Validation of classical modeling of single-photon pulse propagation

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(Received 15 September 2022; accepted 10 January 2023; published 18 January 2023)

"It is well-known to those who know it" that single-photon interference experiments can be modeled classically [S. Barnett, Phys. Scr. **97**, 114004 (2022)]. When a single-photon light pulse was split by a biprism, good agreement with a classical fit was obtained and the photon was counted only once, consistent with a probabilistic interpretation [V. Jacques *et al.*, Eur. Phys. J. D **35**, 561 (2002)]. A justification for this "well know result of quantum optics" is implicit in the work by Hawton [Phys. Rev A **104**, 052211 (2021)] where a real covariant field describing a single photon is first quantized. Here the theoretical basis of this result is reviewed and the theory is extended to multiphoton states. The crucial role of the charge-parity-time theorem in coupling to charged matter and resolution of the photon localization problem is discussed.

DOI: 10.1103/PhysRevA.107.013711

I. INTRODUCTION

Since classical electromagnetic (EM) fields are real and covariant, precise justification of their surprising success in the interpretation of single-photon experiments [1–3] requires quantized fields that are also real and covariant. In Ref. [4] the real classical EM field was first quantized to give a quantum mechanical (QM) description of one-photon states in which their state space is augmented with a scalar product and operators describing the momentum, energy, position, and angular momentum observables.

The initial work on photon wave mechanics was based on positive energy fields [5], but restriction to positive energy is inconsistent with causal pulse propagation. According to the Hegerfeldt theorem, a positive energy field localized in a finite region for an instant spreads immediately throughout space [6]. A technique that allows inclusion of negative energy fields was devised by Mostafazadeh and co-workers [7]. They defined an operator that multiplies the negative energy antiphoton terms by -1. This operator performs the same function as the 4×4 matrix $\beta = \begin{pmatrix} \widehat{1}_2 & 0 \\ 0 & -\widehat{1}_2 \end{pmatrix}$ in the Dirac theory of electrons and positrons. (Here $\widehat{1}_2$ is a 2×2 unit matrix.) Number density cannot be derived directly from a Lagrangian since it is not a conserved quantity, but modification of the sign of the antiparticle term in the conjugate momentum converts the conserved quantity that is generated by a phase change to photon number.

Real fields are, in fact, required for consistency with the charge-parity-time (CPT) theorem of quantum field theory (QFT). Charge conjugation C exchanges all particles with their antiparticles. The fermion four-current is odd under charge conjugation since electrons are exchanged with positrons. To maintain invariance of the current-field interaction and the Dirac equation, the photon four-potential should also be odd under charge conjugation. If A_j^+ is a positive energy photon four-potential and $A_j^- = A_j^{+*}$ is a negative energy antiphoton four-potential, QFT requires that $A_j = (A_j^+ - A_j^-)/\sqrt{2}i$ [8].

To maintain the classical form that we seek, a covariant approach to first and second quantization will be used here. The usual textbook choice in quantum electrodynamics (QED) and quantum optics includes a factor $\omega_k^{-1/2}$ in $\widehat{A}$ where ω_k is the angular frequency at wave vector $\mathbf{k}$, but here we will follow the covariant treatment in Refs. [9,10].

In the next section the fields in Ref. [4] will be generalized to include circularly polarized (CP) light for which rotation of the field vectors mixes sine and cosine terms. For completeness, descriptions of the covariant notation, scalar product, and momentum and position eigenvectors will be included. It will be verified that only the odd field is coupled to charged matter and localizable in a finite region. The probability amplitude to find a photon at $\mathbf{x}$ on the t hyperplane will be calculated and it will be verified that the Born rule is satisfied. In Sec. III, multiphoton states and the classical large photon number limit will be discussed, and in Sec. IV we conclude.

II. ONE-PHOTON FIELDS, SCALAR PRODUCT, AND OBSERVABLES

SI units will be used. The contravariant space-time, wave vector, and momentum four-vectors are $x = x^\mu = (ct, \mathbf{x})$, $k = (\omega_k/c, \mathbf{k})$, and $p = \hbar k$, where $kx = \omega_k t - \mathbf{k}\cdot\mathbf{x}$ is invariant, the matter four-current is J_m, the four-gradient is $\partial = (\partial_{ct}, -\nabla)$, $\Box \equiv \partial_\mu \partial^\mu = \partial_{ct}^2 - \nabla^2$, the four-potential is $A(t,\mathbf{x}) = A^\mu = (\frac{\phi}{c}, \mathbf{A})$ or $a(t,\mathbf{k}) = (a_0, \mathbf{a})$, and $a_\lambda(\mathbf{k})$ denotes a Lorentz invariant scalar describing a state with definite helicity λ. When not written explicitly, the space-time dependence of A and wave vector dependence a is implied. The covariant four-vector corresponding to $U^\mu = (U_0, \mathbf{U})$ is $U_\mu = g_{\mu\nu}U^\nu = (U_0, -\mathbf{U})$, where $g_{\mu\nu} = g^{\mu\nu}$ is a 4×4 diagonal matrix with diagonal $(1, -1, -1, -1)$. With the mutually

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orthogonal polarization unit vectors e^μ defined such that 0 is time-like, 1 and 2 are transverse, and 3 is longitudinal, $e_0 = n^\mu = (1,0,0,0)$, $\mathbf{e}_3(\mathbf{k}) = \mathbf{e}_\mathbf{k} = \mathbf{k}/|\mathbf{k}|$, and the definite helicity transverse unit vectors are

$$\mathbf{e}_\lambda(\mathbf{k}) = \frac{1}{\sqrt{2}}(\mathbf{e}_\theta + i\lambda\mathbf{e}_\phi) \quad (1)$$

for $\lambda = \pm 1$, where $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, and $\mathbf{e}_\mathbf{k}$ are orthonormal $\mathbf{k}$-space spherical polar unit vectors on the t hyperplane.

The four-potential describing single photons and antiphotons in position space will be written as

$$A(x) = \sqrt{\frac{\hbar}{2\epsilon_0}}\int_t \frac{d\mathbf{k}}{(2\pi)^3\omega_k}\{[(a_{1x'}(\mathbf{k}) + a_{-1x'}(\mathbf{k}))] \times e^{-ikx} + [a^*_{1x'}(\mathbf{k}) - a^*_{-1x'}(\mathbf{k})]e^{ikx}\}, \quad (2)$$

where

$$a_{rx'}(\mathbf{k}) = a_r(\mathbf{k})e(\mathbf{k})e^{ikx'}, \quad (3)$$

the subscript -1 denotes a series that is odd under exchange of photons and antiphotons so that $A \to -A$, the subscript 1 denotes an even series so $A \to A$, and $a_r(\mathbf{k})$ for $r = \pm 1$, a real Lorentz scalar. If $a_1 = 0$ this is an odd series, if $a_{-1} = 0$ it is an even series, so

$$A(x) = A_1(x) + iA_{-1}(x), \quad (4)$$

where A_1 and A_{-1} are real. If $a_1 = a_{-1}$, $A(x)$ is a positive energy photon term, while if $a_{-1} = -a_1$, it is a negative energy antiphoton term. The four-vector x' defines the space-time origin. The real four-vectors A_1 and A_{-1} in (4) replace A_c and A_s in Ref. [4] to allow for rotation of CP light that mixes the sine and cosine terms. The subscript t on the integral denotes evaluation at a fixed time t, and $d\mathbf{k} \equiv d^3k$ is an infinitesimal volume in $\mathbf{k}$ space. The above form was selected because $\lim_{V\to\infty}\Delta\mathbf{n}/V = d\mathbf{k}/(2\pi)^3$, where $\Delta\mathbf{n}$ is the number of states and $\int d^4k\delta(\omega_k^2/c^2 - |\mathbf{k}|^2) = \int_t \frac{d\mathbf{k}}{2\omega_k/c}$ is invariant. The electric and magnetic fields are

$$\mathbf{E}(x) = -\partial_t\mathbf{A}(x) - \nabla\phi(x), \quad \mathbf{B}(x) = \nabla \times \mathbf{A}(x). \quad (5)$$

The Mostafazadeh sign of energy operator that is useful for application to linear combinations of positive and negative energy fields is [7]

$$\widehat{\epsilon} \equiv i(-\nabla^2)^{-1/2}\partial_{ct}. \quad (6)$$

In this expression the operator $(-\nabla^2)^{-1/2}$ extracts a factor $|\mathbf{k}^2|^{-1/2}$ from the plane wave $e^{-i\epsilon kx}$, while $i\partial_t e^{-i\epsilon kx} = \epsilon\omega_k e^{-i\epsilon kx}$ so that the operator $\widehat{\epsilon}$ gives the sign of energy, $\epsilon = \pm$.

The Lagrangian describing the real fields A_1 and A_{-1} can be written in the complex form (4) provided that this field and its complex conjugate are treated as formally independent [11]. The standard Lagrangian density $\mathcal{L} = \epsilon_0(\mathbf{E}\cdot\mathbf{E}^* - c^2\mathbf{B}\cdot\mathbf{B}^*) - J_m^{\mu *}A_\mu - J_m^\mu A_\mu^*$, with matter four-current J_m purely imaginary since it is odd, gives the classical Maxwell equations and conservation laws for energy, momentum, and total angular momentum. In the Coulomb gauge in which $\mathbf{A} = \mathbf{A}_\perp$ is transverse, the canonical momentum conjugate to $\mathbf{A}_\perp$ is $-\epsilon_0\mathbf{E}^*_\perp$, the momentum conjugate to $\mathbf{A}^*_\perp$ is $-\epsilon_0\mathbf{E}_\perp$, and the conserved density generated by a global phase change is $-\epsilon_0(\mathbf{E}^*_\perp\cdot\mathbf{A}_\perp - \mathbf{E}_\perp\cdot\mathbf{A}^*_\perp) = 2\epsilon_0\mathbf{E}_\perp\cdot\mathbf{A}^*_\perp$. If the Coulomb gauge is specified, the subscript $\perp$ is redundant, but it is retained here since the transverse part of A is gauge independent. Writing the transverse part of (2) as

$$\mathbf{A}_\perp(x) = \sqrt{\frac{\hbar}{2\epsilon_0}}\sum_{\lambda=\pm1}\int_t \frac{d\mathbf{k}}{(2\pi)^3\omega_k} \times [\mathbf{a}_{\lambda+}(\mathbf{k})e^{-ikx} + \mathbf{a}_{\lambda-}(\mathbf{k})e^{ikx}] \quad (7)$$

for brevity and evaluating $\mathbf{E}_\perp(x) = -\partial_t\mathbf{A}_\perp(x)$, the gauge-invariant conserved quantity becomes

$$2\epsilon_0\int d\mathbf{x}\mathbf{E}_\perp\cdot\mathbf{A}^*_\perp = \sum_{\lambda=\pm1}\int_t \frac{d\mathbf{k}}{(2\pi)^3}[|\mathbf{a}_{\lambda+}(\mathbf{k})|^2 - |\mathbf{a}_{\lambda-}(\mathbf{k})|^2]. \quad (8)$$

For the real potentials $A_1(x)$ and $A_{-1}(x)$ this equals zero. Creation or annihilation of photon/antiphoton pairs is consistent with this conservation law, while creation or annihilation of unaccompanied positive frequency photons or negative frequency antiphotons violates it.

The interpretation of (8) as a conservation law is new, but its form motivated the definition of scalar product used in Ref. [4] and previous work. If $\mathbf{E}_\perp$ is replaced with $\widehat{\epsilon}\mathbf{E}_\perp \equiv \widetilde{\mathbf{E}}_\perp$ so that the sign of the antiphoton terms is changed, the positive definite number density

$$\rho(x) = \frac{\epsilon_0}{\hbar}\widetilde{\mathbf{E}}^*_\perp(x)\cdot\mathbf{A}_\perp(x) \quad (9)$$

is obtained. Details of the contribution of the longitudinal and scalar components to the number density and scalar product will not be presented here, but in Ref. [12] it was found that in the Coulomb gauge only transverse waves propagate, while in the Lorenz gauge the contributions of longitudinal and scalar photons to number density cancel. As in Ref. [4], the scalar product of states A_1 and A_2 will be defined as

$$(A_1, A_2)_t = \frac{\epsilon_0}{\hbar}\sum_{\lambda=\pm}\int_t d\mathbf{x}\widetilde{\mathbf{E}}^*_{1\lambda}(x)\cdot\mathbf{A}_{2\lambda}(x). \quad (10)$$

In bra-ket notation with substitution of (2) and use of the Parseval-Plancherel identity, the $r = \pm 1$, $\lambda = \pm 1$ terms of (10) can be written as

$$\langle\widetilde{\mathbf{E}}_{1r\lambda}\cdot\mathbf{A}_{2r'\lambda'}\rangle = \langle\widetilde{E}_{1r\lambda}|A_{2r'\lambda}\rangle\delta_{\lambda\lambda'}\delta_{rr'} \quad (11)$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^3\omega_k}a_{1r\lambda}(\mathbf{k})a_{2r'\lambda'}(\mathbf{k})e^{ik(x_1-x_2)}\delta_{\lambda\lambda'}\delta_{rr'}, \quad (12)$$

where $A_{jr\lambda} \equiv |\mathbf{A}_{jr\lambda}|$ and $\widetilde{E}_{jr\lambda} \equiv |\widetilde{\mathbf{E}}_{jr\lambda}|$. Inspection of (10)–(12) shows that these expressions for the scalar product involve both the vector potential and the electric field rather than a single function. QM based on scalar products of this form can be described within the formalism of biorthogonal QM [13–15].

The one-photon Hilbert space will be defined as the space of all four-potentials of the form (2) with scalar product (10)–(12). Eigenvectors of observables will be written in positive energy form in real space so that the basis includes both even and odd fields. It can be verified by substitution in (12) that the transverse plane waves with definite momenta $\hbar\mathbf{k}'$ and helicity

λ', defined covariantly as

$$\mathbf{a}_{r'\lambda'\mathbf{k}'}(\mathbf{k}) = (2\pi)^3\omega_k\delta(\mathbf{k}-\mathbf{k}')\mathbf{e}_{\lambda'}(\mathbf{k}) \tag{13}$$

for $r' = \pm 1$ with $\omega_k = c|\mathbf{k}|$, are biorthogonal in the sense that $(A_{r\lambda\mathbf{k}}, A_{r'\lambda'\mathbf{k}'}) = \delta_{rr'}\delta_{\lambda\lambda'}(2\pi)^3\omega_k\delta(\mathbf{k}-\mathbf{k}')$. In position space substitution in (2) gives $\mathbf{A}_{\lambda'\mathbf{k}'}(\mathbf{x}) = \sqrt{\frac{2\hbar}{\epsilon_0}}e^{i\mathbf{k}'\cdot\mathbf{x}}\mathbf{e}_{\lambda'}(\mathbf{k}')$. Position is also an observable. The Fourier transform of the localized state $\delta(\mathbf{x}-\mathbf{x}')$ at $\mathbf{x}'$ is the plane wave $\exp(-i\mathbf{k}\cdot\mathbf{x}')$, so the photon position eigenvectors in the Schrödinger picture (SP) should be of the form (3) with

$$\mathbf{a}_{r\lambda'\mathbf{x}'}(\mathbf{k}) = \mathbf{e}_{\lambda'}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}'}, \quad a_{r\lambda'x'}(\mathbf{k}) = 1 \tag{14}$$

for $r = \pm 1$ [16].

Pulse propagation takes place in real space. The projection of an arbitrary physical state of the form (3) onto the $A_{x\lambda}$ basis, evaluated using (12),

$$\phi_{r\lambda}(x) = (A_{x\lambda}, A_r) = \int_t \frac{d\mathbf{k}}{(2\pi)^3\omega_k}a_{r\lambda}(\mathbf{k})e^{-ikx}, \tag{15}$$

is the probability amplitude for the photon in state $|A_r\rangle$ to be at $\mathbf{x}$ at time t. Setting $a_{r\lambda}(\mathbf{k})$ in (15) equal to $a_{r\lambda x'}(\mathbf{k})$, with $\Delta t \equiv t - t'$ and $R \equiv |\mathbf{x}-\mathbf{x}'|$, an explicit expression for its time evolution can be obtained by taking sums and differences of

$$\int_t \frac{d\mathbf{k}}{(2\pi)^3\omega_k}e^{-ik(x-x')} = \frac{1}{4\pi^2 r}\sum_{\gamma=\pm}\left[i\gamma\pi\delta(R-\gamma c\Delta t) + P\left(\frac{1}{R-\gamma c\Delta t}\right)\right], \tag{16}$$

where P is the principal value and the sum over γ comes from integration over the $\mathbf{k}$-space polar angle and represents a sum over incoming and outgoing spherical waves. For the even field A_1, $\phi_{1\lambda}(x) = (A_{x\lambda}, A_{1x'\lambda})$ is nonlocal and since J_m is odd and hence purely imaginary, there is no source term in its equation of motion. It is completely decoupled from charge matter and thus, if it exists at all, cannot be detected. Only the imaginary odd term in (16),

$$\phi_{-1\lambda x'}(x) = (A_{x\lambda}, A_{-1x'\lambda}) = \frac{1}{4\pi r}[\delta(R+c\Delta t) - \delta(R-c\Delta t)], \tag{17}$$

couples to charged matter. In a source-free region there is no absorption or emission and the photon just passes through $\mathbf{x}'$ at time t'.

Expression (17) satisfies the homogeneous Klein Gordon (KG) equation and equals the advanced minus the retarded potential. The retarded potential is important in classical EM, and (17) shows that it can be calculated for one-photon states. In the presence of a source such as an atom or a quantum dot, the wave equation describing propagating transverse photons is

$$\Box\phi_{-1\lambda}(x) = J_{m\lambda}(x), \tag{18}$$

where $J_{m\lambda}$ is the λ component of the matter four-current. The general solution to this wave equation is a particular solution determined by $J_{m\lambda}$ plus a general solution to the homogeneous wave equation $\Box\phi_{-1\lambda}(x) = 0$. Schweber [17] inverted $\Box$ and found that the unique Green's function solving $\Box G_{\lambda x'}(x) = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ is

$$G_{\lambda x'}(x) = \frac{1}{4\pi R}[\delta(R+c\Delta t) + \delta(R-c\Delta t)], \tag{19}$$

where $t' = t - R/c < t$ is the retarded time and $t' = t + R/c > t$ is the advanced time. He concluded that the retarded potential is determined by boundary conditions. The particular solution to (18) for a source of helicity λ is [17]

$$\phi^{(p)}_{-1\lambda}(x) = \int \frac{d\mathbf{x}'}{4\pi r}[H(\Delta t - R/c)J_{m\lambda}(\mathbf{x}', \Delta t - R/c) + H(-\Delta t - R/c)J_{m\lambda}(\mathbf{x}', \Delta t + R/c)], \tag{20}$$

where $H(s) = 0$ for $s < 0$ and 1 for $s \geqslant 0$ is the Heaviside step function. The one-photon probability amplitude emitted instantaneously at t' by a localized source at $\mathbf{x}'$ is $\frac{1}{2}[G_{\lambda x'}(x) - \phi_{-1\lambda x'}(x)]$, given by (17) and (19), while $\frac{1}{2}[G_{\lambda x'}(x) + \phi_{-1\lambda x'}(x)]$ describes absorption. Equation (20) in combination with a generalization of (17) based on (15) can be used to describe emission or absorption by a more realistic one-photon source.

The potential $\phi_{-1\lambda x'}(x)$ given by (17) is a Lorentz scalar whose time derivative, $i\partial_t\phi_{-1\lambda x'}(x)$, is a density. At $t = t'$,

$$\psi_{-1\lambda\mathbf{x}'}(t,\mathbf{x}) = \int_t \frac{d\mathbf{k}}{(2\pi)^3}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}') \tag{21}$$

forms a localized basis. The Born rule gives a probability interpretation of the state vector. Expanding $\psi_{-1\lambda}$ in the δ basis at time t as

$$\psi_{-1\lambda}(t,\mathbf{x}) = \int_t d\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')\psi_{-1\lambda}(t,\mathbf{x}'), \tag{22}$$

it can be seen that $\psi_{-1\lambda}(x)$ is the probability amplitude for a photon to be in the state $\delta(\mathbf{x}-\mathbf{x}')$ on the t hyperplane. The λ-helicity $\mathbf{x}$-space and $\mathbf{k}$-space probability densities are

$$\rho_{-1\lambda}(t,\mathbf{x}) = |\psi_{-1\lambda}(t,\mathbf{x})|^2, \tag{23}$$

$$\rho_{-1\lambda}(\mathbf{k}) = |a_{-1\lambda}(\mathbf{k})|^2. \tag{24}$$

where

$$\sum_{\lambda=\pm1}\int d\mathbf{x}|\psi_{-1\lambda}(t,\mathbf{x})|^2 = \sum_{\lambda=\pm1}\int \frac{d\mathbf{k}}{(2\pi)^3}|a_{-1\lambda}(\mathbf{k})|^2 = 1 \tag{25}$$

for normalized physical states.

III. QED FOCK SPACE AND MULTIPHOTON STATES

The complete photon state space is determined by QED. Photons are bosons, so there is no exclusion principle and n-photon states are allowed for $n = 0$ or any positive integral value of n. In a plane wave basis the covariant photon commutation relations are

$$[\widehat{a}_\lambda(\mathbf{k}), \widehat{a}_{\lambda'}(\mathbf{k}')] = 0, \quad [\widehat{a}^\dagger_\lambda(\mathbf{k}), \widehat{a}^\dagger_{\lambda'}(\mathbf{k}')] = 0,$$
$$[\widehat{a}_\lambda(\mathbf{k}), \widehat{a}^\dagger_{\lambda'}(\mathbf{k}')] = \delta_{\lambda,\lambda'}(2\pi)^3\omega_k\delta(\mathbf{k}-\mathbf{k}'). \tag{26}$$

where the operator $\widehat{a}_\lambda(\mathbf{k})$ annihilates a photon with wave vector $\mathbf{k}$ and helicity λ and $\widehat{a}^\dagger_\lambda(\mathbf{k})$ creates one. Using these commutation relations, it can be verified that

$$|a_{\lambda\mathbf{k}n}\rangle = \frac{[\widehat{a}^{\dagger}_{\lambda n}(\mathbf{k})]^n}{\sqrt{n!}}|0\rangle \tag{27}$$

are normalized n-photon states where $|0\rangle$ is the zero-photon (vacuum) state.

The field operators can be obtained by second quantization of any real field, so we choose the odd field,

$$\widehat{\mathbf{A}}(x) = -i\sqrt{\frac{\hbar}{\epsilon_0}}\sum_{\lambda=\pm1}\int_t \frac{d\mathbf{k}}{(2\pi)^3\omega_{\mathbf{k}}}[\widehat{a}_\lambda(\mathbf{k})\mathbf{e}_{\lambda'}(\mathbf{k})e^{-ikx} - \widehat{a}^{\dagger}_\lambda(\mathbf{k})\mathbf{e}^*_{\lambda'}(\mathbf{k})e^{ikx}]. \tag{28}$$

The transverse electric and magnetic field operators can then be obtained by differentiation as

$$\widehat{\mathbf{E}}_\perp(x) = -\partial_t\widehat{\mathbf{A}}(x), \quad \widehat{\mathbf{B}}(x) = \nabla\times\widehat{\mathbf{A}}(x). \tag{29}$$

In QED causality is enforced by the commutation relations. Defining

$$\widehat{C}_\lambda(x,x') \equiv i\frac{\epsilon_0}{\hbar}[\widehat{\mathbf{A}}_\lambda(t,\mathbf{x})\cdot\widehat{\mathbf{E}}_\lambda(t',\mathbf{x}') - \widehat{\mathbf{E}}_\lambda(t',\mathbf{x}')\cdot\widehat{\mathbf{A}}_\lambda(t,\mathbf{x})], \tag{30}$$

it can be verified by substitution at $t=t'$ that $\widehat{C}_\lambda(t,\mathbf{x};t,\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$ and

$$\langle 0|\widehat{C}_\lambda(x,x')|0\rangle = \phi_{-1\lambda}(x). \tag{31}$$

Thus, the causal propagation described by (17) is consistent with QED, where in QED the sign change of the antiphoton term is a consequence of the bosonic commutation relations.

The QED positive and negative energy annihilation and creation operators define the plane wave and localized bases, but they do no extend to the description of real photon fields in an obvious way. I found first quantization to be more convenient for this purpose. First quantized one-photon states were the subject of the previous section and, following the rules of QM for bosons, multiphoton states can be written as symmetrized products of one-photon states. For photons at $\mathbf{x}_1$ and $\mathbf{x}_2$ in odd states A_{-1j} and A_{-1k}, the symmetrized two-photon state is

$$A_{jk}(\mathbf{x}_1,\mathbf{x}_2,t) = \frac{1}{\sqrt{2}}[A_{-1j}(\mathbf{x}_1,t)A_{-1k}(\mathbf{x}_2,t) + A_{-1j}(\mathbf{x}_2,t)A_{-1k}(\mathbf{x}_1,t)] \tag{32}$$

at time t. This ensures that the scalar product $(A_4A_3, A_2A_1) = (A_4,A_2)(A_3,A_1) + (A_4,A_1)(A_3,A_2)$ does not depend on photon order in the two-photon state (32). If both photons are in the same state, $A_{jj}(\mathbf{x}_1,\mathbf{x}_2,t) = A_{-j}(\mathbf{x}_1,t)A_{-j}(\mathbf{x}_2,t)$ is symmetric.

As an example, we consider photon pulses traveling in the $+$ or $-$ direction in a one-dimensional wave guide, $A_{-1\lambda}(x\pm ct)$. For a photon propagating in one dimension, states with definite helicity λ are just circularly polarized. A single photon passed through a beam splitter at $x=0$ described by

$$A_{-1\lambda}(x,t) = \frac{1}{\sqrt{2}}[A_{-1\lambda}(x-ct) + A_{-1\lambda}(x+ct)] \tag{33}$$

is equally likely to be counted on the positive or negative x axis. For entangled photons with total linear and angular momentum zero, perhaps created by position annihilation at $x=0$, the two-photon state

$$A(x_1,x_2,t) = \frac{1}{\sqrt{2}}\sum_{\lambda=\pm1}[A_{-1\lambda}(x_1-ct)A_{-1-\lambda}(x_2+ct) + A_{-1-\lambda}(x_1-ct)A_{-1\lambda}(x_2+ct)] \tag{34}$$

is symmetrized under exchange of photons at $\mathbf{x}_1$ and $\mathbf{x}_2$ by the sum over λ. Detection of a photon with helicity λ at $x_1 = ct$ collapses the entangled state (34) to $A_{-1-\lambda}(x_1+ct)$.

A coherent state with helicity λ, wave vector $\mathbf{k}$, and average photon number $\alpha^*_\lambda\alpha_\lambda$ is $|\alpha_{\lambda\mathbf{k}}\rangle = \sum_{n=0}^{\infty}\alpha^n_{\mathbf{k}\lambda}|a_{\lambda\mathbf{k}n}\rangle$, where $\widehat{a}_\lambda(\mathbf{k})|\alpha_{\mathbf{k}\lambda}\rangle = \alpha_{\lambda\mathbf{k}}|\alpha_{\lambda\mathbf{k}}\rangle$ and $\langle\alpha_{\lambda\mathbf{k}}|\widehat{a}^{\dagger}_\lambda(\mathbf{k}) = \alpha^*_{\lambda\mathbf{k}}\langle\alpha_{\lambda\mathbf{k}}|$. If the plane waves (13) are in coherent states $|\{\alpha_{\mathbf{k}\lambda}\}\rangle$ for all basis states $(\mathbf{k},\lambda)$, the expectation value of the vector potential operator is

$$\mathbf{A}_{cl} \equiv \langle\{\alpha_{\lambda\mathbf{k}}\}|\widehat{\mathbf{A}}(x)|\{\alpha_{\lambda\mathbf{k}}\}\rangle = i\sqrt{\frac{\hbar}{\epsilon_0}}\sum_{\lambda=\pm1}\int_t\frac{d\mathbf{k}}{(2\pi)^3\omega_k}[\alpha_{\lambda\mathbf{k}}\mathbf{e}_{\lambda'}e^{-ikx} - \alpha^*_{\lambda\mathbf{k}}(\mathbf{k})\mathbf{e}^*_{\lambda'}(\mathbf{k})e^{ikx}]. \tag{35}$$

After taking expectation values of the operator (28) to get (35), detailed information about the distribution over photon number is lost and only its average value is retained. The averaging process just counts transitions between states that differ by a photon number of 1. For states containing a definite number of photons, in particular for one-photon states, expectation values of the field operators are 0.

IV. SUMMARY AND CONCLUSION

Photon quantum mechanics as described here preserves the classical form of the EM potential and fields when first and second quantized. Only the interpretation need be changed—from real observable classical fields, to probability amplitudes, and then to operators that create and annihilate photons. The real potentials are even and odd under QFT charge conjugation, but only those that are odd couple to charged matter and can be localized in a finite region. Here even and odd fields were written as the real and imaginary parts of a complex field whose use simplifies the mathematics and facilitates use of the standard Lagrangian and relativistic scalar product. The number density if $(\epsilon_0/\hbar)\widetilde{\mathbf{E}}(x)\cdot\mathbf{A}(x)$, where in $\widetilde{\mathbf{E}}$ the sign of the antiphoton term, is changed analogous to the effect of the β matrix in the Dirac theory of electrons and positrons, and the scalar product is based on its spatial integral. Equations (17)–(35) provide a new scalar description of single-photon states with a well-defined physical interpretation that may prove to be useful in applications.

In conclusion, one-photon fields of the classical form combined with a probabilistic interpretation can be used in quantum optics, quantum computing, and quantum information without loss of rigor. This can be extended to multiphoton states, possibly entangled, by construction of a symmetrized product of one-photon states. In the opinion of this author, this conclusion has important practical and fundamental implications.

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