

Quantum measurements constrained by the third law of thermodynamicsM. Hamed Mohammady^{1,*} and Takayuki Miyadera^{2,†}¹*QuIC, École Polytechnique de Bruxelles, CP 165/59, Université Libre de Bruxelles, 1050 Brussels, Belgium*²*Department of Nuclear Engineering, Kyoto University, Nishikyo-ku, Kyoto 615-8540, Japan*

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In the quantum regime, the third law of thermodynamics implies the unattainability of pure states. As shown recently, such unattainability implies that a unitary interaction between the measured system and a measuring apparatus can never implement an ideal projective measurement. In this paper, we introduce an operational formulation of the third law for the most general class of physical transformations, the violation of which is both necessary and sufficient for the preparation of pure states. Subsequently, we investigate how such a law constrains measurements of general observables, or positive operator-valued measures. We identify several desirable properties of measurements which are simultaneously enjoyed by ideal projective measurements—and are hence all ruled out by the third law in such a case—and determine if the third law allows for these properties to obtain for general measurements of general observables and, if so, under what conditions. It is shown that while the third law rules out some of these properties for all observables, others may be enjoyed by observables that are sufficiently “unsharp.”

DOI: [10.1103/PhysRevA.107.022406](https://doi.org/10.1103/PhysRevA.107.022406)**I. INTRODUCTION**

One of the standard assumptions of textbook quantum mechanics is the “Lüders rule,” which states that when an observable—represented by a self-adjoint operator with a nondegenerate spectrum—is measured in a system, the state of the system collapses to the eigenstate of the observable associated with the observed eigenvalue [1–3]. But assuming the universal validity of quantum theory, such a state change must be consistent with a description of the measurement process as a physical interaction between the system to be measured and a given (quantum) measuring apparatus. In his groundbreaking contribution to quantum theory in 1932 [4], von Neumann introduced just such a model for the measurement process, where the system and apparatus interact unitarily.

While Lüders measurements, and von Neumann’s model for their realization, are always available within the formal framework of quantum theory, they may not always be feasible in practice—technological obstacles and fundamental physical principles must also be accounted for. One such principle is that of conservation laws and, as shown by the Wigner-Araki-Yanase theorem [5–14], only observables commuting with the conserved quantity admit a Lüders measurement. This observation naturally raises the following question: Do other physical principles constrain quantum measurements, and if so, how? Given that von Neumann’s model for the measurement process assumes that the measuring apparatus is initially prepared in a pure state, an obvious candidate for consideration immediately presents itself: The third law of thermodynamics, or Nernst’s unattainability prin-

ciple, which states that a system cannot be cooled to absolute zero temperature with finite time, energy, or control complexity; in the quantum regime, the third law prohibits the preparation of pure states [15–25]. As argued by Guryanova *et al.*, such unattainability rules out Lüders measurements for any self-adjoint operator with a nondegenerate spectrum [26].

The more modern quantum theory of measurement [27–30] states that the properties of a quantum system are not exhausted by its sharp observables, i.e., observables represented by self-adjoint operators. Indeed, observables can be fundamentally unsharp, and are properly represented as positive operator valued measures (POVMs) [31,32]. Similarly, the state change that results from measurement is more properly captured by the notion of instruments [33], which need not obey the Lüders rule. Moreover, the interaction between system and apparatus during the measurement process is not necessarily unitary and is more generally described as a channel, which more accurately describes situations where the interaction with the environment cannot be neglected. Therefore, how the third law constrains general measurements should be addressed; in this paper, we shall thoroughly examine this in the finite-dimensional setting.

First, we provide a minimal operational formulation of the third law by constraining the class of permissible channels so that the availability of a channel not so constrained is both necessary and sufficient for the preparation of pure states. The considered class of channels include those whose input and output spaces are not the same, which is the case when the process considered involves composing and discarding systems and is more general than the class of rank nondecreasing channels, such as unitary channels. Indeed, the rank nondecreasing concept can be properly applied only to the limited cases where the input and output systems of a channel are the same.

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Subsequently, we consider the most general class of measurement schemes that are constrained by the third law. That is, we do not assume that the measured observable is sharp, or that the pointer observable is sharp, or that the measurement interaction is rank nondecreasing. Next, we determine if the instruments realized by such measurement schemes may satisfy several desirable properties and, if so, under what conditions. These properties are the following:

(i) *Nondisturbance*: A nonselective measurement does not affect the subsequent measurement statistics of any observable that commutes with the measured observable.

(ii) *First-kindness*: A nonselective measurement of an observable does not affect its subsequent measurement statistics.

(iii) *Repeatability*: Successive measurements of an observable are guaranteed to produce the same outcome.

(iv) *Ideality*: Whenever an outcome is certain from the outset, the measurement does not change the state of the measured system.

(v) *Extremality*: The instrument cannot be written as a probabilistic mixture of distinct instruments.

Lüders measurements of sharp observables simultaneously satisfy the above properties. In general, however, these properties can be satisfied by instruments that do not obey the Lüders rule, and also for observables that are not necessarily sharp. Moreover, they are in general not equivalent: An instrument can enjoy one while not another [34–36]. We therefore investigate each such property individually, providing necessary and sufficient conditions for their fulfillment by a measurement constrained by the third law.

We show that the third law prohibits a measurement of any *small-rank* observable—an observable that has at least one rank-1 effect or POVM element—from satisfying any of the above properties. On the other hand, extremality is shown to be permitted for an observable if each effect has sufficiently large rank, but only if the interaction between the system and apparatus is nonunitary. Finally, we show that while repeatability and ideality are forbidden for all observables, nondisturbance and first-kindness are permitted for observables that are *completely unsharp*: The effects of such observables do not have either eigenvalue 1 or 0, and so such observables do not enjoy the “norm-1” property. That is, nondisturbance and first-kindness are permitted only for observables that cannot have a definite value in any state. Our results are summarized in Table I.

II. OPERATIONAL FORMULATION OF THE THIRD LAW FOR CHANNELS

The third law of thermodynamics states that in the absence of infinite resources of time, energy, or control complexity, a system cannot be cooled to absolute zero temperature. Assuming the universal validity of this law, then it must also hold in the quantum regime [15–24]. Throughout, we shall consider only quantum systems with a finite-dimensional Hilbert space \mathcal{H} . When such a system is in thermal equilibrium at some temperature, it is in a Gibbs state, and whenever the temperature is nonvanishing, such states are full-rank. Conversely, at absolute zero temperature the system will be in a low-rank state, i.e., it will not have full rank. In the special case of a nondegenerate Hamiltonian, the system will in fact be in a

TABLE I. The possibility (\checkmark) or impossibility (\times) of an observable to admit the properties (i)–(v) outlined above are indicated for four classes of observables: Small-rank observables have at least one rank-1 effect; sharp observables are such that all effects are projections; norm-1 observables are such that every effect has eigenvalue 1; and completely unsharp observables are such that no effect has eigenvalue 1 or 0.

	Observable			
	Small-rank	Sharp	Norm-1	Completely unsharp
(i)	\times	\times	\times	\checkmark
(ii)	\times	\times	\times	\checkmark
(iii)	\times	\times	\times	\times
(iv)	\times	\times	\times	\times
(v)	\times	\checkmark	\checkmark	\checkmark

pure state. A minimal operational formulation of the third law in the quantum regime can therefore be phrased as follows: The possible transformations of quantum systems must be constrained so that the only attainable states have full rank.

In the Schrödinger picture, the most general transformations of quantum systems are represented by channels $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$, i.e., completely positive trace-preserving maps from the algebra of linear operators on an input Hilbert space \mathcal{H} to that of an output space \mathcal{K} . In the special case where $\mathcal{H} = \mathcal{K}$, we say that Φ acts in \mathcal{H} . But in general \mathcal{H} need not be identical to \mathcal{K} , and the two systems may have different dimensions. This is because physically permissible transformations include the composition of multiple systems, and discarding of subsystems.

Previous formulations of the third law (see, for example, Proposition 5 of Ref. [17] and Appendix B of Ref. [24]) have restricted the class of available channels to those with the same input and output system, and where the channel does not reduce the rank of the input state of such a system: These are referred to as rank nondecreasing channels, with unitary channels constituting a simple example. An intuitive argument for such restriction is as follows. Consider the case where we wish to cool the system of interest by an interaction with an infinitely large heat bath. But to utilize all degrees of freedom of such a bath one must either manipulate them all at once, which requires an infinite resource of control complexity, or one must approach the quasistatic limit, which requires an infinite resource of time. It stands to reason that, in a realistic protocol, only finitely many degrees of freedom of the bath can be accessed and so the system of interest effectively interacts with a finite, bounded, thermal bath. Such a bath is represented by a Gibbs state with a nonvanishing temperature, which has full rank. It is a simple task to show that if the interaction between the system of interest and the finite thermal bath is a rank nondecreasing channel—such as a unitary channel—acting in the compound of system-plus-bath, then the rank of the system cannot be reduced unless infinite energy is spent. It follows that if the input state of the system is full-rank, for example, if it is a Gibbs state with a nonvanishing temperature, then the third law thus construed will allow only for such a state to be transformed to another full-rank state.

The above formulation has some drawbacks, however. First, the argument relies on the strong assumption that the system interacts with a thermal environment, which is not justified under purely operational grounds; the environment may in fact be an out-of-equilibrium system. Second, the rank nondecreasing condition can be properly applied only to channels with an identical input and output: The rank of a state on \mathcal{H} has meaning only in relation to the dimension of \mathcal{H} . Indeed, the partial trace channel (describing the process by which one subsystem is discarded) and the composition channel (describing the process by which the system of interest is joined with an auxiliary system initialized in some fixed state) are physically relevant transformations that must also be addressed, but lead to absurdities when the change in the state's rank is examined. The partial trace channel is rank-decreasing, but tracing out one subsystem of a global full-rank state can prepare only a state that has full rank in the remaining subsystem. On the other hand, the composition channel is rank-increasing. But it is simple to show that if the rank of the auxiliary state is sufficiently small, then a unitary channel can be applied on the compound so as to purify the system of interest. We thus propose the following minimal definition for channels constrained by the third law, which is conceptually sound, and which does not rely on any assumptions regarding the environment and how it interacts with the system under study, and accounts for the most general class of channels.

Definition 1. A channel $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ is constrained by the third law if for every full-rank state ρ on \mathcal{H} , $\Phi(\rho)$ is a full-rank state on \mathcal{K} .

Properties of channels obeying the above definition are given in Appendix B, and as shown in Appendix C, if we are able to implement any channel that is constrained by the third law, then the added ability to implement a channel not so constrained, that is, a channel that may map some full-rank state to a low-rank state, is both necessary and sufficient for preparing a system in a pure state, given any unknown initial state ρ . Moreover, note that while a rank nondecreasing channel acting in \mathcal{H} satisfies Definition 1, a channel acting in \mathcal{H} and which satisfies such a definition need not be rank nondecreasing: A channel constrained by the third law may reduce the rank of some input state, but only if such a state is not full-rank. Finally, Definition 1 has the benefit that for any pair of channels Φ_1 and Φ_2 satisfying such property, where the output of the former corresponds with the input of the latter, so too does their composition $\Phi_2 \circ \Phi_1$; the set of channels constrained by the third law is thus closed under composition.

Definition 1 also allows us to re-examine the constraints imposed by the third law on state preparations, without modeling a finite-dimensional environment prepared in a Gibbs state, or assuming that the system interacts with such an environment by a rank nondecreasing channel. A state preparation is a physical process so that, irrespective of what input state is given, the output is prepared in a unique state ρ ; indeed, an operational definition of a state is precisely the specification of procedures, or transformations, that produce it. As stated in Ref. [37], “A quantum state can be understood as a preparation channel, sending a trivial quantum system to a nontrivial one prepared in a given state.” That is, state preparations on a Hilbert space \mathcal{H} may be identified with the set of preparation

channels

$$\mathcal{P}(\mathcal{H}) := \{\Phi : \mathcal{L}(\mathbb{C}^1) \rightarrow \mathcal{L}(\mathcal{H})\}.$$

Here the input space is a one-dimensional Hilbert space $\mathbb{C}^1 \equiv \mathbb{C}|\Omega\rangle$, and the only state on such a space is the rank-1 projection $|\Omega\rangle\langle\Omega|$. The triviality of the input space captures the notion that the output of the channel Φ is independent of the input, and so the prepared state $\rho = \Phi(|\Omega\rangle\langle\Omega|)$ is uniquely identified with the channel itself. Without any constraints, all states ρ on \mathcal{H} may be prepared by some $\Phi \in \mathcal{P}(\mathcal{H})$. But now we may restrict the class of preparations by the third law as follows: $\mathcal{P}(\mathcal{H})$ is constrained by the third law if all $\Phi \in \mathcal{P}(\mathcal{H})$ map full-rank states to full-rank states as per Definition 1. But note that $|\Omega\rangle\langle\Omega|$ has full rank in \mathbb{C}^1 , and so ρ is guaranteed to be full-rank in \mathcal{H} .

III. QUANTUM MEASUREMENT

Before investigating how the third law constrains quantum measurements, we shall first cover briefly some basic elements of quantum measurement theory which will be used in the sequel [27–30].

A. Observables

Consider a quantum system \mathcal{S} with a Hilbert space $\mathcal{H}_{\mathcal{S}}$ of finite dimension $2 \leq \dim(\mathcal{H}_{\mathcal{S}}) < \infty$. We denote by $\mathbb{0}$ and $\mathbb{1}_{\mathcal{S}}$ the null and identity operators on $\mathcal{H}_{\mathcal{S}}$, respectively, and an operator E on $\mathcal{H}_{\mathcal{S}}$ is called an *effect* if it holds that $\mathbb{0} \leq E \leq \mathbb{1}_{\mathcal{S}}$. An observable of \mathcal{S} is represented by a normalized positive operator valued measure (POVM) $\mathbf{E} : \Sigma \rightarrow \mathcal{E}(\mathcal{H}_{\mathcal{S}})$, where Σ is a sigma algebra of some value space \mathcal{X} , representing the possible measurement outcomes, and $\mathcal{E}(\mathcal{H}_{\mathcal{S}})$ is the space of effects on $\mathcal{H}_{\mathcal{S}}$. We restrict ourselves to discrete observables for which $\mathcal{X} := \{x_1, x_2, \dots\}$ is countable. In such a case we may identify an observable with the set $\mathbf{E} := \{\mathbf{E}_x : x \in \mathcal{X}\}$, where $\mathbf{E}_x \equiv \mathbf{E}(\{x\})$ are the (elementary) effects of \mathbf{E} (also called POVM elements) which satisfy $\sum_{x \in \mathcal{X}} \mathbf{E}_x = \mathbb{1}_{\mathcal{S}}$. The probability of observing outcome x when measuring \mathbf{E} in the state ρ is given by the Born rule as $p_{\rho}^{\mathbf{E}}(x) := \text{tr}[\mathbf{E}_x \rho]$.

Without loss of generality, we shall always assume that $\mathbf{E}_x \neq \mathbb{0}$, since for any x such that $\mathbf{E}_x = \mathbb{0}$, the outcome x is never observed, i.e., it is observed with probability zero; in such a case we may simply replace \mathcal{X} with the smaller value space $\mathcal{X} \setminus \{x\}$. Additionally, we shall always assume that the observable is nontrivial, as trivial observables cannot distinguish between any states, and are thus uninformative; an effect is trivial if it is proportional to the identity, and an observable is nontrivial if at least one of its effects is not trivial.

We shall employ the short-hand notation $[\mathbf{E}, A] = \mathbb{0}$ to indicate that the operator A commutes with all effects of \mathbf{E} , and $[\mathbf{E}, \mathbf{F}] = \mathbb{0}$ to indicate that all the effects of observables \mathbf{E} and \mathbf{F} mutually commute. An observable \mathbf{E} is commutative if $[\mathbf{E}, \mathbf{E}] = \mathbb{0}$, and a commutative observable is also sharp if additionally $\mathbf{E}_x \mathbf{E}_y = \delta_{x,y} \mathbf{E}_x$, i.e., if \mathbf{E}_x are mutually orthogonal projection operators. Sharp observables are also referred to as projection valued measures, and by the spectral theorem a sharp observable may be represented by a self-adjoint operator $A = \sum_x \lambda_x \mathbf{E}_x$, where $\{\lambda_x\} \subset \mathbb{R}$ satisfies $\lambda_x \neq \lambda_y$ for $x \neq y$. An observable that is not sharp will be called unsharp. An

observable \mathbf{E} has the norm-1 property if it holds that $\|\mathbf{E}_x\| = 1$ for all x , where $\|\cdot\|$ denotes the operator norm. In finite dimensions, each effect of a norm-1 observable has at least one eigenvector with eigenvalue 1. While sharp observables are trivially norm-1, this property may also be enjoyed by some unsharp observables.

We now introduce definitions for classes of observables that are of particular significance to our results.

Definition 2. An observable $\mathbf{E} := \{\mathbf{E}_x : x \in \mathcal{X}\}$ is called “small-rank” if there exists some $x \in \mathcal{X}$ such that \mathbf{E}_x has rank 1. An observable is called “large-rank” if it is not small-rank.

In particular, a subclass of small-rank observables are called rank-1, for which every effect has rank 1 [38,39]. For example, the effects of a sharp observable represented by a nondegenerate self-adjoint operator $A = \sum_x \lambda_x |\psi_x\rangle\langle\psi_x|$ are the rank-1 projections $\mathbf{E}_x = |\psi_x\rangle\langle\psi_x|$. Such observables are therefore rank-1, and hence small-rank. On the other hand, a sharp observable represented by a degenerate self-adjoint operator such that the eigenspace corresponding to each (distinct) eigenvalue has dimension larger than 1 is large-rank, as each effect is a projection with rank larger than 1.

Definition 3. An observable $\mathbf{E} := \{\mathbf{E}_x : x \in \mathcal{X}\}$ is called “nondegenerate” if there exists some $x \in \mathcal{X}$ such that there are no multiplicities in the strictly positive eigenvalues of \mathbf{E}_x . An observable is called “degenerate” if it is not nondegenerate.

An example of a nondegenerate observable is a small-rank observable, since in such a case there exists an effect that has exactly one strictly positive eigenvalue. On the other hand, a large-rank sharp observable is degenerate, since in such a case each effect has more than one eigenvector with eigenvalue 1.

Definition 4. An observable $\mathbf{E} := \{\mathbf{E}_x : x \in \mathcal{X}\}$ is called “completely unsharp” if for each $x \in \mathcal{X}$, the spectrum of \mathbf{E}_x does not contain either 1 or 0.

Completely unsharp observables evidently do not have the norm-1 property, since it holds that $\|\mathbf{E}_x\| < 1$ for all x . But since the effects also do not have eigenvalue 0, then the effects are in fact full-rank. It follows that completely unsharp observables are also large-rank. But a completely unsharp observable may be either degenerate or nondegenerate.

As a simple illustrative example of when an observable may or may not satisfy the aforementioned properties, let us consider the case where the system is a qubit, $\mathcal{H}_S = \mathbb{C}^2$, with the family of binary observables $\mathbf{E}^{(\lambda)} := \{\mathbf{E}_+^{(\lambda)}, \mathbf{E}_-^{(\lambda)}\}$ defined by

$$\mathbf{E}_\pm^{(\lambda)} := \frac{1}{2}(\mathbb{1}_S \pm \lambda \sigma_z), \quad (1)$$

where $0 < \lambda \leq 1$ and σ_z is the Pauli-Z operator. Note that if $\lambda = 0$, then $\mathbf{E}^{(\lambda)}$ is a trivial observable since in such a case $\mathbf{E}_\pm^{(\lambda)} = \mathbb{1}_S/2$. Since $\mathbf{E}^{(\lambda)}$ are binary, so that $\mathbf{E}_-^{(\lambda)} = \mathbb{1}_S - \mathbf{E}_+^{(\lambda)}$, then they are always commutative, i.e., $[\mathbf{E}_+^{(\lambda)}, \mathbf{E}_-^{(\lambda)}] = \mathbb{0}$. Now note that the spectrum of each effect is $\{(1 + \lambda)/2, (1 - \lambda)/2\}$. If $\lambda = 1$, then the spectrum simplifies to $\{1, 0\}$, i.e., each effect has one eigenvector with eigenvalue 1, and one eigenvector with eigenvalue 0. In such a case, $\mathbf{E}^{(\lambda)}$ is a norm-1, sharp, small-rank, and nondegenerate observable. On the other hand, for any $0 < \lambda < 1$, $\mathbf{E}^{(\lambda)}$ is a completely unsharp, large-rank, and nondegenerate observable. Note that for qubits, the only situation where an effect can be degenerate is when the effect is trivial, i.e., when $\lambda = 0$.

B. Instruments

An instrument [33], or operation valued measure, describes how a system is transformed upon measurement and is given as a collection of operations (completely positive trace non-increasing linear maps) $\mathcal{I} := \{\mathcal{I}_x : x \in \mathcal{X}\}$ such that $\mathcal{I}_{\mathcal{X}}(\cdot) := \sum_{x \in \mathcal{X}} \mathcal{I}_x(\cdot)$ is a channel. Throughout, we shall always assume that the instrument acts in \mathcal{H}_S , i.e., that both the input and output space of \mathcal{I}_x is \mathcal{H}_S . An instrument \mathcal{I} is identified with a unique observable \mathbf{E} via the relation $\text{tr}[\mathcal{I}_x(\rho)] = \text{tr}[\mathbf{E}_x \rho]$ for all outcomes x and states ρ , and we shall refer to such \mathcal{I} as an \mathbf{E} -compatible instrument, or an \mathbf{E} -instrument for short, and to $\mathcal{I}_{\mathcal{X}}$ as the corresponding \mathbf{E} -channel [40]. Note that while every instrument is identified with a unique observable, every observable \mathbf{E} admits infinitely many \mathbf{E} -compatible instruments; the operations of the Lüders instrument \mathcal{I}^L compatible with \mathbf{E} are written as

$$\mathcal{I}_x^L(\cdot) := \sqrt{\mathbf{E}_x} \cdot \sqrt{\mathbf{E}_x}, \quad (2)$$

and it holds that the operations of every \mathbf{E} -compatible instrument \mathcal{I} can be constructed as $\mathcal{I}_x = \Phi_x \circ \mathcal{I}_x^L$, where Φ_x are arbitrary channels acting in \mathcal{H}_S that may depend on outcome x [41,42].

C. Measurement schemes

A quantum system is measured when it undergoes an appropriate physical interaction with a measuring apparatus so that the transition of some variable of the apparatus—such as the position of a pointer along a scale—registers the outcome of the measured observable. The most general description of the measurement process is given by a *measurement scheme*, which is a tuple $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ where \mathcal{H}_A is the Hilbert space for (the probe of) the apparatus \mathcal{A} and ξ is a fixed state of \mathcal{A} , \mathcal{E} is a channel acting in $\mathcal{H}_S \otimes \mathcal{H}_A$ which serves to correlate \mathcal{S} with \mathcal{A} , and $\mathbf{Z} := \{\mathbf{Z}_x : x \in \mathcal{X}\}$ is a POVM acting in \mathcal{H}_A which is referred to as a “pointer observable.” Throughout, we shall always assume that $2 \leq \dim(\mathcal{H}_A) < \infty$. For all outcomes x , the operations of the instrument \mathcal{I} implemented by \mathcal{M} can be written as

$$\mathcal{I}_x(\cdot) = \text{tr}_{\mathcal{A}}[(\mathbb{1}_S \otimes \mathbf{Z}_x) \mathcal{E}(\cdot \otimes \xi)], \quad (3)$$

where $\text{tr}_{\mathcal{A}}[\cdot]$ is the partial trace over \mathcal{A} . The channel implemented by \mathcal{M} is thus $\mathcal{I}_{\mathcal{X}}(\cdot) = \text{tr}_{\mathcal{A}}[\mathcal{E}(\cdot \otimes \xi)]$. Every \mathbf{E} -compatible instrument admits infinitely many *normal* measurement schemes, where ξ is chosen to be a pure state, \mathcal{E} is chosen to be a unitary channel, and \mathbf{Z} is chosen to be sharp [43]. Von Neumann’s model for the measurement process is one such example of a normal measurement scheme. However, unless stated otherwise, we shall consider the most general class of measurement schemes, where ξ need not be pure, \mathcal{E} need not be unitary, and \mathbf{Z} need not be sharp.

IV. MEASUREMENT SCHEMES CONSTRAINED BY THE THIRD LAW

We now consider how the third law constrains measurement schemes, and subsequently examine how such constraints limit the possibility of a measurement to satisfy the proprieties (i)–(v) outlined in the introduction.

Since the third law pertains only to channels and state preparations, the only elements of a measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$ that will be limited by the third law are the interaction channel \mathcal{E} , and the apparatus state preparation ξ . By Definition 1 and the preceding discussion, we therefore introduce the following definition.

Definition 5. A measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$ is constrained by the third law if the following hold:

- (i) ξ is a full-rank state on \mathcal{H}_A .
- (ii) For every full-rank state ρ on $\mathcal{H}_S \otimes \mathcal{H}_A$, $\mathcal{E}(\rho)$ is also a full-rank state.

Properties of measurement schemes constrained by the third law are given in Appendix D and Appendix E. Note that the third law does not impose any constraints on the measurability of observables; we may always choose \mathcal{M} to be a “trivial” measurement scheme, where $\mathcal{H}_A \simeq \mathcal{H}_S$ and \mathcal{E} is a unitary swap channel, in which case the observable \mathbf{E} measured in the system is identified with the pointer observable \mathcal{Z} of the apparatus, which can be chosen arbitrarily. This is in contrast to the case where a measurement is constrained by conservation laws; by the Yanase condition, the pointer observable is restricted so that it commutes with the apparatus part of the conserved quantity, and it follows that an observable not commuting with the system part of the conserved quantity is measurable only if it is unsharp, and only if the apparatus preparation has a large coherence in the conserved quantity [13].

The measurability of observables notwithstanding, let us note that an instrument implemented by a trivial measurement scheme is also trivial, i.e., it will hold that for all outcomes x and states ρ , the operations of \mathcal{I} satisfy $\mathcal{I}_x(\rho) = \text{tr}[\mathbf{E}_x \rho] \xi$. Irrespective of what outcome is observed and what the initial state is, the final state is always ξ . In such a case, \mathcal{I} fails all the properties (i)–(v) that are the subject of our investigation. Therefore, whether or not an observable admits an instrument—realizable by a measurement scheme constrained by the third law as per Definition 5—with such properties remains to be seen: We shall now investigate this.

A. Nondisturbance

An \mathbf{E} -compatible instrument \mathcal{I} does not disturb an observable $\mathbf{F} := \{F_y : y \in \mathcal{Y}\}$ if it holds that

$$\text{tr}[F_y \mathcal{I}_x(\rho)] = \text{tr}[F_y \rho]$$

for all states ρ and outcomes y [35]. In other words, \mathcal{I} does not disturb \mathbf{F} if the statistics of \mathbf{F} are not affected by a prior nonselective measurement of \mathbf{E} by \mathcal{I} . See Fig. 1 for a schematic representation of a nondisturbing measurement. Nondisturbance is possible only for *jointly measurable* observables, since in such a case the sequential measurement of \mathbf{E} by \mathcal{I} , followed by a measurement of \mathbf{F} , defines a joint observable for \mathbf{E} and \mathbf{F} [40]. In the absence of any constraints, commutation of \mathbf{E} with \mathbf{F} is sufficient for nondisturbance. That is, if \mathbf{F} commutes with \mathbf{E} , there exists an \mathbf{E} -instrument \mathcal{I} that does not disturb \mathbf{F} . Moreover, the Lüders \mathbf{E} -instrument \mathcal{I}^L does not disturb *all* \mathbf{F} commuting with \mathbf{E} [44]. This can be easily shown by the following: If all effects of \mathbf{F} and \mathbf{E}

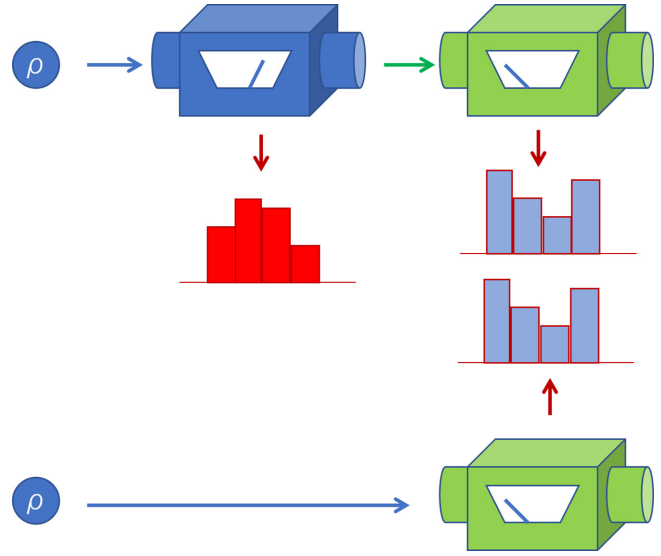


FIG. 1. The top half of the figure represents a sequential measurement of possibly different observables in a system initially prepared in state ρ , with the histograms representing the statistics obtained for each measurement in the sequence. The bottom half shows the case where the first measurement in the sequence is removed, and only the second measurement takes place. When the statistics of such a measurement are the same in both scenarios, for all states ρ , then the first measurement is said to not disturb the second.

mutually commute, then by Eq. (2) we may write

$$\begin{aligned} \text{tr}[F_y \mathcal{I}_x^L(\rho)] &= \sum_x \text{tr}[F_y \sqrt{\mathbf{E}_x} \rho \sqrt{\mathbf{E}_x}] \\ &= \sum_x \text{tr}[\sqrt{\mathbf{E}_x} F_y \sqrt{\mathbf{E}_x} \rho] \\ &= \sum_x \text{tr}[\mathbf{E}_x F_y \rho] = \text{tr}[F_y \rho]. \end{aligned}$$

In the second line we have used the cyclicity of the trace, and in the third line we use $[\mathbf{E}_x, F_y] = 0 \iff [\sqrt{\mathbf{E}_x}, F_y] = 0$.

While commutation is sufficient for nondisturbance, it is in general not necessary; if \mathbf{E} and \mathbf{F} do not commute but are both sufficiently unsharp so as to be jointly measurable [45], then it *may* be possible for a measurement of \mathbf{E} to not disturb \mathbf{F} , but not always: While nondisturbance requires joint measurability, joint measurability does not guarantee nondisturbance. Let us consider an example where nondisturbance is permitted for two noncommuting observables. Consider the case that $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$, with the orthonormal basis $\{|k\rangle \otimes |m\rangle : k, m = 0, 1\}$, and define the following family of operators on \mathbb{C}^2 :

$$\begin{aligned} A_0 &= |0\rangle\langle 0|, & A_1 &= \frac{1}{2}|0\rangle\langle 0|, & A_2 &= \frac{1}{2}|1\rangle\langle 1|, \\ A_3 &= \frac{1}{2}|+\rangle\langle +|, & A_4 &= \frac{1}{2}|-\rangle\langle -|, & A_5 &= |1\rangle\langle 1|, \end{aligned}$$

where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Now consider the binary observables $\mathbf{E} := \{E_0, E_1\}$ and $\mathbf{F} := \{F_0, F_1\}$ acting in \mathcal{H}_S , defined by

$$\begin{aligned} E_0 &= A_0 \otimes |0\rangle\langle 0| + (A_2 + A_4) \otimes |1\rangle\langle 1|, \\ E_1 &= (A_1 + A_3) \otimes |1\rangle\langle 1| + A_5 \otimes |0\rangle\langle 0|, \end{aligned}$$

and

$$\begin{aligned} F_0 &= A_0 \otimes |0\rangle\langle 0| + (A_1 + A_4) \otimes |1\rangle\langle 1|, \\ F_1 &= (A_2 + A_3) \otimes |1\rangle\langle 1| + A_5 \otimes |0\rangle\langle 0|. \end{aligned}$$

One can confirm that $[E, F] \neq 0$. But we can construct an E -instrument \mathcal{I} with operations

$$\begin{aligned} \mathcal{I}_0(\rho) &= \text{tr}[\rho(A_0 \otimes |0\rangle\langle 0| + A_4 \otimes |1\rangle\langle 1|)]|0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ &\quad + \text{tr}[\rho(A_2 \otimes |1\rangle\langle 1|)]|1\rangle\langle 1| \otimes |0\rangle\langle 0|, \\ \mathcal{I}_1(\rho) &= \text{tr}[\rho(A_5 \otimes |0\rangle\langle 0| + A_3 \otimes |1\rangle\langle 1|)]|1\rangle\langle 1| \otimes |0\rangle\langle 0| \\ &\quad + \text{tr}[\rho(A_1 \otimes |1\rangle\langle 1|)]|0\rangle\langle 0| \otimes |0\rangle\langle 0|, \end{aligned}$$

which does not disturb F .

However, we show that under the third law constraint, commutation is in fact necessary for nondisturbance. That is, if an E -instrument \mathcal{I} can be implemented by a measurement scheme constrained by the third law, such that \mathcal{I} does not disturb F , then $[E, F] = 0$ must be satisfied. In Appendix D we show that for any instrument \mathcal{I} , implemented by a measurement scheme constrained by the third law, there exists at least one full-rank state ρ_0 such that $\mathcal{I}_x(\rho_0) = \rho_0$. In such a case, nondisturbance of F_y (i.e., $\text{tr}[F_y \mathcal{I}_x(\rho)] = \text{tr}[F_y \rho]$ for all ρ) implies nondisturbance of a sharp observable $P = \{P_z\}$, where P_z are the spectral projections of F_y . That is, a sequential measurement of E by the instrument \mathcal{I} , followed by a measurement of P , is a joint measurement of E and P . Since joint measurability implies commutation when either observable is sharp, it follows that E must commute with P , and hence with F_y , for all y . In other words, given the existence of a full-rank fixed state ρ_0 , then a measurement of E does not disturb F only if they commute. See also Proposition 4 of Ref. [35].

But when the measurement of E is constrained by the third law, we show that $[E, F] = 0$ is not sufficient for nondisturbance: The properties of E impose further constraints. We now present our first main result.

Theorem 4.1. Under the third law constraint, a completely unsharp observable E admits a measurement that does not disturb any observable F that commutes with E . On the other hand, if an observable E satisfies $\|E_x\| = 1$ for any outcome x , then there exists F which commutes with E but is disturbed by any measurement of E that is constrained by the third law.

That is, an E -compatible instrument \mathcal{I} admits a measurement scheme \mathcal{M} that is constrained by the third law, such that $[E, F] = 0 \Rightarrow \text{tr}[F_y \mathcal{I}_x(\rho)] = \text{tr}[F_y \rho]$ for all y and ρ , if E is completely unsharp and only if $\|E_x\| < 1$ for all outcomes x . Note that an observable can satisfy $\|E_x\| < 1$ for all x without being completely unsharp, since such effects can still have 0 in their spectrum. The proof is presented in Appendix F (Proposition F.1). To show sufficiency of complete unsharpness we prove that, given the third law constraint, an observable admits a Lüders instrument if and only if it is completely unsharp (Proposition D.1). But since Lüders measurements are guaranteed to not disturb any commuting observable, the claim immediately follows. On the other hand, the necessity that the effects have norm smaller than 1 follows from the following: If any effect of E has eigenvalue 1, the projection onto such eigenspace commutes with E but is shown to be disturbed. In particular, this implies that when a norm-1 observable (such as a sharp observable) is measured under the third law constraint,

then there exists some observable F that commutes with E but is nonetheless disturbed.

Of course, even if a sharp or norm-1 observable E fails the strict nondisturbance condition, this does not imply that some nondisturbed observables do not exist. In Appendix F we show that if an observable is small-rank as per Definition 2, then it holds that a third-law-constrained measurement of such an observable will disturb all observables, even if they commute. Second, we show that if E is a nondegenerate observable as per Definition 3, then the class of nondisturbed observables will be commutative, and any pair of nondisturbed observables will commute. That is, for any F and G that are nondisturbed, then it will hold that $[F, G] = [F, F] = [G, G] = 0$. In other words, nondegeneracy of the measured observable will spoil the ‘‘coherence’’ of the measured system. Therefore, to ensure that a measurement of E does not disturb a nontrivial class of (possibly noncommutative) observables, then E must be a large rank (and degenerate) observable.

For the binary qubit observables $E^{(\lambda)}$ introduced in Eq. (1), there exists a third-law-constrained measurement of $E^{(\lambda)}$ such that all commuting observables F are nondisturbed if and only if $\lambda < 1$, in which case $E^{(\lambda)}$ are completely unsharp. But note that since $E^{(\lambda)}$ is always a degenerate observable when $0 < \lambda < 1$, then the nondisturbed observables F must also be commutative. On the other hand, if $\lambda = 1$, then $E^{(\lambda)}$ is a small-rank observable, and so its measurement will disturb all observables. In Appendix F we construct an explicit example where the measurement of a sharp observable that is large-rank, and hence degenerate, will not disturb a nontrivial class of possibly noncommutative observables. This is a binary observable E acting in a two-qubit system $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$, defined by $E_x = \mathbb{1} \otimes |x\rangle\langle x|$. Note that these effects have rank 2, and are hence also degenerate. In such a case, any observable F with effects $F_y \otimes \mathbb{1}$ will be nondisturbed, and it may be the case that $[F, F] \neq 0$.

B. First-kindness

An E -compatible instrument \mathcal{I} is a measurement of the first kind if \mathcal{I} does not disturb E itself, i.e., if it holds that

$$\text{tr}[E_x \mathcal{I}_x(\rho)] = \text{tr}[E_x \rho]$$

for all states ρ and outcomes x [34]. See Fig. 2 for a schematic representation of a measurement of the first kind. In the absence of any constraints, commutativity of an observable is

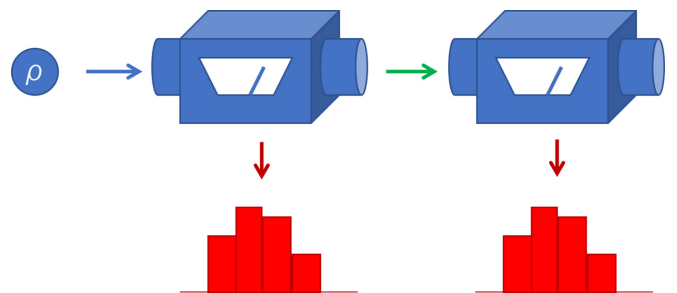


FIG. 2. When the same observable is measured in succession, and when the statistics of the second measurement are the same as those of the first, for all input states ρ , then such a measurement is said to be of the first kind.

sufficient for it to admit a first-kind measurement; for any observable E such that $[E, E] = \mathbb{O}$ holds, the corresponding Lüders instrument is a measurement of the first kind. This follows from analogous reasoning to that given above. But we show that, under the third law constraint, commutativity is necessary for first-kindness, but not sufficient. We now present our second main result.

Theorem 4.2. Under the third law constraint, an observable E admits a measurement of the first kind if and only if E is commutative and completely unsharp.

In particular, note that a third-law-constrained measurement of any norm-1 observable, such as a sharp observable, necessarily disturbs itself. The proof is given in Appendix G (Proposition G.1). The sufficiency follows from the fact that any completely unsharp observable admits a Lüders instrument, as discussed above. On the other hand, the following is a sketch of the proof for the necessity of such a condition: A nonselective measurement constrained by the third law always leaves some full-rank state ρ_0 invariant. Nondisturbance of E therefore demands commutativity, as discussed above. But every commutative observable E is a classical post processing of a sharp observable P , i.e., we may write $E_x = \sum_y p(x|y)P_y$ where $\{p(x|y)\}$ is a family of nonnegative numbers satisfying $\sum_x p(x|y) = 1$ for every y [46]. Given that \mathcal{I}_x has a full-rank fixed state, then if \mathcal{I} is a first-kind measurement, P is also not disturbed [13]. Therefore, a sequential measurement of E by \mathcal{I} followed by measurement of P defines a joint measurement of E and P . By Eq. (3), we obtain for every ρ the following:

$$\text{tr}[P_y E_x P_y \rho] = \text{tr}[P_y \otimes Z_x \mathcal{E}(\rho \otimes \xi)].$$

Now assume that ρ is full-rank. Given that a third-law-constrained measurement employs a full-rank apparatus preparation ξ , while \mathcal{E} obeys Definition 1, then $\mathcal{E}(\rho \otimes \xi)$ is full-rank. It follows that the term on the right-hand side is strictly positive, and hence so too is the term on the left. But this implies that $P_y E_x P_y > \mathbb{O}$, and so $0 < p(x|y) < 1$, for all x, y . Therefore, E is completely unsharp.

For the binary qubit observables $E^{(\lambda)}$ introduced in Eq. (1), there exists a third law-constrained measurement of the first kind if and only if $\lambda < 1$, in which case $E^{(\lambda)}$ are commutative and completely unsharp. In Appendix G we construct an explicit example of a first-kind measurement (not given by a Lüders instrument) of a commutative and completely unsharp observable. We consider a system $\mathcal{H}_S = \mathbb{C}^N$ with orthonormal basis $\{|n\rangle : n = 1, \dots, N\}$, and an observable $E := \{E_x : x = 1, \dots, N\}$ acting in \mathcal{H}_S given by the effects $E_x = \sum_n p(n|x)|n\rangle\langle n|$. Here $p(n|x) = q(x \ominus n)$, where \ominus denotes subtraction modulo N , with $q(n)$ some arbitrary probability distribution satisfying $0 < q(n) < 1$ for all n . Such an observable is commutative and completely unsharp.

C. Repeatability

An E -compatible instrument \mathcal{I} is repeatable if it holds that

$$\text{tr}[E_y \mathcal{I}_x(\rho)] = \delta_{x,y} \text{tr}[E_x \rho]$$

for all states ρ and outcomes x, y [47,48]. In other words, an instrument \mathcal{I} is a repeatable measurement of E if a second measurement of E is guaranteed (with probabilistic certainty) to produce the same outcome as \mathcal{I} . See Fig. 3 for a schematic

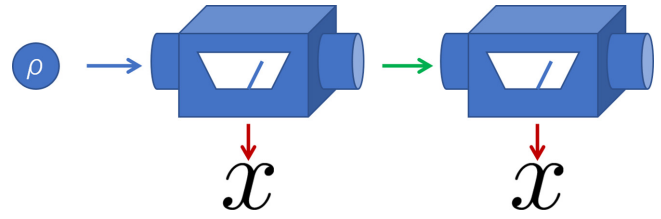


FIG. 3. When the same observable is measured in succession, and when the outcome obtained by the second is guaranteed with probabilistic certainty to coincide with that of the first, for all input states ρ , then such a measurement is said to be repeatable.

representation of a repeatable measurement. It is simple to verify that repeatability implies first-kindness, since if \mathcal{I} is repeatable, then we have

$$\text{tr}[E_y \mathcal{I}_x(\rho)] = \sum_x \text{tr}[E_y \mathcal{I}_x(\rho)] = \text{tr}[E_y \rho].$$

While a first-kind measurement need not be repeatable in general, repeatability and first-kindness coincide for the class of sharp observables (Theorem 1 in Ref. [34]). For example, if E is commutative then the corresponding Lüders instrument is a measurement of the first kind, but such an instrument is repeatable if and only if E is sharp; note that $\text{tr}[E_x \mathcal{I}_x^L(\rho)] = \text{tr}[E_x^2 \rho]$, which satisfies the repeatability condition if and only if $E_x^2 = E_x$.

An observable E admits a repeatable instrument only if it is norm-1, and in the absence of any constraints, all norm-1 observables admit a repeatable instrument. For example, if E is a possibly unsharp observable with the norm-1 property, and if $|\psi_x\rangle$ are eigenvalue-1 eigenvectors of the effects E_x , then an instrument with operations $\mathcal{I}_x(\rho) = \text{tr}[E_x \rho] |\psi_x\rangle\langle \psi_x|$ is repeatable. Note that if the system is a qubit, then only sharp observables admit repeatable measurements. For example, the binary qubit observables $E^{(\lambda)}$ introduced in Eq. (1) are norm-1 if and only if $\lambda = 1$, in which case the observable is also sharp. Now we present our third main result.

Theorem 4.3. Under the third law constraint, no observable admits a repeatable measurement.

This is an immediate consequence of Theorem 4.2 which shows that, under the third law constraint, norm-1 observables do not admit a measurement of the first kind. Since repeatability is admitted only for norm-1 observables, and since repeatability implies first-kindness, then the statement follows. In fact, we can show that for every sequence of outcomes x and y , there exists a state ρ such that $\text{tr}[E_y \mathcal{I}_x(\rho)] > 0$. See Corollary G.1 for further details.

D. Ideality

An instrument \mathcal{I} is said to be an ideal measurement of E if for every outcome x there exists a state ρ such that $\text{tr}[E_x \rho] = 1$, and if for every outcome x and every state ρ the following implication holds:

$$\text{tr}[E_x \rho] = 1 \Rightarrow \mathcal{I}_x(\rho) = \rho.$$

That is, \mathcal{I} is an ideal measurement if it does not change the state of the system whenever the outcome can be predicted with certainty [49]. See Fig. 4 for a schematic

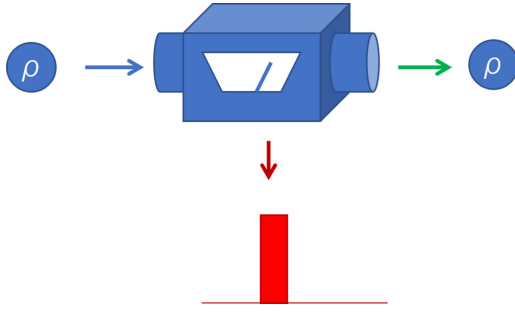


FIG. 4. When an observable is measured in a system such that whenever an outcome can be predicted with certainty, the state of the measured system is unperturbed, then such a measurement is said to be ideal.

representation of an ideal measurement. Note that ideality can be enjoyed only by norm-1 observables; since $\text{tr}[\mathbf{E}_x \rho] \leq \|\mathbf{E}_x\|$, then any \mathbf{E} that does not enjoy the norm-1 property fails the antecedent of the ideality condition, in which case such condition becomes void. Conversely, in the absence of any constraints all norm-1 observables admit an ideal measurement; the condition $\text{tr}[\mathbf{E}_x \rho] = 1$ holds if and only if ρ only has support in the eigenvalue-1 eigenspace of \mathbf{E}_x , which implies that $\mathbf{E}_x \rho = \rho \mathbf{E}_x = \rho$. But in such a case, we obtain $\mathcal{I}_x^L(\rho) = \sqrt{\mathbf{E}_x} \rho \sqrt{\mathbf{E}_x} = \mathbf{E}_x \rho = \rho$, and so the Lüders measurement of a norm-1 observable is ideal.

For the class of sharp observables, the ideal measurements are precisely the Lüders instruments (see Theorem 10.6 in Ref. [30]). Since the third law permits only Lüders instruments for completely unsharp observables, then we may immediately infer that ideal measurements of any sharp observable, even those represented by a possibly degenerate self-adjoint operator, are prohibited by the third law. For example, the binary qubit observables $\mathbf{E}^{(\lambda)}$ introduced in Eq. (1) are norm-1 if and only if $\lambda = 1$, in which case the observable is also sharp. Therefore, such observables never admit an ideal measurement when constrained by the third law.

However, unsharp observables admit ideal measurements that are not given by the Lüders instrument. For example, consider a system $\mathcal{H}_S = \mathbb{C}^3$ with orthonormal basis $\{|-1\rangle, |0\rangle, |1\rangle\}$. Let $\mathbf{E} := \{\mathbf{E}_+, \mathbf{E}_-\}$ be a binary norm-1 observable acting in \mathcal{H}_S , defined by $\mathbf{E}_\pm = |\pm 1\rangle\langle\pm 1| + \frac{1}{2}|0\rangle\langle 0|$. It can easily be verified that an instrument with operations

$$\mathcal{I}_\pm(\cdot) = \langle\pm 1| \cdot |\pm 1\rangle |\pm 1\rangle\langle\pm 1| + \langle 0| \cdot |0\rangle \frac{\mathbb{1}_S}{6}$$

is an ideal measurement of \mathbf{E} . Therefore, the restriction imposed by the third law on the realizability of Lüders instruments does not by itself rule out the possibility of ideal measurements for unsharp norm-1 observables. Now we present our fourth main result.

Theorem 4.4. Under the third law constraint, no observable admits an ideal measurement.

The proof is given in Appendix H (Proposition H.1), and the following is a rough sketch. If \mathcal{I} is an ideal measurement of \mathbf{E} , and if ρ is a state for which outcome x can be predicted with certainty, then $\mathcal{I}_y(\rho) = \mathbb{O}$ for all $y \neq x$, which implies that $\mathcal{I}_X(\rho) = \rho$. But given the third law constraint, for every

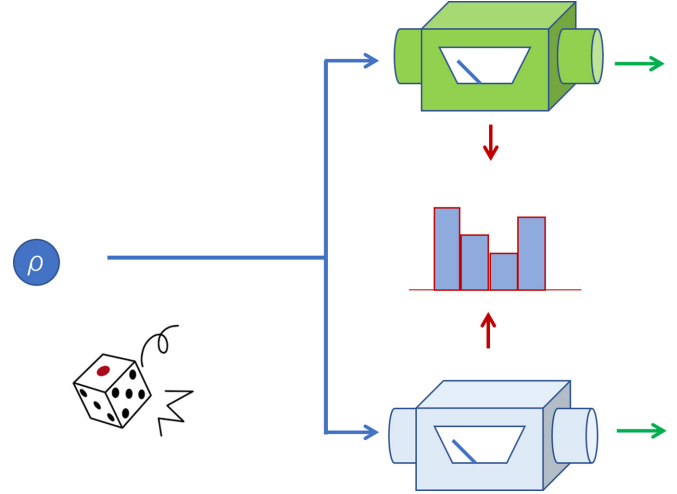


FIG. 5. A system may be measured by an instrument obtained by a probabilistic mixture of two distinct instruments. An extremal instrument is such that cannot be recovered as a probabilistic mixture of two distinct instruments.

state ρ such that $\text{tr}[\mathbf{E}_x \rho] = 1$, it is shown that ρ cannot be a fixed state of \mathcal{I}_X , and so \mathcal{I} cannot be ideal.

E. Extremality

For any fixed value space \mathcal{X} , the set of instruments is convex. That is, given any $\lambda \in [0, 1]$, and any pair of instruments $\mathcal{I}^{(i)} := \{\mathcal{I}_x^{(i)} : x \in \mathcal{X}\}, i = 1, 2$, we can construct an instrument \mathcal{I} with the operations

$$\mathcal{I}_x(\cdot) = \lambda \mathcal{I}_x^{(1)}(\cdot) + (1 - \lambda) \mathcal{I}_x^{(2)}(\cdot).$$

An instrument \mathcal{I} is *extremal* when for any $\lambda \in (0, 1)$ such a decomposition is possible only if $\mathcal{I} = \mathcal{I}^{(1)} = \mathcal{I}^{(2)}$. Intuitively, this implies that an extremal instrument is “pure,” whereas a nonextremal instrument suffers from “classical noise.” See Fig. 5 for a schematic representation of an extremal instrument. For an in-depth analysis of extremal instruments and their properties, see Refs. [36,50]. A simple example of an extremal instrument is the Lüders instrument compatible with an observable with linearly independent effects. Since such linear independence is trivially satisfied for norm-1 observables, then their corresponding Lüders instruments are extremal. But it is also possible for the effects of a completely unsharp observable to be linearly independent. For example, a binary observable $\mathbf{E} := \{\mathbf{E}_0, \mathbf{E}_1\}$ acting in $\mathcal{H}_S = \mathbb{C}^2$, defined as $\mathbf{E}_0 = 3/4|0\rangle\langle 0| + 1/4|1\rangle\langle 1|$ and $\mathbf{E}_1 = \mathbb{1} - \mathbf{E}_0$, is completely unsharp with linearly independent effects. Indeed, the effects of the qubit observables $\mathbf{E}^{(\lambda)}$ defined in Eq. (1) are linearly independent for any $0 < \lambda < 1$. Since the Lüders instruments for such observables are extremal, and can be implemented under the third law constraint, then we can immediately infer that extremality is permitted by the third law. Now we present our final main result.

Theorem 4.5. Under the third law constraint, an observable \mathbf{E} acting in \mathcal{H}_S admits an extremal instrument only if $\text{rank}(\mathbf{E}_x) \geq \sqrt{\dim(\mathcal{H}_S)}$ for all outcomes x , and a measuring apparatus can implement an extremal instrument only if it interacts with the system with a nonunitary channel \mathcal{E} .

The proof is given in Appendix I (Proposition I.1). It follows that under the third law constraint, extremality is permitted only for large-rank observables. Note in particular that since Lüders measurements of completely unsharp observables may be extremal, then the above result indicates that they are realisable under the third law constraint only with nonunitary measurement interactions; indeed, our proof for the sufficiency of complete unsharpness for the realisability of Lüders instruments (Proposition D.1) uses a nonunitary interaction channel. Furthermore, note that in contradistinction to the other properties discussed above, unsharpness of E is not a necessary condition for extremality. Indeed, sharp observables with sufficiently large rank admit an extremal instrument, albeit such instruments cannot be Lüders due to the previous results. In Appendix I, we provide a concrete model for an extremal instrument compatible with a binary sharp observable E acting in $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$, defined by $E_x = \mathbb{1} \otimes |x\rangle\langle x|$. Since $\text{rank}(E_x) = 2 = \sqrt{\dim(\mathcal{H}_S)}$, we see that the bound provided in the above theorem is in fact tight.

V. DISCUSSION

We have generalized and strengthened the results of Ref. [26], in the finite-dimensional setting, in several ways. We have considered the most general class of (discrete) observables—both the observable to be measured and the pointer observable for the measuring apparatus—and not just those that are sharp and rank-1. Moreover, we have considered a more general class of measurement interactions, between the measured system and measuring apparatus, constrained only by our operational formulation of the third law and thus not restricted to the standard unitary or rank nondecreasing framework. Within the extended setting thus described, we have shown that ideal measurements are categorically prohibited by the third law for all observables and, *a fortiori*, we showed that the third law dictates that whenever a measurement outcome can be predicted with certainty, then the state of the measured system is necessarily perturbed upon measurement. Moreover, we showed that the third law also forbids repeatable measurements, where we note that repeatability and ideality coincide only in the case of sharp rank-1 observables. In addition to the aforementioned impossibility statements, however, our results also include possibility statements as regards extremality and nondisturbance: The third law allows for an extremal instrument that measures an observable with sufficiently large rank, and for a measurement of a completely unsharp observable so that such a measurement will not disturb any observable that commutes.

Our results have interesting consequences for the role of unsharp observables in the foundations of quantum theory, and the question: What is real? There are two deeply connected traditional paradigms for the assignment of reality to a system, both of which are formulated with respect to sharp observables: The Einstein-Podolsky-Rosen (EPR) criterion [51], and the macrorealism criteria of Leggett-Garg [52].

The EPR criterion for a physical property to correspond to an element of reality reads: “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this

physical quantity.” In other words, the EPR criterion rests on the possibility of ideal measurements: An eigenvalue of some self-adjoint operator exists in a system when the system is in the corresponding eigenstate, so that an ideal measurement of the observable reveals the eigenvalue while leaving the system in the same state. But the EPR criterion is shown to be in conflict with the third law of thermodynamics: It is in fact *not* possible to ascertain any property of the system, with certainty, without changing its state. As argued by Busch and Jaeger [31,32], however, the EPR criterion is sufficient, but not necessary; a necessary condition for a property of a system to correspond to an element of reality is that it must have the capacity of influencing other systems, such as a measuring apparatus, in a way that is characteristic of such property. Indeed, since the influence the system has on the apparatus may come in degrees—quantified by the probability, or “propensity,” for the apparatus to register that such property obtains a given value in the system—then even an unsharp observable may correspond to an element of “unsharp reality.” But note that this weaker criterion makes no stipulation as to how the state of the system changes upon measurement, and does not rely on the possibility of ideal measurements: A property may exist in a system even if its measurement changes the state of the system. Consequently, our results provide support for the unsharp reality program of Busch and Jaeger from a thermodynamic standpoint, as it is shown to be compatible with the third law.

On the other hand, Leggett and Garg proposed Macrorealism as the conjunction of two postulates: (MR) *Macrorealism per se* and (NI) *Noninvasive measurability*. MR rests on the notion of *definiteness*, i.e., that at any given time, a system can be only in one out of a set of states that are perfectly distinguishable by measurement of the observable describing the system—for example, an eigenstate corresponding to some eigenvalue of a self-adjoint operator. On the other hand, NI requires that measurement of such observable not influence the statistics of other observables at later times. In other words, NI relies on the possibility of a nondisturbing measurement. But we showed that the third law permits nondisturbance only for unsharp observables without the norm-1 property. Since such observables do not admit definite values in any state, i.e., no two states can be perfectly distinguished by a measurement of such observables, the third law is incompatible with the conjunction of MR and NI. It follows that if we want to keep NI, then we must drop MR; once again we are forced to adopt the notion of an unsharp reality.

To be sure, the third law of thermodynamics should not be considered in isolation; a complete analysis of how the laws of thermodynamics constrain channels and quantum measurements demands that the third law be considered in conjunction with the first (conservation of energy) and with the second (no perpetual motion of the second kind). Indeed, our operational formulation of the third law is independent of any notion of temperature, energy, or time. We expect that in the complete picture, that is, when the other laws are also taken into account, our generalized formulation will recover the standard notions of the third law in the literature. It is also an interesting question to ask how our formulation of the third law, and the constraints imposed by such law on measurements, can be extended to the infinite-dimensional setting. A complete

operational formulation of channels constrained by the laws of thermodynamics, and for more general systems than those of finite dimension, is thus still an open problem; our work constitutes one part of such a program, which extends the research discipline devoted to the “thermodynamics of quantum measurements” [53–69]. While our impossibility results are expected to carry over to the more complete framework, the question remains as to how our positive claims must be adapted in light of the other laws of thermodynamics: The combined laws may impose further constraints. Indeed, as witnessed by the Wigner-Araki-Yanase theorem, conservation of energy imposes constraints on the measurability of observables that do not commute with the Hamiltonian [5–14]. This is in contradistinction to the third law, which imposes no constraints on measurability. We leave such open questions for future work.

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APPENDIX A: PRELIMINARIES

Before presenting the proofs for our main results, let us establish some basic notation and definitions. We denote the algebra of linear operators on a finite-dimensional complex Hilbert space \mathcal{H} as $\mathcal{L}(\mathcal{H})$. For any subset $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$, we denote the *commutant* as

$$\mathcal{A}' := \{B \in \mathcal{L}(\mathcal{H}) : [A, B] = 0 \forall A \in \mathcal{A}\}.$$

A “Schrödinger picture” operation is defined as a completely positive (CP), trace-nonincreasing linear map $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$, where \mathcal{H} is the input space and \mathcal{K} is the output space. When both input and output spaces are the same, i.e., $\mathcal{H} = \mathcal{K}$, we say that Φ acts in \mathcal{H} . The associated “Heisenberg picture” dual operation is a completely positive linear map $\Phi^* : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$, defined by the trace duality $\text{tr}[A\Phi(B)] = \text{tr}[\Phi^*(A)B]$ for all $A \in \mathcal{L}(\mathcal{K})$ and $B \in \mathcal{L}(\mathcal{H})$. Φ^* is subunital, i.e., $\Phi^*(\mathbb{1}_{\mathcal{K}}) \leq \mathbb{1}_{\mathcal{H}}$ and is unital when the equality holds, which is the case exactly when Φ is a channel, i.e., when Φ preserves the trace. We shall also refer to unital CP maps Φ^* as channels.

The rank of an operator $A \in \mathcal{L}(\mathcal{H})$ is defined as $\text{rank}(A) := \dim(\text{Im}A) = \dim(A\mathcal{H})$, which coincides with the number of strictly positive eigenvalues of A^*A . A positive operator $A > 0$ is said to be full-rank, or to have full rank in \mathcal{H} , if all of its eigenvalues are strictly positive. That is, A is full-rank in \mathcal{H} when $\text{rank}(A) = \dim(\mathcal{H})$. A state on \mathcal{H} is a positive operator of unit trace, and a full-rank state is also faithful, where a state ρ is called faithful if for any $A \in \mathcal{L}(\mathcal{H})$ the following implication holds: $\text{tr}[\rho A^*A] = 0 \iff A = 0$. Now we show a useful property of full-rank states which shall be frequently employed in this paper.

Lemma A.1. Consider the states ρ and σ on \mathcal{H} . If ρ is full-rank, then there exists $\lambda \in (0, 1)$ such that $\rho \geq \lambda\sigma$.

Proof. Let us first note that $\rho - \lambda\sigma \geq 0$ if and only if $\text{tr}[E(\rho - \lambda\sigma)] \geq 0$ for all $E \in \mathcal{E}(\mathcal{H})$, where we define $\mathcal{E}(\mathcal{H}) := \{E \in \mathcal{L}(\mathcal{H}) : 0 < E \leq \mathbb{1}\}$ as the space of (nonvanishing) effects on \mathcal{H} . Therefore, let us define

$$\epsilon := \inf_{E \in \mathcal{E}(\mathcal{H})} \{\text{tr}[E\rho]\} \in [0, 1],$$

$$\delta := \sup_{E \in \mathcal{E}(\mathcal{H})} \{\text{tr}[E\sigma]\} = 1.$$

If ρ is full-rank, then it holds that $\epsilon > 0$. Choosing $\lambda = \epsilon/\delta$, we thus have $\text{tr}[E(\rho - \lambda\sigma)] \geq \epsilon - \epsilon = 0$ for all $E \in \mathcal{E}(\mathcal{H})$. Therefore, there exists $0 < \lambda < 1$ such that $\rho \geq \lambda\sigma$. ■

APPENDIX B: PROPERTIES OF CHANNELS CONSTRAINED BY THE THIRD LAW

Recall from Definition 1 that a channel is constrained by the third law if it maps full-rank states to full-rank states. We now show that the set of channels constrained by the third law is closed under composition.

Lemma B.1. Consider the channels $\Phi_1 : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$ and $\Phi_2 : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{R})$. If each channel is constrained by the third law, then so too is their composition $\Phi_2 \circ \Phi_1 : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{R})$.

Proof. Assume that Φ_i are constrained by the third law. Let ρ be a full-rank state on \mathcal{H} . It follows that $\Phi_1(\rho)$ is a full-rank state on \mathcal{K} , and so $\Phi_2 \circ \Phi_1(\rho)$ is a full-rank state on \mathcal{R} . As such, $\Phi_2 \circ \Phi_1$ maps all full-rank states on \mathcal{H} to full-rank states on \mathcal{R} , and so by Definition 1 is constrained by the third law. ■

We shall now prove a useful result that will allow for equivalent formulations of channels constrained by the third law.

Lemma B.2. Consider a channel $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$. The following statements are equivalent:

- (i) For every full-rank state ρ on \mathcal{H} , $\Phi(\rho)$ is a full-rank state on \mathcal{K} .
- (ii) There exists a state σ on \mathcal{H} such that $\Phi(\sigma)$ is a full-rank state on \mathcal{K} .
- (iii) For every $A \in \mathcal{L}(\mathcal{K})$, $\Phi^*(A^*A) = 0 \iff A = 0$.
- (iv) In the case that $\mathcal{H} = \mathcal{K}$, there exists a full-rank state ρ_0 on \mathcal{H} such that $\Phi(\rho_0) = \rho_0$.

Proof. (i) \implies (ii): This is trivial.

(ii) \implies (i): By Lemma A.1, for any full-rank state ρ , and for any state σ , there exists $\lambda \in (0, 1)$ such that $\rho \geq \lambda\sigma$. Assume that $\Phi(\sigma)$ is full-rank. It holds that for any full-rank state ρ , and for any unit vector $|\phi\rangle \in \mathcal{K}$, we have $\langle \phi | \Phi(\rho) | \phi \rangle \geq \lambda \langle \phi | \Phi(\sigma) | \phi \rangle > 0$, which implies that $\Phi(\rho)$ is full-rank.

(ii) \implies (iii): For any channel Φ^* , it holds that $\|\Phi^*(A)\| \leq \|A\|$, and so $A = 0 \implies \Phi^*(A^*A) = 0$ follows. Now assume that for some A , it holds that $\Phi^*(A^*A) = 0$. Then $\text{tr}[\Phi(\sigma)A^*A] = \text{tr}[\sigma\Phi^*(A^*A)] = 0$ for any state σ . But if $\Phi(\sigma)$ is full-rank, this holds only if $A = 0$.

(iii) \implies (ii): Consider the complete mixture $\sigma = \mathbb{1}/\dim(\mathcal{H})$, and let $P \leq \mathbb{1}$ be the minimal support projection on $\Phi(\sigma)$, with $P^\perp := \mathbb{1} - P \geq 0$ its orthogonal complement. It follows that $\text{tr}[\Phi^*(P^\perp)\sigma] = \text{tr}[P^\perp\Phi(\sigma)] = 0$. But by (iii),

this holds if and only if $P^\perp = \mathbb{O}$, and hence $P = \mathbb{1}$. That is, Φ maps the complete mixture to a full-rank state.

(i) \Rightarrow (iv): Assume that $\mathcal{H} = \mathcal{K}$, so that for any $n \in \mathbb{N}$ we may define the channel $\Phi^n : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ as n consecutive applications of the channel Φ . Now consider the channel

$$\Phi_{\text{av}}(\cdot) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \Phi^n(\cdot),$$

where we note that this limit exists as $\dim(\mathcal{H}) < \infty$. Let us define the state $\rho_0 := \Phi_{\text{av}}(\sigma)$, where $\sigma = \mathbb{1}/\dim(\mathcal{H})$ is the complete mixture. Since the complete mixture is full-rank, then $\Phi^n(\sigma)$ will be full-rank for all $n \in \mathbb{N}$, and so it holds that ρ_0 is full-rank. But note that

$$\begin{aligned} \Phi(\rho_0) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \Phi^{n+1}(\sigma) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=1}^N \Phi^n(\sigma) + \Phi^{N+1}(\sigma) - \Phi(\sigma) \right) \\ &= \Phi_{\text{av}}(\sigma) = \rho_0. \end{aligned}$$

and so there exists a full-rank state ρ_0 such that $\Phi(\rho_0) = \rho_0$.

(iv) \Rightarrow (ii): This is trivial. \blacksquare

Corollary B.1. The following channels are constrained by the third law:

- (i) Rank nondecreasing channels acting in \mathcal{H} .
- (ii) Bistochastic channels acting in \mathcal{H} .
- (iii) Gibbs-preserving channels acting in \mathcal{H} .
- (iv) Partial trace channels $\text{tr}_A : \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_S)$.
- (5) Composition channels $\Lambda_\xi : \mathcal{L}(\mathcal{H}_S) \rightarrow \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_A)$, $\rho \mapsto \rho \otimes \xi$, where ξ is a full-rank state on \mathcal{H}_A .

Proof. (i) This is trivial.

(ii) Bistochastic channels preserve both the trace and the identity. Consider the complete mixture $\sigma := \mathbb{1}/\dim(\mathcal{H})$. It trivially holds that if Φ is bistochastic, then $\Phi(\sigma) = \sigma$. Such a channel satisfies property (ii) of Lemma B.2 and is hence constrained by the third law.

(iii) A Gibbs-preserving channel Φ acting in \mathcal{H} satisfies $\Phi(\tau_\beta) = \tau_\beta$ for some $\tau_\beta := e^{-\beta H}/\text{tr}[e^{-\beta H}]$, where $H = H^*$, $\beta > 0$. Gibbs states τ_β , with $\beta > 0$, are full-rank. It follows that a Gibbs-preserving channel satisfies property (ii) of Lemma B.2 and is hence constrained by the third law.

(iv) Let ρ and σ be full-rank states on \mathcal{H}_S and \mathcal{H}_A , respectively, so that $\omega = \rho \otimes \sigma$ is full-rank in $\mathcal{H}_S \otimes \mathcal{H}_A$. Given that $\text{tr}_A[\omega] = \rho$, which is full-rank in \mathcal{H}_S , it follows that the partial trace satisfies property (ii) of Lemma B.2 and is hence constrained by the third law.

(v) If ξ is full-rank, then $\Lambda_\xi(\rho) = \rho \otimes \xi$ is full-rank for all full-rank states ρ on \mathcal{H}_S . As such, Λ_ξ is constrained by the third law. \blacksquare

Given that unitary channels preserve the spectrum, they are clearly rank nondecreasing channels. Since unitary channels are a special subclass of bistochastic channels, one may wonder whether or not all bistochastic channels are rank nondecreasing. The following lemma answers such a conjecture in the affirmative.

Lemma B.3. Bistochastic channels Φ acting in \mathcal{H} are rank nondecreasing.

Proof. Consider a state ρ , whose minimal support projection is P . Since ρ and $P/\text{tr}[P]$ are both full-rank states in $P\mathcal{H}$, then by Lemma A.1 there exists $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 P/\text{tr}[P] \leq \rho \leq \lambda_2 P/\text{tr}[P]$. Now consider an arbitrary channel Φ , and define the state $\sigma := \Phi(P/\text{tr}[P])$. By complete positivity, it follows that $\lambda_1 \sigma \leq \Phi(\rho) \leq \lambda_2 \sigma$. Now let us note that $A \geq B \geq \mathbb{O}$ implies that $\text{rank}(A) \geq \text{rank}(B)$. This follows from the fact that for any $A \geq \mathbb{O}$, the condition $|\psi\rangle \in \ker(A)$ is equivalent to $\langle \psi|A|\psi\rangle = 0$. We thus obtain $\text{rank}(\sigma) \leq \text{rank}(\Phi(\rho)) \leq \text{rank}(\sigma)$, which implies that

$$\text{rank}(\Phi(\rho)) = \text{rank}(\sigma). \quad (\text{B1})$$

Assume that Φ is bistochastic, so that it preserves the identity. In such a case, if ρ is full-rank so that $P = \mathbb{1}$ and $\sigma = \Phi(\mathbb{1}/\dim(\mathcal{H})) = \mathbb{1}/\dim(\mathcal{H})$ hold, we find by Eq. (B1) that $\text{rank}(\Phi(\rho)) = \text{rank}(\rho)$. Now let us suppose that ρ is not full-rank. By decomposing the identity as

$$\mathbb{1} = \text{tr}[P] \frac{P}{\text{tr}[P]} + \text{tr}[P^\perp] \frac{P^\perp}{\text{tr}[P^\perp]},$$

we obtain for any bistochastic Φ the following:

$$\mathbb{1} = \Phi(\mathbb{1}) = \text{tr}[P]\sigma + \text{tr}[P^\perp]\mu,$$

where we define $\mu := \Phi(P^\perp/\text{tr}[P^\perp])$, which implies that

$$\text{tr}[P^\perp]\mu = \mathbb{1} - \text{tr}[P]\sigma.$$

Since μ is nonnegative, it must hold that $1 \geq \text{tr}[P]\|\sigma\| = \text{rank}(\rho)\|\sigma\|$, where the equality follows from the fact that $\text{rank}(\rho) = \text{tr}[P]$. This gives $\text{rank}(\rho) \leq 1/\|\sigma\|$. But it holds that $1 = \text{tr}[\sigma] \leq \|\sigma\|\text{rank}(\sigma) = \|\sigma\|\text{rank}(\Phi(\rho))$, where the equality follows from Eq. (B1). This gives $\text{rank}(\Phi(\rho)) \geq 1/\|\sigma\|$. We finally arrive at

$$\text{rank}(\Phi(\rho)) \geq \frac{1}{\|\sigma\|} \geq \text{rank}(\rho),$$

which proves our claim. \blacksquare

Let us note that while any rank nondecreasing channel Φ acting in \mathcal{H} is constrained by the third law, a channel acting in \mathcal{H} that is constrained by the third law may decrease the rank for some input states. To see this, consider $\mathcal{H} \simeq \mathbb{C}^3$, with orthonormal basis $\{|i\rangle : i = 0, 1, 2\}$. Consider the projections $P_i = |i\rangle\langle i|$, and a full-rank state ρ_0 on \mathcal{H} . Now define the channel $\Phi(\rho) = \text{tr}[\rho P_0]\rho_0 + \text{tr}[\rho P_0^\perp]P_1$. This channel clearly maps all full-rank states ρ to a full-rank state, and so is constrained by the third law. However, a rank-2 state $P_0^\perp/2$ is mapped to a rank-1 state P_1 .

APPENDIX C: OPERATIONAL JUSTIFICATION FOR DEFINITION 1

Consider a system \mathcal{H} with a Hamiltonian that has the spectral decomposition $H = \sum_{i=0}^N \epsilon_i P_i$, with the (distinct) energy eigenvalues arranged in increasing order, i.e., $\epsilon_i < \epsilon_{i+1}$. The state of such a system at inverse temperature $\beta = 1/T$ is the Gibbs state $\tau_\beta := e^{-\beta H}/\text{tr}[e^{-\beta H}]$. In such a case, $\text{rank}(\tau_\beta) = \dim(\mathcal{H})$. When cooled to absolute zero temperature, i.e., $\beta = \infty$, the state reads $\tau_\infty = P_0/\text{tr}[P_0]$, and it holds that $\text{rank}(\tau_\infty) < \dim(\mathcal{H})$. Indeed, any state whose support is in $P_0\mathcal{H}$ can be considered as having zero temperature. Therefore, the third law can be seen as prohibiting any channel Φ such that $\Phi(\tau_\beta) = \tau_\infty$. Definition 1 is a generalization of such an

intuitive idea, and below, we shall show that the existence of a channel which is unconstrained by the third law as defined by Definition 1, i.e., a channel that may map a full-rank state on some system to a non-full-rank state of a possibly different system, is both necessary and sufficient for the preparation of some system in a pure state.

Proposition C.1. Let us consider a system \mathcal{S} described by $\mathcal{H}_{\mathcal{S}}$ with $2 \leq \dim(\mathcal{H}_{\mathcal{S}}) < \infty$, and suppose that the system is given with an unknown input state ρ . Assume that we can employ arbitrary channels constrained by the third law as per Definition 1. The following statements are equivalent:

- (i) We may implement some known channel that is unconstrained by the third law.
- (ii) We may prepare the system \mathcal{S} in an arbitrary desired state (in particular, an arbitrary pure state), from any given unknown prior state ρ .

Proof. (ii) \Rightarrow (i): Assume that we can implement only channels that are constrained by the third law. By Lemma B.1, the class of third-law-constrained channels is closed under composition, and so a full-rank input state ρ can be transformed only to another full-rank state. Therefore, the ability to prepare an arbitrary state, such as a pure state, from any input ρ , including one that is full-rank, requires a channel that is unconstrained by the third law.

(i) \Rightarrow (ii): Consider a system $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^N$, with the orthonormal basis $\{|0\rangle, \dots, |N-1\rangle\}$, initially prepared in some unknown state ρ . Consider a channel Φ acting in $\mathcal{H}_{\mathcal{S}}$, defined by $\Phi(\cdot) = \text{tr}[\cdot \mathbb{1}] \rho_0$, where $\rho_0 = \sum_{n=0}^{N-1} \lambda_n |n\rangle\langle n|$ is a known full-rank state. Such a channel maps full-rank states to full-rank states, and so is constrained by the third law. We may therefore use this channel to prepare $\mathcal{H}_{\mathcal{S}}$ in the state ρ_0 , independently of the input ρ .

Now let us assume that we may implement a known channel $\mathcal{E} : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{R})$ which is unconstrained by the third law, i.e., assume that there exists a full-rank state ϱ on \mathcal{K} which is mapped to state ξ on \mathcal{R} that is not full-rank. But preparation channels that prepare a full-rank state are constrained by the third law. As such, we may prepare \mathcal{K} in such a state ϱ , which is known, so that by applying the channel \mathcal{E} , we may prepare \mathcal{R} in the state ξ , which is also known. Let us write such a state as $\xi = \sum_{m=0}^{M-1} p_m |m\rangle\langle m|$, where $\{|0\rangle, \dots, |M-1\rangle\}$ is an orthonormal basis that spans $\mathcal{R} = \mathbb{C}^M$, such that $p_m = 0$ for at least one m .

By preparing D copies of ξ , we may write the state on the total system $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{R}^{\otimes D}$, i.e., $\rho_0 \otimes \xi^{\otimes D} = \rho \otimes \xi \otimes \dots \otimes \xi$, as

$$\rho_0 \otimes \xi^{\otimes D} = \sum_{n=0}^{N-1} \sum_{m_1=0}^{M-1} \dots \sum_{m_D=0}^{M-1} C_{n,m_1,\dots,m_D} \times |n, m_1, \dots, m_D\rangle\langle n, m_1, \dots, m_D|,$$

where we define $C_{n,m_1,\dots,m_D} := \lambda_n p_{m_1} \dots p_{m_D}$. Now choose a unitary channel acting in the total system, with a unitary operator U which permutes the basis $\{|n, m_1, \dots, m_D\rangle\}$. We obtain

$$U(\rho_0 \otimes \xi^{\otimes D})U^* = \sum_{n=0}^{N-1} \sum_{m_1=0}^{M-1} \dots \sum_{m_D=0}^{M-1} C_{\pi(n,m_1,\dots,m_D)} \times |n, m_1, \dots, m_D\rangle\langle n, m_1, \dots, m_D|.$$

By choosing D to be sufficiently large so that $\text{rank}(\xi)^D N \leq M^D$ is satisfied, then the permutation can be chosen so that $C_{\pi(n,m_1,\dots,m_D)} = 0$ for $n \neq 0$, and so the above state will read $U(\rho_0 \otimes \xi^{\otimes D})U^* = |0\rangle\langle 0| \otimes \omega$. The restriction of the final state $|0\rangle\langle 0| \otimes \omega$ on $\mathcal{H}_{\mathcal{S}}$ is thus a pure state $|0\rangle\langle 0|$. To obtain an arbitrary target state σ , we apply a channel $\Lambda(\cdot) = \langle 0| \cdot |0\rangle \sigma + \text{tr}[(\mathbb{1} - |0\rangle\langle 0|)] \frac{\mathbb{1}}{N}$, which maps full-rank states to full-rank states, and so is constrained by the third law. \blacksquare

APPENDIX D: MEASUREMENTS CONSTRAINED BY THE THIRD LAW

Consider again the composition channel $\Lambda_{\xi} : \mathcal{L}(\mathcal{H}_{\mathcal{S}}) \mapsto \mathcal{L}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}})$, $\rho \mapsto \rho \otimes \xi$. We now define the restriction map $\Gamma_{\xi} : \mathcal{L}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}) \rightarrow \mathcal{L}(\mathcal{H}_{\mathcal{S}})$ as dual to the composition channel, i.e., $\Gamma_{\xi} = \Lambda_{\xi}^*$. The restriction map therefore satisfies $\text{tr}[\Gamma_{\xi}(B)\rho] = \text{tr}[B(\rho \otimes \xi)]$ for all $B \in \mathcal{L}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}})$ and states ρ on $\mathcal{H}_{\mathcal{S}}$. Using such a map, and a channel \mathcal{E} acting in $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}$, we define the channel $\Gamma_{\xi}^{\mathcal{E}} : \mathcal{L}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}) \rightarrow \mathcal{L}(\mathcal{H}_{\mathcal{S}})$ as

$$\Gamma_{\xi}^{\mathcal{E}}(\cdot) := \Gamma_{\xi} \circ \mathcal{E}^*(\cdot). \quad (\text{D1})$$

We may thus write the dual of the operations in Eq. (3) implemented by the measurement scheme $\mathcal{M} := (\mathcal{H}_{\mathcal{A}}, \xi, \mathcal{E}, \mathcal{Z})$ as

$$\mathcal{I}_x^*(\cdot) = \Gamma_{\xi}^{\mathcal{E}}(\cdot \otimes \mathbb{Z}_x),$$

to hold for all $x \in \mathcal{X}$. As such, we may write the channel implemented by \mathcal{M} as $\mathcal{I}_x^*(\cdot) = \Gamma_{\xi}^{\mathcal{E}}(\cdot \otimes \mathbb{1}_{\mathcal{A}})$. Moreover, recall that an instrument is compatible with \mathbf{E} if $\text{tr}[\mathcal{I}_x(\rho)] = \text{tr}[\mathbf{E}_x \rho]$ for all outcomes x and for all states ρ , which may equivalently be written as $\mathbf{E}_x = \mathcal{I}_x^*(\mathbb{1}_{\mathcal{S}})$ for all x . We may therefore write the effects of the observable implemented by the measurement scheme \mathcal{M} as

$$\mathbf{E}_x = \Gamma_{\xi}^{\mathcal{E}}(\mathbb{1}_{\mathcal{S}} \otimes \mathbb{Z}_x).$$

Lemma D.1. Let $\mathcal{M} := (\mathcal{H}_{\mathcal{A}}, \xi, \mathcal{E}, \mathcal{Z})$ be a measurement scheme for an \mathbf{E} -compatible instrument \mathcal{I} acting in $\mathcal{H}_{\mathcal{S}}$. Assume that \mathcal{M} is constrained by the third law. The following hold:

- (i) For every $A \in \mathcal{L}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}})$, it holds that $\Gamma_{\xi}^{\mathcal{E}}(A^*A) = 0 \iff A = 0$.
- (ii) There exists at least one full-rank state ρ_0 on $\mathcal{H}_{\mathcal{S}}$ such that $\mathcal{I}_{\mathcal{X}}(\rho_0) = \rho_0$.
- (iii) For every $A \in \mathcal{L}(\mathcal{H}_{\mathcal{S}})$ and x , it holds that $\mathcal{I}_x^*(A^*A) = 0 \iff A = 0$.
- (iv) For every x , let P_x be the minimal projection on the support of \mathbf{E}_x . For every state ρ such that $P_x \rho P_x$ has full rank in $P_x \mathcal{H}_{\mathcal{S}}$, $\mathcal{I}_x(\rho)$ has full rank in $\mathcal{H}_{\mathcal{S}}$.

Proof.

(i) By Definition 5, ξ is full-rank and \mathcal{E} maps full-rank states to full-rank states. Since Γ_{ξ} is dual to the composition channel, then by Corollary B.1 Γ_{ξ} is constrained by the third law. By Lemma B.1, it follows that $\Gamma_{\xi}^{\mathcal{E}} = \Gamma_{\xi} \circ \mathcal{E}^*$ is constrained by the third law. The statement follows from Lemma B.2.

(ii) Since $\mathcal{I}_{\mathcal{X}}^*(\cdot) = \Gamma_{\xi}^{\mathcal{E}}(\cdot \otimes \mathbb{1}_{\mathcal{A}})$, then by the above it follows that $\mathcal{I}_{\mathcal{X}}$ satisfies property (iii), and hence (iv), of Lemma B.2.

(iii) Given that $0 < \|\mathbf{E}_x\| = \|\Gamma_\xi^\mathcal{E}(\mathbb{1}_S \otimes \mathbf{Z}_x)\| \leq \|\mathbf{Z}_x\|$, it holds that $\mathbf{Z}_x > \mathbb{0}$ for all x . Now note that $\mathcal{I}_x^*(A^*A) = \Gamma_\xi^\mathcal{E}(A^*A \otimes \mathbf{Z}_x)$. By item (i), it follows that $\mathcal{I}_x^*(A^*A) = \mathbb{0}$ if and only if $A^*A \otimes \mathbf{Z}_x = \mathbb{0}$, which holds if and only if $A = \mathbb{0}$.

(iv) We may always write $\mathcal{I}_x^*(\cdot) = \sqrt{\mathbf{E}_x} \Phi_x^*(\cdot) \sqrt{\mathbf{E}_x}$ for some channel Φ_x acting in \mathcal{H}_S . It follows that $\mathcal{I}_x^*(\cdot) = P_x \mathcal{I}_x^*(\cdot) P_x$. By (iii), for any $\mathbb{0} \neq A \in \mathcal{L}(\mathcal{H}_S)$, it holds that $\mathcal{L}(P_x \mathcal{H}_S) \ni \mathcal{I}_x^*(A^*A) = P_x \mathcal{I}_x^*(A^*A) P_x > \mathbb{0}$. But for any ρ for which $P_x \rho P_x$ has full-rank in $P_x \mathcal{H}_S$, it follows that $\text{tr}[\mathcal{I}_x^*(A^*A)\rho] > 0$. By writing $\text{tr}[A^*A \mathcal{I}_x(\rho)] = \text{tr}[\mathcal{I}_x^*(A^*A)\rho]$, it follows that $\text{tr}[A^*A \mathcal{I}_x(\rho)] = 0 \iff A = \mathbb{0}$, and so $\mathcal{I}_x(\rho)$ must be full-rank in \mathcal{H}_S . ■

Condition (iv) of the above lemma shows that the third law restricts the possible conditional state transformations by measurements. That is, it is impossible to prepare a system in a state of low rank, given an arbitrary input state, by measurement and selection of an outcome. Moreover, the class of input states that may be prepared in a state of low rank diminishes as the rank of the observable's effects decrease. Indeed, if an effect \mathbf{E}_x has rank 1, so that it may be written as $\mathbf{E}_x = \lambda |\psi\rangle\langle\psi|$ for some unit-vector $|\psi\rangle$ and $\lambda \in (0, 1]$, then for any state ρ such that $\langle\psi|\rho|\psi\rangle > 0$, $\mathcal{I}_x(\rho)$ will be full-rank. In particular, we obtain the following result.

Proposition D.1. Let \mathcal{I}^L be a Lüders instrument compatible with a nontrivial observable \mathbf{E} . \mathcal{I}^L admits a measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$, which is constrained by the third law, if and only if \mathbf{E} is completely unsharp.

Proof. First, let us show the only if statement. By item (iv) of Lemma D.1, $\mathcal{I}_x^L(\cdot) = \sqrt{\mathbf{E}_x} \cdot \sqrt{\mathbf{E}_x}$ must map full-rank states to full-rank states. Consider the complete mixture $\sigma = \mathbb{1}_S / \dim(\mathcal{H}_S)$, which is full-rank. $\mathcal{I}_x^L(\sigma) = \mathbf{E}_x / \dim(\mathcal{H}_S)$ is full-rank if and only if \mathbf{E}_x is full-rank. Since \mathbf{E} is nontrivial, and there must be at least two outcomes x for which $\mathbb{0} < \mathbf{E}_x < \mathbb{1}_S$, then the spectrum of all effects \mathbf{E}_x must be contained in $(0, 1)$. \mathbf{E} is thus completely unsharp.

Now we shall show the if statement. Let us consider a completely unsharp observable $\mathbf{E} = \{\mathbf{E}_x\}_{x=1}^N$ acting in \mathcal{H}_S . Let the apparatus Hilbert space be $\mathcal{H}_A = \mathbb{C}^N$, with an orthonormal basis $\{|y\rangle\}_{y=0}^{N-1}$, and choose the apparatus preparation as the complete mixture $\xi = \mathbb{1}_A / N$. Choose an interaction channel $\mathcal{E}(\cdot) = \sum_x K_x \cdot K_x^*$, with Kraus operators

$$K_x := \sum_a \sqrt{\mathbf{E}_{x \oplus a}} \otimes |x \oplus a\rangle\langle a|, \quad (\text{D2})$$

where \oplus represents summation modulo N . Then for an arbitrary state ρ on \mathcal{H}_S , we obtain

$$\begin{aligned} \mathcal{E}(\rho \otimes \xi) &= \sum_x \sum_a \sum_b \sqrt{\mathbf{E}_{x \oplus a}} \rho \sqrt{\mathbf{E}_{x \oplus b}} \otimes \frac{1}{N} \\ &\quad \times \sum_c |x \oplus a\rangle\langle x \oplus b| \delta_{ac} \delta_{bc} \\ &= \sum_x \sqrt{\mathbf{E}_x} \rho \sqrt{\mathbf{E}_x} \otimes |x\rangle\langle x|. \end{aligned}$$

To show that \mathcal{E} is constrained by the third law, let us note that for $\rho = \mathbb{1}_S / \dim(\mathcal{H}_S)$, we have

$$\mathcal{E}(\rho \otimes \xi) = \frac{1}{\dim(\mathcal{H}_S)} \sum_x \mathbf{E}_x \otimes |x\rangle\langle x|.$$

To show that this state is full rank, we observe that for an arbitrary vector $|\varphi\rangle = \sum_x |\varphi_x\rangle \otimes |x\rangle$, it holds that

$$\langle\varphi|\mathcal{E}(\rho \otimes \xi)|\varphi\rangle = \frac{1}{\dim(\mathcal{H}_S)} \sum_x \langle\varphi_x|\mathbf{E}_x|\varphi_x\rangle,$$

which is nonvanishing if every \mathbf{E}_x is full-rank. By item (ii) of Lemma B.2, \mathcal{E} is constrained by the third law. Finally, choosing the pointer observable as $\mathbf{Z}_x = |x\rangle\langle x|$, then by Eq. (3) we obtain

$$\text{tr}_A[(\mathbb{1}_S \otimes \mathbf{Z}_x)\mathcal{E}(\rho \otimes \xi)] = \sqrt{\mathbf{E}_x} \rho \sqrt{\mathbf{E}_x} =: \mathcal{I}_x^L(\rho),$$

and so the above measurement scheme implements the Lüders instrument. ■

We obtain the following as an immediate consequence of the above.

Corollary D.1. Let \mathbf{E} be a completely unsharp observable. Define \mathcal{S} as the set of all \mathbf{E} -compatible instruments \mathcal{I} such that, for every full-rank state ρ and for every outcome x , $\mathcal{I}_x(\rho)$ is full-rank. It holds that every $\mathcal{I} \in \mathcal{S}$ admits a measurement scheme $\mathcal{M} = (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ that is constrained by the third law.

Proof. The operations of every \mathbf{E} -instrument may be written as $\mathcal{I}_x(\cdot) = \Phi_x \circ \mathcal{I}_x^L(\cdot)$, where \mathcal{I}^L is the Lüders instrument for \mathbf{E} and Φ_x is an arbitrary channel. It is easy to show that $\mathcal{I} \in \mathcal{S}$ if and only if Φ_x is constrained by the third law. That is, $\Phi_x(\omega)$ is full-rank for every full-rank ω . To see this, let $\sigma = \mathbb{1}_S / \dim(\mathcal{H}_S)$ be the complete mixture, and define $\omega := \mathcal{I}_x^L(\sigma) / \text{tr}[\mathcal{I}_x^L(\sigma)] = \mathbf{E}_x / \text{tr}[\mathbf{E}_x]$ which, given complete unsharpness of \mathbf{E} , is guaranteed to be full-rank. Therefore, $\mathcal{I}_x(\sigma)$ is full-rank if and only if $\Phi_x(\omega)$ is full-rank. The claim follows from item (ii) of Lemma B.2.

Now let us define a channel \mathcal{E}_2 by

$$\mathcal{E}_2(A \otimes B) = \sum_x \Phi_x(A) \otimes |x\rangle\langle x| B |x\rangle\langle x|,$$

where Φ_x is an arbitrary channel which is constrained by the third law. In such a case, \mathcal{E}_2 is also constrained by the third law; it is easily verified that $\mathcal{E}_2(\mathbb{1}_S \otimes \mathbb{1}_A)$ has full rank. Therefore, a concatenated channel $\mathcal{E} := \mathcal{E}_2 \circ \mathcal{E}_1$, where \mathcal{E}_1 is a channel defined by the Kraus operators in Eq. (D2), is also constrained by the the third law. But the measurement scheme $\mathcal{M} = (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$, with ξ the complete mixture and $\mathbf{Z}_x = |x\rangle\langle x|$, implements $\mathcal{I}_x(\cdot) = \Phi_x \circ \mathcal{I}_x^L(\cdot)$. ■

APPENDIX E: FIXED-POINT STRUCTURE OF MEASUREMENTS CONSTRAINED BY THE THIRD LAW

We define the fixed-point sets of the \mathbf{E} -channel \mathcal{I}_X and its dual \mathcal{I}_X^* as

$$\begin{aligned} \mathcal{F}(\mathcal{I}_X) &:= \{A \in \mathcal{L}(\mathcal{H}_S) : \mathcal{I}_X(A) = A\}, \\ \mathcal{F}(\mathcal{I}_X^*) &:= \{A \in \mathcal{L}(\mathcal{H}_S) : \mathcal{I}_X^*(A) = A\}. \end{aligned}$$

Now let us define the channels

$$\mathcal{I}_{\text{av}}(\cdot) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (\mathcal{I}_X)^n(\cdot),$$

$$\mathcal{I}_{\text{av}}^*(\cdot) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (\mathcal{I}_X^*)^n(\cdot).$$

$\mathcal{I}_{\text{av}}^*$ is a CP projection on $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) = \mathcal{F}(\mathcal{I}_{\text{av}}^*)$, i.e., it holds that $\mathcal{I}_{\text{av}}^* = \mathcal{I}_{\text{av}}^* \circ \mathcal{I}_{\mathcal{X}}^* = \mathcal{I}_{\mathcal{X}}^* \circ \mathcal{I}_{\text{av}}^* = \mathcal{I}_{\text{av}}^* \circ \mathcal{I}_{\text{av}}^*$. Similarly, \mathcal{I}_{av} is a CP projection on $\mathcal{F}(\mathcal{I}_{\mathcal{X}}) = \mathcal{F}(\mathcal{I}_{\text{av}})$. Now let us assume that the measurement scheme for \mathcal{I} is constrained by the third law. It follows from item (ii) of Lemma D.1 that $\mathcal{F}(\mathcal{I}_{\mathcal{X}})$ contains a full-rank state ρ_0 . This in turn implies that $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ is a von Neumann algebra, i.e., the fixed points of $\mathcal{I}_{\mathcal{X}}^*$ satisfy multiplicative closure [70,71]. But since \mathcal{H}_S is finite-dimensional, then $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ is a finite von Neumann algebra \mathcal{A} , which may have an Abelian nontrivial center $\mathcal{Z} := \mathcal{A} \cap \mathcal{A}'$ generated by the set of ortho-complete projections $\{P_\alpha\}$. That is, every self-adjoint $B \in \mathcal{Z}$ can be written as $B = \sum_\alpha \lambda_\alpha P_\alpha$. We may therefore decompose \mathcal{A} into a finite direct sum $\mathcal{A} = \bigoplus_\alpha \mathcal{A}_\alpha$, where each $\mathcal{A}_\alpha = P_\alpha \mathcal{A}$ is a type I factor (a finite dimensional von Neumann algebra with a trivial center) on $P_\alpha \mathcal{H}_S = \mathcal{K}_\alpha \otimes \mathcal{R}_\alpha$, written as $\mathcal{A}_\alpha = \mathcal{L}(\mathcal{K}_\alpha) \otimes \mathbb{1}_{\mathcal{R}_\alpha}$. It follows that we may write

$$\begin{aligned} \mathcal{F}(\mathcal{I}_{\mathcal{X}}) &= \bigoplus_\alpha \mathcal{L}(\mathcal{K}_\alpha) \otimes \omega_\alpha, \\ \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) &= \bigoplus_\alpha \mathcal{L}(\mathcal{K}_\alpha) \otimes \mathbb{1}_{\mathcal{R}_\alpha}, \end{aligned} \quad (\text{E1})$$

and

$$\begin{aligned} \mathcal{I}_{\text{av}}(\cdot) &= \sum_\alpha \text{tr}_{\mathcal{R}_\alpha}[P_\alpha \cdot P_\alpha] \otimes \omega_\alpha, \\ \mathcal{I}_{\text{av}}^*(\cdot) &= \sum_\alpha \Gamma_{\omega_\alpha}(P_\alpha \cdot P_\alpha) \otimes \mathbb{1}_{\mathcal{R}_\alpha}, \end{aligned} \quad (\text{E2})$$

where ω_α are states on \mathcal{R}_α ; $\Gamma_{\omega_\alpha} : \mathcal{L}(\mathcal{K}_\alpha \otimes \mathcal{R}_\alpha) \rightarrow \mathcal{L}(\mathcal{K}_\alpha)$ are restriction maps; and $\text{tr}_{\mathcal{R}_\alpha} : \mathcal{L}(\mathcal{K}_\alpha \otimes \mathcal{R}_\alpha) \rightarrow \mathcal{L}(\mathcal{K}_\alpha)$ are partial traces [72].

Note that the third-law constraint implies that ω_α are full-rank states on \mathcal{R}_α . This can be immediately inferred by noting that, given the complete mixture $\sigma = \mathbb{1}_S / \dim(\mathcal{H}_S)$, it holds that $\mathcal{I}_{\text{av}}(\sigma) \propto \bigoplus_\alpha \mathbb{1}_{\mathcal{K}_\alpha} \otimes \omega_\alpha$. By property (i) of Lemma B.2, this state must be full-rank, which holds if and only if ω_α are full-rank for all α .

Finally, let us note that since $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ is a von Neumann algebra, then it holds that

$$\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) \subseteq \mathbf{E}' := \{A \in \mathcal{L}(\mathcal{H}_S) : [\mathbf{E}, A] = 0\}, \quad (\text{E3})$$

that is, the fixed points of $\mathcal{I}_{\mathcal{X}}^*$ are contained in the commutant of \mathbf{E} [35]. We now provide a useful result indicating the form that the effects of \mathbf{E} must take in light of the fixed-point structure of the measurement channel.

Lemma E.1. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ be a measurement scheme for a nontrivial observable \mathbf{E} , with instrument \mathcal{I} , acting in \mathcal{H}_S . Assume that \mathcal{M} is constrained by the third law. Then the effects of \mathbf{E} are of the form

$$\mathbf{E}_x = \bigoplus_\alpha \mathbb{1}_{\mathcal{K}_\alpha} \otimes E_{x,\alpha},$$

where $0 < E_{x,\alpha} < \mathbb{1}_{\mathcal{R}_\alpha}$ for all x and α .

Proof. By the channel $\Gamma_\xi^\mathcal{E}$ defined in Eq. (D1), we may write $\mathcal{I}_x^*(\cdot) = \Gamma_\xi^\mathcal{E}(\cdot \otimes \mathbf{Z}_x)$, and so we may write $\mathbf{E}_x = \mathcal{I}_x^*(\mathbb{1}_S) = \Gamma_\xi^\mathcal{E}(\mathbb{1}_S \otimes \mathbf{Z}_x)$ and $\mathcal{I}_x^*(\cdot) = \Gamma_\xi^\mathcal{E}(\cdot \otimes \mathbb{1}_A)$. Since $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ is a von Neumann algebra, for any $A \in \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ it holds that $A^*A, AA^* \in \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$. By the multiplicability theorem [73],

this implies that $A\Gamma_\xi^\mathcal{E}(B) = \Gamma_\xi^\mathcal{E}((A \otimes \mathbb{1}_A)B)$ and $\Gamma_\xi^\mathcal{E}(B)A = \Gamma_\xi^\mathcal{E}(B(A \otimes \mathbb{1}_A))$ for all $B \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_A)$. By choosing $B = \mathbb{1}_S \otimes \mathbf{Z}_x$, we may therefore write

$$\mathcal{I}_x^*(A) = \Gamma_\xi^\mathcal{E}(A \otimes \mathbf{Z}_x) = A\mathbf{E}_x = \mathbf{E}_x A$$

for all $A \in \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$. Now assume that $A\mathbf{E}_x = 0$. By the above equation this implies that $\mathcal{I}_x^*(A^*A) = A^*A\mathbf{E}_x = 0$. By item (iii) of Lemma D.1, it follows that for any $A \in \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$, it holds that $A\mathbf{E}_x = 0 \iff A = 0$.

Now note that the condition $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) \subseteq \mathbf{E}'$ implies that $\mathbf{E} \subseteq \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)'$. By Eq. (E1) it holds that $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)' = \bigoplus_\alpha \mathbb{1}_{\mathcal{K}_\alpha} \otimes \mathcal{L}(\mathcal{R}_\alpha)$. That the effects of \mathbf{E} are decomposed as in the statement of the lemma directly follows. Now assume that $E_{x,\alpha} = 0$ for some α . It will hold that an operator $A = A_\alpha \otimes \mathbb{1}_{\mathcal{R}_\alpha} \in \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ exists, with $A_\alpha \neq 0$, such that $A\mathbf{E}_x = 0$. But this contradicts what we showed above. Therefore, all $E_{x,\alpha}$ must be strictly positive. Finally, since \mathbf{E} is nontrivial, then there exists at least two distinct outcomes, and so by normalization it holds that $E_{x,\alpha} < \mathbb{1}_{\mathcal{R}_\alpha}$. ■

APPENDIX F: NONDISTURBANCE

An observable $\mathbf{F} := \{\mathbf{F}_y : y \in \mathcal{Y}\}$ is nondisturbed by an \mathbf{E} -compatible instrument \mathcal{I} if $\text{tr}[\mathbf{F}_y \mathcal{I}_x(\rho)] = \text{tr}[\mathbf{F}_y \rho]$ holds for all states ρ and outcomes y . This can equivalently be stated as $\mathcal{I}_x^*(\mathbf{F}_y) = \mathbf{F}_y$ for all y , which we denote as $\mathbf{F} \subseteq \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$. If the measurement scheme for \mathcal{I} is constrained by the third law, then as discussed surrounding Eq. (E3) it holds that $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) \subseteq \mathbf{E}'$, and so a necessary condition for nondisturbance of \mathbf{F} is for \mathbf{F} to commute with \mathbf{E} . As we show below, however, commutation is not sufficient; properties of the measured observable impose further constraints.

Proposition F.1. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ be a measurement scheme for an \mathbf{E} -compatible instrument \mathcal{I} acting in \mathcal{H}_S . The following hold:

(i) If $\|\mathbf{E}_x\| = 1$ for any x , then there exists a projection $P \in \mathbf{E}'$ such that $P \notin \mathcal{F}(\mathcal{I}_{\mathcal{X}}^*)$ for any instrument \mathcal{I} that can be implemented by a scheme \mathcal{M} that is constrained by the third law.

(ii) If \mathbf{E} is completely unsharp, then a scheme \mathcal{M} that is constrained by the third law can be chosen so that $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) = \mathbf{E}'$.

Proof.

(i) By Lemma E.1, we may write

$$\mathbf{E}' = \bigoplus_\alpha \mathcal{L}(\mathcal{K}_\alpha) \otimes E'_\alpha,$$

where $E'_\alpha := \{A \in \mathcal{L}(\mathcal{R}_\alpha) : [E_{x,\alpha}, A] = 0 \forall x \in \mathcal{X}\}$. If $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) = \mathbf{E}'$, then by Eq. (E1) it must hold that $E'_\alpha = \mathbb{C}\mathbb{1}_{\mathcal{R}_\alpha}$ for all α .

Recall that for each α , $\{E_{x,\alpha} : x \in \mathcal{X}\}$ is a POVM acting in \mathcal{R}_α , where $0 < E_{x,\alpha} < \mathbb{1}_{\mathcal{R}_\alpha}$ and $\sum_x E_{x,\alpha} = \mathbb{1}_{\mathcal{R}_\alpha}$. Assume that for some α , there exists x such that $E_{x,\alpha}$ has eigenvalue 1. Let P be the projection on the eigenvalue-1 eigenspace of $E_{x,\alpha}$. Since \mathbf{E} is nontrivial, then $P < \mathbb{1}_{\mathcal{R}_\alpha}$. By normalization, it follows that $PE_{x',\alpha} = E_{x',\alpha}P = \delta_{x,x'}P$ for all x' , and so there exists $P \not\propto \mathbb{1}_{\mathcal{R}_\alpha} \in E'_\alpha$. Therefore, $\mathcal{F}(\mathcal{I}_{\mathcal{X}}^*) = \mathbf{E}'$ holds only if $E_{x,\alpha}$ does not have eigenvalue 1, and so $\|\mathbf{E}_x\| < 1$ for all x .

(ii) By Proposition D.1, a completely unsharp observable \mathbf{E} admits a Lüders instrument \mathcal{I}^L , given the third law

constraint. In finite dimensions, for the Lüders instrument compatible with E , it holds that $\mathcal{F}(\mathcal{I}_x^*) = E'$ [44]. ■

In other words, an observable E admits an instrument \mathcal{I} so that $\mathcal{F}(\mathcal{I}_x^*) = E'$, with such instrument realisable by a measurement scheme \mathcal{M} that is constrained by the third law, if E is completely unsharp, and only if $\|E_x\| < 1$ for all x . Note that by item (i), if $\|E_x\| = 1$ for some x , then there exists a POVM $\{P, P^\perp := \mathbb{1}_S - P\}$ that commutes with E , but is disturbed by any realisable E -instrument \mathcal{I} , since $\mathcal{I}_x^*(P) \neq P$. In particular, the above proposition implies that for any possible measurement of a norm-1 observable E , such as a sharp observable, there exists some $F \subset E'$ that is disturbed.

Of course, while a measurement of a norm-1 observable E is guaranteed to disturb some observable that commutes, this does not imply that there are no nondisturbed observables. Below, we provide necessary conditions on E so that its measurement allows for a nontrivial class of nondisturbed observables.

Proposition F.2. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$ be a measurement scheme for an E -compatible instrument \mathcal{I} acting in \mathcal{H}_S , and assume that \mathcal{M} is constrained by the third law. The following hold:

- (i) If E is a small-rank observable, then $\mathcal{F}(\mathcal{I}_x^*) = \mathbb{C}\mathbb{1}_S$.
- (ii) If E is a nondegenerate observable, then $\mathcal{F}(\mathcal{I}_x^*)$ is Abelian.

Proof.

(i) Recall from Definition 2 that a small-rank observable has at least one effect that is rank-1. By Lemma E.1, the rank of every effect of E is bounded as $\text{rank}(E_x) \geq \sum_\alpha \dim(\mathcal{K}_\alpha)$. Therefore, if any effect of E is rank-1, then it must hold that the number of indices α is 1, and that $\dim(\mathcal{K}_\alpha) = 1$, so that by Eq. (E1) we have $\mathcal{F}(\mathcal{I}_x^*) = \mathbb{C}\mathbb{1}_S$.

(ii) Recall from Definition 3 that E is nondegenerate if one of its effects has no multiplicities in its strictly positive eigenvalues. By Lemma E.1, nondegeneracy of such an effect implies that $\dim(\mathcal{K}_\alpha) = 1$ for every α . It follows from Eq. (E1) that $\mathcal{F}(\mathcal{I}_x^*) = \bigoplus_\alpha \mathbb{C}\mathbb{1}_{\mathcal{R}_\alpha}$, i.e., for any $A, B \in \mathcal{F}(\mathcal{I}_x^*)$, it holds that $[A, B] = \mathbb{O}$. ■

In other words, for the class of nondisturbed observables to be nontrivial, then the measured observable must be large-rank. Additionally, for the nondisturbed observables to be noncommutative, then the measured observable must be degenerate. We now provide a concrete example for a measurement scheme constrained by the third law, which measures a sharp observable that is large-rank and hence degenerate, that does not disturb a nontrivial class of possibly noncommutative observables.

Example F.1. Consider $\mathcal{H}_S := \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_A := \mathcal{H}_3$, with $\mathcal{H}_i = \mathbb{C}^2$. Let \mathcal{E} be a unitary channel which acts trivially in \mathcal{H}_1 and implements a swap in $\mathcal{H}_2 \otimes \mathcal{H}_3$, i.e., $\mathcal{E}^*(A \otimes B \otimes C) = A \otimes C \otimes B$. As shown in Corollary B.1 this channel is constrained by the third law. Now define a pointer observable $Z_x := |x\rangle\langle x|$ acting in \mathcal{H}_A . A measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$, with ξ a full-rank state on \mathcal{H}_A , is constrained by the third law, and measures the sharp, large-rank and degenerate observable with effects $E_x = \mathbb{1}_1 \otimes |x\rangle\langle x|$ in \mathcal{H}_S . The fixed-point set of the instrument \mathcal{I} implemented by \mathcal{M} is easily verified to be $\mathcal{F}(\mathcal{I}_x^*) = \mathcal{L}(\mathcal{H}_1) \otimes \mathbb{1}_2 \subset E'$, which is a nontrivial and noncommutative proper subset of E' . That

is, any possibly noncommutative observable F with effects $F_y = F_y \otimes \mathbb{1}_2$ will be nondisturbed.

APPENDIX G: FIRST-KINDNESS AND REPEATABILITY

An E -compatible instrument \mathcal{I} is a measurement of the first kind if $E \subset \mathcal{F}(\mathcal{I}_x^*)$. A subclass of first-kind measurements are repeatable, satisfying the additional condition $\mathcal{I}_y^*(E_x) = \delta_{x,y}E_x$. Only norm-1 observables admit a repeatable instrument. Repeatability implies first-kindness, since $\mathcal{I}_x^*(E_x) = \sum_y \mathcal{I}_y^*(E_x) = E_x$. We now show that the third law permits first-kindness only for completely unsharp observables and so categorically prohibits repeatability.

Proposition G.1. A nontrivial observable E admits a measurement of the first kind, given a measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$ that is constrained by the third law, if and only if E is commutative and completely unsharp.

Proof. Let us first show the only if statement. An E -compatible instrument \mathcal{I} is a measurement of the first kind if $E \subset \mathcal{F}(\mathcal{I}_x^*)$. Now, recall that $\mathcal{F}(\mathcal{I}_x^*) = \mathcal{F}(\mathcal{I}_{\text{av}}^*)$, so that the first-kind condition also reads as $\mathcal{I}_{\text{av}}^*(E_x) = E_x$ for all outcomes x . By Lemma E.1 and Eq. (E2), it follows that $E_x = \bigoplus_\alpha \lambda_\alpha(x) \mathbb{1}_{\mathcal{K}_\alpha} \otimes \mathbb{1}_{\mathcal{R}_\alpha}$ with $\lambda_\alpha(x) := \text{tr}[E_{x,\alpha} \omega_\alpha]$. Since $0 < E_{x,\alpha} < \mathbb{1}_{\mathcal{R}_\alpha}$ and ω_α is full-rank for all α and x , then $\lambda_\alpha(x) \in (0, 1)$. The claim immediately follows.

To show the if statement, recall from Proposition D.1 that a completely unsharp observable admits a Lüders instrument under the third law constraint, and that if such an observable is also commutative, then the Lüders instrument is a first-kind measurement. ■

Corollary G.1. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$ be a measurement scheme for an E -compatible instrument \mathcal{I} . If \mathcal{M} is constrained by the third law, then \mathcal{I} cannot be repeatable. Moreover, for every pair of outcomes x, y , there exists a state ρ such that $\text{tr}[E_y \mathcal{I}_x(\rho)] > 0$.

Proof. If \mathcal{I} is repeatable, then it is also first-kind. By Proposition G.1, given the third law constraint only a completely unsharp observable admits a first-kind measurement. Since repeatability is admitted only for norm-1 observables, and completely unsharp observables lack the norm-1 property, then \mathcal{I} cannot be repeatable. Now note that we may write $\mathcal{I}_x^*(E_y) = \Gamma_\xi^\mathcal{E}(E_y \otimes Z_x)$. Since $E_y \otimes Z_x > \mathbb{O}$ for all x, y , then by item (i) of Lemma D.1 it holds that $\mathcal{I}_x^*(E_y) > \mathbb{O}$, and so $\text{tr}[E_y \mathcal{I}_x(\rho)] = \text{tr}[\mathcal{I}_x^*(E_y) \rho] > 0$ for some ρ . ■

Below we provide a model for a measurement scheme that is constrained by the third law, and which implements a first-kind measurement of a completely unsharp observable. Note that the model does not implement a Lüders instrument.

Example G.1. Consider $\mathcal{H}_S = \mathcal{H}_A = \mathbb{C}^N$ with an orthonormal basis $\{|n\rangle : n = 0, \dots, N-1\}$ for each system. Consider the unitary channel $\mathcal{E}(\cdot) = U \cdot U^*$ acting in $\mathcal{H}_S \otimes \mathcal{H}_A$, with the unitary operator U defined as

$$U = \sum_{m,n} |n\rangle\langle n| \otimes |m \oplus n\rangle\langle m|,$$

where \oplus denotes addition modulo N . As shown in Corollary B.1 this channel is constrained by the third law. Consider a full-rank state on \mathcal{H}_A given as $\xi = \sum_n q(n) |n\rangle\langle n|$ with $q(n) > 0$ for all n . Let $Z_n = |n\rangle\langle n|$ be a pointer observable acting in \mathcal{H}_A . The measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathcal{Z})$

is therefore constrained by the third law. Moreover, the operations of the instrument \mathcal{I} implemented by \mathcal{M} satisfy

$$\mathcal{I}_x^*(A) = \Gamma_\xi^{\mathcal{E}}(A \otimes \mathbf{Z}_x) = \sum_n q(x \ominus n) \langle n|A|n \rangle |n\rangle \langle n|,$$

where \ominus denotes subtraction modulo N . The measured observable is therefore commutative and completely unsharp, with effects $\mathbf{E}_x = \sum_n q(x \ominus n) |n\rangle \langle n|$, whose eigenvalues are in $(0,1)$. Additionally, the fixed-point set of the \mathbf{E} -channel is $\mathcal{F}(\mathcal{I}_x^*) = \oplus_n \mathbb{C}|n\rangle \langle n| \subset \mathbf{E}'$, which is nontrivial. In particular, we have $\mathbf{E} \subset \mathcal{F}(\mathcal{I}_x^*)$, and so this model describes a first-kind measurement for \mathbf{E} .

APPENDIX H: IDEALITY

Proposition H.1. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ be a measurement scheme for a nontrivial norm-1 observable \mathbf{E} , with the instrument \mathcal{I} , acting in \mathcal{H}_S . If \mathcal{M} is constrained by the third law, then \mathcal{I} cannot be ideal.

Proof. Assume that \mathcal{I} is ideal, so that for every state ρ and for every outcome x such that $\text{tr}[\mathbf{E}_x \rho] = 1$, it holds that $\mathcal{I}_x(\rho) = \rho$. Since $\text{tr}[\mathbf{E}_x \rho] = 1$ is equivalent to $\text{tr}[\mathbf{E}_y \rho] = \delta_{x,y}$, and hence $\mathcal{I}_y(\rho) = 0$ for all $y \neq x$, this implies that $\mathcal{I}_x(\rho) = \sum_y \mathcal{I}_y(\rho) = \mathcal{I}_x(\rho) = \rho$, and hence $\rho \in \mathcal{F}(\mathcal{I}_x) = \mathcal{F}(\mathcal{I}_{\text{av}})$. But by Eq. (E2), we have

$$\rho = \mathcal{I}_{\text{av}}(\rho) = \bigoplus_\alpha \sigma_\alpha \otimes \omega_\alpha, \quad (\text{H1})$$

where ω_α are fixed full-rank states on \mathcal{R}_α and $\sigma_\alpha := \text{tr}_{\mathcal{R}_\alpha}[P_\alpha \rho P_\alpha]$ are subunit-trace positive operators on \mathcal{K}_α .

Since \mathbf{E} is norm-1, then by Lemma E.1 it follows that for each outcome x , there exists at least one α such that $\|\mathbf{E}_{x,\alpha}\| = 1$. Any state on $\mathcal{K}_\alpha \otimes \mathcal{R}_\alpha$ written as $\rho = \sigma_\alpha \otimes \mu_\alpha$, such that $\text{tr}[E_{x,\alpha} \mu_\alpha] = 1$, will give $\text{tr}[\mathbf{E}_x \rho] = \text{tr}[(\mathbb{1}_{\mathcal{K}_\alpha} \otimes E_{x,\alpha})(\sigma_\alpha \otimes \mu_\alpha)] = \text{tr}[E_{x,\alpha} \mu_\alpha] = 1$. By Eq. (H1), if \mathcal{I} is ideal we must have

$$\sigma_\alpha \otimes \mu_\alpha = \sigma_\alpha \otimes \omega_\alpha,$$

that is, $\text{tr}[E_{x,\alpha} \mu_\alpha] = 1 \iff \mu_\alpha = \omega_\alpha$. But given that ω_α are full-rank states on \mathcal{R}_α , $\text{tr}[E_{x,\alpha} \omega_\alpha] = 1$ if and only if $\dim(\mathcal{R}_\alpha) = 1$, so that $E_{x,\alpha} = \mathbb{1}_{\mathcal{R}_\alpha}$. But by Lemma E.1, if \mathbf{E} is nontrivial, $E_{x,\alpha} < \mathbb{1}_{\mathcal{R}_\alpha}$ must hold. We therefore have a contradiction, and so \mathcal{I} cannot be ideal. ■

In particular, let us highlight the fact that if \mathbf{E} is a nontrivial norm-1 observable, then for every outcome x , and for every state ρ such that $\text{tr}[\mathbf{E}_x \rho] = 1$, then a third-law-constrained measurement will give $\mathcal{I}_x(\rho) \neq \rho$.

APPENDIX I: EXTREMALITY

Let $\{K_i^{(x)} : i = 1, \dots, M_x\}$ be a minimal Kraus representation for the operation \mathcal{I}_x of an \mathbf{E} -compatible instrument \mathcal{I} , with M_x the Kraus rank of \mathcal{I}_x . The following is a series of necessary conditions for extremality of such an instrument:

Lemma I.1. The instrument \mathcal{I} is extremal only if the following conditions are met:

- (i) The operations \mathcal{I}_x are all extremal.
- (ii) $M_x \leq \text{rank}(\mathbf{E}_x)$ for all x .
- (iii) $\{K_i^{(x)*} K_j^{(x)} : x \in \mathcal{X}; i, j = 1, \dots, M_x\}$ are linearly independent (this condition is necessary and sufficient).

Proof.

(i) This trivially follows from the definition of extremal instruments.

(ii) By Remark 6 of Ref. [74], a subunitary CP map $\Phi^* : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$, with minimal Kraus representation $\{K_i : i = 1, \dots, M\}$, is extremal only if $\{K_i^* K_j \in \mathcal{L}(\mathcal{H})\}$ are linearly independent. Since the cardinality of this set is M^2 , while $\dim(\mathcal{L}(\mathcal{H})) = \dim(\mathcal{H})^2$, it follows that $M \leq \dim(\mathcal{H})$ must hold. Now, since we may always write $\mathcal{I}_x^*(\cdot) = \sqrt{\mathbf{E}_x} \Phi^*(\cdot) \sqrt{\mathbf{E}_x}$, it holds that $\mathcal{I}_x^* : \mathcal{L}(\mathcal{H}_S) \rightarrow \mathcal{L}(P_x \mathcal{H}_S)$, with P_x the minimal projection on the support of \mathbf{E}_x so that $\dim(P_x \mathcal{H}_S) = \text{rank}(\mathbf{E}_x)$. By above, extremality of \mathcal{I}_x implies that $M_x \leq \text{rank}(\mathbf{E}_x)$.

(iii) Theorem 5 of Ref. [36]. ■

Now we shall provide some useful results that will allow us to investigate how the third law constrains extremality in the sequel.

Lemma I.2. Let $\Gamma_\xi : \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_S)$ be a restriction map. Assume that $\Gamma_\xi(A)$ is independent of the state ξ . It holds that $A = B \otimes \mathbb{1}_A$.

Proof. Let $\{|e_n\rangle\}$ and $\{|b_j\rangle\}$ be orthonormal bases that span \mathcal{H}_S and \mathcal{H}_A , respectively. If $\Gamma_\xi(A)$ is independent of ξ , it follows that for any unit vector $|\psi\rangle \in \mathcal{H}_A$,

$$\langle e_m | \Gamma_{|\psi\rangle\langle\psi|}(A) | e_n \rangle = \langle e_m \otimes \psi | A | e_n \otimes \psi \rangle = C_{m,n}.$$

Now, for any choice of j and k , define $|\phi_\pm\rangle := \frac{1}{\sqrt{2}}(|b_j\rangle \pm |b_k\rangle)$ and $|\varphi_\pm\rangle := \frac{1}{\sqrt{2}}(|b_j\rangle \pm i|b_k\rangle)$. We obtain

$$\begin{aligned} \langle e_m | \Gamma_{|\phi_\pm\rangle\langle\phi_\pm|}(A) | e_n \rangle &= C_{m,n} \pm \frac{1 - \delta_{j,k}}{2} \\ &\quad \times (\langle e_m \otimes b_j | A | e_n \otimes b_k \rangle + \langle e_m \otimes b_k | A | e_n \otimes b_j \rangle), \end{aligned}$$

and

$$\begin{aligned} \langle e_m | \Gamma_{|\varphi_\pm\rangle\langle\varphi_\pm|}(A) | e_n \rangle &= C_{m,n} \pm i \frac{1 - \delta_{j,k}}{2} \\ &\quad \times (\langle e_m \otimes b_j | A | e_n \otimes b_k \rangle - \langle e_m \otimes b_k | A | e_n \otimes b_j \rangle). \end{aligned}$$

Thus we conclude that $\langle e_m \otimes b_j | A | e_n \otimes b_k \rangle = \delta_{j,k} C_{m,n}$. It follows that

$$\begin{aligned} A &= \sum_{m,n} \sum_{j,k} \langle e_m \otimes b_j | A | e_n \otimes b_k \rangle |e_m\rangle \langle e_n| \otimes |b_j\rangle \langle b_k|, \\ &= \sum_{m,n} \sum_j C_{m,n} |e_m\rangle \langle e_n| \otimes |b_j\rangle \langle b_j| = B \otimes \mathbb{1}_A. \end{aligned}$$

■

Lemma I.3. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ be a measurement scheme for an instrument \mathcal{I} acting in \mathcal{H}_S . Assume that \mathcal{M} is constrained by the third law. If \mathcal{I} is extremal, then it holds that

$$\mathcal{E}^*(\cdot \otimes \mathbf{Z}_x) = \mathcal{I}_x^*(\cdot) \otimes \mathbb{1}_A$$

for all x .

Proof. If \mathcal{M} is constrained by the third law, then ξ is full-rank. By Lemma A.1, for an arbitrary unit vector $|\phi\rangle \in \mathcal{H}_A$, there exists a $0 < \lambda < 1$ such that $\xi \geq \lambda |\phi\rangle \langle\phi|$. Defining the

state $\sigma := (\xi - \lambda|\phi\rangle\langle\phi|)/(1 - \lambda)$, we may thus decompose ξ as $\xi = \lambda|\phi\rangle\langle\phi| + (1 - \lambda)\sigma$. Using the map $\Gamma_\xi^\mathcal{E}$ defined in Eq. (D1), we may thus write

$$\begin{aligned} \mathcal{I}_x^*(\cdot) &= \Gamma_\xi^\mathcal{E}(\cdot \otimes \mathbf{Z}_x), \\ &= \lambda\Gamma_{|\phi\rangle\langle\phi|}^\mathcal{E}(\cdot \otimes \mathbf{Z}_x) + (1 - \lambda)\Gamma_\sigma^\mathcal{E}(\cdot \otimes \mathbf{Z}_x). \end{aligned}$$

Since \mathcal{I} is assumed to be extremal, it follows that

$$\mathcal{I}_x^*(\cdot) = \Gamma_{|\phi\rangle\langle\phi|}^\mathcal{E}(\cdot \otimes \mathbf{Z}_x) = \Gamma_{|\phi\rangle\langle\phi|} \circ \mathcal{E}^*(\cdot \otimes \mathbf{Z}_x)$$

must hold for arbitrary unit vectors $|\phi\rangle \in \mathcal{H}_A$. The claim follows from Lemma I.2. \blacksquare

We are finally ready to provide necessary conditions for extremality of an instrument constrained by the third law.

Proposition I.1. Let $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ be a measurement scheme for a nontrivial observable \mathbf{E} , with the instrument \mathcal{I} , acting in \mathcal{H}_S . Assume that \mathcal{M} is constrained by the third law, and that \mathcal{I} is extremal. The following hold:

- (i) $\text{rank}(\mathbf{E}_x) \geq \sqrt{\dim(\mathcal{H}_S)}$ for all x .
- (ii) \mathcal{E} cannot be a unitary channel.

Proof. Let us first prove (i). Let P_x be the minimal projection on the support of \mathbf{E}_x , where we note that $\dim(P_x\mathcal{H}_S) = \text{rank}(\mathbf{E}_x)$. Consider a full-rank state $\rho = \sum_{n=1}^{\text{rank}(\mathbf{E}_x)} p_n |\psi_n\rangle\langle\psi_n|$ on $P_x\mathcal{H}_S$, where $p_n > 0$ and $|\psi_n\rangle \in P_x\mathcal{H}_S$ for all n . By item (iv) of Lemma D.1, it holds that $\mathcal{I}_x(\rho)$ has full-rank in \mathcal{H}_S , i.e., $\text{rank}(\mathcal{I}_x(\rho)) = \dim(\mathcal{H}_S)$. Now, for any unit vector $|\psi\rangle \in P_x\mathcal{H}_S$ it holds that

$$\mathcal{I}_x(|\psi\rangle\langle\psi|) = \sum_{i=1}^{M_x} |K_i^{(x)}\psi\rangle\langle K_i^{(x)}\psi|.$$

We thus have $\dim(\mathcal{H}_S) = \text{rank}(\mathcal{I}_x(\rho)) \leq M_x \text{rank}(\mathbf{E}_x)$. By item (ii) of Lemma I.1, extremality implies that $M_x \leq \text{rank}(\mathbf{E}_x)$. It follows that $\dim(\mathcal{H}_S) \leq \text{rank}(\mathbf{E}_x)^2$.

Now we shall prove (ii). If \mathcal{I} is extremal, then by Lemma I.3 it holds that $\mathcal{E}^*(\cdot \otimes \mathbb{1}_A) = \mathcal{I}_x^*(\cdot) \otimes \mathbb{1}_A$. If $\mathcal{E}^*(\cdot) = U^* \cdot U$ is a unitary channel, then \mathcal{I}_x^* is also a unitary channel, and so there exists a unitary operator $V \in \mathcal{L}(\mathcal{H}_S)$ such that $\mathcal{I}_x^*(\cdot) = V^* \cdot V$, and we may thus write

$$U^*(A \otimes \mathbb{1})U = V^* \otimes \mathbb{1}_A (A \otimes \mathbb{1}_A) V \otimes \mathbb{1}_A$$

for all $A \in \mathcal{L}(\mathcal{H}_S)$. This implies that $L^*(A \otimes \mathbb{1}_A)L = A \otimes \mathbb{1}_A$, with $L = U(V^* \otimes \mathbb{1}_A)$, for all A . As such, for all A the following commutation relation must hold:

$$[A \otimes \mathbb{1}_A, U(V^* \otimes \mathbb{1}_A)] = \mathbb{O}.$$

That is, there exists a unitary operator W such that $U = V \otimes W$. Therefore, the unitary channel \mathcal{E} is a product of unitary channels acting separately in \mathcal{H}_S and \mathcal{H}_A , i.e., $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, where $\mathcal{E}_1(\cdot) = V \cdot V^*$ acts in \mathcal{H}_S and $\mathcal{E}_2(\cdot) = W \cdot W^*$ acts in \mathcal{H}_A . It follows that

$$\begin{aligned} \mathbf{E}_x &= \Gamma_\xi \circ \mathcal{E}^*(\mathbb{1}_S \otimes \mathbf{Z}_x) = \Gamma_\xi(\mathcal{E}_1^*(\mathbb{1}_S) \otimes \mathcal{E}_2^*(\mathbf{Z}_x)) \\ &= \Gamma_\xi(\mathbb{1}_S \otimes W^* \mathbf{Z}_x W) = \text{tr}[\mathbf{Z}_x W \xi W^*] \mathbb{1}_S, \end{aligned}$$

which is a trivial observable. \blacksquare

To show that extremality is indeed possible, and that the bound $\text{rank}(\mathbf{E}_x) \geq \sqrt{\dim(\mathcal{H}_S)}$ is tight, let us consider the following model.

Consider $\mathcal{H}_S := \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_A := \mathcal{H}_3$, where $\mathcal{H}_i = \mathbb{C}^2$. Choose the apparatus state preparation as an arbitrary

full-rank state ξ , and choose a pointer observable $\mathbf{Z}_x = |x\rangle\langle x|$. Let the measurement interaction be the channel $\mathcal{E} := \mathcal{E}_2 \circ \mathcal{E}_1$, where

$$\begin{aligned} \mathcal{E}_1(A \otimes B \otimes C) &= A \otimes C \otimes B, \\ \mathcal{E}_2(A \otimes B \otimes C) &= \Phi(A \otimes B) \otimes C. \end{aligned}$$

That is, at first a unitary swap channel is applied in $\mathcal{H}_2 \otimes \mathcal{H}_3$, and subsequently a channel Φ is applied in $\mathcal{H}_1 \otimes \mathcal{H}_2$. The operations of the instrument implemented by the measurement scheme $\mathcal{M} := (\mathcal{H}_A, \xi, \mathcal{E}, \mathbf{Z})$ are thus

$$\mathcal{I}_x^*(A \otimes B) = \Gamma_\xi \circ \mathcal{E}_1^*(\Phi^*(A \otimes B) \otimes |x\rangle\langle x|). \quad (11)$$

Φ is a channel defined by

$$\Phi(A \otimes B) = \sum_{x=0}^1 \sum_{f=0}^1 K_{x,f} (A \otimes B) K_{x,f}^*,$$

with the Kraus operators defined below:

$$K_{x,f} := V_f \otimes |\varphi_f\rangle\langle\varphi_f|,$$

where $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $\{V_f\}$ are Kraus operators for some channel acting in \mathcal{H}_1 , and are

$$\begin{aligned} V_0 &= \begin{bmatrix} \sqrt{\frac{1}{4}} & 0 \\ 0 & \sqrt{\frac{3}{4}} \end{bmatrix}, \\ V_1 &= \sigma_x \begin{bmatrix} \sqrt{\frac{3}{4}} & 0 \\ 0 & \sqrt{\frac{1}{4}} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\frac{1}{4}} \\ \sqrt{\frac{3}{4}} & 0 \end{bmatrix}. \end{aligned}$$

As \mathcal{E}_1 is a unitary channel, it is constrained by the third law. To confirm that \mathcal{E}_2 is constrained by the third law, it is enough to check if Φ is so constrained. We have

$$\begin{aligned} \Phi\left(\frac{\mathbb{1}}{2} \otimes \frac{\mathbb{1}}{2}\right) &= \frac{1}{4} \sum_x \sum_f V_f V_f^* \otimes |\varphi_f\rangle\langle\varphi_f| \\ &= \frac{1}{2} \sum_f V_f V_f^* \otimes |\varphi_f\rangle\langle\varphi_f| \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \otimes (|0\rangle\langle 0| + |+\rangle\langle +|). \end{aligned}$$

As the complete mixture is mapped to a full-rank state, then Φ , and hence also \mathcal{E}_2 , satisfies property (ii) of Lemma B.2 and is thus constrained by the third law. Finally, since third-law-constrained channels are closed under composition, and given that the apparatus preparation is full-rank, then \mathcal{M} is constrained by the third law. Moreover, note that \mathcal{E} is evidently not a unitary channel.

Now note that Φ^* may also be written as

$$\begin{aligned} \Phi^*(A \otimes B) &= \sum_{x,f} K_{x,f}^* A \otimes B K_{x,f} \\ &= \sum_{x,f} \langle\varphi_f| B |\varphi_f\rangle V_f^* A V_f \otimes |x\rangle\langle x| \\ &= \sum_f \langle\varphi_f| B |\varphi_f\rangle V_f^* A V_f \otimes \mathbb{1}. \end{aligned} \quad (12)$$

By Eq. (I1) and Eq. (I2), we may thus write

$$\begin{aligned}
 \mathcal{I}_x^*(A \otimes B) &= \Gamma_\xi \circ \mathcal{E}_1^*(\Phi^*(A \otimes B) \otimes |x\rangle\langle x|) \\
 &= \sum_f \langle \varphi_f | B | \varphi_f \rangle \Gamma_\xi \circ \mathcal{E}_1^*(V_f^* A V_f \otimes \mathbb{1} \otimes |x\rangle\langle x|) \\
 &= \sum_f \langle \varphi_f | B | \varphi_f \rangle \Gamma_\xi (V_f^* A V_f \otimes |x\rangle\langle x| \otimes \mathbb{1}) \\
 &= \sum_f \langle \varphi_f | B | \varphi_f \rangle V_f^* A V_f \otimes |x\rangle\langle x| \\
 &= \sum_f K_{x,f}^* (A \otimes B) K_{x,f}.
 \end{aligned}$$

In particular, the measured observable is

$$E_x = \mathcal{I}_x^*(\mathbb{1} \otimes \mathbb{1}) = \sum_f V_f^* V_f \otimes |x\rangle\langle x| = \mathbb{1} \otimes |x\rangle\langle x|.$$

We see that E is sharp, and that $\text{rank}(E_x) = 2$. But since $\mathcal{H}_S = \mathbb{C}^2 \otimes \mathbb{C}^2$, and so $\dim(\mathcal{H}_S) = 4$, it holds that $\text{rank}(E_x) = \sqrt{\dim \mathcal{H}_S}$. As such, the necessary condition for extremality is satisfied, with the model saturating the bound $\text{rank}(E_x) \geq \sqrt{\dim \mathcal{H}_S}$. But by item (iii) of Lemma I.1, \mathcal{I} is extremal if and only if $\{K_{x,f}^* K_{x,g}\}_{x,f,g}$ is a linearly independent set. These operators are written as

$$K_{x,f}^* K_{x,g} = \langle \varphi_f | \varphi_g \rangle V_f^* V_g \otimes |x\rangle\langle x|,$$

where

$$\begin{aligned}
 V_0^* V_0 &= \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix}, & V_1^* V_1 &= \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}, \\
 V_0^* V_1 &= \begin{bmatrix} 0 & 1/4 \\ 3/4 & 0 \end{bmatrix}, & V_1^* V_0 &= \begin{bmatrix} 0 & 3/4 \\ 1/4 & 0 \end{bmatrix}.
 \end{aligned}$$

The above operators are linearly independent, and so $\{K_{x,f}^* K_{x,g}\}_{x,f,g}$ is a linearly independent set. Therefore our model, which is constrained by the third law, implements an extremal instrument, with the rank of the measured observable's effects taking their smallest possible values.

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