

Work fluctuations and entanglement in quantum batteries

Satoya Imai , Otfried Gühne , and Stefan Nimmrichter 

Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, D-57068 Siegen, Germany



(Received 3 June 2022; accepted 6 February 2023; published 15 February 2023)

We consider quantum batteries given by composite interacting quantum systems in terms of the thermodynamic work cost of local random unitary processes. We characterize quantum correlations by monitoring the average energy change and its fluctuations in the high-dimensional bipartite systems. We derive a hierarchy of bounds on high-dimensional entanglement (the so-called Schmidt number) from the work fluctuations and thereby show that larger work fluctuations can verify the presence of stronger entanglement in the system. Finally, we develop two-point measurement protocols with noisy detectors that can estimate work fluctuations, showing that the dimensionality of entanglement can be probed in this manner.

DOI: [10.1103/PhysRevA.107.022215](https://doi.org/10.1103/PhysRevA.107.022215)

I. INTRODUCTION

At the heart of quantum thermodynamics [1] lies the fundamental question about the emergence of thermodynamic properties in small quantum systems. Quantum thermodynamics has not only established a common playground for statistical mechanics and quantum information theorists, it is now driving experimental efforts to seek and exploit genuine quantum signatures in thermodynamic processes. In particular, quantum correlations have been investigated in terms of their fundamental energetic footprint [2,3] and work cost [4–9], and as a resource in quantum thermal machines [10,11]. Research on quantum batteries [12] highlights the role of correlations for work extraction [13–15] and storage [16–24] in composite quantum systems. Experimental investigations of quantum batteries are already underway [25,26].

In practice, if work is consumed or generated on the quantum scale, strong fluctuations are often inevitable. Whether they are caused by a lack of experimental control, environmental decoherence, or other unknown sources of noise, the fluctuations are not only detrimental to the performance of thermodynamic tasks, but their precise statistics are often inaccessible. It is a common approach in quantum information theory to circumvent this problem by considering—or even deliberately applying—uniformly random unitary operations on the quantum system [27–37]. This operational “worst-case” procedure will override other noise effects by rotating around an arbitrary Hilbert-space direction, which results in a maximally mixed system state on average. Nevertheless, measurement data from a large sample of random unitaries can reveal genuine quantum features of the system state.

In the context of quantum thermodynamics, random unitaries and random Hamiltonians that generate them have been used to characterize the work distribution in chaotic quantum systems [38–42]. Other studies analyzed the thermodynamics of quantum batteries under random unitary rotations [43,44], random repeated collisions [45–47], or random interaction Hamiltonians [48–50].

Here, we show that one can detect bipartite entanglement in a composite interacting working medium through work fluctuations under *local* random unitaries. We derive a hierarchy of bounds on high-dimensional entanglement in terms of the so-called Schmidt number, and we show that stronger work fluctuations can verify the presence of stronger entanglement. Furthermore, we develop noisy two-point energy measurement protocols based on inefficient detectors that can estimate work fluctuations and thereby probe the Schmidt number.

Consider an interacting bipartite quantum system with dimension $d \times d$ and Hamiltonian $H_{AB} = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B + gV$, prepared in a (possibly entangled) quantum state ϱ_{AB} . Its energy content $E = \text{tr}[\varrho_{AB}H_{AB}]$ has contributions from the local Hamiltonians H_A, H_B and from the interaction term V at coupling strength g . The system shall act as a quantum battery that receives or delivers energy through a local (nonentangling) unitary control operation, which we describe by $\varrho'_{AB} = (U_A \otimes U_B)\varrho_{AB}(U_A^\dagger \otimes U_B^\dagger)$ (see also Fig. 1). We assume a pulsed (or cyclic) operation that leaves the system Hamiltonian unchanged, i.e., $H'_{AB} = H_{AB}$.

II. QUANTUM BATTERY

The associated locally extracted work is quantified by the energy difference:

$$W(U_A, U_B) = E - E' = \text{tr}[(\varrho_{AB} - \varrho'_{AB})H_{AB}]. \quad (1)$$

Most studies on quantum battery (dis-)charging focus on the maximum amount of the extractable work, called ergotropy [51], which has recently been linked to quantum correlations [3,52–54]. In this paper, we will not be concerned with the maximization, but rather with the work statistics over a sample of uniformly random local operations, and relate it to the entanglement between the parts of the battery. We consider the average work and its variance over a sample of unitaries U_A, U_B drawn from the unitary groups $\mathcal{U}(d)$:

$$\bar{W} = \int dU_A \int dU_B W(U_A, U_B), \quad (2)$$

$$(\Delta \bar{W})^2 = \overline{W^2} - \bar{W}^2, \quad (3)$$

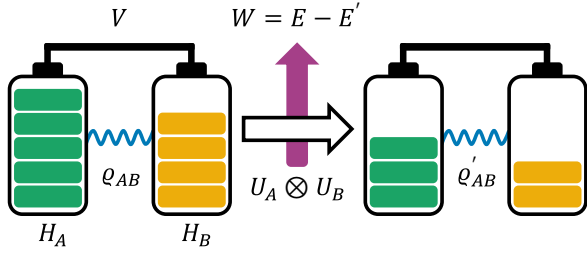


FIG. 1. Sketch of the interacting quantum battery as a composite working medium that can be entangled in a $d \times d$ system. The quantum battery is described by a state ϱ_{AB} and a Hamiltonian $H_{AB} = H_A + H_B + gV$ with coupling strength g . It is transformed by a local random unitary operation $U_A \otimes U_B: \varrho_{AB} \rightarrow \varrho'_{AB} = (U_A \otimes U_B)\varrho_{AB}(U_A^\dagger \otimes U_B^\dagger)$. Then the average extractable work in this process $W(U_A, U_B) = E - E'$ becomes random. The essential thermodynamic quantity to characterize high-dimensional entanglement in this paper is the work variance $(\Delta\bar{W})^2$ over the random unitaries.

where the integrals are taken over the Haar measure (see Appendix A for details), including a commented list of useful identities and known results. We immediately find that $\bar{W} = E - \text{tr}[H_{AB}]/d^2$, since the averaged final battery state is always maximally mixed. On the other hand, we will see that the variance $(\Delta\bar{W})^2$ of work fluctuations can reveal initial quantum correlations in the battery.

III. WORK FLUCTUATIONS

By virtue of the Schur-Weyl duality [55–57], we can carry out the unitary integrals in Eq. (3) and link the work fluctuations to the generalized Bloch decomposition of ϱ_{AB} and H_{AB} . Recall that any $d \times d$ state ϱ_{AB} can be written as

$$\varrho_{AB} = \frac{1}{d^2} \left(\mathbb{1}_{AB} + \sum_{i=1}^{d^2-1} r_i^A \lambda_i \otimes \mathbb{1}_B + \sum_{i=1}^{d^2-1} r_i^B \mathbb{1}_A \otimes \lambda_i + \sum_{i,j=1}^{d^2-1} t_{ij} \lambda_i \otimes \lambda_j \right), \quad (4)$$

with $\lambda_0 = \mathbb{1}_d$ and λ_i the so-called Gell-Mann matrices for $i = 1, \dots, d^2 - 1$ [58–60]. These matrices generalize the Pauli matrices to $SU(d)$, satisfying $\lambda_i^\dagger = \lambda_i$, $\text{tr}[\lambda_i] = 0$, and $\text{tr}[\lambda_i \lambda_j] = d\delta_{ij}$. The coefficient vectors \mathbf{r}^A and \mathbf{r}^B characterize the two reduced battery states, while the matrix (t_{ij}) represents all correlations. Similarly, we can expand the terms of the Hamiltonian as

$$H_X = \sum_{i=0}^{d^2-1} h_i^X \lambda_i, \quad V = \sum_{i,j=1}^{d^2-1} v_{ij} \lambda_i \otimes \lambda_j. \quad (5)$$

This leads to an explicit form for the work fluctuations.

Observation 1. The work variance over local random unitary operations in a $d \times d$ quantum battery described by ϱ_{AB} and H_{AB} can be written in terms of the Bloch representation as

$$(\Delta\bar{W})^2 = \frac{1}{d^2 - 1} \left(r_A^2 h_A^2 + r_B^2 h_B^2 + \frac{t^2 g^2 v^2}{d^2 - 1} \right), \quad (6)$$

where $r_X^2 = |\mathbf{r}^X|^2$, $t^2 = \sum_{i,j} t_{ij}^2$, $h_X^2 = |\mathbf{h}^X|^2$, and $v^2 = \sum_{i,j} v_{ij}^2$, for $X = A, B$.

Proof. First, we can immediately find

$$(\Delta\bar{W})^2 = \overline{(E')^2} - \bar{E}^2. \quad (7)$$

The first term on this right-hand side can be written as

$$\begin{aligned} \overline{(E')^2} &= \int dU_A \int dU_B \{ \text{tr}[\varrho'_{AB} H_{AB}] \}^2 \\ &= \int dU_A \int dU_B \text{tr}[\varrho_{AB}^{\otimes 2} H_{AB}^{\otimes 2}] \\ &= \text{tr} \left[\left(\int dU_A \int dU_B \varrho_{AB}^{\otimes 2} \right) H_{AB}^{\otimes 2} \right] \\ &= \text{tr}[\Phi(\varrho_{AB}) H_{AB}^{\otimes 2}], \end{aligned} \quad (8)$$

where the map $\Phi(\varrho_{AB})$ is given in Appendix A [see Eq. (A9)]. Using the expansion of the Hamiltonian terms in Eq. (5), with help of the properties of Gell-Mann matrices, a long but straightforward calculation leads to the expression (6). ■

Similar quantities have appeared in the notion of sector lengths in quantum information theory [61–65]. Here, the bipartite correlations of the battery state ϱ_{AB} contribute to $(\Delta\bar{W})^2$ via the term t^2 , provided there is a finite coupling $g \neq 0$ between the battery parts. Next, we will characterize the entanglement in ϱ_{AB} based on Eq. (6).

IV. SCHMIDT NUMBER DETECTION

A typical way to describe high-dimensional entanglement in a pure bipartite state $|\psi\rangle$ is to consider its Schmidt decomposition [66], $|\psi\rangle = \sum_{i=1}^r \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle$, with $\langle e_i | e_j \rangle = \langle f_i | f_j \rangle = \delta_{ij}$ and $\sum_{i=1}^r \lambda_i = 1$. The number $r = r(\psi)$ is equal to the rank of $\text{tr}_A[|\psi\rangle\langle\psi|]$, and also called the Schmidt rank, and the state $|\psi\rangle$ is entangled if and only if $r(\psi) > 1$. A high Schmidt rank certifies high-dimensional entanglement of the state, which may imply usefulness for certain information processing tasks [67–70].

The generalization of the Schmidt rank to mixed states ϱ_{AB} is known as the Schmidt number [71]:

$$N_S(\varrho_{AB}) = \inf_{\mathcal{D}(\varrho_{AB})} \max_{\{\psi_i\}} r(\psi_i), \quad (9)$$

where $\mathcal{D}(\varrho_{AB}) = \{p_i, \psi_i : \varrho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|\}$ is the set of all ensemble realizations of ϱ_{AB} . The sets \mathcal{S}_k of all bipartite states with $N_S = k$ form a hierarchy of convex and compact subsets in state space, $\mathcal{S}_k \subset \mathcal{S}_{k+1}$, where \mathcal{S}_1 is the set of separable states. A higher Schmidt number thus indicates stronger entanglement, augmenting the separability problem [72,73]. Several methods to witness the Schmidt number are already known [74–78]. We now formulate a criterion based on work fluctuations, which elucidates the role of entanglement in work exchange processes.

Observation 2. Any $d \times d$ composite quantum battery described by ϱ_{AB} and H_{AB} with $N_S(\varrho_{AB}) = k$ obeys

$$(\Delta\bar{W})^2 \leq \frac{1}{d^2 - 1} \left(r_A^2 h_A^2 + r_B^2 h_B^2 + \frac{g^2 v^2 s_k}{d^2 - 1} \right), \quad (10)$$

with the function $s_k = s(k, d, r_A^2, r_B^2) = kd - 1 + \frac{kd-2}{2}(r_A^2 + r_B^2) - \frac{kd}{2}|r_A^2 - r_B^2|$.

Proof. Let us begin by considering a map given by

$$M_k(X) = \text{tr}[X] \mathbb{1} - \frac{X}{k}, \quad (11)$$

for an operator $X \in \mathcal{H}^d$ and an integer k . Reference [71] showed that, if a two-qudit state ϱ_{AB} has Schmidt number $N_S(\varrho_{AB}) = k$, then $(M_k \otimes \mathbb{1}_B)(\varrho_{AB})$ is positive:

$$(M_k \otimes \mathbb{1}_B)(\varrho_{AB}) = \varrho_A \otimes \mathbb{1}_B - \frac{1}{k} \varrho_{AB} \geq 0, \quad (12)$$

where $\varrho_A = \text{tr}_B[\varrho_{AB}]$. Noting that $\text{tr}[\varrho_{AB}O] \geq 0$ for any positive operator O , and taking $O = (M_k \otimes \mathbb{1}_B)(\varrho_{AB})$, we have

$$\text{tr}[\varrho_{AB}^2] \leq k \text{tr}[\varrho_A^2]. \quad (13)$$

Similarly, we can show that $\text{tr}[\varrho_{AB}^2] \leq k \text{tr}[\varrho_B^2]$. In summary, any $d \times d$ quantum state ϱ_{AB} with Schmidt number k obeys

$$\text{tr}[\varrho_{AB}^2] \leq k \min \{ \text{tr}[\varrho_A^2], \text{tr}[\varrho_B^2] \}. \quad (14)$$

For $k = 1$, this inequality becomes equivalent to the well-known entropic separability criterion [30,79].

Here we note that

$$\text{tr}[\varrho_{AB}^2] = \frac{1}{d^2} (1 + r_A^2 + r_B^2 + t^2). \quad (15)$$

Using Eq. (15) and $\min(a, b) = (a + b - |a - b|)/2$, we can rewrite the above condition as

$$t^2 \leq kd - 1 + \frac{kd - 2}{2} (r_A^2 + r_B^2) - \frac{kd}{2} |r_A^2 - r_B^2|. \quad (16)$$

In the above observation, the right-hand side of (16) is subsumed as $s_k \equiv s(k, d, r_A^2, r_B^2)$. A violation of this inequality implies that the state has a Schmidt number of at least $(k + 1)$. Observation 2 follows by applying the inequality to the work fluctuations $(\Delta \bar{W})^2$ in Eq. (6). We remark that a similar proof technique was employed in Ref. [36]. ■

A violation of Eq. (10) implies that the battery state ϱ_{AB} has a Schmidt number of at least $(k + 1)$. Hence, observing stronger work fluctuations from local random unitaries on a composite quantum battery allows us to detect high-dimensional entanglement.

Note that the converse argument can be also true in the case of pure states. To see this, we begin by noting that the purity constraint $\text{tr}[\varrho_{AB}^2] = 1$ is equivalent to $r_A^2 + r_B^2 = d^2 - 1 - t^2$. For the sake of simplicity, assuming $h_A^2 = h_B^2 = h^2$, we can then express $(\Delta \bar{W})^2$ as

$$(\Delta \bar{W})^2 = h^2 + \frac{Gt^2}{d^2 - 1}, \quad (17)$$

where $G = (g^2 v^2)/(d^2 - 1) - h^2$. Also, we can rewrite the Schmidt number criterion as $t^2 \leq d^2 + 1 - \frac{2d}{k}$. If the interaction is sufficiently strong, that is, $G > 0$, then we get an upper bound on $(\Delta \bar{W})^2$ from the Schmidt number criterion and arrive at the same conclusion as Observation 2. On the other hand, if the interaction is weak, $G < 0$, then a *lower* bound on $(\Delta \bar{W})^2$ is obtained, and hence *weaker* work fluctuations would certify higher entanglement.

We remark that our approach to detect high-dimensional entanglement by observing random fluctuations can be applied not only to energy, but also to other observables measuring bipartite correlations.

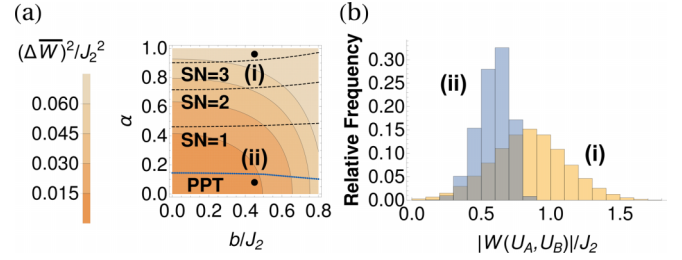


FIG. 2. Schmidt number detection through local work fluctuations in an Ising-type battery of $2 + 2$ qubits. (a) Variance of average work extracted by local random unitaries acting on each battery half as a function of the field strength b and the mixing ratio α between a maximally entangled and a product Gibbs state. All energies are in units of the interaction strength J_2 , and we fix $J_{1,3} = 0.5J_2$ and $T = 1.5J_2$. Quantum states with $N_S = 1, 2, 3$ are contained in the areas below the respective dashed lines, according to Eq. (10), so above a line allows us to detect N_S . For comparison, we also indicate a bottom blue threshold given by the PPT criterion. (b) Exemplary histograms of negative work values from a sample of 10^6 unitaries for the two marked cases (i) and (ii) at $b = 0.45$, corresponding to an entangled state of $N_S = 4$ at $\alpha = 0.96$ and a state at $\alpha = 0.08$, compatible with separable states, respectively. Work values are divided into bins of size $0.1J_2$.

V. EXAMPLE

We shall test our criterion with the family of states

$$\varrho_\alpha = \alpha |\phi\rangle\langle\phi| + (1 - \alpha) \tau_A \otimes \tau_B. \quad (18)$$

They are mixtures between the product of local Gibbs states at temperature T , $\tau_X = \exp(-H_X/T)/Z_X$, and the pure entangled state $|\phi\rangle$ that is locally indistinguishable from the Gibbs states, $\text{tr}_A(|\phi\rangle\langle\phi|) = \tau_B$ and $\text{tr}_B(|\phi\rangle\langle\phi|) = \tau_A$. Note that, in the limit $T \rightarrow \infty$, the Gibbs states are maximally mixed, and hence the ϱ_α are isotropic states.

As a simple example, consider an interacting four-qubit battery based on the Ising-type Hamiltonian

$$H_I = \sum_{i=1,2,3} J_i Z_i \otimes Z_{i+1} + b \sum_{i=1}^4 Z_i, \quad (19)$$

with Z_i being the Pauli-Z matrix acting on the i th qubit, b the homogeneous field strength, and J_i the nearest-neighbor couplings. Assuming the bipartition $(A|B) = (1, 2|3, 4)$, we can identify $h_A^2 = J_1^2 + 2b^2$, $h_B^2 = J_3^2 + 2b^2$, and $g^2 v^2 = J_2^2$. We illustrate the work fluctuations for an exemplary choice of strong-coupling parameters in Fig. 2. Figure 2(a) shows the work variance as a function of (b, α) and the Schmidt-number thresholds for $k = 1, 2, 3$, while Fig. 2(b) shows two selected histograms of suitably binned work values Eq. (1) associated with the Haar-random local unitaries. In practice, these values could be inferred from joint local measurements in the Z basis on sufficiently many identical copies of each unitary sample, and the statistical significance can be evaluated according to Ref. [37]. In the following, however, we will proceed to introduce two different measurement schemes to estimate the work fluctuations.

VI. NOISY TWO-POINT ENERGY MEASUREMENT PROTOCOL

The projective two-point measurement (TPM) protocol [80–82] defines a quantum notion of fluctuating work in analogy to classical stochastic thermodynamics, for trajectories of an arbitrary system state subject to a given isentropic process U . In this protocol, one first performs a projective measurement in the system's energy eigenbasis, lets the postmeasurement state evolve under U , and then performs a second projective energy measurement. The difference between both outcomes can be seen as a random realization of work under U , and the so defined work statistics obey the Jarzynski equality [80–82].

However, the protocol has two major downsides. First, it is highly invasive since the first measurement voids all the coherence between energy levels that the initial system state might have. Genuine quantum signatures such as entanglement between different system parts may thus be destroyed. Second, ideal projective measurements may not be achievable due to limited accuracy and unavoidable noise in experiments. These two problems have motivated recent efforts to generalize the TPM protocol [83–93].

We alleviate both problems by employing a TPM protocol with noisy detectors, first introduced in Ref. [93]. We adapt it to our setting of composite quantum batteries and have A, B each apply the protocol for a *local* energy measurement. To this end, we expand $H_X = \sum_{i=1}^d E_i^X \Pi_i^X$, with the energy eigenvalues E_i and the projectors Π_i^X to the corresponding eigenspaces. Moreover, we write the interaction term as $V = \sum_{i,j=1}^d D_{ij} \Pi_i^A \otimes \Pi_j^B + V_{\text{OD}}$ with $\text{tr}[V_{\text{OD}} \Pi_i^A \otimes \Pi_j^B] = 0$ for all i, j . This separates mere level shifts of the joint diagonal energy spectrum, $E_{ij} = E_i^A + E_j^B + gD_{ij}$, from the actual change of the energy eigenbasis via the off-diagonal part V_{OD} . The following results are based on estimating the E_{ij} spectrum from noisy measurements in the basis of the $\Pi_i^A \otimes \Pi_j^B$. We stress that, for $V_{\text{OD}} \neq 0$, the E_{ij} values are not the battery energies and the measurement does not constitute an actual energy measurement (though it approximates one for small V_{OD}).

The population of the diagonal spectrum (E_{ij}) can be probed straightforwardly by combining the outcomes of local energy measurements. Suppose these measurements are erroneous in that they detect the correct local energy state only with probability ε , while producing a completely random outcome with probability $1 - \varepsilon$. Assuming the same ε for both sides, the corresponding positive operator-valued measures (POVMs) are

$$P_i^X = \varepsilon \Pi_i^X + \frac{1 - \varepsilon}{d} \mathbb{1}_X, \quad \sum_{i=1}^d P_i^X = \mathbb{1}_X. \quad (20)$$

Here we assume that the Π_i^X are rank-1 projectors, so that the entire POVM has d outcomes. On average, we can obtain an unbiased estimator for (E_{ij}) from them by assigning to each joint outcome (ij) occurring with probability $m_{ij} = \text{tr}[P_i^A \otimes P_j^B \varrho_{AB}]$ the rescaled and shifted energy value [93]:

$$e_{ij} = \frac{E_i^A + E_j^B}{\varepsilon} + \frac{gD_{ij}}{\varepsilon^2} - \frac{1 - \varepsilon}{d\varepsilon} (\text{tr}[H_A] + \text{tr}[H_B]). \quad (21)$$

For $\varepsilon = 1$, we have noiseless projective measurements and $e_{ij} = E_{ij}$, whereas small values $\varepsilon \ll 1$ correspond to a weak measurement dominated by errors. Note that the case of different errors $\varepsilon_A, \varepsilon_B$ is also discussed in Appendix B.

We subject the postmeasurement state to local random unitaries $U_A \otimes U_B$,

$$\sigma_{ij} = \frac{U_A \sqrt{P_i^A} \otimes U_B \sqrt{P_j^B} \varrho_{AB} \sqrt{P_i^A} U_A^\dagger \otimes \sqrt{P_j^B} U_B^\dagger}{m_{ij}}, \quad (22)$$

before applying the same local measurement again. The probability to obtain e'_{kl} if the first outcome was e_{ij} is $m_{kl|ij} = \text{tr}[P_k^A \otimes P_l^B \sigma_{ij}]$, to which we associate a presumed work value $w_{ijkl} = e_{ij} - e'_{kl}$. (It may only approximate the extracted work if $V_{\text{OD}} \neq 0$, but small.) Averaged over many repetitions at fixed $U_A \otimes U_B$, we define $W_{\text{TPM}}(\varepsilon) \equiv W_{\text{TPM}}(\varepsilon, U_A, U_B) = \sum_{i,j,k,l} m_{ij} m_{kl|ij} w_{ijkl}$, which in turn can be averaged over a large sample of unitaries to yield

$$\overline{W_{\text{TPM}}(\varepsilon)} = \int dU_A \int dU_B W_{\text{TPM}}(\varepsilon), \quad (23)$$

$$[\Delta \overline{W_{\text{TPM}}(\varepsilon)}]^2 = \overline{W_{\text{TPM}}(\varepsilon)^2} - \overline{W_{\text{TPM}}(\varepsilon)}^2. \quad (24)$$

In general, these TPM cumulants do not coincide with the previously defined ones in Eqs. (2) and (3). However, we can still obtain an explicit relation between the variances.

Observation 3. For any $d \times d$ composite quantum battery described by ϱ_{AB} and H_{AB} , the local noisy TPM protocol results in the presumed work variance:

$$[\Delta \overline{W_{\text{TPM}}(\varepsilon)}]^2 = n_0(\varepsilon) (\Delta \overline{W})_D^2 + n_1(\varepsilon) (\Delta \overline{W}_{\text{Proj}})^2 + [1 - n_0(\varepsilon) - n_1(\varepsilon)] (\Delta \overline{W}_{\text{Noisy}})^2, \quad (25)$$

where the functions $n_{0,1}(\varepsilon) \in [0, 1]$ for any $\varepsilon \in [0, 1]$ are explicitly given. The term $(\Delta \overline{W})_D^2$ is the theoretical work variance in Eq. (6) evaluated for $V_{\text{OD}} = 0$. The $(\Delta \overline{W}_{\text{Proj}})^2$ and $(\Delta \overline{W}_{\text{Noisy}})^2$ represent the variance for a noiseless projective TPM and an additional contribution at finite noise $\varepsilon \in (0, 1)$, respectively, both also at $V_{\text{OD}} = 0$.

See Observation 6 in Appendix B for the proof and the lengthy explicit expressions for $(\Delta \overline{W}_{\text{Proj}})^2$, $(\Delta \overline{W}_{\text{Noisy}})^2$, and $n_{0,1}(\varepsilon)$. There we also show that the noisy TPM variance obeys $[\Delta \overline{W_{\text{TPM}}(\varepsilon)}]^2 \leq (\Delta \overline{W})_D^2$, which saturates in the limit $\varepsilon \rightarrow 0$, where $n_0 \rightarrow 1$ and $n_1 \rightarrow 0$. In the opposite limit $\varepsilon \rightarrow 1$ where $n_1 \rightarrow 1$ and $n_0 \rightarrow 0$, we have a local projective TPM which does not detect any entanglement. We compare the measured work variance at various noise levels to the theoretical values for our example states Eq. (18) in Fig. 3, demonstrating that the noisy local TPM can detect entanglement.

VII. NOISY ENERGY COINCIDENCE MEASUREMENT PROTOCOL

In order to estimate the work variance in Eq. (24), the noisy TPM scheme still relies on subjecting many copies of the battery state to the same randomly drawn local unitary. We can reduce this overhead by performing local coincidence measurements on merely two state copies $\varrho_{AB} \otimes \varrho_{A'B'}$ subjected to the same local unitary $U_A \otimes U_B$. Ideally, a joint dichotomic projective measurement $\Pi_{AA'} \otimes \Pi_{BB'}$ would act locally on

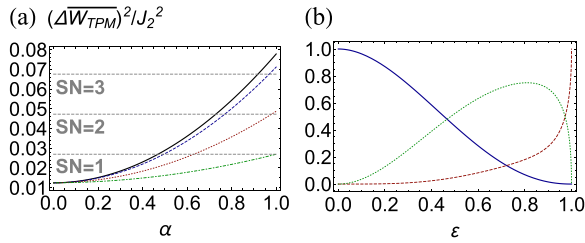


FIG. 3. (a) Comparison between the theoretical work variance $(\Delta \overline{W})_D^2$ (black solid) and the variance $[\Delta \overline{W}_{\text{TPM}}(\epsilon)]^2$ resulting from a local TPM protocol at various noise levels $\epsilon = 0.2, 0.5$, and 1.0 (respectively, dashed blue, dotted red, and dash-dotted green), for the Ising battery of Fig. 2 at fixed $b = 0.45J_2$ and varying mixing ratio α . The dashed horizontal lines show the bounds compatible with Schmidt numbers 1,2,3. (b) Weight functions $n_0(\epsilon)$ (blue solid), $n_1(\epsilon)$ (dashed red), and $1 - n_0(\epsilon) - n_1(\epsilon)$ (dotted green) vs noise level ϵ .

both A copies and on both B copies, with $\Pi_{XX'} = \sum_i \Pi_i^X \otimes \Pi_i^{X'}$ projecting onto the subspace spanned by energy product states with the same eigenvalues $E_i^X = E_i^{X'}$. By repeating this measurement with a large sample of Haar-random unitaries, we could estimate the average probability $\overline{\mathcal{C}}$ that the two copies' local energies on the A and on the B side both coincide.

More generally, we can define a dichotomic energy coincidence POVM based on noisy local energy measurements according to Eq. (20):

$$P_{XX'} = \sum_i P_i^X \otimes P_i^{X'} = \epsilon^2 \Pi_{XX'} + \frac{1 - \epsilon^2}{d} \mathbb{1}_{XX'} \quad (26)$$

The probability for local energy coincidence between the copies on both sides is then $\mathcal{C}(\epsilon) = \text{tr}[P_{AA'} \otimes P_{BB'} \rho'_{AB} \otimes \rho'_{A'B'}]$. Averaged over the unitaries,

$$\overline{\mathcal{C}(\epsilon)} = \frac{1}{d^2} \left[1 + \frac{(r_A^2 + r_B^2)\epsilon^2}{d+1} + \frac{t^2\epsilon^4}{(d+1)^2} \right], \quad (27)$$

which we can directly relate to the entanglement-sensitive work variance $(\Delta \overline{W})^2$ from Eq. (3). Equation (27) can be derived more generally, using different errors ϵ_A, ϵ_B for measurements on the A, B sides.

Proof. First, we can immediately find

$$\overline{\mathcal{C}(\epsilon_A, \epsilon_B)} = \text{tr}[\Phi(Q_{AB}) P_{AA'} \otimes P_{BB'}],$$

where $\Phi(Q_{AB})$ is defined in Eq. (A9). Since

$$\text{tr}[P_{XX'}] = \epsilon_X^2 \text{tr}[\Pi_{XX'}] + \frac{1 - \epsilon_X^2}{d} \text{tr}[\mathbb{1}_{XX'}] = d, \quad (28)$$

$$\begin{aligned} \text{tr}[S_X P_{XX'}] &= \epsilon_X^2 \text{tr}[S_X \Pi_{XX'}] + \frac{1 - \epsilon_X^2}{d} \text{tr}[S_X \mathbb{1}_{XX'}] \\ &= (d-1)\epsilon_X^2 + 1, \end{aligned} \quad (29)$$

we find

$$\overline{\mathcal{C}(\epsilon_A, \epsilon_B)} = \frac{1}{d^2} \left[1 + \frac{r_A^2 \epsilon_A^2}{d+1} + \frac{r_B^2 \epsilon_B^2}{d+1} + \frac{t^2 \epsilon_A^2 \epsilon_B^2}{(d+1)^2} \right].$$

For $\epsilon_A = \epsilon_B = \epsilon$, we arrive at Eq. (27). ■

Expressing the battery interaction strength as $g^2 v^2 = (d-1)(h^2 \epsilon^2 + c)$, with $h^2 = \min(h_A^2, h_B^2)$ and a new term c , we find the following.

Observation 4. In the noisy energy coincidence measurement protocol, we have

$$\overline{\mathcal{C}(\epsilon)} \leq \frac{1}{d^2} \left[1 + \frac{(d-1)\epsilon^2}{h^2} (\Delta \overline{W})^2 + \frac{t^2 \epsilon^2 (|c| - c)}{2(d+1)^2 h^2} \right]. \quad (30)$$

Hence, the energy coincidence measurement protocol on two identical copies gives access to nonlinear functions of the battery state such as the work variance, which allows us to detect the Schmidt number by virtue of Observation 2.

The proven influence of the Schmidt number on work fluctuations exemplifies the observable thermodynamic implications of high-dimensional bipartite entanglement. Our assessment in terms of the work variance with respect to Haar-random samples of unitaries extends previous studies on the direct estimation of nonlinear functions [94–99], experimental lower bounds on the concurrence [100–103], and protocols for randomized measurements [27,31,32].

VIII. CONCLUSION

We have analyzed the role of entanglement in local work exchange with a composite quantum battery, as described by an interacting bipartite quantum system. Specifically, we found that the variance of the average extracted work over a Haar-random sample of local unitary processes obeys a hierarchy of inequalities detecting the Schmidt number of the battery state—a criterion for high-dimensional entanglement. While we saw that these bounds cannot be probed directly in a standard projective two-point measurement scheme, we could show that Schmidt number detection is possible in a two-point measurement with noisy detectors as well as in an energy coincidence measurement.

Our results can be used to probe the influence of entanglement in a quantum working medium and, more generally, elucidate the interplay between entanglement and energy fluctuations in random processes. It would be interesting to verify our results on experimental platforms for quantum thermal machines and batteries. Moreover, the randomized two-point measurement approach could be extended to nonunitary, dissipative processes, facilitating the detection of heat leaks and nonunitary dynamics in complex open quantum systems [104,105].

ACKNOWLEDGMENTS

We would like to thank Roope Uola, Zhen-Peng Xu, and Xiao-Dong Yu for discussions. This work was supported by the Deutsche Forschungsgemeinschaft (Grants No. 447948357 and No. 440958198), the Sino-German Center for Research Promotion (Grant No. M-0294), the European Research Council (Consolidator Grant No. 683107/TempoQ), the German Ministry of Education and Research (Project QuKuK, BMBF Grant No. 16KIS1618K), and the DAAD.

APPENDIX A: USEFUL FORMULAS

Here we summarize useful formulas related to the SWAP operation and to integrals over Haar-random unitaries.

(i) For all operators A and B , $\text{tr}[A \otimes B] = \text{tr}[A]\text{tr}[B]$ and $\text{tr}[A^{\otimes k}] = \text{tr}[A]^k$.

(ii) Let S be the SWAP (flip) operator acting on $d \times d$ -dimensional systems, defined as $S|a\rangle|b\rangle = |b\rangle|a\rangle$. The SWAP operator S can be written as $S = d|\Psi^+\rangle\langle\Psi^+|^T$, where $|\Psi^+\rangle = (1/\sqrt{d})\sum_{i=0}^{d-1}|ii\rangle$ is the maximally entangled state and T_A is the partial transposition on A . That is, $S = \sum_{i,j=0}^{d-1}|ij\rangle\langle ji|$ with ± 1 eigenvalues. Another and useful expression of the SWAP operator is given by

$$S = \frac{1}{d} \sum_{i=0}^{d-1} \lambda_i \otimes \lambda_i. \quad (\text{A1})$$

For the two-qubits Bell state

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{4}(\mathbb{1}_2 \otimes \mathbb{1}_2 + X \otimes X - Y \otimes Y + Z \otimes Z), \quad (\text{A2})$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and with X, Y, Z the Pauli matrices, this can be seen since Pauli Y satisfies $(Y \otimes Y)^T = -Y \otimes Y$. Here we note the important property:

$$\text{tr}[S(A \otimes B)] = \text{tr}[AB]. \quad (\text{A3})$$

(iii) Let dU be the Haar measure, that is, the uniformly random measure on the group of unitary operations $\mathcal{U}(d)$, normalized to $\int dU = 1$. For any integrand function $f(U)$ on $U \in \mathcal{U}(d)$, the Haar measure is both left and right invariant under shifts by any unitary operation $V \in \mathcal{U}(d)$:

$$\int dU f(U) = \int dU f(VU) = \int dU f(UV). \quad (\text{A4})$$

(iv) For an operator $X \in (\mathcal{H}^D)^{\otimes k}$, let us consider

$$\Lambda_k(X) = \int dU U^{\otimes k} X (U^\dagger)^{\otimes k}. \quad (\text{A5})$$

It is known that the unitary integral can be evaluated using the Schur-Weyl duality and the Weingarten calculus (see Refs. [55–57]). In the cases of $k = 1, 2$, they are given by

$$\Lambda_1(X) = \int dU U X U^\dagger = \frac{\text{tr}[X]}{D} \mathbb{1}_D, \quad (\text{A6})$$

$$\begin{aligned} \Lambda_2(X) &= \int dU U^{\otimes 2} X (U^\dagger)^{\otimes 2} \\ &= \frac{1}{D^2 - 1} \left\{ \left[\text{tr}(X) - \frac{\text{tr}(SX)}{D} \right] \mathbb{1}_D^{\otimes 2} \right. \\ &\quad \left. + \left[\text{tr}(SX) - \frac{\text{tr}(X)}{D} \right] S \right\}. \end{aligned} \quad (\text{A7})$$

One lesson from this result is that $\Lambda_2(X)$ realizes a SWAP operation. That is, taking integrals over the Haar unitary for second moments can yield an indirect application of the SWAP operation.

(v) For a two-qudit state ρ_{AB} , let us consider

$$\Phi(\rho_{AB}) = \int dU_A dU_B (U_A^{\otimes 2} \otimes U_B^{\otimes 2}) \rho_{AB}^{\otimes 2} (U_A^\dagger)^{\otimes 2} \otimes (U_B^\dagger)^{\otimes 2}. \quad (\text{A8})$$

Using the generalized Bloch representation of ρ_{AB} and the above formulas, we can obtain

$$\begin{aligned} \Phi(\rho_{AB}) &= \frac{1}{d^4} \left\{ \mathbb{1}_{AB}^{\otimes 2} + \frac{1}{d^2 - 1} \left[r_A^2 (dS_A - \mathbb{1}_A^{\otimes 2}) \otimes \mathbb{1}_B^{\otimes 2} \right. \right. \\ &\quad \left. \left. + r_B^2 \mathbb{1}_A^{\otimes 2} \otimes (dS_B - \mathbb{1}_B^{\otimes 2}) \right. \right. \\ &\quad \left. \left. + \frac{t^2 (dS_A - \mathbb{1}_A^{\otimes 2}) \otimes (dS_B - \mathbb{1}_B^{\otimes 2})}{d^2 - 1} \right] \right\}, \end{aligned} \quad (\text{A9})$$

where S_A and S_B respectively are the SWAP operators acting on the two-copy system of $\rho_{AB}^{\otimes 2}$.

APPENDIX B: NOISY TWO-POINT ENERGY MEASUREMENT PROTOCOL
1. Noisy energy measurement and general observations

We consider noisy local energy measurements on A and B with errors $\varepsilon_A, \varepsilon_B$:

$$P_i^A = \varepsilon_A \Pi_i^A + \frac{1 - \varepsilon_A}{d} \mathbb{1}_A, \quad P_i^B = \varepsilon_B \Pi_i^B + \frac{1 - \varepsilon_B}{d} \mathbb{1}_B. \quad (\text{B1})$$

In the main text, we assumed $\varepsilon_A = \varepsilon_B$. The probability to obtain the local measurement outcomes i, j on ρ_{AB} is given by $m_{ij} = \text{tr}[P_i^A \otimes P_j^B \rho_{AB}]$. Following the notion of quantum instruments [106], the normalized postmeasurement state can be described by

$$\sigma_{ij} = \frac{1}{\text{tr}[\mathcal{J}_{ij}(\rho_{AB})]} \mathcal{J}_{ij}(\rho_{AB}), \quad (\text{B2})$$

where \mathcal{J}_{ij} is a linear completely positive and trace-preserving map satisfying

$$m_{ij} = \text{tr}[\mathcal{J}_{ij}(\rho_{AB})]. \quad (\text{B3})$$

Like most studies on two-point measurement protocols, we employ the so-called von Neumann-Lüders instrument in the main text:

$$\mathcal{J}_{ij}^{\text{vN-L}}(\rho_{AB}) = \sqrt{P_i^A} \otimes \sqrt{P_j^B} \rho_{AB} \sqrt{P_i^A} \otimes \sqrt{P_j^B}. \quad (\text{B4})$$

For the sake of simplicity, let us now define the diagonal Hamiltonian H_D as an effective description:

$$H_D = H_{AB} - gV_{\text{OD}}, \quad (\text{B5})$$

where $V = \sum_{i,j=1}^d D_{ij} \Pi_i^A \otimes \Pi_j^B + V_{\text{OD}}$ and V_{OD} is the off-diagonal part of the interaction Hamiltonian (with vanishing diagonal elements) in the eigenenergy basis of the local Hamiltonian. On the one hand, the Hamiltonian H_D can be decomposed using the corresponding projectors $\Pi_i^A \otimes \Pi_j^B$:

$$H_D = \sum_{i,j} E_{ij} \Pi_i^A \otimes \Pi_j^B, \quad (\text{B6})$$

with the joint diagonal energy spectrum $E_{ij} = E_i^A + E_j^B + gD_{ij}$, given in the main text. On the other hand, the Hamiltonian H_D can also be decomposed into the measurement operators P_i^A, P_j^B ,

$$H_D = \sum_{i,j} e_{ij} P_i^A \otimes P_j^B, \quad (\text{B7})$$

with appropriate energy values e_{ij} assigned to each pair of measurement outcomes (i, j) :

$$e_{ij} = e_i^A + e_j^B + g d_{ij}, \quad (\text{B8})$$

$$e_i^A = \frac{1}{\varepsilon_A} E_i^A - \frac{1 - \varepsilon_A}{d \varepsilon_A} \text{tr}[H_A],$$

$$e_j^B = \frac{1}{\varepsilon_B} E_j^B - \frac{1 - \varepsilon_B}{d \varepsilon_B} \text{tr}[H_B], \quad (\text{B9})$$

$$d_{ij} = \frac{1}{\varepsilon_A \varepsilon_B} D_{ij}.$$

The POVM decomposition of the Hamiltonian is motivated by the research in Ref. [93]. For $\varepsilon_A, \varepsilon_B = 1$, we have noiseless projective measurements and $e_{ij} = E_{ij}$, whereas small values $\varepsilon_A, \varepsilon_B \ll 1$ correspond to a weak measurement dominated by errors.

Similarly with the main text, we define the average work over the noisy TPM protocol for independent errors as

$$W_{\text{TPM}}(\varepsilon_A, \varepsilon_B) \equiv W_{\text{TPM}}(\varepsilon_A, \varepsilon_B, U_A, U_B) = \sum_{i,j,k,l} m_{ij} m_{kl} w_{ijkl}. \quad (\text{B10})$$

Here we recall that $m_{kl|ij} = \text{tr}[P_k^A \otimes P_l^B \sigma'_{ij}]$ is the conditional probability to obtain the outcomes k, l associated to the energy value e'_{kl} in the second measurement, given that we obtained (i, j) and e_{ij} in the first measurement. The second measurement receives the state $\sigma'_{ij} = (U_A \otimes U_B) \sigma_{ij} (U_A \otimes U_B)^\dagger$, which is the state transformed by a local random unitary operation

after the first noisy energy measurement. We associate the presumed work value $w_{ijkl} = e_{ij} - e'_{kl}$ to the outcomes. Taking an average over a large sample of local unitaries yields

$$\overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)} = \int dU_A \int dU_B W_{\text{TPM}}(\varepsilon_A, \varepsilon_B), \quad (\text{B11})$$

$$[\Delta \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}]^2 = \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2} - \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}^2. \quad (\text{B12})$$

In the following, we evaluate and simplify the unitary integrals:

Observation 5. For any $d \times d$ composite quantum battery described by ϱ_{AB} and H_D , the local noisy TPM protocol with ε_A and ε_B for the von Neumann-Lüders instrument results in the average which can be expressed as

$$\overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)} = \text{tr}[\varrho_{AB} H_D] - \frac{\text{tr}[H_D]}{d^2}. \quad (\text{B13})$$

Observation 6. For any $d \times d$ composite quantum battery described by ϱ_{AB} and H_D with $\text{tr}[H_D] = 0$, the local noisy TPM protocol with ε_A and ε_B for the von Neumann-Lüders instrument results in the presumed work variance which can be expressed as

$$[\Delta \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}]^2 = \Upsilon_{\text{Ideal}} + \Upsilon_{\text{Proj}} + \Upsilon_{\text{Noisy}}, \quad (\text{B14})$$

where Υ_{Ideal} , Υ_{Proj} , and Υ_{Noisy} , respectively, represent the effects of the ideal theoretical work variance, the variance from a noiseless projective TPM, and the noisy additional measurements at finite noise. They are given by

$$\Upsilon_{\text{Ideal}} \equiv \kappa_{AB}^2 (\Delta \overline{W})_D^2, \quad (\text{B15})$$

$$\Upsilon_{\text{Proj}} \equiv \frac{1}{d^2 - 1} \left\{ [(f_{\varepsilon_A}^4 f_{\varepsilon_B}^4 + \kappa_A^2)(d p_A^2 - 1) + \kappa_B^2 r_A^2] h_A^2 + [(f_{\varepsilon_A}^4 f_{\varepsilon_B}^4 + \kappa_B^2)(d p_B^2 - 1) + \kappa_A^2 r_B^2] h_B^2 \right. \\ \left. + \frac{g^2 v^2}{d^2 - 1} \left[f_{\varepsilon_A}^4 f_{\varepsilon_B}^4 (d^2 p_{AB}^2 - d p_A^2 - d p_B^2 + 1) + \kappa_A^2 \sum_{a,b,c} t_{ab} t_{cb} \zeta_{ac}^A + \kappa_B^2 \sum_{a,b,c} t_{ab} t_{ac} \zeta_{bc}^B \right] \right\}, \quad (\text{B16})$$

$$\Upsilon_{\text{Noisy}} \equiv \frac{2}{d^2 - 1} \left\{ [\gamma_A (d p_A^2 - 1) + \kappa_B \kappa_{AB} r_A^2] h_A^2 + [\gamma_B (d p_B^2 - 1) + \kappa_A \kappa_{AB} r_B^2] h_B^2 \right. \\ \left. + \frac{g^2 v^2}{d^2 - 1} [\gamma_{AB} (d^2 p_{AB}^2 - d p_A^2 - d p_B^2 + 1)] + \kappa_A \kappa_{AB} \sum_{a,b,c} t_{ab} t_{cb} \zeta_{ac}^A + \kappa_B \kappa_{AB} \sum_{a,b,c} t_{ab} t_{ac} \zeta_{bc}^B \right\}. \quad (\text{B17})$$

Here, $(\Delta \overline{W})_D^2$ is the ideal theoretical work variance, Eq. (6) in the main text, evaluated for the diagonal Hamiltonian H_D [i.e., for $V_{\text{OD}} = 0$ and $v^2 = (1/d^2) \sum_{i,j} D_{ij}^2$]. In the above expressions, we introduce the short-hand notations

$$\gamma_A \equiv f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 (\kappa_A + \kappa_B + \kappa_{AB}) + \kappa_A (\kappa_B + \kappa_{AB}), \quad (\text{B18})$$

$$\gamma_B \equiv f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 (\kappa_A + \kappa_B + \kappa_{AB}) + \kappa_B (\kappa_A + \kappa_{AB}), \quad (\text{B19})$$

$$\gamma_{AB} \equiv f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 (\kappa_A + \kappa_B + \kappa_{AB}) + \kappa_A \kappa_B, \quad (\text{B20})$$

$$\kappa_{AB} \equiv \kappa_A \kappa_B / (f_{\varepsilon_A}^2 f_{\varepsilon_B}^2), \quad \kappa_A \equiv f_{\varepsilon_A}^2 g_{\varepsilon_B} (2f_{\varepsilon_B} + d g_{\varepsilon_B}),$$

$$\kappa_B \equiv f_{\varepsilon_B}^2 g_{\varepsilon_A} (2f_{\varepsilon_A} + d g_{\varepsilon_A}), \quad (\text{B21})$$

$$f_{\varepsilon_X} \equiv \sqrt{\varepsilon_X + \frac{1 - \varepsilon_X}{d}} - \sqrt{\frac{1 - \varepsilon_X}{d}}, \quad (\text{B22})$$

$$g_{\varepsilon_X} \equiv \sqrt{\frac{1 - \varepsilon_X}{d}}, \quad (\text{B23})$$

$$p_{AB}^2 \equiv \sum_{i,j} (p_{ij}^{AB})^2, \quad p_A^2 \equiv \sum_i (p_i^A)^2, \quad p_B^2 \equiv \sum_j (p_j^B)^2, \quad (\text{B24})$$

$$p_{ij}^{AB} \equiv \text{tr}[\Pi_i^A \otimes \Pi_j^B \varrho_{AB}], \quad p_i^A \equiv \sum_j p_{ij}^{AB}, \quad p_j^B \equiv \sum_i p_{ij}^{AB}, \quad (\text{B25})$$

$$\zeta_{ab}^A \equiv \sum_i \frac{\text{tr}(\Pi_i^A \lambda_a \Pi_i^A \lambda_b)}{d}, \quad \zeta_{ab}^B \equiv \sum_i \frac{\text{tr}(\Pi_i^B \lambda_a \Pi_i^B \lambda_b)}{d}, \quad (\text{B26})$$

with the normalization condition

$$f_{\varepsilon_X}^2 + 2f_{\varepsilon_X}g_{\varepsilon_X} + dg_{\varepsilon_X}^2 = 1, \quad (\text{B27})$$

for $X = A, B$. Let us define

$$n_0(\varepsilon_A, \varepsilon_B) \equiv \kappa_{AB}^2, \quad (\text{B28})$$

$$n_1(\varepsilon_A, \varepsilon_B) \equiv f_{\varepsilon_A}^4 f_{\varepsilon_B}^4 + \kappa_A^2 + \kappa_B^2, \quad (\text{B29})$$

$$n_{\text{Noisy}}(\varepsilon_A, \varepsilon_B) \equiv 2[f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 (\kappa_A + \kappa_B + \kappa_{AB}) + \kappa_A \kappa_B + \kappa_A \kappa_{AB} + \kappa_B \kappa_{AB}], \quad (\text{B30})$$

where $n_0(\varepsilon_A, \varepsilon_B)$, $n_1(\varepsilon_A, \varepsilon_B)$, and $n_{\text{Noisy}}(\varepsilon_A, \varepsilon_B)$ are explicitly known functions obeying

$$0 \leq n_0(\varepsilon_A, \varepsilon_B), n_1(\varepsilon_A, \varepsilon_B), n_{\text{Noisy}}(\varepsilon_A, \varepsilon_B) \leq 1, \quad (\text{B31})$$

$$n_0(\varepsilon_A, \varepsilon_B) + n_1(\varepsilon_A, \varepsilon_B) + n_{\text{Noisy}}(\varepsilon_A, \varepsilon_B) = 1. \quad (\text{B32})$$

Then we also have

$$\begin{aligned} [\Delta \overline{W}_{\text{TPM}}(\varepsilon_A, \varepsilon_B)]^2 & \equiv n_0(\varepsilon_A, \varepsilon_B) (\Delta \overline{W})_D^2 + n_1(\varepsilon_A, \varepsilon_B) (\Delta \overline{W}_{\text{Proj}})^2 \\ & + [1 - n_0(\varepsilon_A, \varepsilon_B) - n_1(\varepsilon_A, \varepsilon_B)] (\Delta \overline{W}_{\text{Noisy}})^2, \end{aligned} \quad (\text{B33})$$

where

$$(\Delta \overline{W}_{\text{Proj}})^2 \equiv \frac{1}{n_0(\varepsilon_A, \varepsilon_B)} \Upsilon_{\text{Proj}}, \quad (\text{B34})$$

$$(\Delta \overline{W}_{\text{Noisy}})^2 \equiv \frac{1}{1 - n_0(\varepsilon_A, \varepsilon_B) - n_1(\varepsilon_A, \varepsilon_B)} \Upsilon_{\text{Noisy}}. \quad (\text{B35})$$

Remark. In the case of symmetric errors, $\varepsilon_A = \varepsilon_B = \varepsilon$, we arrive at Observation 3 in the main text.

Remark. For any $\varepsilon_A, \varepsilon_B$ and any dimension d , we find the inequality

$$[\Delta \overline{W}_{\text{TPM}}(\varepsilon_A, \varepsilon_B)]^2 \leq (\Delta \overline{W})_D^2, \quad (\text{B36})$$

which is saturated by the limit $\varepsilon_A, \varepsilon_B \rightarrow 0$.

To see this, we first show that

$$\begin{aligned} p_A^2 & = \sum_i (p_i^A)^2 = \sum_i \text{tr}[\Pi_i^A \varrho_A]^2 = \sum_i \text{tr}[\Pi_i^A \varrho_A \Pi_i^A \varrho_A] \\ & \leq \sum_i \text{tr}[\Pi_i^A \varrho_A^2] = \text{tr}[\varrho_A^2], \end{aligned} \quad (\text{B37})$$

where we employ that $\text{tr}[ABAB] \leq \text{tr}[A^2B^2]$, for any Hermitian operators A, B . This result directly yields $dp_A^2 - 1 \leq r_A^2$. Similarly we can have that $dp_B^2 - 1 \leq r_B^2$ and $d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1 \leq t^2$. Also, we find

$$\begin{aligned} \sum_{a,b,c} t_{ab} t_{cb} \zeta_{ac}^A & = \frac{1}{d^2} \sum_{a,b,c,d} \sum_i t_{ab} t_{cd} \text{tr}(\Pi_i^A \lambda_a \Pi_i^A \lambda_c) \text{tr}(\lambda_b \lambda_d) \\ & = \frac{1}{d^2} \sum_{a,b,c,d} \sum_i t_{ab} t_{cd} \text{tr}[(\Pi_i^A \otimes \mathbb{1}_B)(\lambda_a \otimes \lambda_b) \\ & \quad \times (\Pi_i^A \otimes \mathbb{1}_B)(\lambda_c \otimes \lambda_d)] \\ & = \frac{1}{d^2} \sum_i \text{tr}[(\Pi_i^A \otimes \mathbb{1}_B) T_2 (\Pi_i^A \otimes \mathbb{1}_B) T_2] \\ & \leq \frac{1}{d^2} \sum_i \text{tr}[(\Pi_i^A \otimes \mathbb{1}_B) T_2^2] = \frac{1}{d^2} \text{tr}[T_2^2] = t^2, \end{aligned} \quad (\text{B38})$$

where we employ that $\text{tr}[ABAB] \leq \text{tr}[A^2B^2]$, for any Hermitian operators A, B . Similarly, we have that $\sum_{a,b,c} t_{ab} t_{ac} \zeta_{bc}^B \leq t^2$. Substituting these results into the expression $[\Delta \overline{W}_{\text{TPM}}(\varepsilon_A, \varepsilon_B)]^2$ given in Observation 6 and using the condition $f_{\varepsilon_X}^2 + 2f_{\varepsilon_X}g_{\varepsilon_X} + dg_{\varepsilon_X}^2 = 1$ for $X = A, B$, we can straightforwardly complete the proof.

2. Proof of Observation 5

We begin by writing the TPM work average for a fixed unitary as

$$\begin{aligned} W_{\text{TPM}}(\varepsilon_A, \varepsilon_B) & = \sum_{i,j,k,l} m_{ij} m_{kl} |ij\rangle w_{ijkl} \\ & = \sum_{i,j} m_{ij} e_{ij} - \sum_{i,j,k,l} m_{ij} m_{kl} |ij\rangle e'_{kl} \\ & = \sum_{i,j} \text{tr}[P_i^A \otimes P_j^B \varrho_{AB}] e_{ij} \\ & \quad - \sum_{i,j,k,l} m_{ij} \text{tr}[P_k^A \otimes P_l^B \sigma'_{ij}] e'_{kl} \\ & = \text{tr}[\varrho_{AB} H_D] - \sum_{i,j} \text{tr}[(U_A \otimes U_B) \sqrt{P_i^A} \\ & \quad \otimes \sqrt{P_j^B} \varrho_{AB} \sqrt{P_i^A} \otimes \sqrt{P_j^B} (U_A \otimes U_B)^\dagger H_D], \end{aligned} \quad (\text{B39})$$

by virtue of Eq. (B3). In order to derive the unitary average $\overline{W}_{\text{TPM}}(\varepsilon_A, \varepsilon_B)$, we first note that

$$\sqrt{P_i^A} = f_{\varepsilon_A} \Pi_i^A + g_{\varepsilon_A} \mathbb{1}_A, \quad \sqrt{P_i^B} = f_{\varepsilon_B} \Pi_i^B + g_{\varepsilon_B} \mathbb{1}_B, \quad (\text{B40})$$

with $\mathbb{1}_X = \Pi_i^X + \sum_{j \neq i} \Pi_j^X$. We then define

$$f_{\varepsilon_X} \equiv \sqrt{\varepsilon_X + \frac{1 - \varepsilon_X}{d}} - \sqrt{\frac{1 - \varepsilon_X}{d}}, \quad g_{\varepsilon_X} \equiv \sqrt{\frac{1 - \varepsilon_X}{d}}, \quad (\text{B41})$$

with the normalization condition

$$f_{\varepsilon_X}^2 + 2f_{\varepsilon_X}g_{\varepsilon_X} + dg_{\varepsilon_X}^2 = 1. \quad (\text{B42})$$

Abbreviating $\Pi_{ij}^{AB} \equiv \Pi_i^A \otimes \Pi_j^B$, a straightforward calculation leads to

$$\begin{aligned} \sum_{i,j} \sqrt{P_i^A} \otimes \sqrt{P_j^B} \varrho_{AB} \sqrt{P_i^A} \otimes \sqrt{P_j^B} \\ = f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \xi_{AB} + \kappa_A \xi_A + \kappa_B \xi_B + \kappa_{AB} \varrho_{AB}, \end{aligned} \quad (\text{B43})$$

where we define

$$\xi_{AB} \equiv \sum_{i,j} \Pi_{ij}^{AB} \varrho_{AB} \Pi_{ij}^{AB}, \quad \xi_A \equiv \sum_i \Pi_i^A \otimes \mathbb{1}_B \varrho_{AB} \Pi_i^A \otimes \mathbb{1}_B,$$

$$\xi_B \equiv \sum_j \mathbb{1}_A \otimes \Pi_j^B \varrho_{AB} \mathbb{1}_A \otimes \Pi_j^B, \quad (\text{B44})$$

$$\kappa_{AB} \equiv \kappa_A \kappa_B / (f_{\varepsilon_A}^2 f_{\varepsilon_B}^2), \quad \kappa_A \equiv f_{\varepsilon_A}^2 g_{\varepsilon_B} (2f_{\varepsilon_B} + dg_{\varepsilon_B}),$$

$$\kappa_B \equiv f_{\varepsilon_B}^2 g_{\varepsilon_A} (2f_{\varepsilon_A} + dg_{\varepsilon_A}). \quad (\text{B45})$$

From this follows

$$\overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)} = \text{tr}[\varrho_{AB} H_D] - \int dU_A \int dU_B \text{tr} \times [(f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \xi'_{AB} + \kappa_A \xi'_A + \kappa_B \xi'_B + \kappa_{AB} \varrho'_{AB}) H_D], \quad (\text{B46})$$

$$\overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2} = \text{tr}[\varrho_{AB} H_D]^2. \quad (\text{B48})$$

For the first term $\overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2}$, let us consider the expansion

$$\begin{aligned} \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2} &= \int dU_A \int dU_B \{ \text{tr}[\varrho_{AB} H_D] - \text{tr}[(f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \xi'_{AB} + \kappa_A \xi'_A + \kappa_B \xi'_B + \kappa_{AB} \varrho'_{AB}) H_D] \}^2 \\ &= \text{tr}[\varrho_{AB} H_D]^2 + \int dU_A \int dU_B \{ \text{tr}[(f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \xi'_{AB} + \kappa_A \xi'_A + \kappa_B \xi'_B + \kappa_{AB} \varrho'_{AB}) H_D] \}^2 \\ &\quad - 2 \text{tr}[\varrho_{AB} H_D] \int dU_A \int dU_B \{ \text{tr}[(f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \xi'_{AB} + \kappa_A \xi'_A + \kappa_B \xi'_B + \kappa_{AB} \varrho'_{AB}) H_D] \}. \end{aligned} \quad (\text{B49})$$

By virtue of Eq. (B47) and the assumption $\text{tr}[H_D] = 0$, the third line vanishes. Expanding the second term in the second line, we identify ten types of unitary integrals:

$$\Xi_{\xi_{AB}} = f_{\varepsilon_A}^4 f_{\varepsilon_B}^4 \int dU_A \int dU_B \text{tr}[\xi'_{AB} H_D]^2, \quad (\text{B50})$$

$$\Xi_{\xi_A} = \kappa_A^2 \int dU_A \int dU_B \text{tr}[\xi'_A H_D]^2, \quad (\text{B51})$$

$$\Xi_{\xi_B} = \kappa_B^2 \int dU_A \int dU_B \text{tr}[\xi'_B H_D]^2, \quad (\text{B52})$$

$$\Xi_{\varrho_{AB}} = \kappa_{AB}^2 \int dU_A \int dU_B \text{tr}[\varrho'_{AB} H_D]^2, \quad (\text{B53})$$

$$\Xi_{c_1} = f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_A \int dU_A \int dU_B \text{tr}[\xi'_{AB} H_D] \text{tr}[\xi'_A H_D], \quad (\text{B54})$$

$$\Xi_{c_2} = f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_B \int dU_A \int dU_B \text{tr}[\xi'_{AB} H_D] \text{tr}[\xi'_B H_D], \quad (\text{B55})$$

$$\Xi_{c_3} = \kappa_A \kappa_B \int dU_A \int dU_B \text{tr}[\xi'_A H_D] \text{tr}[\xi'_B H_D], \quad (\text{B56})$$

$$\Xi_{c_4} = f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_{AB} \int dU_A \int dU_B \text{tr}[\xi'_{AB} H_D] \text{tr}[\varrho'_{AB} H_D], \quad (\text{B57})$$

$$\Xi_{c_5} = \kappa_A \kappa_{AB} \int dU_A \int dU_B \text{tr}[\xi'_A H_D] \text{tr}[\varrho'_{AB} H_D], \quad (\text{B58})$$

$$\Xi_{c_6} = \kappa_B \kappa_{AB} \int dU_A \int dU_B \text{tr}[\xi'_B H_D] \text{tr}[\varrho'_{AB} H_D]. \quad (\text{B59})$$

where $\chi' = (U_A \otimes U_B) \chi (U_A^\dagger \otimes U_B^\dagger)$ for any $\chi = \xi_{AB}, \xi_A, \xi_B, \varrho_{AB}$. With help of Eq. (A6) in Appendix A, we straightforwardly arrive at

$$\int dU_A \int dU_B \text{tr}[\chi' H_D] = \frac{\text{tr}[H_D]}{d^2}, \quad (\text{B47})$$

provided that $\text{tr}[\chi'] = 1$. Finally, by applying the normalization condition in Eq. (B42), the proof of Observation 5 is completed.

3. Proof of Observation 6

We begin by recalling that $[\Delta \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}]^2 = \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2} - \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}^2$. Based on the assumption $\text{tr}[H_D] = 0$ and the result of Observation 5, the second term simplifies to

Hence we have

$$\begin{aligned} [\Delta \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}]^2 &= \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)^2} - \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}^2 \\ &= \Xi_{\xi_{AB}} + \Xi_{\xi_A} + \Xi_{\xi_B} + \Xi_{\varrho_{AB}} + 2 \sum_{i=1}^6 \Xi_{c_i}. \end{aligned} \quad (\text{B60})$$

We notice that the fourth term $\Xi_{\varrho_{AB}} / \kappa_{AB}^2$ is equal to the theoretical work variance $(\Delta \overline{W})_D^2$ on the diagonal Hamiltonian H_D in Observation 1 in the main text. The first three terms, $\Xi_{\xi_{AB}}, \Xi_{\xi_A}, \Xi_{\xi_B}$, can be attributed to the noiseless local TPM, and their sum corresponds to the variance $(\Delta \overline{W}_{\text{Proj}})^2$ in Observation 3 in the main text. Finally, all the cross terms Ξ_{c_i} for $i = 1, 6$ vanish in the limits $\varepsilon_A, \varepsilon_B \rightarrow 0, 1$; they constitute the additional noise contribution $(\Delta \overline{W}_{\text{Noisy}})^2$ in Observation 3.

In order to find the explicit form of $[\Delta \overline{W_{\text{TPM}}(\varepsilon_A, \varepsilon_B)}]^2$, we must evaluate all these terms. We begin by recalling the generalized Bloch representation of ϱ_{AB} :

$$\varrho_{AB} = \frac{1}{d^2} (\mathbb{1}_{AB} + R_1^A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes R_1^B + T_2), \quad (\text{B61})$$

introducing the traceless Hermitian operators

$$R_1^A = \sum_{i=1}^{d^2-1} r_i^A \lambda_i, \quad R_1^B = \sum_{i=1}^{d^2-1} r_i^B \lambda_i, \quad T_2 = \sum_{i,j=1}^{d^2-1} t_{ij} \lambda_i \otimes \lambda_j. \quad (\text{B62})$$

For these expressions, we define the quantities

$$\begin{aligned} r_A^2 &= \frac{1}{d} \text{tr}[(R_1^A)^2] = \sum_{i=1}^{d^2-1} (r_i^A)^2, \\ r_B^2 &= \frac{1}{d} \text{tr}[(R_1^B)^2] = \sum_{i=1}^{d^2-1} (r_i^B)^2, \\ t^2 &= \frac{1}{d^2} \text{tr}[T_2^2] = \sum_{i,j=1}^{d^2-1} t_{ij}^2, \end{aligned} \quad (\text{B63})$$

which capture the magnitude of the one- and two-body quantum correlations of ϱ_{AB} . With these expressions, we rewrite

the state ξ_{AB} in Eq. (B44) as

$$\xi_{AB} = \sum_{i,j} \Pi_{ij}^{AB} \varrho_{AB} \Pi_{ij}^{AB} = \sum_{i,j} p_{ij}^{AB} \Pi_{ij}^{AB}, \quad (\text{B64})$$

where $p_{ij}^{AB} \equiv \text{tr}[\Pi_{ij}^{AB} \varrho_{AB}] = (1/d^2)\{1 + \text{tr}[\Pi_i^A R_1^A] + \text{tr}[\Pi_j^B R_1^B] + \text{tr}[\Pi_{ij}^{AB} T_2]\}$. Also, we define $p_i^A \equiv \sum_j p_{ij}^{AB}$ and $p_j^B \equiv \sum_i p_{ij}^{AB}$.

Letting $p_A^2 = \sum_i (p_i^A)^2$, $p_B^2 = \sum_j (p_j^B)^2$, $p_{AB}^2 = \sum_{i,j} (p_{ij}^{AB})^2$, and employing the formulas (A6) and (A7), a long calculation leaves us with

$$\Xi_{\xi_{AB}} = \frac{f_{\varepsilon_A}^4 f_{\varepsilon_B}^4}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{(d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1)g^2 v^2}{d^2 - 1} \right], \quad (\text{B65})$$

$$\Xi_{\xi_A} = \frac{\kappa_A^2}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + r_B^2 h_B^2 + \frac{g^2 v^2}{d^2 - 1} \sum_{a,b,c} t_{ab} t_{cb} \zeta_{ac}^A \right], \quad (\text{B66})$$

$$\Xi_{\xi_B} = \frac{\kappa_B^2}{d^2 - 1} \left[r_A^2 h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{g^2 v^2}{d^2 - 1} \sum_{a,b,c} t_{ab} t_{ac} \zeta_{bc}^B \right], \quad (\text{B67})$$

$$\Xi_{c_1} = \frac{f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_A}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{(d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1)g^2 v^2}{d^2 - 1} \right], \quad (\text{B68})$$

$$\Xi_{c_2} = \frac{f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_B}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{(d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1)g^2 v^2}{d^2 - 1} \right], \quad (\text{B69})$$

$$\Xi_{c_3} = \frac{\kappa_A \kappa_B}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{(d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1)g^2 v^2}{d^2 - 1} \right], \quad (\text{B70})$$

$$\Xi_{c_4} = \frac{f_{\varepsilon_A}^2 f_{\varepsilon_B}^2 \kappa_{AB}}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{(d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1)g^2 v^2}{d^2 - 1} \right], \quad (\text{B71})$$

$$\Xi_{c_5} = \frac{\kappa_A \kappa_{AB}}{d^2 - 1} \left[(dp_A^2 - 1)h_A^2 + r_B^2 h_B^2 + \frac{g^2 v^2}{d^2 - 1} \sum_{a,b,c} t_{ab} t_{cb} \zeta_{ac}^A \right], \quad (\text{B72})$$

$$\Xi_{c_6} = \frac{\kappa_B \kappa_{AB}}{d^2 - 1} \left[r_A^2 h_A^2 + (dp_B^2 - 1)h_B^2 + \frac{g^2 v^2}{d^2 - 1} \sum_{a,b,c} t_{ab} t_{ac} \zeta_{bc}^B \right]. \quad (\text{B73})$$

Here we introduced

$$\zeta_{ab}^A \equiv \sum_i \frac{\text{tr}(\Pi_i^A \lambda_a \Pi_i^A \lambda_b)}{d}, \quad \zeta_{ab}^B \equiv \sum_i \frac{\text{tr}(\Pi_i^B \lambda_a \Pi_i^B \lambda_b)}{d}. \quad (\text{B74})$$

Summarizing these terms, we can complete the proof of Observation 6. Here, it might be useful for some readers to note that

$$\begin{aligned} \sum_{a,b,c,d} t_{ab} t_{cd} \zeta_{ac}^A \zeta_{bd}^B &= \frac{1}{d^2} \sum_{i,j} \text{tr}[\Pi_{ij}^{AB} T_2 \Pi_{ij}^{AB} T_2] = \frac{1}{d^2} \sum_{i,j} \{\text{tr}[\Pi_{ij}^{AB} T_2]\}^2 \\ &= \frac{1}{d^2} \sum_{i,j} (d^2 p_{ij}^{AB} - dp_i^A - dp_j^B + 1)^2 = d^2 p_{AB}^2 - dp_A^2 - dp_B^2 + 1, \end{aligned} \quad (\text{B75})$$

where we use that $\text{tr}[\Pi_{ij}^{AB} T_2] = d^2 p_{ij}^{AB} - dp_i^A - dp_j^B + 1$.

- [1] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, *J. Phys. A: Math. Theor.* **49**, 143001 (2016).
- [2] A. Mukherjee, A. Roy, S. S. Bhattacharya, and M. Banik, *Phys. Rev. E* **93**, 052140 (2016).
- [3] M. Alimuddin, T. Guha, and P. Parashar, *Phys. Rev. A* **99**, 052320 (2019).
- [4] D. E. Bruschi, M. Perarnau-Llobet, N. Friis, K. V. Hovhannisyán, and M. Huber, *Phys. Rev. E* **91**, 032118 (2015).
- [5] M. Perarnau-Llobet, K. V. Hovhannisyán, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, *Phys. Rev. X* **5**, 041011 (2015).
- [6] M. Huber, M. Perarnau-Llobet, K. V. Hovhannisyán, P. Skrzypczyk, C. Klöckl, N. Brunner, and A. Acín, *New J. Phys.* **17**, 065008 (2015).
- [7] N. Friis, M. Huber, and M. Perarnau-Llobet, *Phys. Rev. E* **93**, 042135 (2016).
- [8] M. Brunelli, M. G. Genoni, M. Barbieri, and M. Paternostro, *Phys. Rev. A* **96**, 062311 (2017).
- [9] E. McKay, N. A. Rodríguez-Briones, and E. Martín-Martínez, *Phys. Rev. E* **98**, 032132 (2018).
- [10] N. Brunner, M. Huber, N. Linden, S. Popescu, R. Silva, and P. Skrzypczyk, *Phys. Rev. E* **89**, 032115 (2014).
- [11] J. Bohr Brask, G. Haack, N. Brunner, and M. Huber, *New J. Phys.* **17**, 113029 (2015).
- [12] F. Campaioli, F. A. Pollock, and S. Vinjanampathy, *Thermodynamics in the Quantum Regime, Fundamental Aspects and New Directions* (Springer, New York, 2018).
- [13] R. Alicki and M. Fannes, *Phys. Rev. E* **87**, 042123 (2013).
- [14] K. V. Hovhannisyán, M. Perarnau-Llobet, M. Huber, and A. Acín, *Phys. Rev. Lett.* **111**, 240401 (2013).
- [15] R. Salvia, G. De Palma, and V. Giovannetti, *Phys. Rev. A* **107**, 012405 (2023).
- [16] F. C. Binder, S. Vinjanampathy, K. Modi, and J. Goold, *New J. Phys.* **17**, 075015 (2015).
- [17] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, *Phys. Rev. Lett.* **118**, 150601 (2017).
- [18] N. Friis and M. Huber, *Quantum* **2**, 61 (2018).
- [19] T. P. Le, J. Levinsen, K. Modi, M. M. Parish, and F. A. Pollock, *Phys. Rev. A* **97**, 022106 (2018).
- [20] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, *Phys. Rev. Lett.* **120**, 117702 (2018).
- [21] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, *Phys. Rev. Lett.* **122**, 047702 (2019).
- [22] S. Julià-Farré, T. Salamon, A. Riera, M. N. Bera, and M. Lewenstein, *Phys. Rev. Res.* **2**, 023113 (2020).
- [23] J. Q. Quach and W. J. Munro, *Phys. Rev. Appl.* **14**, 024092 (2020).
- [24] J.-Y. Gyhm, D. Šafránek, and D. Rosa, *Phys. Rev. Lett.* **128**, 140501 (2022).
- [25] J. Q. Quach, K. E. McGhee, L. Ganzer, D. M. Rouse, B. W. Lovett, E. M. Gauger, J. Keeling, G. Cerullo, D. G. Lidzey, and T. Virgili, *Sci. Adv.* **8**, eabk3160 (2022).
- [26] C.-K. Hu, J. Qiu, P. J. P. Souza, J. Yuan, Y. Zhou, L. Zhang, J. Chu, X. Pan, L. Hu, J. Li, Y. Xu, Y. Zhong, S. Liu, F. Yan, D. Tan, R. Bachelard, C. J. Villas-Boas, A. C. Santos, and D. Yu, *Quantum Sci. Technol.* **7**, 045018 (2022).
- [27] S. J. van Enk and C. W. J. Beenakker, *Phys. Rev. Lett.* **108**, 110503 (2012).
- [28] M. C. Tran, B. Dakić, F. Arnault, W. Laskowski, and T. Paterek, *Phys. Rev. A* **92**, 050301(R) (2015).
- [29] M. C. Tran, B. Dakić, W. Laskowski, and T. Paterek, *Phys. Rev. A* **94**, 042302 (2016).
- [30] A. Elben, B. Vermersch, M. Dalmonte, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **120**, 050406 (2018).
- [31] A. Elben, B. Vermersch, C. F. Roos, and P. Zoller, *Phys. Rev. A* **99**, 052323 (2019).
- [32] A. Elben, B. Vermersch, R. van Bijnen, C. Kokail, T. Brydges, C. Maier, M. K. Joshi, R. Blatt, C. F. Roos, and P. Zoller, *Phys. Rev. Lett.* **124**, 010504 (2020).
- [33] T. Brydges, A. Elben, P. Jurcevic, B. Vermersch, C. Maier, B. P. Lanyon, P. Zoller, R. Blatt, and C. F. Roos, *Science* **364**, 260 (2019).
- [34] A. Ketterer, N. Wyderka, and O. Gühne, *Phys. Rev. Lett.* **122**, 120505 (2019).
- [35] A. Ketterer, N. Wyderka, and O. Gühne, *Quantum* **4**, 325 (2020).
- [36] S. Imai, N. Wyderka, A. Ketterer, and O. Gühne, *Phys. Rev. Lett.* **126**, 150501 (2021).
- [37] A. Ketterer, S. Imai, N. Wyderka, and O. Gühne, *Phys. Rev. A* **106**, L010402 (2022).
- [38] I. García-Mata, A. J. Roncaglia, and D. A. Wisniacki, *Phys. Rev. E* **95**, 050102(R) (2017).
- [39] M. Łobejko, J. Łuczka, and P. Talkner, *Phys. Rev. E* **95**, 052137 (2017).
- [40] A. Chenu, J. Molina-Vilaplana, and A. del Campo, *Sci. Rep.* **8**, 12634 (2018).
- [41] A. Chenu, J. Molina-Vilaplana, and A. del Campo, *Quantum* **3**, 127 (2019).
- [42] R. Salvia and V. Giovannetti, *Quantum* **5**, 514 (2021).
- [43] F. Caravelli, G. Coulter-De Wit, L. P. García-Pintos, and A. Hamma, *Phys. Rev. Res.* **2**, 023095 (2020).
- [44] S. F. E. Oliviero, L. Leone, F. Caravelli, and Alioscia Hamma, *SciPost Phys.* **10**, 076 (2021).
- [45] G. Gennaro, G. Benenti, and G. M. Palma, *Phys. Rev. A* **79**, 022105 (2009).
- [46] G. De Chiara and M. Antezza, *Phys. Rev. Res.* **2**, 033315 (2020).
- [47] V. Shaghghi, G. M. Palma, and G. Benenti, *Phys. Rev. E* **105**, 034101 (2022).
- [48] D. Rossini, G. M. Andolina, D. Rosa, M. Carrega, and M. Polini, *Phys. Rev. Lett.* **125**, 236402 (2020).
- [49] D. Rosa, D. Rossini, G. M. Andolina, M. Polini, and M. Carrega, *J. High Energy Phys.* **11** (2020) 067.
- [50] Y. Jia and J. J. M. Verbaarschot, *J. High Energy Phys.* **07** (2020) 193.
- [51] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, *Europhys. Lett.* **67**, 565 (2004).
- [52] G. Francica, J. Goold, F. Plastina, and M. Paternostro, *npj Quantum Inf.* **3**, 12 (2017).
- [53] F. Bernardis, M. Kleinmann, O. Gühne, and M. Paternostro, *Entropy* **21**, 771 (2019).
- [54] G. Francica, *Phys. Rev. E* **105**, L052101 (2022).
- [55] R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups* (Cambridge University Press, Cambridge, England, 1998).

- [56] D. A. Roberts and B. Yoshida, *J. High Energy Phys.* **04** (2017) 121.
- [57] M. Kliesch and I. Roth, *PRX Quantum* **2**, 010201 (2021).
- [58] G. Kimura, *Phys. Lett. A* **314**, 339 (2003).
- [59] R. A. Bertlmann and P. Kramer, *J. Phys. A: Math. Theor.* **41**, 235303 (2008).
- [60] J. Siewert, *J. Phys. Commun.* **6**, 055014 (2022).
- [61] H. Aschauer, J. Calsamiglia, M. Hein, and H. J. Briegel, *Quantum Inf. Comp.* **4**, 383 (2004).
- [62] J. I. de Vicente and M. Huber, *Phys. Rev. A* **84**, 062306 (2011).
- [63] C. Klöckl and M. Huber, *Phys. Rev. A* **91**, 042339 (2015).
- [64] N. Wyderka and O. Gühne, *J. Phys. A: Math. Theor.* **53**, 345302 (2020).
- [65] C. Eltschka and J. Siewert, *Quantum* **4**, 229 (2020).
- [66] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
- [67] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, *Phys. Rev. Lett.* **88**, 127902 (2002).
- [68] J. Barrett, A. Kent, and S. Pironio, *Phys. Rev. Lett.* **97**, 170409 (2006).
- [69] F. Buscemi and N. Datta, *Phys. Rev. Lett.* **106**, 130503 (2011).
- [70] M. Huber and M. Pawłowski, *Phys. Rev. A* **88**, 032309 (2013).
- [71] B. M. Terhal and P. Horodecki, *Phys. Rev. A* **61**, 040301(R) (2000).
- [72] O. Gühne and G. Tóth, *Phys. Rep.* **474**, 1 (2009).
- [73] N. Friis, G. Vitagliano, M. Malik, and M. Huber, *Nat. Rev. Phys.* **1**, 72 (2019).
- [74] A. Sanpera, D. Bruß, and M. Lewenstein, *Phys. Rev. A* **63**, 050301(R) (2001).
- [75] J. Sperling and W. Vogel, *Phys. Rev. A* **83**, 042315 (2011).
- [76] J. Sperling and W. Vogel, *Phys. Scr.* **83**, 045002 (2011).
- [77] M. Huber, L. Lami, C. Lancien, and A. Müller-Hermes, *Phys. Rev. Lett.* **121**, 200503 (2018).
- [78] S. Liu, N. Fadel, Q. He, M. Huber, and G. Vitagliano, *arXiv:2208.04909*.
- [79] R. Horodecki and M. Horodecki, *Phys. Rev. A* **54**, 1838 (1996).
- [80] P. Talkner, E. Lutz, and P. Hänggi, *Phys. Rev. E* **75**, 050102(R) (2007).
- [81] M. Esposito, U. Harbola, and S. Mukamel, *Rev. Mod. Phys.* **81**, 1665 (2009).
- [82] M. Campisi, P. Hänggi, and P. Talkner, *Rev. Mod. Phys.* **83**, 771 (2011).
- [83] A. J. Roncaglia, F. Cerisola, and J. P. Paz, *Phys. Rev. Lett.* **113**, 250601 (2014).
- [84] G. De Chiara, A. J. Roncaglia, and J. P. Paz, *New J. Phys.* **17**, 035004 (2015).
- [85] P. Talkner and P. Hänggi, *Phys. Rev. E* **93**, 022131 (2016).
- [86] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyán, M. Huber, and A. Acín, *Phys. Rev. Lett.* **118**, 070601 (2017).
- [87] E. Bäumer, M. Lostaglio, M. Perarnau-Llobet, and R. Sampaio, Fluctuating work in coherent quantum systems: Proposals and limitations, in *Thermodynamics in the Quantum Regime, Fundamental Aspects and New Directions* (Springer, New York, 2018).
- [88] G. De Chiara, P. Solinas, F. Cerisola, and A. J. Roncaglia, Ancilla-assisted measurement of quantum work, in *Thermodynamics in the Quantum Regime, Fundamental Aspects and New Directions* (Springer, New York, 2018).
- [89] M. Lostaglio, *Phys. Rev. Lett.* **120**, 040602 (2018).
- [90] A. Bednorz and W. Belzig, *Phys. Rev. Lett.* **105**, 106803 (2010).
- [91] Y. Guryanova, N. Friis, and M. Huber, *Quantum* **4**, 222 (2020).
- [92] T. Debarba, G. Manzano, Y. Guryanova, M. Huber, and N. Friis, *New J. Phys.* **21**, 113002 (2019).
- [93] K. Beyer, R. Uola, K. Luoma, and W. T. Strunz, *Phys. Rev. E* **106**, L022101 (2022).
- [94] P. Horodecki and A. Ekert, *Phys. Rev. Lett.* **89**, 127902 (2002).
- [95] P. Horodecki, *Phys. Rev. Lett.* **90**, 167901 (2003).
- [96] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwék, *Phys. Rev. Lett.* **88**, 217901 (2002).
- [97] H. A. Carteret, *Phys. Rev. Lett.* **94**, 040502 (2005).
- [98] F. A. Bovino, G. Castagnoli, A. Ekert, P. Horodecki, C. M. Alves, and A. V. Sergienko, *Phys. Rev. Lett.* **95**, 240407 (2005).
- [99] C. Schmid, N. Kiesel, W. Wieczorek, H. Weinfurter, F. Mintert, and A. Buchleitner, *Phys. Rev. Lett.* **101**, 260505 (2008).
- [100] F. Mintert, M. Kuś, and A. Buchleitner, *Phys. Rev. Lett.* **95**, 260502 (2005).
- [101] L. Aolita and F. Mintert, *Phys. Rev. Lett.* **97**, 050501 (2006).
- [102] F. Mintert and A. Buchleitner, *Phys. Rev. Lett.* **98**, 140505 (2007).
- [103] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, *Phys. Rev. A* **75**, 032338 (2007).
- [104] D. Kafri and S. Deffner, *Phys. Rev. A* **86**, 044302 (2012).
- [105] B. Gardas and S. Deffner, *Sci. Rep.* **8**, 17191 (2018).
- [106] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement* (Cambridge University Press, Cambridge, England, 2012).