Bipartite entanglement and the arrow of time

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We provide a different perspective on the close relationship between entanglement and time. Our main focus is on bipartite entanglement, where this connection is foreshadowed both in the positive partial transpose criterion due to Peres [A. Peres, Phys. Rev. Lett. **77**, 1413 (1996)] and in the classification of quantum within more general nonsignaling bipartite correlations [M. Frembs and A. Döring, Phys. Rev. A **106**, 062420 (2022)]. Extracting the relevant common features, we identify a necessary and sufficient condition for bipartite entanglement in terms of a compatibility condition with respect to time orientations in local observable algebras, which express the dynamics in the respective subsystems. As an outlook and a program for future work, we discuss the relevance of the latter in the broader context of von Neumann algebras and the thermodynamical notion of time naturally arising within the latter.

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I. INTRODUCTION

The connection between quantum entanglement and the arrow of time has been the subject of numerous research enterprises; some recent ones include Refs. [1–6]. Here we mainly focus on bipartite entanglement. It has been surmised that the operation of partial transposition in the positive partial transpose (PPT) criterion[1] for bipartite entanglement due to Peres [8] is related to time reversal [9]. However, this relationship seems not to have been made precise before.

A related area of research, where time unexpectedly enters the picture, is the classification of quantum from nonsignaling bipartite correlations. More precisely, the present author and Döring have recently shown that quantum states are characterized by a compatibility condition with respect to time orientations (roughly, the unitary evolution) in local observable algebras [10].

We review the basics of and extract some key insights from these results in the following sections. Building on those, in Sec. II A we prove a necessary and sufficient criterion for bipartite entanglement. In Sec. II B we show that this too can be recast as a compatibility condition with respect to time orientations. Our work opens up various directions for future research, including practical considerations of our entanglement criterion as well as its generalization to von Neumann algebras [11] (see Secs. II C and III).

A. The PPT criterion

Throughout, we write $\mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A})$ and $\mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})$ with $\dim(\mathcal{H}_\mathcal{A})$ and $\dim(\mathcal{H}_\mathcal{B})$ finite, and $\mathbb{1}_\mathcal{A}$ and $\mathbb{1}_\mathcal{B}$ for the respective identity matrices. Peres noted that the operation of partial transposition transforms separable states into separable states. In turn, any bipartite quantum state[2] $\rho = \sum_{ij} c_{ij}\rho_{\mathcal{A},i} \otimes \rho_{\mathcal{B},j} \in \mathcal{S}(\mathcal{A}\otimes\mathcal{B})$, with $c_{ij} \in \mathbb{C}$, $\rho_{\mathcal{A},i} \in \mathcal{S}(\mathcal{A})$, and $\rho_{\mathcal{B},j} \in \mathcal{S}(\mathcal{B})$, whose partial transpose $\rho^{T_\mathcal{A}} := \sum_{ij} c_{ij}\rho^T_{\mathcal{A},i} \otimes \rho_{\mathcal{B},j}$ has at least one negative eigenvalue, is necessarily entangled [8]. The partial-transpose criterion is necessary and sufficient in low dimensions $\dim(\mathcal{H}_\mathcal{A}) = 2$ and $\dim(\mathcal{H}_\mathcal{B}) = 2, 3$, but is merely sufficient in higher dimensions [7]. Driven mainly by practical considerations, the result has been sharpened in various ways (see Ref. [9] and references therein). This development, while rich and still active, has overshadowed the physical significance of Peres's insight. In contrast, here we will only be concerned with the conceptual importance, leaving the practical value of our work for future study.

1. Pure and purified mixed states

We recall the following simple fact.

Proposition 1. The PPT criterion is necessary and sufficient for pure bipartite states.

Proof. Let $|\psi\rangle \in \mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B}$ and consider the Schmidt decomposition $|\psi\rangle = \sum_i \alpha_i|ii\rangle$, $\alpha_i \in \mathbb{R}_+$, with density matrix $\rho_\psi = |\psi\rangle\langle\psi| = \sum_{ij}\alpha_i\alpha_j|i\rangle\langle j| \otimes |i\rangle\langle j| \in \mathcal{S}(\mathcal{A}\otimes\mathcal{B})$. Note that ρ_ψ is separable if and only if the sum in the Schmidt decomposition collapses to a single term. Applying partial transposition on system $\mathcal{A}$, we obtain $\rho_\psi^{T_\mathcal{A}} = \sum_{ij}\alpha_i\alpha_j|j\rangle\langle i| \otimes |i\rangle\langle j|$. It is easy to see that this operator has a negative eigenvalue for every pair of nonzero coefficients $\alpha_i, \alpha_j \neq 0$. ■

This is of course well known. The following observation is equally straightforward: We can apply Proposition 1 to any bipartite state by considering purifications. Furthermore, a version of the Schrödinger-Hughston-Jozsa-Wootters theorem

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[1]Sometimes the criterion is also referred to as the Peres-Horodecki criterion [7,8].

[2]We will identify states $\sigma \in \mathcal{S}(\mathcal{A})$ with their associated density matrices ρ via $\sigma(a) = \mathrm{tr}[\rho a]$ for all $a \in \mathcal{A}$.

ensures independence of the choice of purification [12,13] (see also Ref. [14]). The PPT criterion thus becomes necessary and sufficient with respect to purifications. Of course, for pure states there are easier ways to check whether a state is entangled or separable.[3] Nevertheless, as we will see below, the criterion works *because* it works on the level of purifications. This is best expressed in terms of channels.

2. Transposition vs Hermitian adjoint

Note that transposition reverses the order of matrix multiplication: Letting $a \in M_{k\times l}(\mathbb{C})$ and $b \in M_{l\times m}(\mathbb{C})$, then

$$((ab)_{ki})^T = (ab)_{ik} = \sum_{j=1}^{l} a_{ij}b_{jk}$$

$$= \sum_{j=1}^{l} (b_{kj})^T (a_{ji})^T = (b^T a^T)_{ki}.$$

This fact is somewhat left implicit from the perspective of bipartite states $\rho \in \mathcal{S}(\mathcal{A}\otimes\mathcal{B})$. In order to make it explicit, we identify a bipartite state ρ with its quantum channel $\phi_\rho : \mathcal{A} \to \mathcal{B}$ under the Choi-Jamiołkowski isomorphism [15,16] (see also Ref. [17]): Recall that every quantum channel $\phi : \mathcal{A} \to \mathcal{B}$, i.e., every completely positive linear map, determines a bipartite state ρ_ϕ (up to normalization) by[4]

$$\rho_\phi = \sum_{ij} E_{ij} \otimes \phi(E_{ij}), \tag{1}$$

where E_{ij} is the matrix with 1 in the entry (i, j) and 0 elsewhere.

Conversely, every bipartite state $\rho \in \mathcal{S}(\mathcal{A}\otimes\mathcal{B})$ corresponds to the quantum channel[5]

$$\phi_\rho(a) = \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[\rho(a^T \otimes \mathbb{1}_\mathcal{B})]. \tag{2}$$

Clearly, with respect to the choice of basis in Eq. (2) we have $(E_{ij})^T = E^*_{ij}$. This allows us to replace transposition with the (Hermitian) adjoint, which we will denote by $*$ (not $\dagger$).

Lemma 1. Let $\rho \in \mathcal{S}(\mathcal{A}\otimes\mathcal{B})$, let ϕ_ρ be the map under the linear isomorphism in Eq. (2), and let $(\Phi_\rho, v, \mathcal{K})$ be a Stinespring dilation of ϕ_ρ, i.e., $\phi_\rho = v^*\Phi_\rho v$ with $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}$ linear and $\Phi_\rho : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ a C^*-homomorphism [18]. Then $\phi_{\rho^{T_\mathcal{A}}} = \phi^*_\rho = v^*\Phi^*_\rho v$.[6,7]

Proof. We have

$$\sum_{ij} E_{ij} \otimes \phi_{\rho^{T_\mathcal{A}}}(E_{ij}) = \rho^{T_\mathcal{A}} = (\rho^*)^{T_\mathcal{A}} = \sum_{ij} E_{ij} \otimes \phi^*_\rho(E_{ij})$$

$$= \sum_{ij} E_{ij} \otimes v^*\Phi^*_\rho(E_{ij})v,$$

where we used Eq. (1) in the first and third steps and $\rho^* = \rho$ in the second. ∎

Partial transposition therefore assumes a more natural interpretation in terms of the adjoint operation on the local system $\mathcal{B}$ (cf. Ref. [17]). In Sec. II C we will see that this encodes a difference between time orientations on the system $\mathcal{B}$. The latter also play a crucial role in selecting quantum from more general nonsignaling bipartite correlations [10].

B. Quantum from nonsignaling correlations

It is instructive to view the problem of entanglement classification from the broader perspective of classifying quantum from nonsignaling correlations. In general, nonsignaling distributions are far from being quantum [19]. Considering product quantum observables, a Gleason-type argument restricts nonsignaling bipartite correlations to normalized linear functionals that are positive on pure tensors, yet not necessarily positive [20–23]. To further single out quantum correlations among the latter, various additional physical principles have been proposed (see, e.g., Ref. [24]). While successful in some instances, none has been shown to recover the quantum state space in general [25].

Recently, a classification of quantum states from more general nonsignaling bipartite correlations was obtained: Quantum states correspond to those correlations, which satisfy (i) an extension of the no-signaling principle to dilations and (ii) a relative consistency condition between the canonical unitary evolution in the respective subsystems (see Definition 1 and Theorem 2 as well as Definition 2 and Theorem 3 in Ref. [10]). Correlations under (i) [but not necessarily (ii)] correspond to decomposable maps under the Choi-Jamiołkowski isomorphism in Eq. (2).[8]

Recall that a linear map $\phi : \mathcal{A} \to \mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})$ is decomposable if there exists a Hilbert space $\mathcal{K}$, a bounded linear operator $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}$, and a Jordan $*$-homomorphism Φ, i.e., $\Phi(aa' + a'a) = \Phi(a)\Phi(a') + \Phi(a')\Phi(a)$ and $\Phi^*(a) = \Phi(a^*)$ for all $a, a' \in \mathcal{A}$ (for details, see Sec. II B) such that $\phi = v^*\Phi v$. Such maps are more general than quantum channels $\phi : \mathcal{A} \to \mathcal{B}$, which are of similar form $\phi = v^*\Phi v$, with Φ a C^*-homomorphism. By Stinespring's theorem [18], the latter is equivalent to ϕ being completely positive: If $x_{ij} \in M_n(\mathcal{A})_+ = (M_n(\mathbb{C})\otimes\mathcal{A})_+$, then $\phi(x_{ij}) := \mathrm{id}_{M_n(\mathbb{C})} \otimes \phi(x_{ij}) \in M_n(\mathcal{B})_+$. Similarly, decomposable maps can be characterized by a weaker positivity condition [28]: If $x_{ij} \in M_n(\mathcal{A})_+$ *and* $x_{ji} \in M_n(\mathcal{A})_+$, then $\phi(x_{ij}) \in M_n(\mathcal{B})_+$. Let $\mathcal{S}_\mathrm{D}(\mathcal{A}\otimes\mathcal{B})$ denote the class of bipartite states corresponding to decomposable maps

[3] For instance, note that we have used the Schmidt rank in Proposition 1.

[4] Sometimes this is written as $\rho_\phi = (\mathrm{id}\otimes\phi)(|\Phi\rangle\langle\Phi|)$, where $|\Phi\rangle = \sum_i |i\rangle\otimes|i\rangle$ is a maximally mixed state.

[5] There are different versions of this isomorphism; for a detailed discussion, see Ref. [17].

[6] Notably, this is different than so-called copositive maps, i.e., maps $\phi^T := T_\mathcal{B}\circ\phi$, where $\phi : \mathcal{A} \to \mathcal{B}$ is a completely positive map. By a similar computation we obtain $\phi_{\rho^{T_\mathcal{A}}} = \phi^T_\rho$, yet $\phi^T_\rho \neq v^*\Phi^T_\rho v$ in general.

[7] We remark that $\phi^* := *\circ\phi$ denotes the adjoint of the image of the channel ϕ, not its (Heisenberg) dual.

[8] We remark that for $\dim(\mathcal{H}_\mathcal{A}) = 2$ and $\dim(\mathcal{H}_\mathcal{B}) = 2, 3$ every positive map $\phi : \mathcal{A} \to \mathcal{B}$ is decomposable [26,27], which implies the necessity of the PPT criterion in those dimensions [7] (see also Theorem 2 below).

under the Choi-Jamiołkowski isomorphism. Interestingly, (the weaker positivity condition in) $\mathcal{S}_{\mathrm{D}}(\mathcal{A}\otimes\mathcal{B})$ is preserved under partial transposition.

Proposition 2. Let $\rho\in\mathcal{S}_{\mathrm{D}}(\mathcal{A}\otimes\mathcal{B})$, i.e., ρ corresponds to a decomposable map under the Choi-Jamiołkowski isomorphism in Eq. (1). Then $\rho^{T_{\mathcal{A}}}\in\mathcal{S}_{\mathrm{D}}(\mathcal{A}\otimes\mathcal{B})$.

Proof. By an argument similar to the one in Lemma 1, $\phi_{\rho^{T_{\mathcal{A}}}}=\phi_\rho^*=v^*\Phi_\rho^*v$, where Φ_ρ is a Jordan $*$-homomorphism, and hence, $\Phi_\rho^*:=*\circ\Phi_\rho=\Phi_\rho\circ *$. However, then so is Φ_ρ^*: For all $a_1,a_2\in\mathcal{A}$,

$$\begin{aligned}\Phi_\rho^*(\{a_1,a_2\})&=\Phi_\rho(\{a_1,a_2\}^*)=\Phi_\rho(\{a_1^*,a_2^*\})\\&=\{\Phi_\rho(a_1^*),\Phi_\rho(a_2^*)\}=\{\Phi_\rho^*(a_1),\Phi_\rho^*(a_2)\}.\end{aligned}$$

Consequently, $\phi_{\rho^{T_{\mathcal{A}}}}$ is decomposable and $\rho^{T_{\mathcal{A}}}\in\mathcal{S}_{\mathrm{D}}(\mathcal{A}\otimes\mathcal{B})$. ■

Summarizing this motivational prologue, in Sec. I A we remarked that the PPT criterion becomes necessary and sufficient when applied to purifications and used the Choi-Jamiołkowski isomorphism to translate the criterion from bipartite states to bipartite channels. In particular, we recast partial transposition into the (Hermitian) adjoint in Lemma 1, which further suggests to lift the PPT criterion from the level of bipartite quantum channels ϕ to C^*-homomorphisms Φ in Stinespring dilations $\phi=v^*\Phi v$.

More generally, in Sec. I B we considered the PPT criterion with respect to dilations of decomposable maps. Proposition 2 shows that partial transpose preserves the respective positivity condition of linear functionals corresponding to such maps. We deduce that the PPT criterion is sensitive precisely to the difference between decomposable and completely positive maps. In the context of classifying quantum from nonsignaling bipartite correlations, this is achieved by enforcing a compatibility condition with respect to the relative time orientation between systems $\mathcal{A}$ and $\mathcal{B}$ [10]. Building on this motivation, in the next section we identify a necessary and sufficient condition for bipartite separability in terms of a compatibility condition with respect to different time orientations between $\mathcal{A}$ and $\mathcal{B}$.

II. ENTANGLEMENT AND THE ARROW OF TIME

In this main part, we combine the insights gained in previous sections. In Sec. II A we identify a necessary and sufficient criterion for bipartite entanglement (Theorem 1). In Sec. II B we employ the structure theory of Jordan and (associative) C^*-algebras to translate this criterion into a compatibility condition between canonical time orientations on local subsystems (Theorem 2). Finally, in Sec. II C we interpret our results in light of the intrinsic flow of time in von Neumann algebras by means of Tomita-Takesaki theory, Connes cocyles, and the background-independent thermodynamical arrow of time [11].

A. A necessary and sufficient criterion for bipartite entanglement

The PPT criterion translates between bipartite states and bipartite channels as follows:

$$\begin{aligned}\rho^{T_{\mathcal{A}}}\text{ positive}\quad&\overset{\text{Choi's theorem}}{\Longleftrightarrow}\quad\phi_{\rho^{T_{\mathcal{A}}}}\text{ completely positive}\\&\overset{\text{Lemma 1}}{\Longleftrightarrow}\quad\phi_\rho^*\text{ completely positive}\\&\overset{\text{Stinespring's theorem}}{\Longleftrightarrow}\quad\phi_\rho^*=(v')^*\Phi'_\rho v',\ \text{ where }(v',\Phi'_\rho,\mathcal{K}')\text{ is a Stinespring dilation.}\end{aligned}$$

Now since ϕ_ρ is completely positive, it also has a Stinespring dilation $\phi_\rho=v^*\Phi_\rho v$. We may thus strengthen the PPT criterion as follows: Rather than ϕ_ρ^* admitting any Stinespring dilation, we ask when $v'=v$, $\Phi'_\rho=\Phi_\rho^*$, i.e., when Φ_ρ^* is a C^*-homomorphism.

Note first that this condition does not depend on the choice of Stinespring dilation.

Lemma 2. Let $\phi=v_1\Phi_1v_1^*=v_2\Phi_2v_2^*$ be two Stinespring dilations of $\phi:\mathcal{A}\to\mathcal{B}$. Then $\Phi_1^*:\mathcal{A}\to\mathcal{L}(\mathcal{K}_1)$ is a C^*-homomorphism if and only if $\Phi_2^*:\mathcal{A}\to\mathcal{L}(\mathcal{K}_2)$ is a C^*-homomorphism.

Proof. There is a partial isometry $W:\mathcal{K}_1\to\mathcal{K}_2$ defined by $W\Phi_1(a)v_1|\psi\rangle=\Phi_2(a)v_2|\psi\rangle$ for all $a\in\mathcal{A}$ and $|\psi\rangle\in\mathcal{H}_{\mathcal{B}}$ such that $\Phi_1=W^*\Phi_2W$.[9] Hence, $\Phi_1^*=W^*\Phi_2^*W$ and the claim follows. ■

Next we have the following important characterization.

Lemma 3. Let $\Phi:\mathcal{A}\to\mathcal{L}(\mathcal{K})$ be a C^*-homomorphism. Then $\Phi^*:\mathcal{A}\to\mathcal{L}(\mathcal{K})$ is a C^*-homomorphism if and only if $\Phi(\mathcal{A})\subset\mathcal{L}(\mathcal{K})$ is a commutative subalgebra,[10] equivalently, $\Phi=\Phi|_V$ for $V\subset\mathcal{A}$ a commutative subalgebra.

Proof. If Φ^* is a C^*-homomorphism, then for all $a_1,a_2\in\mathcal{A}$,

$$\begin{aligned}\Phi(a_1)\Phi(a_2)&=\Phi(a_1a_2)=\Phi^*((a_1a_2)^*)=\Phi^*(a_2^*a_1^*)\\&=\Phi^*(a_2^*)\Phi^*(a_1^*)=\Phi(a_2)\Phi(a_1),\end{aligned}$$

and hence $\Phi(\mathcal{A})\subset\mathcal{L}(\mathcal{K})$ is a commutative subalgebra. Conversely, if $\Phi(\mathcal{A})\subset\mathcal{L}(\mathcal{K})$ is a commutative subalgebra, then for all $a_1^*,a_2^*\in\mathcal{A}$,

$$\begin{aligned}\Phi^*(a_2^*a_1^*)&=\Phi^*((a_1a_2)^*)=\Phi(a_1a_2)=\Phi(a_1)\Phi(a_2)\\&=\Phi(a_2)\Phi(a_1)=\Phi^*(a_2^*)\Phi^*(a_1^*).\end{aligned}$$

Hence, Φ^* is a C^*-homomorphism.

Since $\Phi(\mathcal{A})\subset\mathcal{L}(\mathcal{K})$ is a commutative subalgebra, there exists a maximal commutative subalgebra $V\subset\mathcal{A}$ such that

[9]Note that for minimal Stinespring dilations W is unitary.

[10]It is interesting to note that Bell's theorem holds for states over C^*-algebras as long as one of them is commutative [29]. We discuss the relation with Bell's theorem and Bell nonlocality elsewhere [30].

$\Phi(V) = \Phi(\mathcal{A})$. We show that $\Phi(a) = 0$ for all $a \perp V$ (with respect to the Hilbert-Schmidt inner product $(a_1, a_2) := \mathrm{tr}[a_2^* a_1]$). Without loss of generality, we may assume that V is the commutative subalgebra generated by diagonal matrices. We want to show that $\Phi(a) = 0$ for all $a \in V^\perp$. The latter implies $\mathrm{tr}[a] = 0$; hence, $a = [b, c]$ for some $b, c \in \mathcal{A}$ [31]. We have $\Phi(a) = \Phi([b, c]) = [\Phi(b), \Phi(c)] = 0$, since Φ is a homomorphism and $\Phi(\mathcal{A})$ is commutative. Consequently, Φ acts nontrivially only on V and is zero otherwise. ∎

The following key result shows that this property is equivalent to separability.

Theorem 1. Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$, let ϕ_ρ be the map under the isomorphism in Eq. (2), and let $\phi_\rho = v^* \Phi_\rho v$ be a Stinespring dilation of ϕ_ρ. Then ρ is separable if and only if $\phi_\rho^* = v^* \Phi_\rho^* v$ is a Stinespring dilation of ϕ_ρ^*, i.e., if and only if Φ_ρ^* is a C^*-homomorphism.

Proof. First, assume that Φ^* is a C^*-homomorphism. By Lemma 2 we can choose the Stinespring dilation $\phi_\rho = v^* \Phi_\rho v$ to be of the simple form[11] $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}$ for $\mathcal{K} = \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_\mathcal{A}$ and $\Phi_\rho = \mathbb{1}_\mathcal{B} \otimes \mathrm{id}_{\mathrm{supp}(\Phi_\rho)}$, with $\mathrm{supp}(\Phi_\rho) \subset \mathcal{A}$ the support of Φ_ρ.[12] In fact, if Φ_ρ^* is a C^*-homomorphism, then $\phi_{\rho^{T_\mathcal{A}}} = \phi_\rho^* = v^* \Phi_\rho^* v$ is a Stinespring dilation (see Lemma 1).[13] By Lemma 3, $\Phi_\rho(\mathcal{A}) \subset \mathcal{L}(\mathcal{K})$ is a commutative subalgebra and there exists a commutative subalgebra $V \subset \mathcal{A}$ such that $\Phi_\rho(V) = \Phi_\rho(\mathcal{A}) \subset \mathcal{L}(\mathcal{K})$. We thus have $\Phi_\rho(a) = \Phi_\rho|_V(a) = \mathbb{1}_\mathcal{B} \otimes \pi_V a \pi_V$, where π_V denotes the projection onto V.

Next let $(|\xi_k\rangle)_k$ be a basis of $\mathcal{H}_\mathcal{A}$ such that $|\xi_k\rangle\langle\xi_k| = \pi_k$ for all one-dimensional projections $\pi_k \in \mathcal{P}_1(V)$ and let $(|e_i\rangle)_i$ be an orthonormal basis of $\mathcal{H}_\mathcal{B}$. We can decompose $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}$ by its action on basis states $v(|e_j\rangle) := \sum_{ik} c_{ij}^k |e_i\rangle \otimes |\xi_k\rangle$ with $c_{ij}^k \in \mathbb{C}$; hence, $v = \sum_{ijk} c_{ij}^k |e_i\rangle\langle e_j| \otimes |\xi_k\rangle = \sum_k X_k \otimes |\xi_k\rangle$ with $X_k = \sum_{ij} c_{ij}^k |e_i\rangle\langle e_j|$. Consequently, for all $a = \sum_r a_r \pi_r \in V$,

$$\begin{aligned}\phi_\rho(a) &= v^* \Phi_\rho|_V(a) v = v^*(\mathbb{1}_\mathcal{B} \otimes a) v \\ &= \sum_{kl} X_l^* X_k \, \langle \xi_l | \sum_r a_r \pi_r | \xi_k \rangle \\ &= \sum_k E_k \, \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[|\xi_k\rangle\langle\xi_k| \, a]. \end{aligned} \tag{3}$$

Clearly, $E_k = X_k^* X_k \in \mathcal{B}_+$; hence (after normalization $E_k \to E_k/\mathrm{tr}[E_k]$ and $|\xi_k\rangle\langle\xi_k| \to \mathrm{tr}[E_k]|\xi_k\rangle\langle\xi_k|$) ϕ_ρ is an entanglement-breaking channel[14] and equivalently ρ is a separable state [35].

Conversely, let ρ be a separable state.[15] Then ϕ_ρ is an entanglement-breaking channel [35], i.e., there exist states $E_k \in \mathcal{S}(\mathcal{B})$ and positive operators $0 \leqslant F_k \in \mathcal{A}$ such that

$$\phi_\rho(a) = \sum_{k=1}^K E_k \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[F_k a].$$

We may extend F_k to a positive-operator-valued measure $(F_k)_{k=0}^K$, $F_k \in \mathcal{A}_+$ by setting $F_0 := \mathbb{1}_\mathcal{A} - \sum_{k=1}^K F_k$ such that $\sum_{k=0}^K F_k = \mathbb{1}_\mathcal{A}$. By Naimark's theorem [18,32], F admits a dilation $F = \tilde{v}^* \pi \tilde{v}$, where $\tilde{v} : \mathcal{H}_\mathcal{A} \to \mathcal{H}_{\tilde{\mathcal{A}}}$ is a linear map and $\pi = (\pi_k)_{k=0}^K$, $\pi_k \in \mathcal{P}(\mathcal{H}_{\tilde{\mathcal{A}}})$, a projection-valued measure. Consequently, $\phi_\rho(a) = \tilde{\phi}_\rho(a) - E_0 \, \mathrm{tr}[F_0 a]$ (where, e.g., $E_0 \propto \mathbb{1}_\mathcal{B}$) and

$$\begin{aligned}\tilde{\phi}_\rho(a) &= \sum_{k=0}^K E_k \, \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[\tilde{v}^* \pi_k \tilde{v} a] = \sum_{k=0}^K E_k \, \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[\tilde{v}^* \pi_k \tilde{\Phi}_\rho(a) \tilde{v}] \\ &= \sum_{k=0}^K E_k \, \mathrm{tr}_{\mathcal{H}_{\tilde{\mathcal{A}}}}[\pi_k \tilde{\Phi}_\rho(a)].\end{aligned}$$

Here $\tilde{\Phi}_\rho : \mathcal{A} \to \tilde{\mathcal{A}} = \mathcal{L}(\mathcal{H}_{\tilde{\mathcal{A}}})$, $\tilde{\Phi}_\rho(a) := \tilde{v} a \tilde{v}^*$ is the natural embedding under the isometry $\tilde{v}$ (that is, $\tilde{v}^* \tilde{v} = \mathbb{1}_\mathcal{A}$) and we used that $\tilde{\Phi}_\rho \tilde{v} \tilde{v}^* = \tilde{\Phi}_\rho$, where $\tilde{v}\tilde{v}^* \in \mathcal{P}(\mathcal{H}_{\tilde{\mathcal{A}}})$ and $(\tilde{v}\tilde{v}^*)\mathcal{H}_{\tilde{\mathcal{A}}} \cong \mathcal{H}_\mathcal{A}$ in the last step. Note that $(a_1, a_2) := \mathrm{tr}_{\mathcal{H}_{\tilde{\mathcal{A}}}}[a_2^* a_1]$ defines an inner product on $\tilde{\mathcal{A}}$. We can thus restrict the action of $\tilde{\Phi}_\rho$ to the preimage of the commutative subalgebra $W := \langle \pi_k \rangle_{k=0}^K \subset \tilde{\mathcal{A}}$, spanned by the projections π_k. More precisely, we define $\Phi_\rho : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ for $\mathcal{K} = \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_{\tilde{\mathcal{A}}}$ by

$$\Phi_\rho(a) = \begin{cases} \mathbb{1}_\mathcal{B} \otimes \tilde{\Phi}_\rho & \text{for all} a \in \tilde{\Phi}_\rho^{-1}(W) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, Φ_ρ is a C^*-homomorphism since $\tilde{\Phi}_\rho|_{\tilde{\Phi}_\rho^{-1}(W)}$ is. Moreover, $\Phi_\rho(\mathcal{A}) \subset \mathcal{L}(\mathcal{K})$ is a commutative subalgebra by construction; hence, Φ_ρ^* is a C^*-homomorphism by Lemma 3. Finally, we obtain Stinespring dilations $\phi_\rho = v^* \Phi_\rho v$ and $\phi_\rho^* = v^* \Phi_\rho^* v$ as before: Define $v = \sum_{k=1}^K X_k \otimes |\xi_k\rangle$ with $X_k^* X_k = E_k$ and with $(|\xi_k\rangle)_{k=0}^K$ the orthonormal basis of $\mathcal{H}_{\tilde{\mathcal{A}}}$, corresponding to the commutative subalgebra $W \subset \tilde{\mathcal{A}}$, i.e., $|\xi_k\rangle\langle\xi_k| = \pi_k$. ∎

We have used the fact that a state is separable if and only if its corresponding quantum channel in Eq. (2) is a measure-prepare channel [35]. The latter requires a decomposition of v as in Eq. (3). Clearly, such a decomposition exists for any commutative subalgebra $V \subset \mathcal{A}$. In turn, Theorem 1 shows that such a decomposition exists for all of $\mathcal{A}$ if and only if $\Phi_\rho(\mathcal{A})$ is a commutative subalgebra.

Moreover, Theorem 1 sheds light on the reason why the PPT criterion is not necessary in general: If we let $\phi_\rho = v^* \Phi_\rho v$ be a Stinespring dilation and let $\phi_\rho^* = \phi_{\rho^{T_\mathcal{A}}}$ be completely positive, this does not imply that $\phi_\rho^* = v^* \Phi_\rho^* v$ is a Stinespring dilation for ϕ_ρ^*.

We record the following corollary of Theorem 1.

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[11]For a general Stinespring dilation, one needs $\mathcal{K} = \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A}$ (cf. Refs. [18,32]). However, as it turns out in the case that both Φ and Φ^* are both C^*-homomorphisms, $\mathcal{K}$ can be reduced by one factor of $\mathcal{H}_\mathcal{A}$.

[12]This representation allows us to interpret a quantum channel ϕ as a coarse-grained unitary bipartite channel on the target and some unknown ancillary system, after tracing out the latter. In particular, generalized measurements can be understood as projective measurements on a larger system [33].

[13]In particular, $\rho^{T_\mathcal{A}}$ is positive in this case.

[14]Entanglement-breaking channels are also called measure-prepare channels or in Holevo form [34,35].

[15]Clearly, $\rho^{T_\mathcal{A}}$ is positive; equivalently, $\phi_{\rho^{T_\mathcal{A}}}$ is completely positive in this case.

Corollary 1. Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$, let ϕ_ρ be the map under the isomorphism in Eq. (2), and let $\phi_\rho = v^* \Phi_\rho v$ be a Stinespring dilation. Then ρ is separable if and only if $\phi_\rho = \phi_\rho|_V$ for a commutative subalgebra $V \subset \mathcal{A}$.

Proof. This follows immediately from Lemma 2, Theorem 1, and Lemma 3. ■

We surmise that Theorem 1, especially in the form of Corollary 2, entails improvements of existing protocols for practical verification of entanglement, e.g., in the form of semi-definite linear programs in [36,37]. We leave this as an exciting direction for future research. In the remainder, we focus on the physical content of Theorem 1 in terms of the arrow of time.

B. Entanglement and time orientations

Comparing Proposition 2 with Theorem 1, it is natural to study the difference between Jordan $*$-homomorphisms and C^*-homomorphisms. To this end, we first review some basic facts about Jordan algebras and their dynamics in terms of one-parameter groups of automorphisms, before proving a reformulation of Theorem 1 in terms of local time orientations.

1. Jordan algebras

We recall that an abstract Jordan algebra $\mathcal{J}$ is an algebra over a field with a product $\circ$ that satisfies $a \circ b = b \circ a$ and $(a \circ b) \circ (a \circ a) = a \circ (b \circ (a \circ a))$ for all $a, b \in \mathcal{J}$.[16] Given an associative algebra $\mathcal{A}$, one obtains a Jordan algebra $\mathcal{J}(\mathcal{A})$ under the symmetrized product $a_1 \circ a_2 := \frac{1}{2}(a_1 a_2 + a_2 a_1)$ for all $a_1, a_2 \in \mathcal{A}$. If $\mathcal{J} \subseteq \mathcal{J}(\mathcal{A})$ for an associative algebra $\mathcal{A}$, then the Jordan algebra is called special; otherwise it is called exceptional.[17] In particular, every C^* (and von Neumann[18]) algebra defines a JB(W) algebra: A JB(W) algebra is a (weakly closed) Jordan algebra that is also a Banach space ($\|a \circ b\| \leqslant \|a\| \cdot \|b\|$) such that $\|a^2\| = \|a\|^2 \leqslant \|a^2 + b^2\|$. For simplicity, here we only consider matrix algebras over the complex numbers, $\mathcal{A} = M_n(\mathbb{C})$, $n \in \mathbb{N}$. In this case, the set of Hermitian matrices $H_n(\mathbb{C})$ under the anticommutator $\{a_1, a_2\} := a_1 a_2 + a_2 a_1$ defines a real Jordan algebra $\mathcal{J}(\mathcal{A})_{\mathrm{sa}} := (H_n(\mathbb{C}), \{\cdot, \cdot\})$. We define its complexification by $\mathcal{J}(\mathcal{A}) = \mathcal{J}(\mathcal{A})_{\mathrm{sa}} + i\mathcal{J}(\mathcal{A})_{\mathrm{sa}}$.

Crucially, Jordan products are commutative. As such the Jordan algebra $\mathcal{J}(\mathcal{A})_{\mathrm{sa}}$ is the same as the Jordan algebra $\mathcal{J}(\mathcal{A}^{\mathrm{op}})_{\mathrm{sa}}$ of the opposite algebra $\mathcal{A}^{\mathrm{op}} = (\mathcal{J}(\mathcal{A}), \cdot_-)$, i.e., the algebra obtained from $\mathcal{A} = (\mathcal{J}(\mathcal{A}), \cdot_+)$ by reversing the order of composition (matrix multiplication), where for all $a_1, a_2 \in \mathcal{J}(\mathcal{A})$,

$$\begin{aligned} a_1 \cdot_+ a_2 &= \tfrac{1}{2}\{a_1, a_2\} + \tfrac{1}{2}[a_1, a_2], \\ a_1 \cdot_- a_2 &= \tfrac{1}{2}\{a_1, a_2\} - \tfrac{1}{2}[a_1, a_2]. \end{aligned} \tag{4}$$

The difference between the associative algebras $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ therefore lies in the antisymmetric part or commutator. In order to extract from this a notion of time directionality, we relate commutators to (infinitesimal) symmetries of $\mathcal{J}(\mathcal{A})_{\mathrm{sa}}$.

2. Time orientations

Dynamics is naturally expressed in terms of one-parameter groups of Jordan automorphisms $\mathbb{R} \ni t \mapsto \mathrm{Aut}[\mathcal{J}(\mathcal{A})_{\mathrm{sa}}]$. Recall that for $\mathcal{A} = \mathcal{L}(\mathcal{H}_A)$ [in particular, for $\mathcal{A} = M_n(\mathbb{C})$] every such one-parameter group is given by conjugation with a unitary or antiunitary operator by Wigner's theorem [43,44]. In fact, Wigner's theorem holds on the level of Jordan algebras [30,45]. In $\mathcal{A}$ we obtain one-parameter groups of the form

$$e^{t\,\mathrm{ad}(ia_1)}(a_2) = e^{ita_1} a_2 e^{-ita_1} \;\forall\, t \in \mathbb{R},\, a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}. \tag{5}$$

If we interpret a_1 as the Hamiltonian of the system, then Eq. (5) is just the standard expression for unitary evolution, in which t plays the role of a time parameter. More generally, Eq. (5) defines a one-parameter group for every element $a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$. In particular, note that for every $a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$ and $\lambda \in \mathbb{R}_+$ also $\lambda a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$. Hence, we cannot give physical meaning to the absolute value of t without first specifying a Hamiltonian a.

Nevertheless, the sign of t carries physical meaning independent of the choice of $a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$. To see this, we remark that intrinsic to Eq. (5) is the canonical identification between self-adjoint operators (observables) and generators of Jordan automorphisms (symmetry generators), $a \mapsto \mathrm{ad}(ia)$ for all $a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$ [46–48]. Moreover, note that changing this identification to $a \mapsto \mathrm{ad}(-ia)$ results in a sign change for the parameter t in Eq. (5) or equivalently to a change in the commutator and thus to a change in the order of composition from $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}$ in Eq. (4) (see also Lemma 4 below).

In contrast, in $\mathcal{J}(\mathcal{A})$ there is no canonical identification between self-adjoint operators and generators of Jordan automorphisms [40,46]. Consequently, in $\mathcal{J}(\mathcal{A})$ we cannot interpret the sign of the parameter t in the corresponding one-parameter groups independently of the choice of Hamiltonian $a \in \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$. By comparison, lifting $\mathcal{J}(\mathcal{A})$ to $\mathcal{A}$ thus equips the latter with an intrinsic direction of time, mediated by the identification $a \mapsto \mathrm{ad}(ia)$.[19] To emphasize this distinction, we define the canonical time orientation $\Psi_{\mathcal{A}}$ on $\mathcal{J}(\mathcal{A})$ by[20]

$$\Psi_{\mathcal{A}} := \mathrm{Ad} : \mathbb{R} \times \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \ni (t, a) \mapsto e^{t\,\mathrm{ad}(ia)} \tag{6}$$

and call $\mathcal{A}_+ := (\mathcal{J}(\mathcal{A}), \Psi_{\mathcal{A}})$ the observable Jordan algebra together with its canonical time orientation. Similarly, we define the reverse time orientation by

$$\Psi^*_{\mathcal{A}} := * \circ \Psi_{\mathcal{A}} : \mathbb{R} \times \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \ni (t, a) \mapsto e^{-t\,\mathrm{ad}(ia)}, \tag{7}$$

such that $\Psi^*_{\mathcal{A}}(t, a) = \Psi_{\mathcal{A}}(-t, a)$, and we set $\mathcal{A}_- := (\mathcal{J}(\mathcal{A}), \Psi^*_{\mathcal{A}})$.[21]

[16] For an extensive study of Jordan algebras, see Refs. [38–41].

[17] The prototypical exceptional Jordan algebra is the so-called Albert algebra $H_3(\mathbb{O})$ [42].

[18] Recall that a C^*-algebra is an involutive Banach algebra (closed in norm) satisfying the defining C^* property $\|x^* x\| = \|x\|^2$. A von Neumann algebra is a C^*-algebra closed in the weak operator topology.

[19] Generalizing the mapping $a \mapsto \mathrm{ad}(ia)$, Ref. [49] characterizes those maps which lift JB(W) to C^* (von Neumann) algebras. By their physical interpretation, such maps are called dynamical correspondences.

[20] The notion of time orientation was introduced in Refs. [10,50].

[21] These are the only time orientations on $\mathcal{J}(\mathcal{A})$, which correspond to $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$, respectively (cf. Ref. [49]).

3. Entanglement and time orientation

Returning to Theorem 1, we are interested in the difference between Jordan $*$-homomorphism and C^*-homomorphism. Recall that a Jordan $*$-homomorphism $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ is a linear map preserving the Hermitian adjoint $* \circ \Phi = \Phi \circ *$ or, equivalently, $\Phi|_{\mathcal{J}(\mathcal{A})_{\rm sa}} : \mathcal{J}(\mathcal{A})_{\rm sa} \to \mathcal{J}(\mathcal{B})_{\rm sa}$, and the Jordan product, i.e., $\Phi(\{a_1, a_2\}) = \{\Phi(a_1), \Phi(a_2)\}$ for all $a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\rm sa}$. Consequently, $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ lifts to a C^*-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ if and only if it preserves commutators, $\Phi([a_1, a_2]) = [\Phi(a_1), \Phi(a_2)]$ for all $a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\rm sa}$. Using Eq. (6), we reexpress this condition in terms of one-parameter groups of Jordan automorphisms.

Lemma 4. Let $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ be a Jordan $*$-homomorphism. Then $\Phi : \mathcal{A} \to \mathcal{B}$ lifts to a C^*-homomorphism if and only if it preserves the canonical time orientations $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{B}}$,

$$\Phi \circ \Psi_{\mathcal{A}}(t, a) = \Psi_{\mathcal{B}}(t, \Phi(a)) \circ \Phi \; \forall \, t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{A})_{\rm sa}. \tag{8}$$

Proof. Clearly, a C^*-homomorphism Φ satisfies Eq. (8). Conversely, by differentiation,

$$\frac{d}{dt}\bigg|_{t=0} (\Phi \circ \Psi_{\mathcal{A}}(t, a_1))(a_2) = \frac{d}{dt}\bigg|_{t=0} (\Psi_{\mathcal{B}}(t, \Phi(a_1)) \circ \Phi)(a_2) \Leftrightarrow \Phi\left(\frac{d}{dt}\bigg|_{t=0} e^{ita_1} a_2 e^{-ita_1} \right)$$

$$= \frac{d}{dt}\bigg|_{t=0} e^{it\Phi(a_1)} \Phi(a_2) e^{-it\Phi(a_1)} \Leftrightarrow \Phi([a_1, a_2]) = [\Phi(a_1), \Phi(a_2)]$$

for all $a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\rm sa}$. Thus Φ preserves commutators; hence is a C^*-homomorphism. ■

Assume $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ in Eq. (8) is part of a Stinespring dilation $\phi_\rho = v^* \Phi v$ for the image of a bipartite state $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ under the isomorphism in Eq. (2). Since $\mathcal{B}$ arises from $\mathcal{L}(\mathcal{K})$ by restriction under v, the time orientation $\Psi_{\mathcal{B}}$ on $\mathcal{B}$ uniquely lifts to a time orientation $\Psi'_{\mathcal{B}}$ on $\mathcal{L}(\mathcal{K})$. This motivates the following definition, which first appeared in the context of classifying quantum states from nonsignaling bipartite correlations [10] (see also Sec. I B).

Definition 1. Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$. Here ρ is called time-oriented with respect to $\mathcal{A}_- \cong (\mathcal{J}(\mathcal{A}), \Psi^*_{\mathcal{A}})$ and $\mathcal{B}_+ = (\mathcal{J}(\mathcal{B}), \Psi_{\mathcal{B}})$ if and only if $\Phi_\rho : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ in $\phi_\rho = v^* \Phi_\rho v$ preserves time orientations $\Psi^*_{\mathcal{A}} = * \circ \Psi_{\mathcal{A}}$ and $\Psi'_{\mathcal{B}}$,

$$\Phi_\rho \circ \Psi^*_{\mathcal{A}}(t, a) = \Psi'_{\mathcal{B}}(t, \Phi(a)) \circ \Phi_\rho \; \forall \, t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{A})_{\rm sa}.$$

We remark that the appearance of the reverse time orientation $\Psi^*_{\mathcal{A}}$ in Definition 1 is a consequence of the identification of bipartite quantum states and quantum channels via Choi's theorem [15] (for more details, see Ref. [17]). Definition 1 is the missing piece of physical data to identify bipartite nonsignaling distributions with quantum states [10].

Furthermore, being time-oriented is a genuine quantum effect. In fact, it is directly related with entanglement, as Definition 1 allows us to reformulate the separability criterion in Theorem 1 in terms of time orientations.

Theorem 2. A bipartite state $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ is separable if and only if it is time-oriented with respect to $\mathcal{A}_- = (\mathcal{J}(\mathcal{A}), \Psi^*_{\mathcal{A}})$ and $\mathcal{B}_+ = (\mathcal{J}(\mathcal{B}), \Psi_{\mathcal{B}})$ as well as $\mathcal{A}_-$ and $\mathcal{B}_- = (\mathcal{J}(\mathcal{B}), \Psi^*_{\mathcal{B}})$.

Proof. By Theorem 1, ρ is separable if and only if Φ_ρ and Φ^*_ρ are C^*-homomorphisms for any Stinespring dilation $\phi_\rho = v^* \Phi_\rho v$. Since C^*-homomorphisms preserve time orientations by Lemma 4, ρ is time-oriented with respect to both $\mathcal{A}_-$ and $\mathcal{B}_+$ as well as $\mathcal{A}_-$ and $\mathcal{B}_-$.

Conversely, ρ is time-oriented with respect to $\mathcal{A}_-$ and $\mathcal{B}_+$ by Theorem 3 in Ref. [10], i.e.,

$$\begin{aligned} \Phi_\rho \circ \Psi^*_{\mathcal{A}}(t, a) &= \Phi_\rho \circ \Psi_{\mathcal{A}}(-t, a) \\ &= \Psi'_{\mathcal{B}}(t, \Phi(a)) \circ \Phi_\rho \; \forall \, t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{A})_{\rm sa}, \end{aligned} \tag{9}$$

where $\Psi^*_{\mathcal{A}}(t, a) = * \circ \Psi_{\mathcal{A}}(t, a) = \Psi_{\mathcal{A}}(-t, a)$ by Eq. (7). In particular, Φ_ρ in $\phi_\rho = v^* \Phi_\rho v$ is a C^*-homomorphism [18]. If ρ is also time-oriented with respect to $\mathcal{A}_-$ and $\mathcal{B}_-$, then by Definition 1,

$$\begin{aligned} & \Phi_\rho \circ \Psi^*_{\mathcal{A}}(t, a) = \Psi'^*_{\mathcal{B}}(t, \Phi(a)) \circ \Phi_\rho \; \forall \, t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{A})_{\rm sa} \\ \Longleftrightarrow \; & \Phi_\rho \circ \Psi_{\mathcal{A}}(t, a) = \Psi'_{\mathcal{B}}(t, \Phi(a)) \circ \Phi_\rho \; \forall \, t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{A})_{\rm sa}. \end{aligned} \tag{10}$$

Differentiating Eqs. (9) and (10) yields $[\Phi_\rho(a_1), \Phi_\rho(a_2)] = -[\Phi_\rho(a_1), \Phi_\rho(a_2)] = 0$ for all $a_1, a_2 \in \mathcal{J}(\mathcal{A})_{\rm sa}$ (cf. Lemma 4). It follows that $\Phi_\rho(\mathcal{A}) \subset \mathcal{B}$ is a commutative subalgebra; by Lemma 3, Φ^*_ρ is therefore a C^*-algebra homomorphism and by Theorem 1 ρ is separable. ■

Finally, note that the separability condition in terms of time orientations in Theorem 2 is symmetric, that is, entanglement encodes a relative time orientation between subsystems. To see this, note that if $\Phi^*_\rho(a) = \Phi_\rho(a^*)$ for all $a \in \mathcal{A}$, then Φ_ρ preserves time orientations $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{B}}$ if and only if it preserves time orientations $\Psi^*_{\mathcal{A}}$ and $\Psi^*_{\mathcal{B}}$; similarly, Φ_ρ preserves time orientations $\Psi^*_{\mathcal{A}}$ and $\Psi_{\mathcal{B}}$ if and only if it preserves time orientations $\Psi_{\mathcal{A}}$ and $\Psi^*_{\mathcal{B}}$.

C. Time orientations and the arrow of time

In this section we embed the classification of bipartite entanglement in terms of compatibility with time orientations in local observable algebras (Theorem 2) into a wider context. We especially focus on time orientations as a complex structure on $\mathcal{J}(\mathcal{A})$, as well as their role within the intrinsic thermodynamic arrow of time in von Neumann algebras.

1. Time orientations and complex structure

Following Refs. [10,50], we have expressed the difference between $\mathcal{J}(\mathcal{A})$ and $\mathcal{A}$ in terms of time orientations in Definition 1. As exponentials of dynamical correspondences [49], time orientations highlight the double role played by self-adjoint operators: As observables and as generators of dynamics in quantum mechanics [46,48]. This perspective has some appeal when considering axiomatic reconstructions of quantum mechanics and possible generalizations they suggest.

For instance, note that a quantum formalism can be defined also over the real instead of the complex numbers (see, e.g., Refs. [51,52]). More generally, several results aiming to reconstruct quantum mechanics arrive at the level of (special) Jordan algebras corresponding to associative algebras over the real, complex, and quaternionic numbers [48,53–56]. In this context, it is interesting to note that time orientations define a complex structure on (the order derivations of) $\mathcal{J}(\mathcal{A})$ [49,57]. Compare this with Eq. (6), where we used the complex structure of the associative algebra $\mathcal{A}$ implicitly to define the canonical time orientation $\Psi_{\mathcal{A}}$. In this way, dynamical correspondences can be seen as a justification for the prominence of complex numbers in quantum mechanics [58].[22] By Theorem 2, these arguments are further inherently connected with quantum entanglement.

2. Outlook: Intrinsic dynamics and thermal time

One of the deepest insights into the emergence of time from purely algebraic considerations arises in infinite dimensions and the structure theory of (hyperfinite) von Neumann algebras [59–62]. The latter heavily rests on the foundational insights by Tomita and Takesaki [63,64].

Given a von Neumann algebra $\mathcal{N}$ and a faithful normal state $\omega \in \mathcal{S}(\mathcal{N})$, ω becomes a cyclic and separating vector Ω in its GNS representation. The operator defined by $S_\omega a\Omega := a^*\Omega$ for all $a \in \mathcal{N}$ is closable and hence has a polar decomposition $S_\omega = J_\omega \Delta_\omega^{1/2}$, where J_ω is an antiunitary involution and Δ_ω is a self-adjoint positive operator. The fundamental results of Tomita-Takesaki theory are summarized in the statements $J_\omega \mathcal{N} J_\omega = \mathcal{N}'$, where $\mathcal{N}'$ is the commutant of $\mathcal{N}$, and $\Delta_\omega^{it}\mathcal{N}\Delta_\omega^{-it} = \mathcal{N}$ for all $t \in \mathbb{R}$ [63]. The latter implies that every faithful normal state $\omega \in \mathcal{S}(\mathcal{N})$ defines a one-parameter group of automorphisms $\sigma^\omega : \mathbb{R} \to \mathrm{Aut}(\mathcal{N})$, $\sigma_t^\omega(a) \mapsto \Delta^{it} a \Delta^{-it}$, called the modular automorphism group of ω.

Crucially, S_ω and thus σ^ω are state dependent since they are defined with respect to the support of the state ω. Despite this fact, the difference between σ_t^ω and $\sigma_t^{\omega'}$ is merely an inner automorphism $\sigma_t^\omega(a) = u_t \sigma_t^{\omega'}(a) u_t^{-1}$ for all $a \in \mathcal{N}$, where the unitaries $(u_t)_{t\in\mathbb{R}}$ satisfy Connes's cocycle condition $\sigma_{s+t}^\omega = u_s \sigma_t^\omega(u_t)$ [65]. As a consequence, $\mathcal{N}$ carries an intrinsic, i.e., state-independent, notion of dynamics, given by (the subgroup of) the automorphism group generated by the σ^ω. In contrast, one can also study the operators S_ω in JBW algebras. However, without the existence of a dynamical correspondence (or, equivalently, time orientation), a JBW algebra cannot distinguish between the one-parameter families of automorphisms σ_t^ω and σ_{-t}^ω [66].

Furthermore, the intrinsic dynamics in von Neumann algebras is further exemplified in the study of statistical mechanics in a background-independent setting [67,68]. Here $\omega \in \mathcal{S}(\mathcal{N})$ is understood as a state in thermodynamic equilibrium. In the setting of quantum statistical mechanics such states are characterized by the Kubo-Martin-Schwinger (KMS) condition [69]. It is a remarkable fact that ω satisfies the KMS condition for every faithful normal state $\omega \in \mathcal{S}(\mathcal{N})$ [63]. In contrast, no analog of this condition holds for Jordan algebras [66]. In effect, time orientations in von Neumann algebras allow one to interpret time from a thermodynamical standpoint, encoded in a state of thermodynamic equilibrium [11].

The crucial role played by time orientations (or, equivalently, dynamical correspondences) in von Neumann algebras and in Theorem 2 is hardly coincidental. In particular, it is tempting to "explain" the intrinsic dynamics and (thermodynamic) origin of the arrow of time more fundamentally in terms of the entanglement structure of a given faithful normal state. To this end, one would like to generalize Theorem 2 to the setting of general von Neumann algebras. We leave this and similar considerations for future work.

III. CONCLUSION

We found a necessary and sufficient criterion for bipartite entanglement using Stinespring dilations in Theorem 1. The latter adopts a clear physical meaning in terms of a compatibility condition with respect to time orientations (Definition 1) on the respective local observable algebras in Theorem 2. Moreover, we highlighted the key role time orientations play within the broader picture of the intrinsic flow of time in von Neumann algebras.

More explicitly, our results are motivated by and bear close resemblance to the PPT criterion [7,8]. As such, it would be interesting to study the practical relevance of Theorem 1. For example, it seems possible that existing results on marginal extension problems, e.g., those in Refs. [36,37], can be strengthened using Corollary 1.

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