Single photons versus coherent-state input in waveguide quantum electrodynamics: Light scattering, Kerr, and cross-Kerr effect

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While the theoretical studies in waveguide quantum electrodynamics predominate with single-photon and two-photon Fock-state (photon number states) input, the experiments are primarily carried out using a faint coherent light. We create a theoretical toolbox to compare and contrast linear and nonlinear light scattering by a two-level or a three-level emitter embedded in an open waveguide carrying Fock-state or coherent-state inputs. We particularly investigate light transport properties and the Kerr and cross-Kerr nonlinearities of the medium for the two types of inputs. A generalized description of the Kerr and cross-Kerr effect for different types of inputs is formulated using the first-order correlation function to compare the Kerr and cross-Kerr nonlinearity between two photons in these models.

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I. INTRODUCTION

A decade of experimental activities has established waveguide quantum electrodynamics (QED) as an emergent research discipline [1,2]. One principal aim of this discipline is to explore strong light-matter interactions between a few propagating photons without any cavity along the direction of propagation. The waveguide QED systems promise to overcome many limitations of cavity QED systems for building quantum networks of light [1]. Many exciting phenomena [1–10] as well as fascinating all-optical devices [11–17] have been demonstrated in these systems. These have improved our fundamental understanding of quantum and nonlinear optics and led to higher sensitivity in quantum metrology and sensing.

Many of the early theoretical proposals in the waveguide QED systems [1,18–38] are utilizing single-photon and two-photon Fock-state (photon number states) inputs. Coherent-state inputs have also been investigated in some theoretical studies [23,39–47]. However, the experimental studies [4,6,7,12,14] in these systems predominately apply a weak light beam in a coherent state to explore the physics of single- or few-photon scattering from single or multiple emitters. It is, therefore, in many cases challenging to compare results for different types of inputs. It also remains unclear if there will be any fundamental difference in those studied phenomena if a true (antibunched) single-photon source is applied in comparison to a weak coherent state. In this paper, one of our goals is to develop a theoretical toolbox to compare linear and nonlinear scattering properties of light in the waveguide QED systems for Fock-state and weak or attenuated coherent-state inputs.

We primarily consider the scattering of one or two light beams by a two-level emitter (2LE) or a ladder-type three-level emitter (3LE) embedded in an open waveguide (see Fig. 1). The input beams consist of either one or two Fock-state photons or a weak coherent state. We apply two different theories for two types of inputs. For example, scattering theory is used for input photons in the Fock state. For weak coherent-state inputs, we write the Heisenberg equations for the time evolution of operators. Our theory for coherent-state input also works for an arbitrary strength of input beam(s). We mainly investigate the linear and nonlinear scattering of input beams and nonlinear interactions between photons of a single beam (Kerr effect) and multiple beams (cross-Kerr effect) generated by correlated scattering by the emitters. Both Kerr and cross-Kerr media allow for effective control of motion of one photon by another, which is an essential requirement in many all-optical devices and optical quantum gates, e.g., nonlinear optical diodes [17,22,44,48,49] and transistors [19]. Therefore, it is essential to quantify and understand which medium creates higher nonlinearity or effective interactions (either Kerr or cross-Kerr type) between two photons as that would be more useful in controlling the motion of one photon by another photon. Our main findings in this paper are the following.

(a) We show how to identify and compare the linear and nonlinear contributions in transport coefficient for a few-photon Fock-state input and a faint coherent-state input.

(b) We formulate a generalized description of the Kerr and cross-Kerr effect using a first-order correlation function for different types of inputs.

(c) We compare the Kerr nonlinearity between two single photons by a 2LE to the cross-Kerr nonlinearity between them by a 3LE.

In the following three sections, we explain our results in detail. We also add three appendices to give mathematical details of our derivations.

II. LIGHT SCATTERING BY 2LE

We first consider a 2LE side coupled to a one-dimensional continuum of photon modes inside an open waveguide [see Fig. 1(a)]. The difference in energy between the excited level

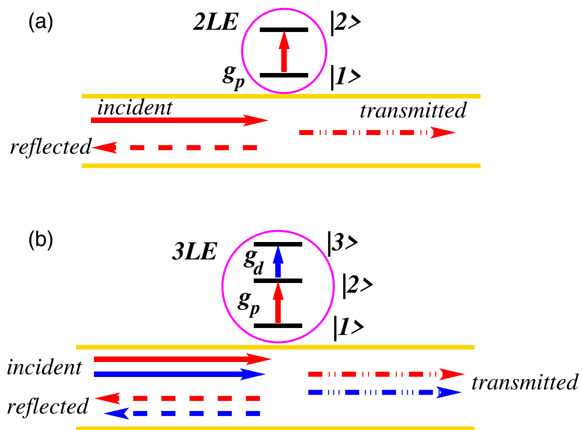

FIG. 1. Schematics of (a) a 2LE and (b) a ladder-type 3LE side coupled to an open waveguide carrying (a) a probe or (b) a probe and a drive beam. The coupling strengths of the probe beam (red) and the drive beam (blue) with the emitter are g_p and g_d, respectively.

$|2\rangle$ and ground level $|1\rangle$ of the emitter is $\hbar\omega_{21}$. Our calculation for the scattering of weak coherent-state inputs is within the Heisenberg picture of quantum mechanics; we then consider photon modes in momentum space and evaluate the time evolution of operators. The scattering of single photons in the Fock state is derived within the Schrödinger picture, and it is then convenient to take a real-space description of photon modes. Within the Schrödinger picture, the operators are time independent, and we find scattering states of the entire system using scattering theory. The photon modes in real space are related by the Fourier transform to those in the momentum space. The Hamiltonian of the whole system with a linearized energy-momentum dispersion (e.g., $\omega_k = v_g k$) of photons reads in momentum space as [50]

$$\frac{\mathcal{H}_2^k}{\hbar} = \omega_{21}\sigma^\dagger\sigma + \sum_k [v_g k(a_k^\dagger a_k - b_k^\dagger b_k) + g_p\sigma^\dagger(a_k + b_k) + g_p(a_k^\dagger + b_k^\dagger)\sigma], \tag{1}$$

where $\sigma^\dagger \equiv |2\rangle\langle 1|$ $(\sigma \equiv |1\rangle\langle 2|)$, and v_g is the group velocity of photons. Here, $a_k^\dagger$ $[b_k^\dagger]$ is the creation operator of right-moving (left-moving) photon modes. Within the rotating-wave approximation and the dipole approximation (a linear light-matter interaction), the coupling strength of the photon modes with the 2LE is given by g_p. To obtain a real-space description of propagating photons at position $x \in [-\mathcal{L}/2, \mathcal{L}/2]$, we define $\tilde{a}_x(t) = \sum_k e^{ikx} a_k(t)/\sqrt{\mathcal{L}}$ and $\tilde{b}_x(t) = \sum_k e^{ikx} b_k(t)/\sqrt{\mathcal{L}}$. Here, $\mathcal{L}$ is the length of the waveguide which can also be considered our quantization length. Thus, we get the following real-space version of the full Hamiltonian:

$$\frac{\mathcal{H}_2^x}{\hbar} = \omega_{21}\sigma^\dagger\sigma - iv_g\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx(\tilde{a}_x^\dagger\partial_x\tilde{a}_x - \tilde{b}_x^\dagger\partial_x\tilde{b}_x) + \bar{g}_p\sigma^\dagger(\tilde{a}_0 + \tilde{b}_0) + \bar{g}_p(\tilde{a}_0^\dagger + \tilde{b}_0^\dagger)\sigma, \tag{2}$$

where $\bar{g}_p = \sqrt{\mathcal{L}}g_p$.

We take incident light in the right-moving channel injected from the left of the emitter. The coherent-state input $|E_p, \omega_p\rangle$ is a monochromatic, continuous-wave beam of frequency ω_p $(\omega_p = v_g k_p)$ and amplitude E_p (which we here assume to be real). It is an eigenstate of a_k: $a_k(t_0)|E_p, \omega_p\rangle = (\sqrt{\mathcal{L}}E_p/v_g)\delta_{k,\omega_p/v_g}|E_p, \omega_p\rangle$, where t_0 is an initial time before the interaction of the input beam with the emitter. The intensity (total number of photons per unit length) of the incident coherent beam is $I_{cp} = \langle E_p, \omega_p|\sum_k a_k^\dagger(t_0)a_k(t_0)|E_p, \omega_p\rangle/\mathcal{L} = E_p^2/v_g^2$. The single- and two-photon Fock-state inputs with a wave-vector k_p are, respectively,

$$|k_p\rangle = \frac{1}{\sqrt{\mathcal{L}}}\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx\, e^{ik_p x}\tilde{a}_x^\dagger|\varphi\rangle,$$

$$|\boldsymbol{k}_p\rangle = \frac{1}{\mathcal{L}}\int_{-\mathcal{L}/2}^{\mathcal{L}/2}\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx_1 dx_2\, e^{ik_p(x_1+x_2)}\frac{1}{\sqrt{2}}\tilde{a}_{x_1}^\dagger\tilde{a}_{x_2}^\dagger|\varphi\rangle,$$

where $|\varphi\rangle$ denotes the vacuum of the electromagnetic fields. The intensities of a single-photon and a two-photon incident beam are, respectively, $I_{1p} = \langle k_p|\sum_k a_k^\dagger a_k|k_p\rangle/\mathcal{L} = 1/\mathcal{L}$ and $I_{2p} = \langle \boldsymbol{k}_p|\sum_k a_k^\dagger a_k|\boldsymbol{k}_p\rangle/\mathcal{L} = 2/\mathcal{L}$. We assume the emitter in the ground state at t_0. We evaluate the time evolution of the incident coherent state and the emitter using the Heisenberg equations following Refs. [39,44]. The finding of outgoing scattering states for the single- and two-photon inputs is carried out following Refs. [18,22,23]. The details of the calculation in both cases are given in Appendices A and C.

To quantify linear and nonlinear light scattering, we calculate transport properties such as the reflection and the Kerr and cross-Kerr phase shifts of the transmitted photon(s). For a side-coupled emitter, the reflection of light is a measure for the transfer of photons from the incident right-moving mode(s) to the left-moving mode(s). The reflection of light is generally quantified by the reflection coefficient, which can be obtained from reflection current $\mathcal{J}$ after dividing $\mathcal{J}$ by incident light intensity and group velocity. We define reflection current [44] as

$$\mathcal{J} = i\bar{g}_p\langle(\sigma^\dagger\tilde{b}_0 - \tilde{b}_0^\dagger\sigma)\rangle, \tag{3}$$

where $\langle\dots\rangle$ within the Schrödinger picture is an expectation in the full states (e.g., $|k_p^+\rangle$ and $|\boldsymbol{k}_p^+\rangle$ in Appendix A) of $\mathcal{H}_2^x$ after the scattering of incident light by the 2LE, and the operators in Eq. (3) are time independent. For the Heisenberg picture used in coherent-state input, the expectation is carried out in $|E_p, \omega_p\rangle$ but the operators in Eq. (3) are evolved to a time which is much later after the scattering by 2LE takes place. We denote reflection current for single-photon and two-photon input, respectively, by $\mathcal{J}_1$ and $\mathcal{J}_2$, which are

$$\mathcal{J}_1 = \frac{v_g}{\mathcal{L}}\frac{4\Gamma_p^2}{\Delta_p^2 + 4\Gamma_p^2} = v_g I_{1p}\frac{4\Gamma_p^2}{\Delta_p^2 + 4\Gamma_p^2}, \tag{4}$$

$$\mathcal{J}_2 = v_g I_{2p}\frac{4\Gamma_p^2}{\Delta_p^2 + 4\Gamma_p^2} - (v_g I_{1p})^2\frac{16\Gamma_p^3}{\left(\Delta_p^2 + 4\Gamma_p^2\right)^2}, \tag{5}$$

where the relaxation rate $\Gamma_p = \bar{g}_p^2/(2v_g)$, and the detuning $\Delta_p = \omega_p - \omega_{21}$. The reflection coefficient of a single photon is $\mathcal{R}_{1p} \equiv \mathcal{J}_1/v_g I_{1p} = |r_{1p}|^2 = 4\Gamma_p^2/(\Delta_p^2 + 4\Gamma_p^2)$, which is one for a resonant photon, i.e., $\Delta_p = 0$. The first part of $\mathcal{J}_2$ gives an independent reflection of two individual photons by the 2LE, and the reflection coefficient of this process is the same as $\mathcal{R}_{1p}$. The second part of $\mathcal{J}_2$ denotes the correlated reflection of two photons by 2LE, and its strength for resonant photons decreases with increasing light-matter coupling $\bar{g}_p$. While the probability of a photon inside the waveguide interacting with the 2LE is an order of $1/\mathcal{L}$, that for two photons simultaneously interacting with the 2LE is an order of $1/\mathcal{L}^2$. Therefore, the correlated scattering of two photons is smaller than the individual photon reflection by one order of $\mathcal{L}$. While the strength of correlated scattering of order $1/\mathcal{L}^2$ grows with the increasing number of incident photons, there also appear higher-order terms of $1/\mathcal{L}^m$ with $m > 2$ in correlated reflection of m photons for a finite-length waveguide.

The reflection current for a coherent-state input is found to be

$$\begin{aligned}\mathcal{J}_c &= \frac{2\Gamma_p\Omega_p^2}{\Delta_p^2+4\Gamma_p^2+2\Omega_p^2},\\ &= v_g I_{cp}\frac{4\Gamma_p^2}{\Delta_p^2+4\Gamma_p^2} - (v_g I_{cp})^2\frac{16\Gamma_p^3}{\left(\Delta_p^2+4\Gamma_p^2\right)^2} + O\left(\Omega_p^6\right),\end{aligned} \tag{6}$$

where $\Omega_p = \bar{g}_p E_p/v_g$ is the Rabi frequency of the incident coherent state. Thus, $I_{cp} = \Omega_p^2/(2v_g\Gamma_p)$. For a faint coherent-state input [i.e., $\Omega_p^2/(2\Gamma_p^2) \ll 1$], we find $\mathcal{R}_{1p}$ matches to $\mathcal{R}_{cp} \equiv \mathcal{J}_c/v_g I_{cp}$ when we drop $2\Omega_p^2$ from the denominator of $\mathcal{J}_c$ in Eq. (6) in the limit $\Omega_p \to 0$. We further identify the correlated reflection contribution in $\mathcal{J}_2$ from $\mathcal{J}_c$ by expanding it up to the order of Ω_p^4. The above analysis can be generalized to relate the reflection current $\mathcal{J}_m$ for m number of Fock photons with $\mathcal{J}_c$ through its expansion up to Ω_p^{2m}. We have extended the Fock-state analysis to $m = 3$ to confirm the above generalization. A weak coherent-state input is a superposition (with the corresponding weights) of a vacuum state, a single-photon Fock state, a two-photon Fock state, and so on. Therefore, $\mathcal{J}_2$ is not the same as $\mathcal{J}_c$ by expanding up to Ω_p^4 in the second line of Eq. (6), which is evident from the appearance of I_{1p} and I_{2p} in $\mathcal{J}_2$. Nevertheless, our comparison or identification using explicit formulas for these currents would help one to figure out $\mathcal{J}_2$ by measuring $\mathcal{J}_c$ or vice versa.

III. KERR EFFECT

The nonlinear scattering of an input beam can be characterized by the so-called optical Kerr effect, in which the refractive index of any optical medium depends on the beam's intensity. In such a case, the refractive index n can be separated into linear and nonlinear parts as [51] $n = n_0 + \bar{n}_2 E_p^2$, where n_0 is the weak-beam (or single-photon) linear part of the refractive index, and $\bar{n}_2$ is a coefficient representing the nonlinear refractive index. The linear and nonlinear refractive indices are proportional to the linear and nonlinear susceptibilities. For light scattering by a single emitter inside the waveguide, we can relate the complex susceptibility of the medium to the change in phase $\phi_p = \phi_p^{(1)} + \phi_p^{(2)}$ of coherently scattered photons [51] where $\phi_p^{(1)}$ is the linear change in phase for a weak beam (or a single photon) and $\phi_p^{(2)}$ is the nonlinear (two-photon) contribution. Thus, we have $\Delta n = n - n_0 \propto \phi_p^{(2)}$. In the regime of recent experimental interest with few photons [3,5,6], we can write an approximate relation to define the Kerr coefficient K as $\phi_p^{(2)} \equiv \phi_p - \phi_p^{(1)} = KE_p^2$.

For a coherent-state input, ϕ_p can be computed from the coherent transmission amplitude $\tilde{t}_p$:

$$\begin{aligned}\tilde{t}_p &= \frac{\langle E_p, \omega_p|\tilde{a}_{x>0}(t)|E_p, \omega_p\rangle}{\langle E_p, \omega_p|\tilde{a}_{x>0}(t)|E_p, \omega_p\rangle_{g_p=0}} = 1 + 2i\boldsymbol{\chi}(t)\\ &= 1 - \frac{2i\Gamma_p}{\Omega_p}e^{i\omega_p(t-\frac{x}{v_g}-t_0)}\langle E_p, \omega_p|\sigma\left(t-\frac{x}{v_g}\right)|E_p, \omega_p\rangle,\end{aligned} \tag{7}$$

where $\langle\ldots\rangle_{g_p=0}$ denotes no coupling between the 2LE and the input beam. Here, $\boldsymbol{\chi}(t)$ represents the optical susceptibility of the medium, which includes both linear and nonlinear parts of the susceptibility. The phase ϕ_p associated with $\tilde{t}_p$ is $\phi_p(t) = \tan^{-1}\{2\mathrm{Re}\,\boldsymbol{\chi}(t)/[1 - 2\mathrm{Im}\,\boldsymbol{\chi}(t)]\}$. We can get $\phi_p^{(1)}$ from ϕ_p by taking $\Omega_p \to 0$, and extract $\phi_p^{(2)}$ at any arbitrary Ω_p using $\phi_p^{(2)} = \phi_p|_{\Omega_p\neq 0} - \phi_p|_{\Omega_p\to 0}$. However, it is not clear how to define such transmission amplitude for a two-photon Fock-state input since the scattered states then have amplitudes of two transmitted photons as well as one transmitted and one reflected photons. Instead, we use the first-order correlation function $G^{(1)}(x', x; t) = \langle\tilde{a}_{x'}^\dagger(t)\tilde{a}_x(t)\rangle$ to obtain the change in phase. For a coherent-state input, we find, at $x > 0$ and $x' < 0$, $G_c^{(1)}(x', x; t) = I_{cp}e^{i\omega_p(x-x')/v_g}\tilde{t}_p$, which shows $G^{(1)}(x', x; t)$ is trivially related to $\tilde{t}_p$. Thus, we can compute the coherent change in phase of the incident light from $G^{(1)}(x', x; t)$ at $x > 0$ and $x' < 0$. Below, we demonstrate that $G^{(1)}(x', x; t)$ can also be applied to extract the phase change and the Kerr effect for a multiphoton Fock-state input.

We get the following expressions of $G^{(1)}(x', x; t)$ at $x > 0$ and $x' < 0$ for a coherent-state input and a two-photon Fock-state input with wave vector k_p (check Appendices A and C):

$$\begin{aligned}G_c^{(1)}(x', x; t) &= I_{cp}e^{i\omega_p(x-x')/v_g}\frac{i\Delta_p}{i\Delta_p - 2\Gamma_p}\\ &\quad\times\left(1 - \frac{8\Gamma_p^2 v_g I_{cp}}{i\Delta_p\left(\Delta_p^2+4\Gamma_p^2\right)} + O\left(\Omega_p^4\right)\right),\end{aligned} \tag{8}$$

$$\begin{aligned}G_2^{(1)}(x', x; t) &= I_{2p}e^{i\omega_p(x-x')/v_g}\frac{i\Delta_p}{i\Delta_p - 2\Gamma_p}\\ &\quad\times\left(1 - \frac{8\Gamma_p^2 v_g I_{1p}}{i\Delta_p\left(\Delta_p^2+4\Gamma_p^2\right)} + \frac{2\Gamma_p v_g I_{1p}}{\Delta_p^2+4\Gamma_p^2}\right),\end{aligned} \tag{9}$$

respectively. The expansion in order of Ω_p^2 in Eq. (8) is performed for a weak coherent-state input. Here, $i\Delta_p/(i\Delta_p - 2\Gamma_p) \equiv t_{1p}$ is the transmission amplitude of a single-photon or a faint coherent-state input in the limit $\Omega_p \to 0$. Thus, the linear change in phase $\phi_p^{(1)} = \tan^{-1}(-2\Gamma_p/\Delta_p)$. The nonlinear phase change $\phi_p^{(2)}$ is obtained by the argument of the terms within the round brackets in Eqs. (8) and (9). The second term within the round brackets in Eq. (8) for a coherent state matches that in Eq. (9) for Fock-state input if we replace I_{cp} by I_{1p}. The appearance of I_{1p} in Eq. (9) is comprehensible since a single photon generates the nonlinear phase shift to another for a two-photon input, and it also signals the nonlinear phase shift is smaller than the linear one by a factor of $1/\mathcal{L}$.

Nevertheless, the third term in Eq. (9) does not have an equivalent contribution in Eq. (8), and the term is due to the intermediate amplitudes of the excited emitter with one photon in the waveguide (see Appendix A). Thus, we find that $\phi_p^{(2)}$ for a two-photon Fock-state input matches to that of a faint

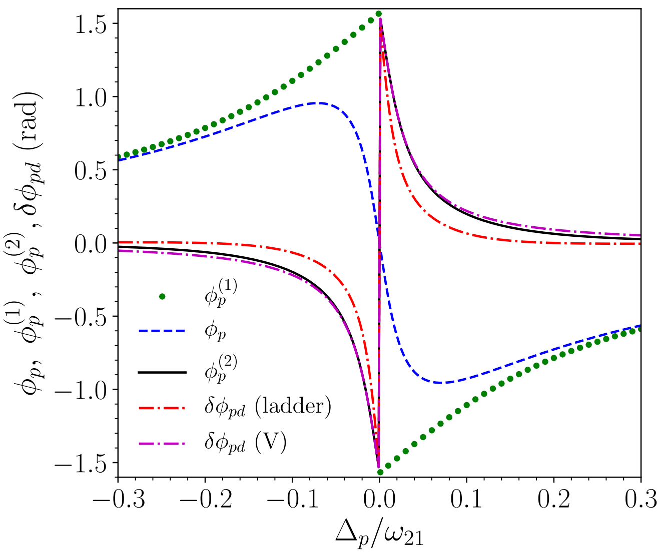

FIG. 2. Kerr vs cross-Kerr effect with Fock-state photons in waveguide QED. The linear, nonlinear (Kerr), and total phase shifts ($\phi_p^{(1)}, \phi_p^{(2)}, \phi_p$) of two transmitted photons inside an open waveguide side coupled to a 2LE. The cross-Kerr phase shift $\delta\phi_{pd}$ of a single-photon probe beam by a single-photon drive beam, where both beams interact with two allowed transitions of a ladder-type or a V-type 3LE. The parameters are $v_g = 1$, $\Gamma_p/\omega_{21} = \Gamma_d/\omega_{21} = 0.1$, $\Delta_d = 0$, and $I_{1p} = I_{1d} = 0.0125\omega_{21}/v_g$.

coherent-state input (along with a replacement of I_{cp} by I_{1p}) if we ignore the third term within the round brackets in Eq. (9). The third term in Eq. (9) has a relatively small contribution to $\phi_p^{(2)}$ in comparison to the second term for the parameters of validity of the expression. From Eq. (8), we find the Kerr coefficient as $K = (1/E_p^2)\tan^{-1}\{8\Gamma_p^2 v_g I_{cp}/[\Delta_p(\Delta_p^2 + \Gamma_p^2)]\} \approx 8\Gamma_p^2/[v_g\Delta_p(\Delta_p^2 + \Gamma_p^2)]$ for $\Omega_p^2/(2\Gamma_p^2) \ll 1$ and $|\Delta_p| \geqslant \Gamma_p$.

A side-coupled 2LE perfectly reflects a resonant single-photon input, and the transmission phase shift $\phi_p^{(1)}$ is then not defined. However, two resonant photons cannot be simultaneously perfectly reflected by a single emitter. Therefore, there will be a finite transmission for a two-photon resonant input pulse, and total phase shift ϕ_p will be zero at $\Delta_p = 0$ due to the photon which passes the emitter without interacting with it. At $\Delta_p = 0\pm$, we find $\phi_p^{(1)} = \mp\pi/2$ from $\phi_p^{(1)} = \tan^{-1}(-2\Gamma_p/\Delta_p)$; thus we get $\phi_p^{(2)} = \pm\pi/2$ as $\phi_p \approx 0$. Magnitude of both $\phi_p^{(1)}$ and $\phi_p^{(2)}$ falls with increasing detuning $|\Delta_p|$. The linearity of $\phi_p^{(2)}$ with E_p^2 is true for small $\phi_p^{(2)}$ at $|\Delta_p| \geqslant \Gamma_p$. We show the above features of $\phi_p^{(1)}$ and $\phi_p^{(2)}$ with probe detuning Δ_p for a two-photon Fock-state input in Fig. 2.

IV. CROSS-KERR EFFECT BY 3LE

Next, we consider a ladder-type 3LE with levels $|1\rangle, |2\rangle$, and $|3\rangle$ as in Fig. 1(b). The energy difference between the excited states is $\hbar\omega_{32}$; thus the energy of excited level $|3\rangle$ is $\hbar(\omega_{32} + \omega_{21})$. Two allowed optical transitions between the levels $|1\rangle \leftrightarrow |2\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ are side coupled to a probe and a drive beam of frequency ω_p and ω_d, respectively. The full Hamiltonian of 3LE, light beams, and their couplings reads [46]

$$\frac{\mathcal{H}_3^k}{\hbar} = \omega_{21}\sigma^\dagger\sigma + (\omega_{32} + \omega_{21})\mu^\dagger\mu + \sum_k \Bigg[v_g k \sum_{\alpha=\pm}(a_{k\alpha}^\dagger a_{k\alpha} - b_{k\alpha}^\dagger b_{k\alpha}) + g_p(\sigma^\dagger\beta_{k+} + \beta_{k+}^\dagger\sigma) + g_d(\mu^\dagger\beta_{k-} + \beta_{k-}^\dagger\mu)\Bigg], \tag{10}$$

where we define $\beta_{k\pm} = (a_{k\pm} + b_{k\pm})$, and $\mu^\dagger \equiv |3\rangle\langle 2|$, $\mu \equiv |2\rangle\langle 3|$, $\nu^\dagger \equiv |1\rangle\langle 3|$, and $\nu \equiv |3\rangle\langle 1|$. Here, $a_{k\alpha}^\dagger$ [$b_{k\alpha}^\dagger$] are creation operators for two different polarizations of right-moving (left-moving) photon modes of the probe and drive beams. The polarizations are denoted by subscript $\alpha = \pm$, and we choose $+$ and $-$ polarization, respectively, for the probe and drive beam. g_p and g_d are the respective coupling strength of the probe and drive beam with the 3LE. We assume that both the probe and drive input beams are incoming from the left of the 3LE.

For both the probe and drive input beams in the coherent states, the initial state at time t_0 is $|\psi\rangle = |E_p, \omega_p\rangle \otimes |E_d, \omega_d\rangle$, which satisfies $a_{k+}(t_0)|\psi\rangle = (\sqrt{\mathcal{L}}E_p/v_g)\delta_{k,\omega_p/v_g}|\psi\rangle$, $a_{k-}(t_0)|\psi\rangle = (\sqrt{\mathcal{L}}E_d/v_g)\delta_{k,\omega_d/v_g}|\psi\rangle$, where E_p and E_d are their respective (real) amplitude. Further, the intensity of probe and drive beam are $I_{cp} = E_p^2/v_g^2$ and $I_{cd} = E_d^2/v_g^2$. For Fock-state input, we consider the probe and drive beams consisting of single photons as

$$|k_p, k_d\rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2}\int_{-\mathcal{L}/2}^{\mathcal{L}/2} \frac{dx_1 dx_2}{\mathcal{L}} e^{i(k_p x_1 + k_d x_2)} \tilde{a}_{x_1+}^\dagger \tilde{a}_{x_2-}^\dagger |\varphi\rangle,$$

where $\tilde{a}_{x\pm}(t) = \sum_k e^{ikx} a_{k\pm}(t)/\sqrt{\mathcal{L}}$ and $\omega_d = v_g k_d$. The intensity of the single-photon probe and drive beam are, respectively, $I_{1p} = 1/\mathcal{L}$ and $I_{1d} = 1/\mathcal{L}$.

A ladder-type 3LE made of a superconducting artificial atom was used to demonstrate an effective interaction between two different light beams at the single-photon quantum regime in Ref. [6]. Such an effective interaction is similar in physical mechanism to the above-explored effective interaction between photons of a single beam induced by the Kerr nonlinearity of the medium (e.g., an emitter). This effective coupling between multiple beams is known as the cross-Kerr effect. The photon-photon interaction in a cross-Kerr medium has been utilized to propose quantum nondemolition measurement of a single propagating microwave photon with high fidelity [52]. Extending the earlier discussion of the optical Kerr effect, the cross-Kerr effect can be interpreted as modulation of refractive index or a phase change of a probe beam due to a drive beam. Thus, we write the total phase shift of the probe beam in a cross-Kerr medium as $\phi_{pd} = \phi_p^{(1)} + \delta\phi_{pd}$, where $\phi_p^{(1)}$ again indicates the linear change in phase of the probe beam in the absence of the drive beam and $\delta\phi_{pd}$ captures the change in phase of the probe beam in the presence of the drive beam. In analogy to the Kerr coefficient, we define the cross-Kerr coefficient K_c using $\delta\phi_{pd} \equiv \phi_{pd}|_{\Omega_d\neq 0} - \phi_{pd}|_{\Omega_d=0} = K_c E_d^2$, where $\Omega_d = \bar{g}_d E_d/v_g$ is the Rabi frequency of the coherent drive beam and $\bar{g}_d = \sqrt{\mathcal{L}}g_d$. We can find ϕ_{pd}

of a probe beam in the presence and absence of a drive beam using $G_{+}^{(1)}(x', x; t) = \langle \tilde{a}_{x'+}^{\dagger}(t)\tilde{a}_{x+}(t)\rangle$ at $x > 0$ and $x' < 0$.

To find the cross-Kerr effect in our system, we derive the outgoing scattered states of the probe and drive beams within the Schrödinger picture for the Fock-state inputs and calculate the time evolution of operators of the full system within the Heisenberg picture for the coherent-state inputs. The first-order coherences $G_{+}^{(1)}(x', x; t)$ at $x > 0$ and $x' < 0$ for coherent-state [46] and Fock-state (see Appendix B) inputs are

$$G_{+c}^{(1)}(x', x; t) = I_{cp} e^{i\omega_p(x-x')/v_g} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} \times \left(1 - \frac{4\Gamma_p\Gamma_d v_g I_{cd}}{i\Delta_p(i\Delta_p - 2\Gamma_p)[i(\Delta_p + \Delta_d) - 2\Gamma_d]} + O(\Omega_d^4)\right), \quad (11)$$

$$G_{+2}^{(1)}(x', x; t) = I_{1p} e^{i\omega_p(x-x')/v_g} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} \times \left(1 - \frac{4\Gamma_p\Gamma_d v_g I_{1d}}{i\Delta_p(i\Delta_p - 2\Gamma_p)[i(\Delta_p + \Delta_d) - 2\Gamma_d]}\right), \quad (12)$$

where we have expanded Eq. (11) in order of Ω_d^2 for a faint coherent-state drive beam. Here, $\Delta_d = \omega_d - \omega_{32}$, $\Gamma_d = \bar{g}_d^2/(2v_g)$. We find from Eqs. (11) and (12) that the leading-order contribution from a faint coherent-state drive beam to $\delta\phi_{pd}$ matches that from a single-photon drive when we identify I_{cd} by I_{1d}. From Eq. (11), we find the cross-Kerr coefficient as $K_c = 4\Gamma_p\Gamma_d[4\Gamma_p\Gamma_d - \Delta_p(\Delta_p + \Delta_d)]/\{v_g\Delta_p(\Delta_p^2 + \Gamma_p^2)[(\Delta_p + \Delta_d)^2 + \Gamma_d^2]\}$ for $|\Delta_p| \geqslant \Omega_d^2/(2\Gamma_d)$. While the cross-Kerr phase shift $\delta\phi_{pd}$ depends on both probe and drive photon detuning, let us study it for $\Delta_d = 0$ when it shows a relatively large value. The features of $\delta\phi_{pd}$ with Δ_p are similar to $\phi_p^{(2)}$ for single photons as shown in Fig. 2. However, the value of $\delta\phi_{pd}$ at $\Delta_d = 0$ in Fig. 2 is always smaller than $\phi_p^{(2)}$ at any finite Δ_p for single photons (see Appendix B). Thus, the Kerr nonlinearity K by a 2LE between two single photons is relatively higher than the cross-Kerr nonlinearity K_c by a ladder-type 3LE between them. Nevertheless, the value of K_c depends on the type of 3LE, which is determined by the optical transitions used for the drive and probe beams [46]. For example, we find $\delta\phi_{pd}$ for a single-photon probe and a single-photon drive beam can be higher for a V-type 3LE than a ladder-type 3LE as shown in Fig. 2. Further, $\delta\phi_{pd}$ by a V-type 3LE can be slightly higher than $\phi_p^{(2)}$ by a 2LE between two Fock-state photons (see Fig. 2). In Appendix B, we give an expression of $G_{+2}^{(1)}(x', x; t)$ for a V-type 3LE, and we discuss the above comparison in detail there.

V. SUMMARY AND OUTLOOK

Our findings for comparing various linear and nonlinear light scattering by a single emitter embedded in an open waveguide for different light sources will benefit the rapid progress of waveguide QED. By deriving explicit formulas using different approaches, we show how to identify few-photon reflection currents by measuring the reflection current for a faint coherent-state input or vice versa. We further develop a generalized description using the first-order correlation function to extract coherent phase shifts of transmitted light in waveguide QED systems for different light sources at low intensity. The last achievement helps to compare and contrast the Kerr and cross-Kerr effects in various waveguide QED systems with few photons, which would be particularly useful for selecting the right Kerr and cross-Kerr medium for nonlinear quantum devices, e.g., nonlinear optical diodes [17,22,44,48,49] and transistors [19]. Experiments with superconducting circuits can verify our theoretical predictions in this paper. In future studies, we aim to compare different light sources for more complex waveguide QED setups, such as with giant atoms, separated emitters, and topological waveguides.

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APPENDIX A: SCATTERING OF SINGLE- AND TWO-PHOTON FOCK STATES BY A TWO-LEVEL EMITTER

The real-space Hamiltonian of a 2LE side coupled to a linear waveguide with right-moving and left-moving photon modes [Eq. (2)] reads as

$$\frac{\mathcal{H}_2^x}{\hbar} = \omega_{21}\sigma^\dagger\sigma - iv_g\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx(\tilde{a}_x^\dagger\partial_x\tilde{a}_x - \tilde{b}_x^\dagger\partial_x\tilde{b}_x) + \bar{g}_p\sigma^\dagger(\tilde{a}_0 + \tilde{b}_0) + \bar{g}_p(\tilde{a}_0^\dagger + \tilde{b}_0^\dagger)\sigma. \quad (A1)$$

First, we consider a single-photon input state with wave vector k_p and frequency $\omega_p = v_g k_p$ from the left of the emitter:

$$|k_p\rangle = \frac{1}{\sqrt{\mathcal{L}}}\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx\, e^{ik_p x}\tilde{a}_x^\dagger|\varphi\rangle, \quad (A2)$$

which satisfies $I_{1p} = \langle k_p|\int dk\, a_k^\dagger a_k|k_p\rangle/\mathcal{L} = 1/\mathcal{L}$. The full single-photon state $|k_p^+\rangle$ of the Hamiltonian is derived from $\mathcal{H}_2^x|k_p^+\rangle = \hbar v_g k_p|k_p^+\rangle$ using the initial conditions in Eq. (A2). It is given as

$$|k_p^+\rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx[g_R(x)\tilde{a}_x^\dagger + g_L(x)\tilde{b}_x^\dagger + \delta(x)\tilde{e}_p\sigma^\dagger]|\varphi\rangle|g\rangle, \quad (A3)$$

the amplitudes of which are

$$g_R(x) = \frac{e^{ik_p x}}{\sqrt{\mathcal{L}}}[\theta(-x) + t_{1p}\theta(x)],\ g_L(x) = \frac{e^{-ik_p x}}{\sqrt{\mathcal{L}}} r_{1p}\theta(-x),\ e_p = \frac{\bar{g}_p}{\Delta_p + 2i\Gamma_p},\ t_{1p} = \frac{\Delta_p}{\Delta_p + 2i\Gamma_p}. \tag{A4}$$

Here, t_{1p} and $r_{1p} = t_{1p} - 1$ are, respectively, single-photon transmission and reflection amplitude, and the amplitude of the emitter's excitation is $\tilde{e}_p = e_p/\sqrt{\mathcal{L}}$.

The normalized two-photon incident Fock state with degenerate wave vectors $\mathbf{k}_p = (k_p, k_p)$ in the right-moving channels reads as

$$|\boldsymbol{k}_p\rangle = \frac{1}{\mathcal{L}} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx_1 dx_2\, e^{ik_p(x_1+x_2)} \frac{1}{\sqrt{2}} \tilde{a}^{\dagger}_{x_1} \tilde{a}^{\dagger}_{x_2} |\varphi\rangle, \tag{A5}$$

which has intensity $I_{2p} = 2/\mathcal{L}$. The two-photon state of the Hamiltonian including the scattered and incident parts is

$$|\boldsymbol{k}_p^+\rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx_1 dx_2 \bigg[g_{RR}(x_1, x_2) \frac{1}{\sqrt{2}} \tilde{a}^{\dagger}_{x_1} \tilde{a}^{\dagger}_{x_2} + g_{RL}(x_1, x_2) \tilde{a}^{\dagger}_{x_1} \tilde{b}^{\dagger}_{x_2} + g_{LL}(x_1, x_2) \frac{1}{\sqrt{2}} \tilde{b}^{\dagger}_{x_1} \tilde{b}^{\dagger}_{x_2}$$
$$+ [e_R(x_1)\tilde{a}^{\dagger}_{x_1} + e_L(x_1)\tilde{b}^{\dagger}_{x_1}]\delta(x_2)\sigma^{\dagger} \bigg] |\varphi\rangle |g\rangle, \tag{A6}$$

the amplitudes of which can be found by solving a set of linear, coupled, inhomogeneous different equations obtained from $\mathcal{H}_2^x |\boldsymbol{k}_p^+\rangle = 2\hbar v_g k_p |\boldsymbol{k}_p^+\rangle$ with the initial conditions set by Eq. (A5). These amplitudes are

$$g_{RR}(x_1, x_2) = g_R(x_1) g_R(x_2) + \left[\frac{2\Gamma_p}{v_g} \tilde{e}_p^2\, e^{i(2v_g k_p - \omega_{21} + 2i\Gamma_p)x_2/v_g} e^{i(\omega_{21} - 2i\Gamma_p)x_1/v_g} \theta(x_2 - x_1)\theta(x_1) + (x_1 \leftrightarrow x_2) \right],$$
$$g_{RL}(x_1, x_2) = \sqrt{2} g_R(x_1) g_L(x_2) + \left[\frac{2\sqrt{2}\Gamma_p}{v_g} \tilde{e}_p^2\, e^{-i(2v_g k_p - \omega_{21} + 2i\Gamma_p)x_2/v_g} e^{i(\omega_{21} - 2i\Gamma_p)x_1/v_g} \theta(|x_2| - x_1)\theta(x_1)\theta(-x_2) + (x_1 \leftrightarrow x_2) \right],$$
$$g_{LL}(x_1, x_2) = g_L(x_1) g_L(x_2) + \left[\frac{2\Gamma_p}{v_g} \tilde{e}_p^2\, e^{-i(2v_g k_p - \omega_{21} + 2i\Gamma_p)x_2/v_g} e^{-i(\omega_{21} - 2i\Gamma_p)x_1/v_g} \theta(|x_2 - x_1|)\theta(-x_1)\theta(-x_2) + (x_1 \leftrightarrow x_2) \right],$$
$$e_R(x) = \sqrt{2}\, g_R(x)\tilde{e}_p + \frac{\sqrt{2} i \bar{g}_p}{v_g} \tilde{e}_p^2\, e^{i(2v_g k_p - \omega_{21} + 2i\Gamma_p)x/v_g} \theta(x),$$
$$e_L(x) = \sqrt{2}\, g_L(x)\tilde{e}_p + \frac{\sqrt{2} i \bar{g}_p}{v_g} \tilde{e}_p^2\, e^{-i(2v_g k_p - \omega_{21} + 2i\Gamma_p)x/v_g} \theta(-x), \tag{A7}$$

where $\theta(x)$ is the Heaviside step function.

1. First-order coherence

Using the single-photon and two-photon states of the Hamiltonian, we can find the first-order coherence $G^{(1)}(x', x; t)$ of the transmitted photon(s) for incident photon(s) in the right-moving channel(s). We get for the single-photon case with $x > 0, x' < 0$

$$G_1^{(1)}(x', x; t) = \langle k_p^+ | \tilde{a}^{\dagger}_{x'} \tilde{a}_x | k_p^+ \rangle = I_{1p} t_{1p} e^{ik_p(x - x')} \equiv I_{1p} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} e^{ik_p(x - x')}, \tag{A8}$$

which gives the linear change in phase $\phi_p^{(1)} = \tan^{-1}(-2\Gamma_p/\Delta_p)$. For two photons, we find again for $x > 0, x' < 0$

$$G_2^{(1)}(x', x; t) = \langle \boldsymbol{k}_p^+ | \tilde{a}^{\dagger}_{x'} \tilde{a}_x | \boldsymbol{k}_p^+ \rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy [2 g^*_{RR}(x', y) g_{RR}(x, y) + g^*_{RL}(x', y) g_{RL}(x, y)] + e^*_R(x') e_R(x). \tag{A9}$$

Below we give each term in the above relation explicitly to show how to get the final result. We find

$$2\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy\, g_{RR}^*(x',y)g_{RR}(x,y) = \frac{1}{\mathcal{L}}e^{ik_p(x-x')}\left[(1+|t_{1p}|^2)t_{1p} - \frac{4}{\mathcal{L}}r_{1p}t_{1p}^*e_p^2 + \frac{2}{\mathcal{L}}r_{1p}t_{1p}^*e_p^2\left(e^{i(\Delta_p+2i\Gamma_p)x/v_g} + e^{-i(\Delta_p+2i\Gamma_p)(x-\mathcal{L}/2)/v_g}\right)\right], \tag{A10}$$

$$\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy\, g_{RL}^*(x',y)g_{RL}(x,y) = \frac{1}{\mathcal{L}}e^{ik_p(x-x')}\left[|r_{1p}|^2t_{1p} - \frac{2}{\mathcal{L}}|r_{1p}|^2e_p^2\left(1-e^{-i(\Delta_p+2i\Gamma_p)(x-\mathcal{L}/2)/v_g}\right) + \frac{2}{\mathcal{L}}|r_{1p}|^2e_p^2\left(e^{i(\Delta_p+2i\Gamma_p)x/v_g}-1\right)\right], \tag{A11}$$

$$e_R^*(x')e_R(x) = \frac{2}{\mathcal{L}^2}|e_p|^2e^{ik_p(x-x')}\left[t_{1p} - r_{1p}e^{i(\Delta_p+2i\Gamma_p)x/v_g}\right]. \tag{A12}$$

We add Eqs. (A10)–(A12) to find that the terms with a factor $e^{i(\Delta_p+2i\Gamma_p)x/v_g}$ disappear. The terms with a factor $e^{-i(\Delta_p+2i\Gamma_p)(x-\mathcal{L}/2)/v_g}$ also vanish when $\mathcal{L} \gg v_g/\Gamma_p$. Thus, we finally get $G_2^{(1)}(x',x;t)$ as in the main text:

$$\begin{aligned} G_2^{(1)}(x',x;t) &= \frac{1}{\mathcal{L}}e^{ik_p(x-x')}\left[(1+|t_{1p}|^2+|r_{1p}|^2)t_{1p} - \frac{4}{\mathcal{L}}r_{1p}t_{1p}^*e_p^2 - \frac{4}{\mathcal{L}}|r_{1p}|^2e_p^2 + \frac{2}{\mathcal{L}}|e_p|^2t_{1p}\right] \\ &= I_{2p}\frac{i\Delta_p}{i\Delta_p-2\Gamma_p}e^{ik_p(x-x')}\left[1 - \frac{8\Gamma_p^2v_gI_{1p}}{i\Delta_p\left(\Delta_p^2+4\Gamma_p^2\right)} + \frac{2\Gamma_pv_gI_{1p}}{\Delta_p^2+4\Gamma_p^2}\right]. \end{aligned} \tag{A13}$$

2. Reflection current

We evaluate the expectation of the reflection current operator [44] in the full single-photon and two-photon states. We remind the reader that the contributions in reflection current arise solely from scattered photons by the emitter. We find

$$\begin{aligned} \mathcal{J}_1 &= i\bar{g}_p\langle k_p^+|(\sigma^\dagger\tilde{b}_0 - \tilde{b}_0^\dagger\sigma)|k_p^+\rangle = \frac{v_g}{\mathcal{L}}|r_{1p}|^2, \\ \mathcal{J}_2 &= i\bar{g}_p\langle \boldsymbol{k}_p^+|(\sigma^\dagger\tilde{b}_0 - \tilde{b}_0^\dagger\sigma)|\boldsymbol{k}_p^+\rangle = 2\mathrm{Re}\left\{i\bar{g}_p\int_{-\mathcal{L}/2}^{\mathcal{L}/2}dx\,[e_R^*(x)g_{RL}(x,0) + \sqrt{2}e_L^*(x)g_{LL}(x,0)]\right\}. \end{aligned} \tag{A14}$$

Each part of the two-photon reflection current is given as following:

$$\begin{aligned} i\bar{g}_p\int_{-\mathcal{L}/2}^{\mathcal{L}/2}dx e_R^*(x)g_{RL}(x,0) &= \frac{i\bar{g}_p}{2\mathcal{L}}e_p^*r_{1p} + \frac{\Gamma_p}{\mathcal{L}}|t_{1p}|^2|e_p|^2 + \frac{\Gamma_p}{\mathcal{L}^2}|e_p|^4\left(1-e^{-2\Gamma_p\mathcal{L}/v_g}\right) \\ &\quad + \frac{2\Gamma_p}{\mathcal{L}^2}\left[e_p^3e_p^*t_{1p}^*\left(e^{i(\Delta_p+2i\Gamma_p)\mathcal{L}/(2v_g)}-1\right) + e_p^{*3}e_pt_{1p}\left(e^{i(-\Delta_p+2i\Gamma_p)\mathcal{L}/(2v_g)}-1\right)\right], \end{aligned} \tag{A15}$$

$$\begin{aligned} i\bar{g}_p\int_{-\mathcal{L}/2}^{\mathcal{L}/2}dx\sqrt{2}e_L^*(x)g_{LL}(x,0) &= \frac{i\bar{g}_p}{2\mathcal{L}}|r_{1p}|^2e_p^*r_{1p} + \frac{\Gamma_p}{\mathcal{L}^2}|e_p|^4\left(1-e^{-2\Gamma_p\mathcal{L}/v_g}\right) + \frac{2\Gamma_pv_g}{i\mathcal{L}^2\bar{g}_p}\left[r_{1p}^2e_p^{*3}\left(1-e^{i(-\Delta_p+2i\Gamma_p)\mathcal{L}/(2v_g)}\right)\right. \\ &\quad \left.- r_{1p}^{*2}e_p^3\left(1-e^{i(\Delta_p+2i\Gamma_p)\mathcal{L}/(2v_g)}\right)\right]. \end{aligned} \tag{A16}$$

In the limit of $\mathcal{L} \gg v_g/\Gamma_p$, we find the following by adding Eqs. (A15) and (A16):

$$\mathcal{J}_2 = \frac{2v_g}{\mathcal{L}}|r_{1p}|^2 - \frac{16v_g^2\Gamma_p^3}{\mathcal{L}^2\left(\Delta_p^2+4\Gamma_p^2\right)^2} = \frac{2}{\mathcal{L}}\frac{4v_g\Gamma_p^2}{\Delta_p^2+4\Gamma_p^2} - \frac{1}{\mathcal{L}^2}\frac{16v_g^2\Gamma_p^3}{\left(\Delta_p^2+4\Gamma_p^2\right)^2}. \tag{A17}$$

APPENDIX B: SCATTERING OF PROBE AND DRIVE SINGLE-PHOTON FOCK STATES BY A LADDER-TYPE THREE-LEVEL EMITTER

We consider a ladder-type 3LE with two allowed transitions being side coupled to a linear waveguide carrying right-moving and left-moving photon modes of a probe and a drive beam. The real-space Hamiltonian [46] is given by

$$\begin{aligned} \frac{\mathcal{H}_3^x}{\hbar} &= \omega_{21}\sigma^\dagger\sigma + (\omega_{32}+\omega_{21})\mu^\dagger\mu - iv_g\int_{-\mathcal{L}/2}^{\mathcal{L}/2}dx\sum_{\alpha=\pm}(\tilde{a}_{x\alpha}^\dagger\partial_x\tilde{a}_{x\alpha} - \tilde{b}_{x\alpha}^\dagger\partial_x\tilde{b}_{x\alpha}) + \bar{g}_p\sigma^\dagger(\tilde{a}_{0+}+\tilde{b}_{0+}) + \bar{g}_p(\tilde{a}_{0+}^\dagger+\tilde{b}_{0+}^\dagger)\sigma \\ &\quad + \bar{g}_d\mu^\dagger(\tilde{a}_{0-}+\tilde{b}_{0-}) + \bar{g}_d(\tilde{a}_{0-}^\dagger+\tilde{b}_{0-}^\dagger)\mu, \end{aligned} \tag{B1}$$

where $\bar{g}_d = \sqrt{\mathcal{L}}g_d$. Here, $\tilde{a}_{x\alpha}^\dagger(t) = \int dk\, e^{-ikx}a_{k\alpha}^\dagger(t)/\sqrt{\mathcal{L}}$ and $\tilde{b}_{x\alpha}^\dagger(t) = \int dk\, e^{-ikx}b_{k\alpha}^\dagger(t)/\sqrt{\mathcal{L}}$ are creation operators at position $x \in [-\mathcal{L}/2, \mathcal{L}/2]$ of right-moving and left-moving photon modes of the probe ($\alpha = +$) and drive ($\alpha = -$) beams.

The input state of a single-photon probe and a single-photon drive beam with respective wave vector k_p and k_d (with corresponding frequencies $\omega_p = v_g k_p$ and $\omega_d = v_g k_d$) in the right-moving channel is

$$|k_p, k_d\rangle = \frac{1}{\mathcal{L}} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx_1 dx_2\, e^{i(k_p x_1 + k_d x_2)} \tilde{a}^\dagger_{x_1+} \tilde{a}^\dagger_{x_2-} |\varphi\rangle, \tag{B2}$$

which satisfies $I_{1p} = \langle k_p, k_d| \int dk\, a^\dagger_{k+} a_{k+} |k_p, k_d\rangle/\mathcal{L} = 1/\mathcal{L}$ and $I_{1d} = \langle k_p, k_d| \int dk\, a^\dagger_{k-} a_{k-} |k_p, k_d\rangle/\mathcal{L} = 1/\mathcal{L}$.

The full two-photon state of the Hamiltonian $\mathcal{H}_3^x$ including the scattered and incident photons is

$$\begin{aligned}|k_p, k_d^+\rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dx_1 dx_2 \{&\tilde{g}_{RR}(x_1, x_2) \tilde{a}^\dagger_{x_1+} \tilde{a}^\dagger_{x_2-} + \tilde{g}_{RL}(x_1, x_2) \tilde{a}^\dagger_{x_1+} \tilde{b}^\dagger_{x_2-} + \tilde{g}_{LR}(x_1, x_2) \tilde{b}^\dagger_{x_1+} \tilde{a}^\dagger_{x_2-} \\ &+ \tilde{g}_{LL}(x_1, x_2) \tilde{b}^\dagger_{x_1+} \tilde{b}^\dagger_{x_2-} + [\tilde{e}_R(x_2) \tilde{a}^\dagger_{x_2-} + \tilde{e}_L(x_2) \tilde{b}^\dagger_{x_2-}] \delta(x_1) \sigma^\dagger\} |\varphi\rangle |g\rangle,\end{aligned} \tag{B3}$$

the amplitudes of which can be found by solving a set of linear, coupled, inhomogeneous different equations obtained from $\mathcal{H}_3^x |k_p, k_d^+\rangle = \hbar v_g (k_p + k_d) |k_p, k_d^+\rangle$ with the initial conditions set by Eq. (B2). These amplitudes are

$$\begin{aligned}
\tilde{g}_{RR}(x_1, x_2) &= g_R(x_1) \frac{e^{ik_d x_2}}{\sqrt{\mathcal{L}}} - \left[\frac{2\sqrt{\Gamma_p \Gamma_d}}{v_g} \tilde{e}_p \tilde{e}_d\, e^{i(k_p x_1 + k_d x_2)} e^{i(\Delta_p + 2i\Gamma_p)(x_2 - x_1)/v_g} \theta(x_2 - x_1) \theta(x_1) \right],\\
\tilde{g}_{LR}(x_1, x_2) &= g_L(x_1) \frac{e^{ik_d x_2}}{\sqrt{\mathcal{L}}} - \left[\frac{2\sqrt{\Gamma_p \Gamma_d}}{v_g} \tilde{e}_p \tilde{e}_d\, e^{i(-k_p x_1 + k_d x_2)} e^{i(\Delta_p + 2i\Gamma_p)(x_2 + x_1)/v_g} \theta(x_2 + x_1) \theta(-x_1) \right],\\
\tilde{g}_{RL}(x_1, x_2) &= -\frac{2\sqrt{\Gamma_p \Gamma_d}}{v_g} \tilde{e}_p \tilde{e}_d\, e^{i(k_p x_1 - k_d x_2)} e^{-i(\Delta_p + 2i\Gamma_p)(x_2 + x_1)/v_g} \theta(-x_2 - x_1) \theta(x_1),\\
\tilde{g}_{LL}(x_1, x_2) &= -\frac{2\sqrt{\Gamma_p \Gamma_d}}{v_g} \tilde{e}_p \tilde{e}_d\, e^{-i(k_p x_1 + k_d x_2)} e^{-i(\Delta_p + 2i\Gamma_p)(x_2 - x_1)/v_g} \theta(-x_2 + x_1) \theta(-x_1),\\
\tilde{e}_R(x) &= \frac{e^{ik_d x}}{\sqrt{\mathcal{L}}} \tilde{e}_p - \frac{i\bar{g}_d}{v_g} \tilde{e}_p \tilde{e}_d\, e^{i(v_g(k_p + k_d) - \omega_{21} + 2i\Gamma_p)x/v_g} \theta(x),\\
\tilde{e}_L(x) &= -\frac{i\bar{g}_d}{v_g} \tilde{e}_p \tilde{e}_d\, e^{-i(v_g(k_p + k_d) - \omega_{21} + 2i\Gamma_p)x/v_g} \theta(-x), \quad e_d = \frac{\bar{g}_d}{\Delta_p + \Delta_d + 2i\Gamma_d},
\end{aligned} \tag{B4}$$

where $\Gamma_d = \bar{g}_d^2/(2v_g)$, $\Delta_d = \omega_d - \omega_{32}$, and $\tilde{e}_d = e_d/\sqrt{\mathcal{L}}$.

Next, we calculate the first-order coherence $G^{(1)}_{+2}(x', x; t)$ of the transmitted probe photon for incident probe and drive photons in the right-moving channels. We find for $x > 0, x' < 0$

$$G^{(1)}_{+2}(x', x; t) = \langle k_p, k_d^+ | \tilde{a}^\dagger_{x'+} \tilde{a}_{x+} | k_p, k_d^+\rangle = \int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy [\tilde{g}^*_{RR}(x', y) \tilde{g}_{RR}(x, y) + \tilde{g}^*_{RL}(x', y) \tilde{g}_{RL}(x, y)]. \tag{B5}$$

We find $\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy\, \tilde{g}^*_{RL}(x', y) \tilde{g}_{RL}(x, y) = 0$, and

$$\int_{-\mathcal{L}/2}^{\mathcal{L}/2} dy\, \tilde{g}^*_{RR}(x', y) \tilde{g}_{RR}(x, y) = \frac{1}{\mathcal{L}} e^{ik_p(x - x')} \left[t_{1p} - \frac{i\bar{g}_d}{v_g} e_d \tilde{e}_p^2 \left(1 - e^{-i(\Delta_p + 2i\Gamma_p)(x - \mathcal{L}/2)/v_g}\right) \right]. \tag{B6}$$

The term with a factor $e^{-i(\Delta_p + 2i\Gamma_p)(x - \mathcal{L}/2)/v_g}$ in Eq. (B6) vanishes when $\mathcal{L} \gg v_g/\Gamma_p$. Thus, we finally get $G^{(1)}_{+2}(x', x; t)$ as

$$G^{(1)}_{+2}(x', x; t) = I_{1p} e^{i\omega_p(x - x')/v_g} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} \left(1 - \frac{4\Gamma_p \Gamma_d v_g I_{1d}}{i\Delta_p (i\Delta_p - 2\Gamma_p)(i(\Delta_p + \Delta_d) - 2\Gamma_d)}\right). \tag{B7}$$

Next, we compare the cross-Kerr phase shift $\delta\phi_{pd}$ of a probe photon due to a drive photon with the Kerr phase shift $\phi_p^{(2)}$ between two Fock-state photons. For this, we write Eq. (B7) in the limit of $\Delta_d = 0$, $\Gamma_d = \Gamma_p$, and $I_{1d} = I_{1p}$:

$$G^{(1)}_{+2}(x', x; t) = I_{1p} e^{i\omega_p(x - x')/v_g} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} \left(1 - \frac{4\Gamma_p^2 v_g I_{1p}}{i\Delta_p (i\Delta_p - 2\Gamma_p)^2}\right). \tag{B8}$$

The second term within the round brackets in Eq. (B8) is a bit similar to the second term within the round brackets in Eq. (A13) except a half-factor difference in the numerator and the appearance of $(i\Delta_p - 2\Gamma_p)^2$ instead of $|i\Delta_p - 2\Gamma_p|^2$ in the denominator.

The first-order coherence $G^{(1)}_{+2}(x', x; t)$ of the transmitted probe photon for incident probe and drive single-photon Fock state in the right-moving channels for a V-type 3LE (see Ref. [46] for a Hamiltonian of the system) is

$$G^{(1)}_{+2}(x', x; t) = I_{1p} e^{i\omega_p(x - x')/v_g} \frac{i\Delta_p}{i\Delta_p - 2\Gamma_p} \left(1 - \frac{4\Gamma_p \Gamma_d v_g I_{1d} [i(\Delta_p + \Delta_d) - 2(\Gamma_p + \Gamma_d)]}{i\Delta_p (i\Delta_p - 2\Gamma_p)\left(\Delta_d^2 + 4\Gamma_d^2\right)} + \frac{2\Gamma_d v_g I_{1d}}{\Delta_d^2 + 4\Gamma_d^2}\right). \tag{B9}$$

The total phase shift ϕ_{pd} for a single-photon probe beam by a V-type 3LE in the presence of a single-photon drive beam is exactly the same as total phase shift ϕ_p by a 2LE for a two-photon probe beam when we set $\Delta_d = \Delta_p$, $\Gamma_d = \Gamma_p$, and $I_{1d} = I_{1p}$. Nevertheless, we can separately tune Δ_p and Δ_d in a V-type 3LE since the probe and drive beams are coupled to two different transitions. The freedom to separately control Δ_p and Δ_d can give rise to a higher value of cross-Kerr phase shift $\delta\phi_{pd}$ by a V-type 3LE for a single probe and drive photon than the Kerr phase shift $\phi_p^{(2)}$ by a 2LE for two probe photons. In Fig. 2, we show a comparison between $\delta\phi_{pd}$ by a V-type and a ladder-type 3LE and $\phi_p^{(2)}$ by a 2LE for two input photons in the Fock state.

APPENDIX C: SCATTERING OF A COHERENT-STATE INPUT BY A TWO-LEVEL EMITTER

The momentum-space Hamiltonian of a 2LE side coupled to a waveguide with linearized energy-momentum dispersion of photons [Eq. (1)] is

$$\frac{\mathcal{H}_2^k}{\hbar} = \omega_{21}\sigma^\dagger\sigma + \sum_k [v_g k(a_k^\dagger a_k - b_k^\dagger b_k) + g_p\sigma^\dagger(a_k + b_k) + g_p(a_k^\dagger + b_k^\dagger)\sigma]. \tag{C1}$$

We first write the Heisenberg equations for different operators appearing in the above Hamiltonian. Then, we formally solve these equations for the photon field operators as

$$a_k(t) = e^{-iv_g kt}a_k(t_0) - ig_p\int_{t_0}^t dt' e^{-iv_g k(t-t')}\sigma(t'), \tag{C2}$$

$$b_k(t) = e^{iv_g kt}b_k(t_0) - ig_p\int_{t_0}^t dt' e^{iv_g k(t-t')}\sigma(t'), \tag{C3}$$

where $a_k(t_0)$ and $b_k(t_0)$ are initial photon fields at $t = t_0$ before the fields interacting with the 2LE. We plug $a_k(t)$ and $b_k(t)$ in Eqs. (C2) and (C3) in the Heisenberg equations of the emitter operators and rewrite these equations as

$$\frac{d\sigma(t)}{dt} = (-i\omega_{21} - 2\Gamma_p)\sigma(t) - ig_p[1 - 2\sigma^\dagger(t)\sigma(t)][\eta_a(t) + \eta_b(t)], \tag{C4}$$

$$\frac{d\sigma^\dagger(t)\sigma(t)}{dt} = -4\Gamma_p\sigma^\dagger(t)\sigma(t) - ig_p\sigma^\dagger(t)[\eta_a(t) + \eta_b(t)] + ig_p[\eta_a^\dagger(t) + \eta_b^\dagger(t)]\sigma(t), \tag{C5}$$

where $\eta_a(t) = \sum_k e^{-iv_g k(t-t_0)}a_k(t_0)$ and $\eta_b(t) = \sum_k e^{iv_g k(t-t_0)}b_k(t_0)$. For a coherent-state input $|E_p, \omega_p\rangle$ from the left of the 2LE, we have $a_k(t_0)|E_p, \omega_p\rangle = (\sqrt{\mathcal{L}}E_p/v_g)\delta_{k,\omega_p/v_g}|E_p, \omega_p\rangle$ and $b_k(t_0)|E_p, \omega_p\rangle = 0$, where $\delta_{k,\omega_p/v_g}$ is a Kronecker delta function. We take expectation of the operators in Eqs. (C4) and (C5) in $|E_p, \omega_p\rangle$, and rewrite these equations as

$$\frac{d\mathcal{S}_1(t)}{dt} = (i\Delta_p - 2\Gamma_p)\mathcal{S}_1(t) - i\Omega_p + 2i\Omega_p\mathcal{S}_2(t), \tag{C6}$$

$$\frac{d\mathcal{S}_2(t)}{dt} = -4\Gamma_p\mathcal{S}_2(t) + i\Omega_p[\mathcal{S}_1(t) - \mathcal{S}_1^*(t)], \tag{C7}$$

where $\Omega_p = \bar{g}_pE_p/v_g$, $\mathcal{S}_1(t) = \langle E_p, \omega_p|\sigma(t)|E_p, \omega_p\rangle e^{i\omega_p(t-t_0)}$, and $\mathcal{S}_2(t) = \langle E_p, \omega_p|\sigma^\dagger(t)\sigma(t)|E_p, \omega_p\rangle$. The coupled differential equations of $\mathcal{S}_1(t)$, $\mathcal{S}_1^*(t)$, and $\mathcal{S}_2(t)$ can be solved for some initial conditions of these variables, and the long-time steady-state solutions are independent of the initial conditions for the 2LE. We find at steady state

$$\mathcal{S}_1(t\to\infty) = \frac{i\Omega_p(-i\Delta_p - 2\Gamma_p)}{\Delta_p^2 + 4\Gamma_p^2 + 2\Omega_p^2},\ \mathcal{S}_2(t\to\infty) = \frac{\Omega_p^2}{\Delta_p^2 + 4\Gamma_p^2 + 2\Omega_p^2},$$

which can be used to determine the reflection current $\mathcal{J}_c$ at steady state for a coherent-state input:

$$\begin{aligned}\mathcal{J}_c &= i\bar{g}_p\langle E_p, \omega_p|[\sigma^\dagger(t)\tilde{b}_0(t) - \tilde{b}_0^\dagger(t)\sigma(t)]|E_p, \omega_p\rangle\\ &= 2\Gamma_p\mathcal{S}_2(t\to\infty) = \frac{2\Gamma_p\Omega_p^2}{\Delta_p^2 + 4\Gamma_p^2 + 2\Omega_p^2} = \frac{\Omega_p^2}{2v_g\Gamma_p}\frac{4v_g\Gamma_p^2}{\Delta_p^2 + 4\Gamma_p^2} - \left(\frac{\Omega_p^2}{2v_g\Gamma_p}\right)^2\frac{16v_g^2\Gamma_p^3}{\left(\Delta_p^2 + 4\Gamma_p^2\right)^2} + O\left(\Omega_p^6\right),\end{aligned} \tag{C8}$$

where the last expansion in the order of Ω_p^2 is obtained for a weak coherent-state input, i.e., $\Omega_p^2/(2\Gamma_p^2) \ll 1$. We can further identify $I_{cp} = \langle E_p, \omega_p|\sum_k a_k^\dagger(t_0)a_k(t_0)|E_p, \omega_p\rangle/\mathcal{L} = E_p^2/v_g^2 = \Omega_p^2/(2v_g\Gamma_p)$.

Taking Fourier transform to real space, we get from Eq. (C2)

$$\tilde{a}_x(t) = \frac{1}{\sqrt{\mathcal{L}}}\eta_a\left(t - \frac{x}{v_g}\right) - \frac{i\bar{g}_p}{v_g}\sigma\left(t - \frac{x}{v_g}\right)\theta(x)\theta(v_g t - x), \tag{C9}$$

where we set $t_0 = 0$. Thus, we find for the first-order coherence

$$G_c^{(1)}(x', x; t) = \langle E_p, \omega_p | \tilde{a}^\dagger_{x'}(t)\tilde{a}_x(t)|E_p, \omega_p\rangle$$
$$= I_{cp}e^{i\omega_p(x-x')/v_g} + \frac{\bar{g}_p^2}{v_g^2}\langle E_p, \omega_p|\sigma^\dagger\left(t - \frac{x'}{v_g}\right)\sigma\left(t - \frac{x}{v_g}\right)|E_p, \omega_p\rangle\theta(x)\theta(x')\theta(v_g t - x')\theta(v_g t - x)$$
$$- \frac{i\bar{g}_p E_p}{v_g^2}\langle E_p, \omega_p|\sigma\left(t - \frac{x}{v_g}\right)|E_p, \omega_p\rangle e^{i\omega_p(t - x'/v_g)}\theta(x)\theta(v_g t - x)$$
$$+ \frac{i\bar{g}_p E_p}{v_g^2}\langle E_p, \omega_p|\sigma^\dagger\left(t - \frac{x'}{v_g}\right)|E_p, \omega_p\rangle e^{-i\omega_p(t - x/v_g)}\theta(x')\theta(v_g t - x'). \quad \text{(C10)}$$

For $x' < 0, x > 0$, the above expression becomes

$$G_c^{(1)}(x' < 0, x > 0; t) = \frac{E_p^2}{v_g^2}e^{i\omega_p(x-x')/v_g}\left[1 - \frac{2i\Gamma_p}{\Omega_p}\mathcal{S}_1\left(t - \frac{x}{v_g}\right)\right]. \quad \text{(C11)}$$

At very long time when $t \gg x/v_g$, we can replace $\mathcal{S}_1(t - x/v_g)$ by $\mathcal{S}_1(t \to \infty)$ to obtain

$$G_c^{(1)}(x' < 0, x > 0; t) = I_{cp}e^{i\omega_p(x-x')/v_g}\frac{\Delta_p^2 - 2i\Gamma_p\Delta_p + 2\Omega_p^2}{\Delta_p^2 + 4\Gamma_p^2 + 2\Omega_p^2}$$
$$= I_{cp}e^{i\omega_p(x-x')/v_g}\frac{i\Delta_p}{i\Delta_p - 2\Gamma_p}\left(1 - \frac{8v_g\Gamma_p^2 I_{cp}}{i\Delta_p\left(\Delta_p^2 + 4\Gamma_p^2\right)} + O\left(\Omega_p^4\right)\right), \quad \text{(C12)}$$

where we again employ an expansion in Ω_p^2 for a weak coherent-state input.

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