS}|_{f=1} = \frac{1}{2}\left(\sqrt{\omega^2 + \omega_0^2} - \omega_0\right) > 0. \quad (29)$$

This means that at the superradiant quantum phase transition the ground-state energy is above the potential barrier. Thus, the quasiclassical motion with total energy $E_{\rm GS}$ in this regime is described by the Jacobi cn function, see Eq. (24). As the coupling constant grows further, at some value of $f > 1$, the energy of the ground state drops below the top of the potential barrier, i.e., $E_{\rm GS} < 0$. This is illustrated in Fig. 1. By substituting (29) in (22) and in limit $\omega_0 \gg \omega$ we obtain the following expression for the probability of Landau-Zener tunneling for the quasiclassical dynamics with the energy, defined by the energy of the ground state of the quantum model at the point of the superradiant phase transition,

$$P = \exp\left\{-\frac{\pi\omega_0^2}{\sqrt{2}\omega^2}\right\}. \quad (30)$$

This expression defines the upper bound for the probability of Landau-Zener tunneling. The tunneling is suppressed for $\omega_0 > \omega$, which is an agreement with the result for applicability of adiabatic approximation [14] according to which the nonadiabatic coefficient,

$$C = \langle S, -S; q|\hat{p}|S, -S+1; q\rangle|_{q=0} = \frac{f}{2}\sqrt{\frac{\omega}{\omega_0}} \quad (31)$$

should be small.

V. MEASURING NORMAL PHASE BREAKDOWN VIA THE TRANSMISSION LINE COUPLED TO THE PHOTONIC CONDENSATE

The normal phase of the Dicke model would correspond to the case of a single-well potential in Eq. (11) when $f^2 < 1$ and $q = 0$ at the minimum of the effective potential energy. The emergence of the superradiant condensate at $f^2 > 1$ causes transition to a bifurcating equilibria, i.e., to the double-well potentials in Eq. (11), thus, making the normal phase unstable. Simultaneously, the semiclassical motions of the photonic condensate Eqs. (24) and (25) could be measured, e.g., using transmission line setup [5]. In this setup, an electromagnetic wave $Q(y,t)$ propagating in the transmission line (along axis y) is described by the Maxwell propagation equation of an electromagnetic wave coupled linearly to the superradiant condensate via semiclassical coordinate $q(t)$,

$$\frac{\partial^2 Q(y,t)}{c^2\partial t^2} - \frac{\partial^2 Q(y,t)}{\partial y^2} = \kappa\delta(y - y_0)q(t), \quad (32)$$

$$H = H_a(S, -S) - \kappa q(t)Q(y_0, t), \quad (33)$$

where c is the electromagnetic wave propagation velocity and κ is the strength of inductive coupling between the transmission line and the microwave cavity that contains the condensate described by the unperturbed Hamiltonian Eq. (9). Considering now the last term in Eq. (33) as a perturbation, one finds a response $q_1(t)$ of the condensate, linear in $\kappa Q(y_0,t)$, that follows from Eq. (11):

$$\frac{\ddot{q}_1}{\omega} + \frac{\partial^2 U_{dw}(q_0)}{\partial q^2}q_1 \equiv \hat{L}q_1 = \kappa Q(y_0, t), \quad (34)$$

where unperturbed condensate motions $q_0(t)$, expressed in Eqs. (24)–(26), obey Eq. (11). Allowing for the latter, one finds that Eq. (34) is of the Lamé type with "external force" $\kappa Q(y_0,t)$ and, hence, its solution looks like,

$$q_1(t) = \int_0^t G^R(t - t')\kappa Q(y_0, t')dt' \quad (35)$$

$$\hat{L}G^R(t - t') = \delta(t - t'); \quad G^R(t - t' < 0) \equiv 0, \quad (36)$$

where retarded Green's function G^R of the Lamé differential operator $\hat{L}$ is introduced. Substituting Eq. (35) into the right-hand side of Maxwell's equation Eq. (32) and making its Fourier transform with respect to time-variable t, one finds a Schrödinger-like equation for a scattering from the Dirac-δ function potential barrier, which results in a transmission coefficient $D(\tilde{\omega})$ of the transmission line for a wave traveling

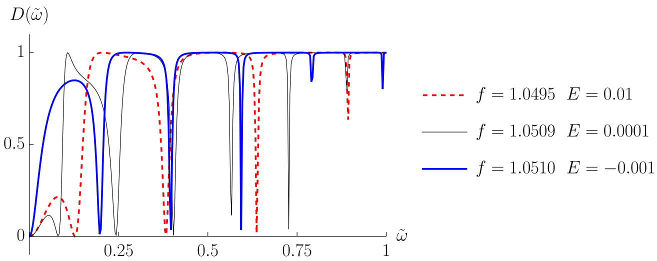

FIG. 2. Frequency dependences of the transmission coefficient $D(\tilde{\omega})$ Eq. (38) for a transmission line inductively coupled to the microwave cavity with the superradiant condensate at different values of the coupling strength f that enters Eqs. (24) and (25). Propagation velocity c and coupling constant κ in Eq. (33) obey relation: $c\kappa^2 = 0.0225$. Other parameters are the same as in Fig. 4.

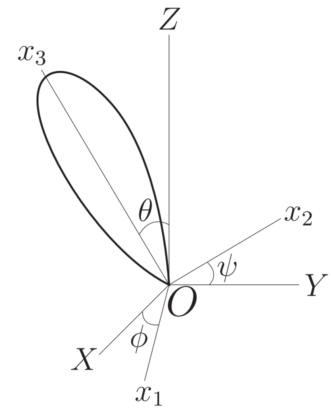

FIG. 3. Scheme of the Euler top with pivoted point of the x_3 axis and the definition of the Euler angles.

with the frequency $\tilde{\omega}$ [23],

$$-\partial_y^2 Q_{\tilde{\omega}} - \kappa^2 G^R(\tilde{\omega})\delta(y-y_0)Q_{\tilde{\omega}} = \frac{\tilde{\omega}^2}{c^2}Q_{\tilde{\omega}}, \qquad (37)$$

$$D(\tilde{\omega}) = \left|\frac{2\tilde{\omega}}{2\tilde{\omega} - ic\kappa^2 G^R(\tilde{\omega})}\right|^2. \qquad (38)$$

The Green's function $G^R(t)$ is easily constructed allowing for the fact that zero modes of Lamé differential operator $\hat{L}\,\partial_t q_0 = 0$ are just the first time derivatives of the corresponding unperturbed solutions, $q_0(t)$ of the Hamiltonian dynamics Eq. (11) expressed via the Jacobi elliptic functions in Eqs. (24) and (25),

$$G^R(t \geqslant 0) = \begin{cases} -\dfrac{\omega}{\Omega^2}\partial_t \mathrm{cn}(\Omega t, k), & E > 0 \\ -\dfrac{\omega}{\Omega^2}\partial_t dn(\Omega t, k), & E < 0, \end{cases}$$
$$G^R(t < 0) \equiv 0. \qquad (39)$$

Using Fourier series expansions of the Jacobi elliptic functions [22] $\mathrm{cn}(\Omega t, k), dn(\Omega t, k)$ one finds $G^R(\tilde{\omega})$ as the sum of the Green's functions $G^{0R}_{\Omega_n}(\tilde{\omega})$ of harmonic oscillators with frequencies forming (half)integer multiples of photonic condensate frequency $\pi\Omega/K$,

$$G^R_{E>0}(\tilde{\omega}) = \frac{\pi^2\omega}{2\Omega k K^2}\sum_{n=0}^{\infty}\frac{(2n+1)G^{0R}_{\Omega_n}(\tilde{\omega})}{\mathrm{ch}[(2n+1)\pi K'/2K]},$$

$$\Omega_n = \frac{(2n+1)\pi\Omega}{2K}, \qquad (40)$$

$$G^R_{E<0}(\tilde{\omega}) = \frac{\pi^2\omega}{\Omega K^2}\sum_{n=1}^{\infty}\frac{nG^{0R}_{\tilde{\Omega}_n}(\tilde{\omega})}{\mathrm{ch}[n\pi K'/K]}, \quad \tilde{\Omega}_n = \frac{n\pi\Omega}{K}, \qquad (41)$$

$$G^{0R}_{\Omega}(\tilde{\omega}) = \frac{1}{2}\left[\frac{1}{\tilde{\omega}+\Omega+i\delta} - \frac{1}{\tilde{\omega}-\Omega+i\delta}\right],$$
$$K' = K(\sqrt{1-k^2}). \qquad (42)$$

Corresponding frequency dependences of the transmission coefficient $D(\tilde{\omega})$ for the transmission line are presented in Fig. 2. The presence of the superradiant condensate is reflected by the sharp transmission drops at the frequencies belonging to the Fourier spectrum of the semiclassical motion of the condensate, the latter being marked by the poles of the Green's function in Eqs. (40) and (41). Relative narrowing of the intervals between transmission coefficient sharp drops in Fig. 2 when one moves from $E = 0.01$ to $E = 0.0001$, i.e., more close to the saddle-point energy of photonic condensate $E = 0$, is remarkable. It is related with the fact that solutions for k values in Eqs. (24) and (25) provide the limit $k \to 1$ when $E \to 0$. Since the complete elliptic integral of the first kind $K(k \to 1) \to \infty$ [22] the frequencies spectrum Ω_n of the superradiant condensate given in Eqs. (40) and (41), condenses in the direction of zero frequency.

VI. MAPPING ON DYNAMICS OF THE SPINNING LAGRANGE TOP

Using definition of the θ angle in Eq. (4) in combination with equation of motion of the photon field coordinate q in Eq. (11) in the limit Eq. (10), it is straightforward to find a striking coincidence of the photonic coordinate dynamics with the dynamics of the Euler θ angle of a symmetric top in gravitational field [15], the Lagrange top. For this purpose, we remind this fundamental problem in classical mechanics. It is well known that the Euler-Poisson equations of motion of a rigid symmetric top that moves about a fixed point under the action of a gravitational force (the Lagrange case) are integrable [15]. The energy conservation law for this case is written as

$$\tilde{E} = \frac{\tilde{I}_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \mu g l\cos\theta. \qquad (43)$$

Here the Euler angles $\theta,\ \phi,$ and ψ are introduced as usual [15], relative to the laboratory $x,\ y,$ and z coordinate system, and gravity acts in the negative direction of the polar z-axis direction, see Fig. 3.

Then, angle θ is formed by the z axis and the axis x_3 of the rotating top, i.e., one of its three principal axes $x_{1\text{–}3}$ with the corresponding moments of inertia $I_1 = I_2, I_3$. The pivoting point and the center of mass both lie on the x_3 axis, separated by distance l.

Constants μ and g are the mass and the acceleration of the top in the gravitational field, and $\tilde{I}_1 \equiv I_1 + \mu l^2$. Since the

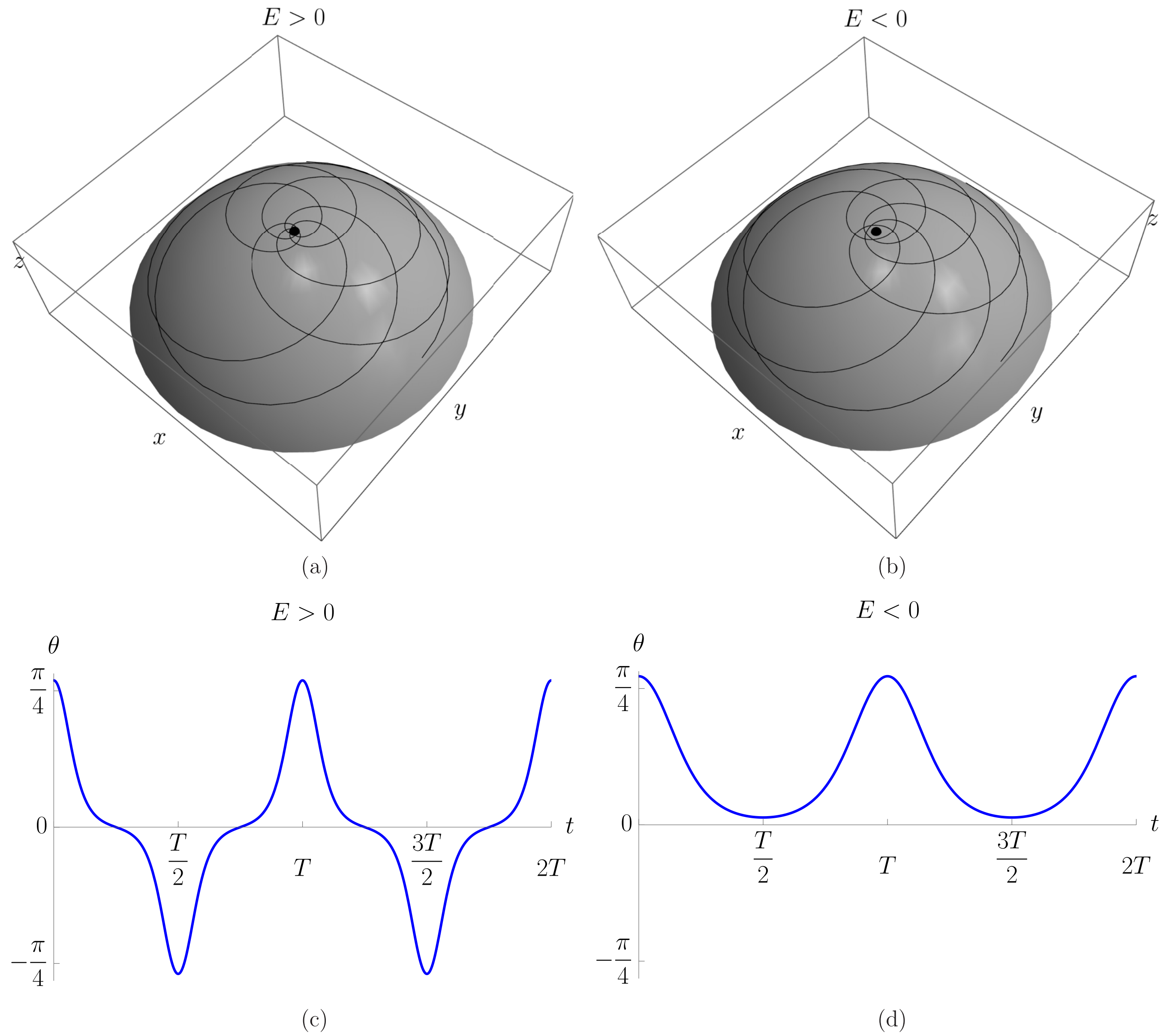

FIG. 4. Trajectories on the sphere of the free end of the x_3 axis of the spinning Lagrange top with: (a) positive energy $E = 0.01$ and $f = 1.21$, (b) negative energy $E = -0.03$ and $f = 1.22$. Other parameters of the Dicke model Hamiltonian Eq. (1): $\omega = 1,\ \omega_0 = 5,\ S = 10$. Corresponding time evolution of the angle $\theta(t)$, (c) and (d) are taken from Dicke model solutions Eqs. (24) and (25) and definition Eq. (4) of angle θ. T is the oscillation time period of the corresponding Jacobi function.

conjugate momenta corresponding to the cyclic angles ϕ, ψ are conserved, one obtains the following extra integrals of motion:

$$p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = M_3, \tag{44}$$

$$p_\phi = (\tilde{I}_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\,\cos\theta = M_z, \tag{45}$$

where integrals of motion M_3, M_z are the angular moments of the top along the axes x_3 and z, respectively, counted with respect to the fixed point O of the top. Now, excluding variables $\dot{\phi}, \dot{\psi}$ from expression in Eq. (43) using Eqs. (44) and (45), and considering a particular case $M_3 = M_z \equiv M$, one finds instead of Eq. (43),

$$\tilde{E} - \frac{M^2}{2I_3} - \mu gl = \frac{\tilde{I}_1}{2}\dot{\theta}^2 + \frac{M^2}{2\tilde{I}_1}\tan^2\frac{\theta}{2} - 2\mu gl\,\sin^2\frac{\theta}{2}$$
$$\approx 2\tilde{I}_1\dot{Q}^2 + \frac{M^2}{2\tilde{I}_1}\left[Q^4 + Q^2\left(1 - \frac{4\mu gl\tilde{I}_1}{M^2}\right)\right], \tag{46}$$

$$\dot{\phi} = \frac{M}{2\tilde{I}_1(1-Q^2)}, \quad \dot{\psi} = \frac{M[I_3 + 2(\tilde{I}_1 - I_3)(1-Q^2)]}{2\tilde{I}_1 I_3(1-Q^2)}, \tag{47}$$

where smallness of the Euler angle θ is assumed: $\sin\theta/2 \equiv Q \ll 1$. Now, a direct comparison of Eq. (46) with Eq. (11), allowing for the inequality Eq. (10), leads to the conclusion that the pseudospin rotation angle θ defined in Eq. (4) possesses the same dynamics as the Euler angle of the Lagrange top θ that enters dynamics Eq. (46). Namely, the one-to-one correspondence between Eqs. (11) and (46) is achieved under the following conditions:

$$\mu gl/\tilde{I}_1 = \omega^2(2f^2-1), \qquad M/2\tilde{I}_1 = \omega f^{-1},$$
$$\frac{E}{2\omega_0 S} = \left[\tilde{E} - \frac{M^2}{2I_3} - \mu gl\right](M^2/2\tilde{I}_1)^{-1}. \tag{48}$$

The analytical solutions Eqs. (24) and (25) expressed via e angle θ as defined in Eq. (4), are plotted in the form of trajectories of the spinning Lagrange top(sleeping top noddings) on the sphere, see Fig. 4. The Euler angle ϕ depends linearly on time t according to Eqs. (47) and (48): $\phi \approx \omega f^{-1} t$. This time dependence of the rotation angle ϕ around Z axis was considered as a fast one with respect to $\theta(t)$ and averaging over $\phi(t)$ was made in the equations of motion in Ref. [24].

VII. CONCLUSIONS

In conclusion, we have considered the Dicke model in a superradiant state and found analytic solutions of the semiclassical dynamics of the photonic condensate in the adiabatic limit with respect to the bare cavity mode coupled to the two-level atomic array. We have also discovered one-to-one correspondence between photonic condensate dynamics and nodding motion of the Lagrange sleeping top. Finally, we found an analytic expression for the frequency dependence of the transmission coefficient of a transmission line inductively coupled to the microwave cavity with the superradiant photonic condensate. Predicted sharp transmission drops reflect Fourier spectrum of the semiclassical motion of photonic condensate and of a nodding sleeping top. This opens a way to observe directly the fingerprints of photonic condensates emerging in the quantum metamaterials, as well as to use QED circuits for a simulation of dynamics of the classical mechanical systems.

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