Sharing preparation contextuality in a Bell experiment by an arbitrary pair of sequential observers

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Based on the quantum violation of the bipartite Bell inequality, it has been established that the sharing of nonlocality can be demonstrated for at most two sequential observers at one end and at most one pair of observers at both ends. In this work we study the sharing of nonlocality and preparation contextuality based on a bipartite Bell inequality, involving arbitrary n measurements by one party and 2^{n-1} measurements by another party. Such a Bell inequality has two bounds, the local bound and the preparation noncontextual bound, which is smaller than the local bound. We show that while nonlocality can be shared only by the first pair of the sequential observers, the preparation contextuality can be shared by an arbitrary pair of independent sequential observers at both ends.

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I. INTRODUCTION

Bell's theorem [1] is one of the most famous discoveries of quantum theory. It is the first no-go theorem that demonstrates that an ontological model satisfying locality cannot reproduce all the statistics of quantum theory. Such a feature is widely known as quantum nonlocality, which is demonstrated through the quantum violation of suitable Bell inequalities. It is one of the elegant routes to demonstrate the supremacy of quantum theory over its classical counterpart. In addition to the extensive importance of the Bell inequality in quantum foundations, it plays a pivotal role in quantum information processing such as secure quantum key distribution [2–5], randomness certification [6–9], witnessing Hilbert space dimension [10–17], and achieving advantages in communication complexity tasks [18].

There is another important no-go theorem, the Kochen-Specker (KS) theorem [19], which provides a way of discriminating quantum theory from classical noncontextual theories. Over the years, many simpler proofs of the KS theorem have been proposed [20–23]. Importantly, contextuality can be revealed for a single system in contrast to nonlocality, which requires spatially separated entangled systems. However, the KS theorem only involves the measurement noncontextuality and it is not applicable to unsharp measurement. Later, Spekkens [24] generalized the notion of noncontextuality for arbitrary operational theories and extended the formulation to preparation, transformation, and unsharp measurement. In recent times, the quantum preparation contextuality has been extensively studied [25–33].

The aim of this work is to demonstrate the sharing of preparation contextuality by multiple independent sequential observers. Sharing of various forms of quantum correlations has gained considerable attention in recent times. Based on the quantum violation of the Clauser-Horne-Shimony-Halt (CHSH) inequality [34], Silva *et al.* [35] first demonstrated that at most two independent observers can share the nonlocality sequentially when sharing is considered for one party. Let a bipartite two-qubit entangled state be shared by two distant observers, Alice and Bob, such that one qubit is in possession of Alice and the other qubit is with Bob, who performs unsharp measurements and relays the qubit to the second Bob (next sequential observer), who does the same. Then the sharing of nonlocality implies that by recycling the qubit of sequential Bobs, the Bell inequality is violated by independent sequential observers. If two sequential Bobs violate Bell's inequality, both of them cannot get the optimal quantum nonlocality; rather, they share the nonlocality. In a recent work Brown and Colbeck [36] demonstrated that the CHSH nonlocality can be shared by an arbitrarily long sequence of independent observers. However, for every sequential observer, the unsharpness parameter and a new set of observables have to be chosen. This is in contrast to earlier works [35], where every sequential observer performs the same set of observables. Very recently, Cheng *et al.* [37,38] showed that at most one pair of observers can share nonlocality while considering the sharing for both parties. In recent years, a flurry of works have examined the number of independent sequential observers sharing different quantum correlations, viz., entanglement [39], steering [40–42], nonlocality [35,36,43–47], and preparation contextuality [32,48].

In this paper we study the sharing of nonlocality and preparation contextuality by multiple independent sequential observers at both ends. For this we use a suitable Bell inequality proposed in [27] involving n measurements by one party and 2^{n-1} measurements by another party. Such a Bell inequality has a local bound and a preparation noncontextual bound. The preparation noncontextual bound is considerably lower than the local bound. Such an inequality arises from a communication game known as the parity-oblivious random access code (PORAC). We show that the sharing of nonlocality is restricted only to a single pair of observers, but

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sharing of preparation contextuality can be demonstrated for an unbounded pair of sequential observers at both ends. For simplicity, we consider the symmetric case sharing when the kth observer at both ends can share preparation contextuality.

This paper is organized as follows. In Sec. II we demonstrate the PORAC game, which is used as a tool to demonstrate our results. In Sec. III we discuss the Bell expression and its local, preparation noncontextual, and quantum bounds. In Sec. IV we derive the sequential quantum value of the Bell expression. The condition of sharing of nonlocality and preparation contextuality at both ends are demonstrated in Secs. V and VI, respectively. In Sec. VII we provide the analytical proof of sharing by an unbounded pair of sequential observers at both ends. In Sec. VIII we discuss our results.

II. ARBITRARY INPUT BELL INEQUALITY AND ITS LOCAL, PREPARATION NONCONTEXTUAL, AND QUANTUM BOUND

We briefly discuss the notion of the PORAC, which is a two-party one-way communication game, used as a tool to demonstrate our results. An n-bit random access code [25,49] involves a sender Alice and a receiver Bob. Alice has an n-bit string x chosen randomly from $x \in \{0,1\}^n$ and Bob receive inputs $y \in \{1, 2, \dots, n\}$ uniformly at random and outputs b. Bob's task is to recover the yth bit of Alice's input with a probability, i.e., the winning condition of the game is $b = x_y$. The average success probability of the game is defined as

$$P = \frac{1}{2^n n} \sum_{x,y} p(b = x_y|x, y). \tag{1}$$

The task of Alice and Bob is to maximize the success probability. To help Bob, Alice can communicate some information with him. However, there is a parity-oblivious condition which dictates that Alice can communicate any number of bits to Bob but no information about the parity of x should be transmitted.

Spekkens *et al.* [25] defined the parity-oblivious condition with respect to a parity set $\mathbb{P}_n = \{x \mid x \in \{0,1\}^n, \sum_r x_r \geqslant 2\}$ with $r \in \{1, 2, \dots, n\}$. For any $s \in \mathbb{P}_n$, no information about $s x = \oplus_r s_r x_r$ (s parity) is to be transmitted to Bob, where $\oplus$ is the sum modulo 2. The maximum average success probability in such a classical n-bit PORAC is [25]

$$P_{\text{PNC}} \leqslant \frac{1}{n} + \frac{n-1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right). \tag{2}$$

It has been demonstrated in [24,25,29,33] that the parity obliviousness at the operational level must be satisfied at the level of ontic states if the ontological model of quantum theory is preparation noncontextual. Thus, in a preparation noncontextual model, the classical bound remains the same as that given in Eq. (2). Quantum violation of this bound thus demonstrates a form of nonclassicality, the preparation contextuality. Throughout this paper, by quantum preparation contextuality we refer to the violation of the preparation noncontextuality inequality in Eq. (2).

In quantum mechanics, Alice encodes her n-bit string of $x \in \{1, 2, \dots, 2^n\}$ into quantum states ρ_x. Here we consider an entanglement assisted version of the PORAC so that Alice and Bob share a suitable entangled state. By performing 2^{n-1} dichotomic measurements, Alice can steer the state ρ_x to Bob. After receiving the state ρ_x, for every $y \in \{1, 2, \dots, n\}$, Bob performs a dichotomic measurement and reports the outcome b as his output. The quantum success probability can be written as [27]

$$p_Q = \frac{1}{2} + \frac{1}{2^n n} \sum_{y=1}^{n} \sum_{i=1}^{2^{n-1}} (-1)^{x_y^i} \langle A_i \otimes B_y \rangle. \tag{3}$$

It is seen from Eq. (3) that the quantum success probability is dependent solely on the Bell expression

$$\mathcal{B}_n = \sum_{y=1}^{n} \sum_{i=1}^{2^{n-1}} (-1)^{x_y^i} A_{n,i} \otimes B_{n,y}, \tag{4}$$

where n is an arbitrary integer with $n > 2$. For $n = 2$ and 3, the Bell expressions $\mathcal{B}_n$ become the well-known CHSH [34] and Gisin elegant Bell [50] expressions. Using an elegant sum-of-squares decomposition [27], the optimal quantum value of the Bell expression was derived as [17]

$$\left(\mathcal{B}_n^{\text{opt}}\right)_Q = 2^{n-1}\sqrt{n}. \tag{5}$$

This is achieved when Alice and Bob share a number $\lfloor \frac{n-1}{2} \rfloor$ of two-qubit maximally entangled states and Bob performs the measurements of a number n of mutually anticommuting observables. Then the optimal quantum success probability for an n-bit PORAC is

$$P_Q^{\text{opt}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{n}}\right). \tag{6}$$

Since for any n, $p_Q^{\text{opt}} > P_{\text{PNC}}^{\text{opt}} = \frac{1}{2}(1 + \frac{1}{n})$, the quantum preparation contextuality is demonstrated. In other words, using quantum resources, the success probability of the n-bit PORAC exceeds the preparation noncontextual bound. In this paper we use the Bell expression (4) to demonstrate the sharing of nonlocality and preparation contextuality at both ends.

The local bound of the Bell expression (4) is derived as [32]

$$(\mathcal{B}_n)_L \leqslant n\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}. \tag{7}$$

For $n = 2$ and 3 we have $(\mathcal{B}_n)_L \leqslant 2$ (the CHSH inequality) and $(\mathcal{B}_n)_L \leqslant 6$ (Gisin's elegant Bell inequality [50]).

Importantly, the optimal quantum value derived in Eq. (5) automatically satisfy the parity-oblivious condition in Eq. (8) [27]. In quantum theory, the parity-oblivious condition demands that

$$\frac{1}{2^{n-1}} \sum_{x|x.s=0} \rho_x = \frac{1}{2^{n-1}} \sum_{x|x.s=1} \rho_x \,\forall\, s. \tag{8}$$

For the n-bit case, the total $C_n = 2^{n-1} - n$ nontrivial parity-oblivious condition between Alice's observables [27] needs to satisfy

$$\sum_{i=1}^{2^{n-1}} (-1)^{s.x^i} A_{n,i} = 0 \,\forall\, s. \tag{9}$$

The parity-oblivious conditions in Eq. (9) will have equivalent representation in an ontological model, which is, in the premise of the preparation noncontextuality assumption, in an ontological model. Putting the condition (9) into Eq. (4), we have [27]

$$(\mathcal{B}_n)_{\mathrm{PNC}} \leqslant 2^{n-1}, \quad (10)$$

which is preparation noncontextual bound of the Bell expression in Eq. (4). Comparing the local bound in Eq. (7) and the preparation noncontextual bound in Eq. (10), we can see that $(\mathcal{B}_n)_L > (\mathcal{B}_n)_{\mathrm{PNC}}$. Then there may be instances when nonlocality may not be revealed but a nonclassicality in the form of preparation contextuality can be revealed. This indicates that there is a higher chance of sharing the preparation contextuality for more pairs of sequential observers than nonlocality. In this work we examine the maximum number of pairs of Alice and Bob that can share the nonlocality and the preparation contextuality across both Alice's and Bob's ends based on the quantum violations of Eqs. (7) and (10), respectively.

III. SEQUENTIAL QUANTUM VALUE OF THE BELL EXPRESSION

We start by pointing out that the sharing of any quantum correlation protocol requires the prior observers to perform unsharp measurements [51] represented by a set of positive-operator-valued measures (POVMs). Although sharp measurement seems advantageous from an information-theoretic perspective, there remain certain tasks where unsharp measurement showcases its supremacy over sharp measurement [52–54]. An ideal sharp measurement extracts maximum information by collapsing the system state into one of the eigenstates of the measured observable. This causes a maximum disturbance to the initial system. For an entangled state, if a sharp measurement is performed by one party the system becomes a mixed state. No quantum violation of Bell's inequality can be found by a sequential observer. However, if previous observers perform an unsharp measurement, the initial entangled state is partially disturbed. This can be controlled by tuning the degree of the unsharpness parameter. In such a case some amount of entanglement may remain in the system, which can be used by sequential observers. Then, by using this residual entanglement there remains a chance of sharing nonlocality through the quantum violation of Bell's inequality.

In order to demonstrate sharing of nonlocality and preparation contextuality, we consider that an entangled state is shared between multiple independent observers. For the Bell expression $\mathcal{B}_n$ in Eq. (4), each sequential Alice and Bob perform dichotomic measurements upon receiving inputs $x \in \{1, 2, \dots, 2^{n-1}\}$ and $y \in \{, 2, \dots, n\}$, respectively. Here it should be noted that Alice's and Bob's choices of measurement settings are completely random and in each run they use the same set of respective measurement settings. For any arbitrary n, the respective POVMs that are performed by a number k of sequential Alices and a number l of sequential Bobs are represented by

$$E^{\pm}_{n,x,p} = \frac{1 \pm \eta_{n,p}}{2}\Pi^{+}_{A_{n,x,p}} + \frac{1 \mp \eta_{n,p}}{2}\Pi^{-}_{A_{n,x,p}} \quad (11)$$

and

$$E^{\pm}_{n,y,q} = \frac{1 \pm \chi_{n,q}}{2}\Pi^{+}_{B_{n,y,q}} + \frac{1 \mp \chi_{n,q}}{2}\Pi^{-}_{B_{n,y,q}}, \quad (12)$$

with $p = (1, 2, \dots, k)$ and $q = (1, 2, \dots, l)$. Here $\eta_{n,p}$ $(\chi_{n,q})$ is the sharpness parameters of the pth Alice (qth Bob) satisfying $0 \leqslant \eta_{n,p}, \chi_{n,q} \leqslant 1$. Then the postmeasurement state after an unsharp measurement of the $(k-1)$th Alice (Alice$_{k-1}$) and an unsharp measurement of the $(l-1)$th Bob (Bob$_{l-1}$) can be written as

$$\begin{aligned}\rho_{n,k,l} &= \frac{1}{2^{n-1}n}\sum_{a,b\in\{+,-\}}\sum_{x=1}^{2^{n-1}}\sum_{y=1}^{n}\Big[\Big(\sqrt{E^a_{n,x,k-1}}\otimes\sqrt{E^b_{n,y,l-1}}\Big)\rho_{n,k-1,l-1}\Big(\sqrt{E^a_{n,x,k-1}}\otimes\sqrt{E^b_{n,y,l-1}}\Big)\Big]\\ &= \sqrt{1-\eta^2_{n,k-1}}\,\sigma_{n,k-1,l-1} + \frac{1-\sqrt{1-\eta^2_{n,k-1}}}{2^{n-1}}\sum_{a\in\{+,-\}}\sum_{x=1}^{2^{n-1}}\big(\Pi^a_{A_{n,x,k-1}}\otimes\mathbb{I}\big)\sigma_{n,k-1,l-1}\big(\Pi^a_{A_{n,x,k-1}}\otimes\mathbb{I}\big), \end{aligned} \quad (13)$$

where $\sigma_{n,k-1,l-1}$, the reduced state after an unsharp measurement of the $(l-1)$th Bob, is given by

$$\sigma_{n,k-1,l-1} = \sqrt{1-\chi^2_{n,l-1}}\,\rho_{n,k-1,l-1} + \frac{1-\sqrt{1-\chi^2_{n,l-1}}}{n}\sum_{b\in\{+,-\}}\sum_{y=1}^{n}\big(\mathbb{I}\otimes\Pi^b_{B_{n,y,l-1}}\big)\rho_{n,k-1,l-1}\big(\mathbb{I}\otimes\Pi^b_{B_{n,y,l-1}}\big). \quad (14)$$

If the initial system state shared by Alice$_1$ and Bob$_1$ is a maximally entangled state as mentioned earlier, the quantum value of the Bell expression (4) for Alice$_k$ and Bob$_l$ is obtained as

$$\big(\mathcal{B}^{k,l}_n\big)_Q = 2^{n-1}\sqrt{n}\prod_{p=1}^{k-1}\gamma^A_p\prod_{q=1}^{l-1}\gamma^B_q\,\eta_{n,k}\chi_{n,l}, \quad (15)$$

where $\gamma^A_{n,p} = \frac{1+(n-1)\sqrt{1-\eta^2_{n,p}}}{n}$ and $\gamma^B_{n,q} = \frac{1+(n-1)\sqrt{1-\chi^2_{n,q}}}{n}$. Now the nonlocality and the preparation contextuality can be shared by Alice$_k$ and Bob$_l$, if $(\mathcal{B}^{k,l}_n)_Q > (\mathcal{B}_n)_L$ and $(\mathcal{B}^{k,l}_n)_Q > (\mathcal{B}_n)_{\mathrm{PNC}}$ are respectively satisfied.

IV. SHARING NONLOCALITY WITH an EQUAL NUMBER OF SEQUENTIAL ALICEs AND BOBs

We first examine how many sequentially independent pairs of Alices and Bobs can share nonlocality through the quantum violation of $(\mathcal{B}_n)_L$ in Eq. (7). In order to share nonlocality by Alice$_k$ and Bob$_l$, $(\mathcal{B}^{k,l})_Q > n\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$ should be satisfied. For the case of $n=2$, the Bell inequality in Eq. (7) reduces to the CHSH inequality. The quantum expression (13) shared between Alice$_1$ and Bob$_1$ reduces to $(\mathcal{B}_2^{1,1})_Q = 2\sqrt{2}\eta_{2,1}\chi_{2,1}$. Then quantum violation of the Bell inequality by Alice$_1$ and Bob$_1$ requires $\eta_{2,1}=\chi_{2,1}\geqslant 0.841$ and the maximum violation of $2\sqrt{2}$ is obtained for $\eta_{2,1}=\chi_{2,1}=1$. The quantum value of the CHSH expression for Alice$_2$ and Bob$_2$ is obtained as

$$(\mathcal{B}_2^{2,2})_Q = \frac{2\sqrt{2}}{4}\left[\left(1+\sqrt{1-\eta_{2,1}^2}\right)\left(1+\sqrt{1-\chi_{2,1}^2}\right)\eta_{2,2}\chi_{2,2}\right], \tag{16}$$

which is dependent on the sharpness parameters of Alice$_1$ and Bob$_1$. By considering the critical values $\eta^*_{2,1}=\chi^*_{2,1}=0.841$ such that Alice$_1$ and Bob$_1$ just violate the Bell inequality, the maximum quantum value for Alice$_2$ and Bob$_2$ is obtained to be 1.679 when $\eta_{2,2}=\chi_{2,2}=1$. It approaches 2 for either or both of $\eta_{2,2}$ and $\chi_{2,2}$ greater than 1. Hence, within the valid range of sharpness parameters the nonlocality cannot be shared by Alice$_2$ and Bob$_2$. This result has already been studied in [37,38].

For $n=3$, the Bell inequality in Eq. (7) becomes Gisin's elegant Bell inequality with local bound 6. The quantum expression of the Bell inequality for Alice$_1$ and Bob$_1$ is obtained as $4\sqrt{3}\eta_{3,1}\chi_{3,1}$. The quantum violation of the Bell inequality by Alice$_1$ and Bob$_1$ requires $\eta_{3,1}=\chi_{3,1}\geqslant 0.931$ and the maximum violation of $4\sqrt{3}$ is obtained for $\eta_{2,1}=\chi_{2,1}=1$. The quantum value of the elegant Bell expression for Alice$_2$ and Bob$_2$ is

$$(\mathcal{B}_3^{2,2})_Q = \frac{4\sqrt{3}}{9}\left[\left(1+2\sqrt{1-\eta_{3,1}^2}\right)\left(1+\sqrt{1-2\chi_{3,1}^2}\right)\eta_{3,2}\chi_{3,2}\right]. \tag{17}$$

By considering the critical values $\eta^*_{3,1}=\chi^*_{3,1}=0.931$, the maximum quantum value for Alice$_2$ and Bob$_2$ is obtained to be 2.304 at $\eta_{3,2}=\chi_{3,2}=1$, which is much less than the local bound 6. Hence, also for the case of $n=3$, the nonlocality cannot be shared by Alice$_2$ and Bob$_2$.

It is not expected that more than one pair of Alice and Bob can share the nonlocality for large n, but for completeness we examine a higher value of n. In Fig. 1 we have plotted the local bound $(\mathcal{B}_n)_L$ (blue solid line with circles) and maximum quantum values obtained by the Alice$_1$-Bob$_1$ pair, $(\mathcal{B}_n^{11})_Q$ (red dashed line with squares), and the Alice$_2$-Bob$_2$ pair, $(\mathcal{B}_n^{22})_Q$ (orange dashed line with diamonds), up to $n=10$. From Fig. 1 it can be seen that the maximum quantum value of $(\mathcal{B}_n^{2,2})_Q$ for the Alice$_2$-Bob$_2$ pair (orange diamonds) always remains less than the respective local bound $(\mathcal{B}_n)_L$ (blue circles). It can also be seen that the difference between $(\mathcal{B}_n)_L$ (blue circles) and $(\mathcal{B}_n^{2,2})_Q$ (orange diamonds) increases as n increases. We then can say that if both Alice and Bob perform an unsharp measurement, using the violation of the local bound

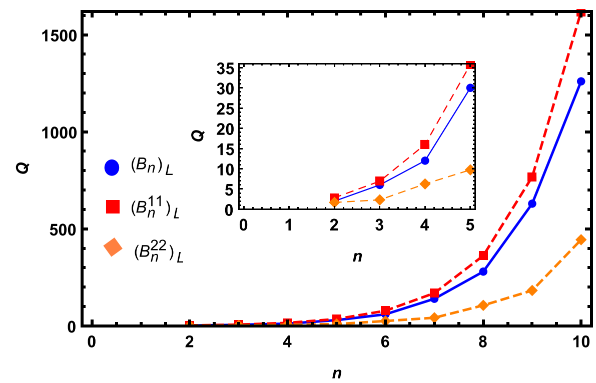

FIG. 1. Local bound (blue line with circles) and maximum quantum value obtained by the Alice$_1$-Bob$_1$ pair (red dashed line with squares) and the Alice$_2$-Bob$_2$ pair (orange dashed line with diamonds) up to $n=10$ bits.

of the Bell expression $\mathcal{B}_n$ in Eq. (4), the possibility of sharing a nonlocality by the second pair of Alice and Bob decreases as n increases, i.e., sharing of nonlocality remain restricted to one pair of sequential observers as claimed in [37,38]. Now, since $(\mathcal{B}_n)_L > (\mathcal{B}_n)_{\text{PNC}}$, we may expect that the sharing of preparation contextuality can be demonstrated by more pairs of sequential Alices and Bobs. In the next section we study this through the violation of the preparation noncontextual inequality in Eq. (10).

V. SHARING PREPARATION CONTEXTUALITY

Note that for the case of $n=2$, the Bell inequality in Eq. (4) reduces to the CHSH inequality for which both the local and preparation noncontextual bounds are the same. Hence, the analysis for $n=2$ remains the same as we did for sharing the nonlocality. As mentioned, the Bell inequality in Eq. (4) has two classical bounds, the local bound and the preparation noncontextual bound, but the quantum bound remains same. The quantum expression for Alice$_1$ and Bob$_1$ for $n=3$ is again $4\sqrt{3}\eta_{3,1}\chi_{3,1}$. The quantum violation of the preparation noncontextuality bound $(\mathcal{B}_n)_{\text{PNC}}\leqslant 4$ requires that $\eta_{3,1}=\chi_{3,1}\geqslant 0.840$, which is less than what is required for sharing the nonlocality. For the case of Alice$_2$ and Bob$_2$, the quantum expression of the Bell inequality is given in Eq. (16). Then, by considering the critical values $\eta^*_{3,1}=\chi^*_{3,1}=0.840$ such that Alice$_1$ and Bob$_1$ just violate the Bell inequality, the maximum quantum value of the Bell expression for Alice$_2$ and Bob$_2$ is obtained to be 4.073 at $\eta_{3,2}=\chi_{3,2}=1$. Thus, for $n=3$, sharing of the preparation contextuality can be demonstrated for the second pair of sequential observers.

Next the quantum value of the elegant Bell expression for Alice$_3$ and Bob$_3$ can be written as

$$(\mathcal{B}_3^{3,3})_Q = \frac{4\sqrt{3}}{81}\left[\left(1+2\sqrt{1-\eta_{3,1}^2}\right)\left(1+2\sqrt{1-\eta_{3,2}^2}\right)\right. \\ \left.\times\left(1+\sqrt{1-2\chi_{3,1}^2}\right)\left(1+\sqrt{1-2\chi_{3,2}^2}\right)\eta_{3,3}\chi_{3,3}\right], \tag{18}$$

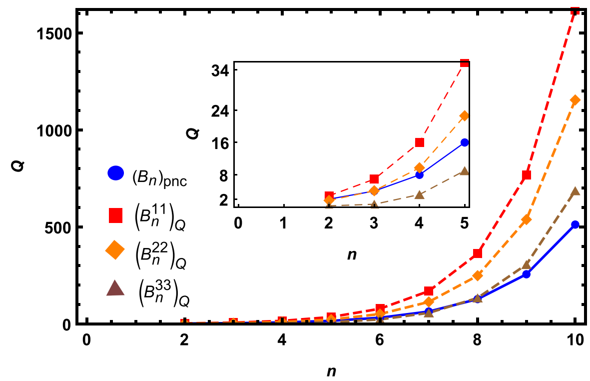

FIG. 2. Preparation noncontextuality bound (blue solid line with circles) and the maximum quantum value obtained by the Alice$_1$-Bob$_1$ pair (red dashed line with squares), the Alice$_2$-Bob$_2$ pair (orange dashed line with diamonds), and the Alice$_3$-Bob$_3$ pair (brown dashed line with triangles) up to $n = 10$ bits.

which depends on the sharpness parameter of all the previous observers Alice$_1$, Bob$_1$, Alice$_2$, and Bob$_2$. Substituting the critical values $\eta^*_{3,1} = \chi^*_{3,1} = 0.840$ and $\eta^*_{3,2} = \chi^*_{3,2} = 9.742$, the maximum quantum value of $(\mathcal{B}_3^{3,3})_Q$ is obtained to be 0.727 at $\eta_{3,3} = \chi_{3,3} = 1$. Hence, for $n = 3$ only two pairs of Alices and Bobs can share preparation contextuality.

Further, we have numerically studied the case for $n = 4$ and found that the quantum value for Alice$_3$ and Bob$_3$ increases with n, but cannot beat preparation noncontextuality bound. From Fig. 2 it can be seen that the sharing of preparation contextuality is restricted for Alice$_2$ and Bob$_2$ up to $n = 7$. For $n = 8$, the preparation noncontextuality bound is 128 and the maximum quantum value obtained for Alice$_3$ and Bob$_3$ is 134.012; thereby Alice$_3$ and Bob$_3$ can also share the preparation contextuality. In Fig. 2 we show the result up to $n = 10$. However, for a higher value of n, the sharing of preparation contextuality can be demonstrated for more sequential Alices and Bobs. We analytically show that for a sufficiently large value of n, an unbounded number of Alices and Bobs can share preparation contextuality.

VI. GENERALIZATION OF SHARING NONTRIVIAL PREPARATION CONTEXTUALITY

In order to show the sharing of preparation contextuality by an arbitrary pair of sequential observers, from Eq. (15) we write the general condition on the sharpness parameters for Alice$_k$ and Bob$_l$ for sequentially sharing preparation contextuality, given by

$$\eta_{n,k}\chi_{n,l} > \frac{1}{\sqrt{n}\prod_{p=1}^{k-1}\gamma^A_{n,p}\prod_{q=1}^{l-1}\gamma^B_{n,q}}. \tag{19}$$

For convenience let us assume that $l = k$ and the unsharpness parameters of Alice$_k$ and Bob$_k$ for a given k are equal, i.e., $\chi_{n,l} = \eta_{n,k}$. Then the condition on the unsharpness parameter for sharing preparation contextuality becomes

$$\eta^2_{n,k} > \frac{1}{\sqrt{n}\left(\prod_{p=1}^{k-1}\gamma_{n,p}\right)^2}. \tag{20}$$

Using Eq. (20), the critical value of the sharpness parameter of Alice$_k$ and Bob$_k$ above which violation of noncontextual inequality in Eq. (10) is obtained is given by

$$\eta^2_{n,k} = \frac{1}{\sqrt{n}\left(\gamma_{n,k-1}\prod_{p=1}^{k-2}\gamma_{n,p}\right)^2}. \tag{21}$$

Then, in order to share the contextuality by Alice$_k$ and Bob$_k$, the sharpness parameter requires

$$\eta^2_{n,k} \geqslant \frac{1}{\sqrt{n}\left(\prod_{p=1}^{k-1}\gamma_{n,p}\right)^2}. \tag{22}$$

Now, for Alice$_{k-1}$ and Bob$_{k-1}$, the critical value of the sharpness parameter is

$$\eta^2_{n,k-1} = \frac{1}{\sqrt{n}\left(\prod_{p=1}^{k-2}\gamma_{n,p}\right)^2}. \tag{23}$$

Using Eq. (23), the condition on the sharpness parameter for sharing nontrivial preparation contextuality for Alice$_k$ and Bob$_k$ in Eq. (22) reduces to

$$\eta_{n,k} \geqslant \frac{\eta_{n,k-1}}{\gamma_{n,k-1}}. \tag{24}$$

Since $\gamma_{n,k-1} = \frac{1+(n-1)\sqrt{1-\eta^2_{n,p}}}{n} > \sqrt{1-\eta^2_{n,k-1}}$, we can rewrite Eq. (24) as

$$\eta_{n,k} \geqslant \frac{\eta_{n,k-1}}{\sqrt{1-\eta^2_{n,k-1}}}. \tag{25}$$

Here it should be noted that the critical value of the unsharpness parameter for Alice$_1$ and Bob$_1$ requires $\eta^{*2}_{n,1} = 1/\sqrt{n}$, i.e., $\eta^*_{n,1} = (1/n)^{1/4}$. Using Eq. (25), we have the critical value of unsharpness parameter for Alice$_2$ and Bob$_2$ as $\eta_{n,2} = \frac{1}{\sqrt{\sqrt{n}-1}}$ and for Alice$_3$ and Bob$_3$, $\eta_{n,3} = \frac{1}{\sqrt{\sqrt{n}-2}}$. Thus, the unsharpness parameter for Alice$_k$ and Bob$_k$ has to satisfy

$$\eta_{n,k} \geqslant \frac{1}{\sqrt{\sqrt{n}-(k-1)}}. \tag{26}$$

In other words, preparation contextuality can be shared by k arbitrary pairs of Alice and Bob if Eq. (26) is satisfied. If the final pair of Alice and Bob performs the sharp measurement, we have the condition

$$n(k) \geqslant k^2 \tag{27}$$

as k is very large. Hence, using the Bell expression for $n = k^2$, the preparation contextuality can be shared by k independent sequential pairs of Alice and Bob. In Fig. 3 we have shown that using quantum violation of the Bell inequality for $n = 100$, at most ten pairs of Alices and Bobs can sequentially share preparation contextuality.

VII. CONCLUSION

In summary, we have examined the maximum number of pairs of Alice and Bob that can share the nonlocality and the preparation contextuality. Our study is based on the quantum violation of suitable bipartite Bell inequality, arising from

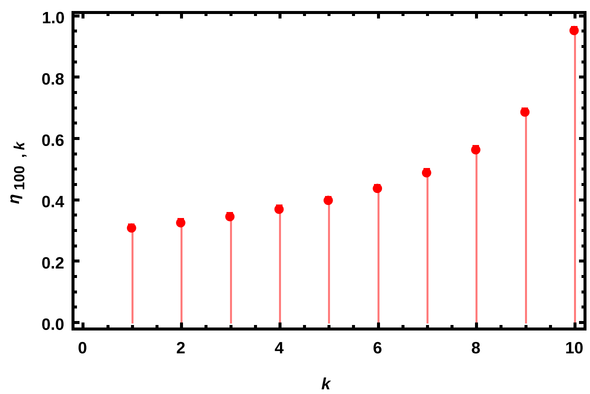

FIG. 3. Minimum value of the sharpness parameter of ten pairs of Alice and Bob required for violating the preparation noncontextual bound for the family of Bell expressions for $n = 100$.

a communication game known as parity-oblivious random access code. Such a Bell inequality involves n arbitrary measurements by one party and 2^{n-1} measurements by the other party. As mentioned, it has two different classical bounds, the local bound and the preparation noncontextual bound, which is lower than the local bound for $n > 2$.

We demonstrated that if the sharing is considered for both parties then at most one pair of Alice and Bob can share the nonlocality irrespective of the value of n. This result is in accordance with previous works [37,38] which considered the $n = 2$ case. Since the preparation noncontextual bound is lower than the local bound and the optimal quantum value remains the same, there may be more pairs that can share the preparation contextuality. Indeed, we have shown that the preparation contextuality can be shared by an arbitrary pair of independent sequential observers at both ends of the bipartite Bell experiment for a sufficiently large value of n.

Our work has a potential application for generating certified device-independent randomness in the sequential scenario following the line of work developed in [54]. Further studies along this line are thus called for. Finally, we note here that the preparation contextuality is a weaker correlation than the nonlocality. It is worthwhile to explore the possibility of formulating a suitable local realistic inequality to investigate the sharing of nonlocality for more than one pair of sequential observers. It would also be interesting to formulate a new preparation noncontextual inequality for multioutcome and multiparty scenarios to demonstrate the sharing of preparation contextuality. Studies along this line could be an interesting avenue of research.

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