Winding number and bulk-boundary correspondence in a one-dimensional non-Hermitian photonic lattice

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A topological phase transition and the related bulk-boundary correspondence in a class of one-dimensional non-Hermitian photonic lattice with chiral symmetry are studied. We find that the corresponding topological number should be defined by adding up winding numbers calculated under the chiral symmetry. The Wilson loop method is used to calculate this topological number. We reveal the topological phase transition in one-dimensional non-Hermitian systems with chiral symmetry and provide a convenient and general pathway to characterize the related topology.

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I. INTRODUCTION

Finding a set of global constants which characterize different topological phases is one of the core contents of topological band theory [1–4]. In some Hermitian systems, different topological phases have been defined. These different phases can be characterized by some topological numbers. In one-dimensional (1D) systems, the Berry phase is used to define the winding number, whose name originates from its geometrical understanding [5–7]. In two-dimensional systems, the Chern number is used to classify the topological phases [4,8,9]. Apart from the topological numbers, symmetries also play an important role in this process. The symmetries of a topological system protect its topological phases, and systems with the same symmetry have significant similarities in the topological phases [10–13].

Recently, there has been more interest in the non-Hermitian system. The question naturally arises how to generalize the theory of the Hermitian systems to that of non-Hermitian systems. For example, what is the bulk-boundary correspondence in non-Hermitian systems? How should we define the topological phases and topological numbers? The difficulties arise from many aspects. For instance, the spectrum of a non-Hermitian Hamiltonian is generally complex. Meanwhile, the eigenstates of a Hamiltonian and of its Hermitian conjugate are not the same. To redefine the topological numbers on the complex spectrum, much effort has been put forth from different perspectives. For instance, biorthogonal quantum mechanics has been used [14] to extend the winding number and the Chern number in non-Hermitian systems and successfully classify the topological phases in some systems [11,12,15,16]. Also, new symmetries based on Hermitian theories have been introduced and a topological classification of non-Hermitian systems has been built. Furthermore, in non-Hermitian systems, there are exceptional points (EPs) [17–19] at which the eigensystems of the Hamiltonian coalesce. Exceptional points can be viewed as the branch points on the spectrum and have singularities, which introduce difficulties in calculating the topological numbers. In Hermitian system, though the eigenspace at the degeneracy point is multidimensional, one can always use the linear combination to reconstruct the eigenstates. Thus, the eigenvalues and eigenstates can always be single valued along an arbitrary path. Then the topological number on the paths can be easily calculated. In non-Hermitian systems, however, the singularity of EPs introduces difficulties in defining topological numbers. If a path goes through an EP, the eigensystems along the path will be multivalued, making the definition of topological numbers ambiguous. In some previous works, the researchers dropped the points nearby the EPs, calculating the topological numbers on the remaining spectrum [13,20]. We find that the singularity of the EPs can be overcome with the help of symmetries and in that way we can have a more precise topological description.

In this work we overcome the difficulties introduced by EPs by building a relation between the symmetries and the spectrum. We use a numerical algorithm based on the Wilson loop method to calculate the topological number. In this way we describe a topological phase transition and the related bulk-boundary correspondence in a class of 1D non-Hermitian photonic lattices. The paper is organized as follows. In Sec. II we define our model and illustrate its topological phase transition. In Sec. III we address the symmetries, especially the chiral symmetry (CS) in the non-Hermitian system, and discuss how the symmetry confines the chosen eigensystems when calculating topological numbers. In Sec. IV we show that systems with CS can have a more precise topological description, which requires an arrangement of the spectrum to follow the CS. We summarize in Sec. V.

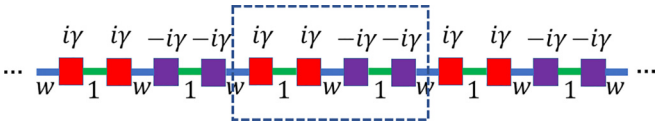

FIG. 1. Open chain described by Eq. (1). The dashed box denotes the unit cell.

II. MODEL AND TOPOLOGICAL EDGE STATES

We begin with a non-Hermitian lattice model, which is based on the Su-Schrieffer-Heeger model [21] with balanced on-site gain and loss. In the open-boundary condition, the Hamiltonian reads

$$\begin{aligned} H_O = &\sum_{m=1}^{M}(c_{m,B}^{\dagger}c_{m,A} + c_{m,C}^{\dagger}c_{m,D} + \text{H.c.}) \\ &+ w\sum_{m=1}^{M-1}(c_{m,C}^{\dagger}c_{m,B} + c_{m+1,A}^{\dagger}c_{m,D} + \text{H.c.}) \\ &+ i\gamma\sum_{m=1}^{M}(c_{m,A}^{\dagger}c_{m,A} + c_{m,B}^{\dagger}c_{m,B} \\ &- c_{m,C}^{\dagger}c_{m,C} - c_{m,D}^{\dagger}c_{m,D}) \quad (w>0,\ \gamma>0), \end{aligned} \tag{1}$$

where A, B, C, and D are sublattices in a unit cell and M is the number of unit cells, as shown in Fig. 1. In addition, $c_{m,X}$ ($c_{m,X}^{\dagger}$) is the annihilation (creation) operator on sublattice X of the mth unit cell. The hopping amplitudes between the sites are w and 1. The on-site gain and loss are determined by γ. In the periodic boundary condition, the bulk Hamiltonian reads

$$H(k) = \begin{pmatrix} i\gamma & 1 & 0 & we^{-ik} \\ 1 & i\gamma & w & 0 \\ 0 & w & -i\gamma & 1 \\ we^{ik} & 0 & 1 & -i\gamma \end{pmatrix}, \tag{2}$$

where k in $[-\pi,\pi)$ is the Bloch wave vector. This Hamiltonian, together with the one in the open-boundary condition, is non-Hermitian because of the diagonal terms $\pm i\gamma$, namely, $H^{\dagger}\neq H$. This means that the system interacts with the environment. The eigenvalues of the bulk Hamiltonian are

$$E(k) = \pm\sqrt{\frac{-A\pm\sqrt{A^2-4B(k)}}{2}}, \tag{3}$$

where $A = 2(\gamma^2 - w^2 - 1)$ and $B = \gamma^4 + 1 + w^4 + 2\gamma^2 - 2\gamma^2w^2 - 2w^2\cos k$.

There is a real gap in the spectrum of the bulk Hamiltonian. Generally, $\text{Re}[E(k)]\neq 0$, except when $\gamma^2 = w^2 - 1$. When the energy gap closes at $\gamma^2 = w^2 - 1$, the system will experience a topological phase transition. For a system in the open-boundary condition with large M and γ smaller (larger) than $\sqrt{w^2-1}$, there will be two (no) eigenstates with eigenvalues $E = \pm iA, A > 0$. Several examples of the spectrum are shown in Fig. 2.

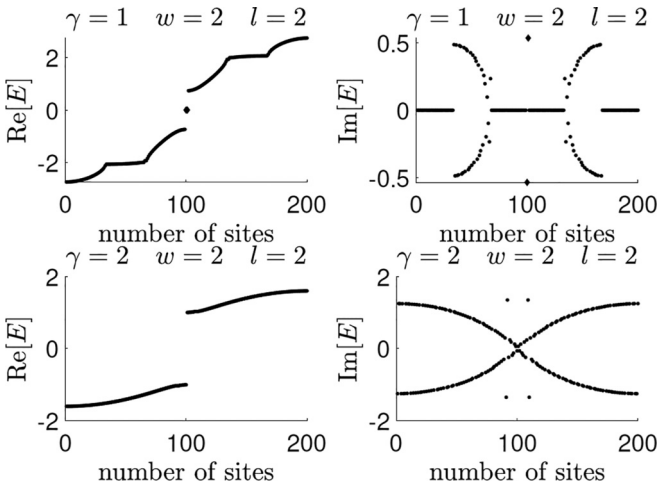

FIG. 2. Spectrum of the open-boundary condition, with $M = 50$. There are two eigenstates with energy $E = \pm iA$, labeled by the diamonds, in the topological phase, and no such edge state in the trivial phase.

III. SYMMETRIES AND TOPOLOGICAL NUMBER OF THE SYSTEM

There are three important symmetries in our model, which are the time-reversal symmetry (TRS†),[1] the particle-hole symmetry (PHS†), and the CS [11]. The TRS† is simply given by the fact that

$$H^{\text{T}}(k) = H(-k). \tag{4}$$

The 1D non-Hermitian systems with TRS† are insensitive to the boundary condition [22,23], namely, the bulk states under either the periodic boundary condition or the open-boundary condition are similar. The PHS† is described by

$$\eta H^{*}(k)\eta^{-1} = -H(-k). \tag{5}$$

Here $\eta = \mathbb{I}_2\otimes\sigma_z$. These two symmetries directly lead to the CS:

$$\eta H^{\dagger}(k)\eta^{-1} = -H(k). \tag{6}$$

The CS connects the left and right eigenstates of the Hamiltonian. Generally, we can write the eigenequations of $H(k)$,

$$\begin{aligned} H(k)|R_n(k)\rangle &= E_n(k)|R_n(k)\rangle, \\ H^{\dagger}(k)|L_n(k)\rangle &= E_n^{*}(k)|L_n(k)\rangle, \end{aligned} \tag{7}$$

where $n = 1, 2, 3, 4$ refers to four eigenstates. According to biorthogonal quantum mechanics [14], we can set $\langle L_n(k)|R_n(k)\rangle = 1$. Due to the CS, which is Eq. (6),

$$\begin{aligned} &\eta H^{\dagger}(k)\eta|L_n(k)\rangle = \eta E_n^{*}|L_n\rangle = -H(k)\eta|L_n(k)\rangle \\ &\rightarrow H(k)\eta|L_n(k)\rangle = -E_n^{*}(k)\eta|L_n(k)\rangle. \end{aligned} \tag{8}$$

[1]The dagger relates to the definition of the symmetry. In non-Hermitian physics, most symmetries have two ramifications because the Hamiltonian and its Hermitian conjugate are different. A dagger is used to specify one of them.

So, for the index n' such that $E_{n'}(k) = -E_n^*(k)$, $\eta|L_n(k)\rangle = C_R(k)|R_{n'}(k)\rangle$, where $C_R(k)$ is a complex number. Similarly, $\eta|R_n(k)\rangle = C_L(k)|L_{n'}(k)\rangle$, where $C_L(k)$ is another complex number.

Lattices in the open-boundary conditions also have these three symmetries, by replacing $H(k)$ with H_O and changing $\eta = \mathbb{I}_2 \otimes \sigma_z$ into $\eta = \mathbb{I}_{2M} \otimes \sigma_z$ in Eqs. (4)–(6). It turns out that the PHS† and the CS become one symmetry because $H^* = H^\dagger$ in the open-boundary condition. When the system is in the topological phase, there are two edge states which we denote by $|\phi_\pm\rangle$ and the two corresponding left eigenstates denoted by $|\chi_\pm\rangle$, namely,

$$H_O|\phi_\pm\rangle = \pm iA|\phi_\pm\rangle,$$
$$H_O^\dagger|\chi_\pm\rangle = \mp iA|\chi_\pm\rangle. \tag{9}$$

Due to the CS and the fact that $-(iA)^* = iA$, we conclude that $\eta|\phi_\pm\rangle = D|\chi_\pm\rangle$, where D is a complex number. That means the two edge states are protected by the CS.

IV. DEFINITION OF THE TOPOLOGICAL NUMBER

In some previous works, the generalized Berry phase was used to define the winding number [11–13,24]

$$Q_m = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\langle L_m(k)|\frac{\partial}{\partial k}|R_m(k)\rangle dk \pmod 1, \tag{10}$$

which is the integral of the general Berry connection $\langle L_m(k)|\frac{\partial}{\partial k}|R_m(k)\rangle$ along the path. Equation (10) can be calculated approximately using the Wilson loop:

$$Q_m \approx \frac{1}{2\pi}\arg[\langle L_m(k_0)|R_m(k_1)\rangle\langle L_m(k_1)|R_m(k_2)\rangle \times\cdots\langle L_m(k_{N-1})|R_m(k_0)\rangle]. \tag{11}$$

The eigensystem whose eigenvalues are $E_n(k)$ is labeled by n. The interval of the integral (10) is $[-\pi,\pi)$, which is a path in the first Brillouin zone. In our system, there are generally four eigenvalues for a given k. Thus, we expect four different Q_m naturally. The question is how to determine the eigenstates on the interval. To calculate the integral in Eq. (10), we need $|R_n(k)\rangle$ and $|L_n(k)\rangle$ to be continuous; however, due to the existence of EPs, there are multiple choices of $|R_m(k)\rangle$ and $|L_m(k)\rangle$. For example, turning the eigenstates into those of $E_m^*(k)$ is also a choice that makes the eigenstates continuous. In some previous works [13,25], researchers dropped the small intervals near the EPs to avoid singularity. Here we introduce a method to deal with this problem. For a fixed n, we label the eigensystem whose eigenvalue is $-E_n^*(k)$ as n'. The eigensystems n and n' are related by the CS. Furthermore, for the same n, an eigensystem is labeled by n_c if there is a k_c where $E_{n_c}(k_c) = E_n(k_c)$; k_c is one of the EPs. An example of this convention is shown in Fig. 3.

After introducing n' and n_c, we use Eq. (10) or (11) to calculate the winding number on paths labeled by n, n', and n_c, whose values are denoted by Q_n, $Q_{n'}$, and Q_{n_c}. To calculate the topological number, we solve the eigenproblem (7) numerically on several discrete k, $-\pi = k_0 < k_1 < k_2 < \cdots < k_N = \pi$, where N is a large integer. Generally, the calculated eigenstates have random phases. According to the biorthogonal quantum mechanics [14], we must make the eigenstates

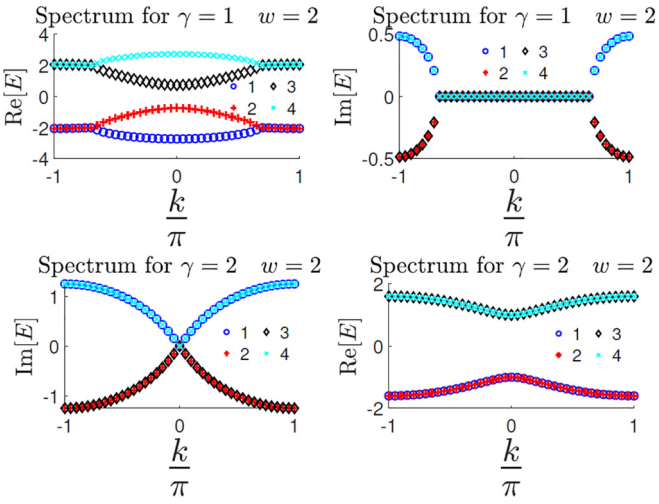

FIG. 3. Spectra of the bulk Hamiltonian with different parameters. Circles, pluses, diamonds, and crosses denote $n = 1$, 2, 3, and 4, respectively, and $n' = 4, 3, 2, 1$ and $n_c = 2, 1, 4, 3$.

satisfy

$$\langle L_n(k_p)|R_m(k_p)\rangle = \delta_{mn}. \tag{12}$$

Here $\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$. After that, we can calculate the Wilson loop (11), without being affected by the random phase of $|R_n(k)\rangle$. To calculate the integral (10) using a difference, however, we must have a smooth gauge along the path. To acquire a smooth gauge, we follow the discussion in [26]. In the beginning, we make

$$\mathrm{Im}[\langle L_n(k_p)|R_n(k_{p+1})\rangle] = 0 \tag{13}$$

by adjusting the phase of the eigenstates. It offers a smooth gauge (the eigenstates along the path vary continuously with k). However, the eigenstates at $k_0 = -\pi$ and $k_N = \pi$ will generally have a different phase. We define this fact as $|R_m(-\pi)\rangle = \exp(i\phi_m)|R_m(\pi)\rangle$, $-\pi < \phi_m \leqslant \pi$. To get a smooth gauge, we need to multiply a continuous phase function $\exp[i\phi(k)]$, making the eigenstates at $k = -\pi$ and π have the same phase. Then we can calculate the winding number and the result yields $Q_m = \frac{\phi_m}{2\pi}$, because the Wilson loop (11) is independent of the gauge and from Eq. (13) $Q_m = \frac{\phi_m}{2\pi}$. The geometric meaning of the Berry connection is the infinitesimal phase change at the Bloch vector k. So

$$Q_m \approx \sum_{p=1}^{N-1}\langle L_m(k_p)|\left(\frac{|R_m(k_{p+1})\rangle}{|\langle L_m(k_p)|R_m(k_{p+1})\rangle|} - |R_m(k_p)\rangle\right). \tag{14}$$

In some previous works, the authors used $\sum_{m=1}^{4} Q_m$ to give the topological number, without the mod 1 [13,25]; however, we have found that there is a more precise description. For any band $E_n(k)$, $Q_{n_c} + Q_{n'}$ will be 0.5 mod 1 in the topological phase and 0 mod 1 in trivial phase. In our system, there will be two 0.5 mod 1 in the topological phase, since there are four energy bands. Each of them refers to an edge state. The topological numbers with different γ are presented in Fig. 4.

The acquirement of the more precise topological description is based on the fact that the arrangement of the energy band is consistent with the chiral symmetry. Because the edge

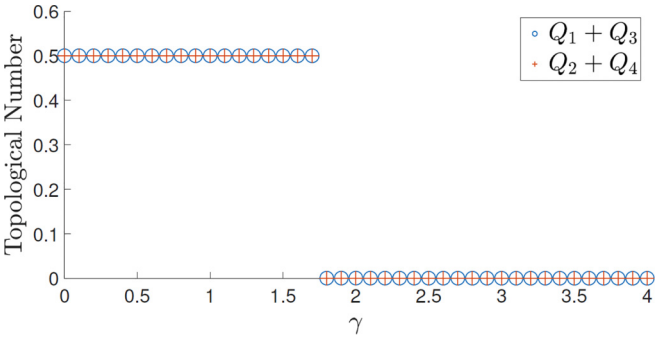

FIG. 4. Topological numbers of systems with $w = 2$ while γ varies from 0 to 4. The change in topological numbers indicates the topological phase transition. There is a slight shift in the blue lines for avoiding overlap. The phase transition point is analytically $\gamma = \sqrt{3}$.

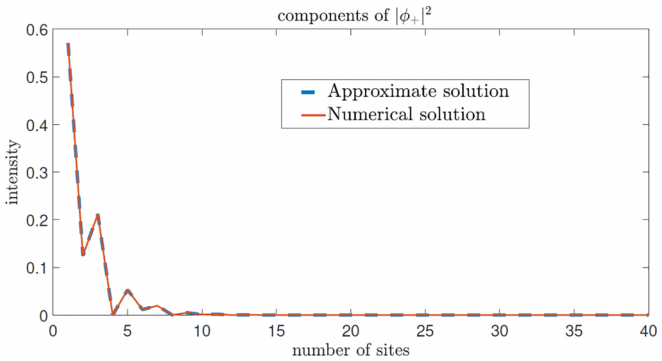

FIG. 5. Numerical solution and theoretical approximate solution for $w = 2$ and $\gamma = 1$. After normalization, the approximate solution is $a_1 = 0.7557$, $b_1 = -0.3528$, $a_2 = -0.4602$, and $\lambda = 0.3045$.

states are protected by the chiral symmetry and in the periodic boundary condition, chiral symmetry relates different eigensystems at fixed k.

V. CONCLUSION

In summary, we have exhibited a non-Hermitian photonic lattice system that has topological phase transition and calculated its topological numbers. We used the CS to overcome the singularity of the EPs, making the paths for calculating topological numbers single valued and well defined. Using the gauge smoothing process, we calculated the topological numbers, which do not rely on the analytical solution of eigenequations. Our result showed how CS affects the definition of the topological numbers and $\mathrm{Re}[E] < 0$ cannot fully describe the topological phase in non-Hermitian systems.

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APPENDIX: EXPONENTIAL BEHAVIOR OF THE EDGE STATES

From our observation of the numerical results, we can use the exponential behavior of the edge states to construct them and find the system's topological phase by seeking the solutions with the form

$$\phi = (a_1, ib_1, a_2, ib_2, \ldots, a_{nM}, b_{nM})^{\mathrm{T}}, \tag{A1}$$

which satisfies $H\phi = iA\phi$. In Eq. (A1), a_j and b_j are real numbers, n is the number of sites in the unit cell, and M is the number of unit cells. We assume that $(a_{mn+1}, ib_{mn+1}, \ldots, a_{(m+1)n}, ib_{(m+1)n}) = \lambda(a_{(m-1)n+1}, ib_{(m-1)n+1}, \ldots, a_{mn}, ib_{mn})$, where $m = 1, 2, \ldots, M-1$ and λ is a constant. This means that the amplitude grows (decays) exponentially along the chain. For simplicity, we set $n = 2$. The eigenequations $(H - iA)\phi = 0$ can be reduced to six independent equations

$$\begin{aligned}
a_1(\gamma - A) + b_1 &= 0,\\
a_1 + b_1(A - \gamma) + a_2 w &= 0,\\
b_1 w - a_2(\gamma + A) + b_2 &= 0,\\
a_2 + b_2(\gamma + A) + a_1\lambda w &= 0,\\
b_2 w + \lambda[a_1(\gamma - A) + b_1] &= 0,\\
\lambda^{M-1}[a_2 + b_2(\gamma + A) + a_1\lambda w] - a_1\lambda^M w &= 0.
\end{aligned} \tag{A2}$$

Other equations are the linear combination of the six equations. There is no exact nontrivial solution. However, an approximate nontrivial solution can be calculated for large M and there are two pathways. The first one is to abort the last equation $\lambda^{M-1}[a_2 v + b_2(\gamma + A) + a_1\lambda w] - a_1\lambda^M w = 0$. Setting $a_1 = 1$ without losing generality, we find that $b_2 = 0$ and there are four remaining unknown variables in the first four equations. After calculation, the aborted equation becomes $-a_1 w\lambda^M = 0$. If $\lambda < 1$, this approximation solution converges for large M, indicating that the solution of ϕ is a good approximation of the edge state we observed before (the edge state exists; see Fig. 5). So $\lambda < 1$ implies that the system is in the topological phase. If $\lambda > 1$, this approximation diverges and thus the edge states do not exist. Hence, $\lambda > 1$ implies that the system is in the trivial phase. The boundary $\lambda = 1$ is regarded as the phase transition point.

The other pathway is to abort the first equation and start from the last question. First, multiple λ^{-M} for every equation to make ϕ's norm converge. In that case, one will get that $a_1 = 0$, A changes its sign compared to the first pathway, $-b_2 w\lambda^{-M} \approx 0$, $\lambda > 1$ refers to the topological phase, and $\lambda < 1$ refers to the topological phase. Since the two approximate solutions' eigenvalues are opposite, we denoted these two by $\phi_\pm$, $H\phi_\pm = \pm iA\phi_\pm$.

Analogously to the Bloch theorem [27], when $M \to +\infty$, one can conclude that $(a_1, ib_1, a_2, ib_2)^{\mathrm{T}}$ is the eigenstate of $H(-i\ln\lambda)$, whose eigenvalue is $\pm iA$. When $\lambda = 1$, $H(-i\ln 1) = H(0)$ can be viewed as a Bloch Hamiltonian, whose Bloch vector is $k = 0$. Based on the previous discussion, $\lambda = 1$ will be the phase transition point. We will show that $A = 0$ and the energy gap will close. To prove these, we address the fact that, at this time, the states represented

by $a_1 = 0$ and $b_{2n} = 0$ degenerate because $\lambda = \lambda' = 1$. Thus, these two states degenerate, implying that $A = A'$ and the real gap must be closed. For the bulk Hamiltonian, the real gap closes at $k = 0$. In conclusion, we have shown that the edge states have exponential behavior and proved that the phase transition happens when the real gap is closed.

[1] L. Lu, J. D. Joannopoulos, and M. Soljačić, Topological photonics, Nat. Photon. **8**, 821 (2014).

[2] J. K. Asbóth, L. Oroszlány, and A. Pályi, A short course on topological insulators, Lect. Notes Phys. **919**, 166 (2016).

[3] Z. Gao and Z. Lan, Flat bands and $\mathbb{Z}_2$ topological phases in a non-Abelian kagome lattice, Phys. Rev. B **102**, 245133 (2020).

[4] C. L. Kane and E. J. Mele, Z_2 Topological Order and the Quantum Spin Hall Effect, Phys. Rev. Lett. **95**, 146802 (2005).

[5] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. London Ser. A **392**, 45 (1984).

[6] Y. Aharonov and D. Bohm, Significance of electromagnetic potentials in the quantum theory, Phys. Rev. **115**, 485 (1959).

[7] J. Zak, Berry's Phase for Energy Bands in Solids, Phys. Rev. Lett. **62**, 2747 (1989).

[8] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. **49**, 405 (1982).

[9] A. A. Soluyanov and D. Vanderbilt, Smooth gauge for topological insulators, Phys. Rev. B **85**, 115415 (2012).

[10] A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Phys. Rev. B **55**, 1142 (1997).

[11] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and Topology in Non-Hermitian Physics, Phys. Rev. X **9**, 041015 (2019).

[12] K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Topological unification of time-reversal and particle-hole symmetries in non-Hermitian physics, Nat. Commun. **10**, 297 (2019).

[13] H. C. Wu, L. Jin, and Z. Song, Topology of an anti-parity-time symmetric non-Hermitian Su-Schrieffer-Heeger model, Phys. Rev. B **103**, 235110 (2021).

[14] D. C. Brody, Biorthogonal quantum mechanics, J. Phys. A: Math. Theor. **47**, 035305 (2014).

[15] S. Ryu and Y. Hatsugai, Topological Origin of Zero-Energy Edge States in Particle-Hole Symmetric Systems, Phys. Rev. Lett. **89**, 077002 (2002).

[16] S.-D. Liang and G.-Y. Huang, Topological invariance and global Berry phase in non-Hermitian systems, Phys. Rev. A **87**, 012118 (2013).

[17] W. Heiss, Phases of wave functions and level repulsion, Eur. Phys. J. D **7**, 1 (1999).

[18] C. Dembowski, B. Dietz, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, and A. Richter, Encircling an exceptional point, Phys. Rev. E **69**, 056216 (2004).

[19] W. Heiss, Exceptional points of non-Hermitian operators, J. Phys. A: Math. Gen. **37**, 2455 (2004).

[20] J. Arkinstall, M. H. Teimourpour, L. Feng, R. El-Ganainy, and H. Schomerus, Topological tight-binding models from nontrivial square roots, Phys. Rev. B **95**, 165109 (2017).

[21] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in Polyacetylene, Phys. Rev. Lett. **42**, 1698 (1979).

[22] S. Yao and Z. Wang, Edge States and Topological Invariants of Non-Hermitian Systems, Phys. Rev. Lett. **121**, 086803 (2018).

[23] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, Adv. Phys. **69**, 249 (2020).

[24] K. Takata and M. Notomi, Photonic Topological Insulating Phase Induced Solely by Gain and Loss, Phys. Rev. Lett. **121**, 213902 (2018).

[25] H. Shen, B. Zhen, and L. Fu, Topological Band Theory for Non-Hermitian Hamiltonians, Phys. Rev. Lett. **120**, 146402 (2018).

[26] M. Wagner, F. Dangel, H. Cartarius, J. Main, and G. Wunner, Numerical calculation of the complex Berry phase in non-Hermitian systems, Acta Polytech. **57**, 470 (2017).

[27] C. Kittel, P. McEuen, and P. McEuen, *Introduction to Solid State Physics* (Wiley, New York, 1996), Vol. 8.