Exact entanglement cost of quantum states and channels under positive-partial-transpose-preserving operations

Xin Wang [1,2,*] and Mark M. Wilde [3,4,†]

[1] *Institute for Quantum Computing, Baidu Research, Beijing 100093, China*
[2] *Joint Center for Quantum Information and Computer Science, University of Maryland, College Park, Maryland 20742, USA*
[3] *Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA*
[4] *School of Electrical and Computer Engineering, Cornell University, Ithaca, New York 14850, USA*

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This paper establishes single-letter formulas for the exact entanglement cost of simulating quantum channels under free quantum operations that completely preserve positivity of the partial transpose (PPT). First, we introduce the κ-entanglement measure for point-to-point quantum channels, based on the idea of the κ entanglement of bipartite states, and we establish several fundamental properties for it, including amortization collapse, monotonicity under PPT superchannels, additivity, normalization, faithfulness, and nonconvexity. Second, we introduce and solve the exact entanglement cost for simulating quantum channels in both the parallel and sequential settings, along with the assistance of free PPT-preserving operations. In particular, we establish that the entanglement cost in both cases is given by the same single-letter formula, the κ-entanglement measure of a quantum channel. We further show that this cost is equal to the largest κ entanglement that can be shared or generated by the sender and receiver of the channel. This formula is calculable by a semidefinite program, thus allowing for an efficiently computable solution for general quantum channels. Noting that the sequential regime is more powerful than the parallel regime, another notable implication of our result is that both regimes have the same power for exact quantum channel simulation, when PPT superchannels are free. For several basic Gaussian quantum channels, we show that the exact entanglement cost is given by the Holevo-Werner formula [Holevo and Werner, Phys. Rev. A **63**, 032312 (2001)], giving an operational meaning of the Holevo-Werner quantity for these channels.

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I. INTRODUCTION

A. Background

Quantum entanglement, the most nonclassical manifestation of quantum mechanics, has found use in a variety of physical tasks in quantum information processing, quantum cryptography, thermodynamics, and quantum computing [1]. A natural and fundamental problem is to develop a theoretical framework to quantify and describe it. In spite of remarkable recent progress in the resource theory of entanglement (for reviews see, e.g., Refs. [1,2]), many fundamental challenges have remained open.

One of the most important aspects of the resource theory of entanglement consists of the interconversions of states, with respect to a class of free operations. In particular, the problem of *entanglement dilution* [3] asks: Given a target bipartite state ρ_{AB} and a canonical unit of entanglement represented by the Bell state (or ebit) $\Phi_2 \equiv |\Phi_2\rangle\langle\Phi_2|$, where $|\Phi_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, what is the minimum rate at which we can produce copies of ρ_{AB} from copies of Φ_2 under a chosen set of free operations?

The *entanglement cost* [4] was introduced to quantify the minimal rate R of converting $\Phi_2^{\otimes nR}$ to $\rho_{AB}^{\otimes n}$ with an arbitrarily high fidelity in the limit as n becomes large. When local operations and classical communication (LOCC) are allowed for free, the authors of Ref. [5] proved that the entanglement cost is equal to the regularized entanglement of formation [4]. When the free operations consist of quantum operations that completely preserve positivity of the partial transpose (the PPT-preserving operations of Refs. [6,7]), it is known that the entanglement cost is not equal to the regularized entanglement of formation [8–10].

The *exact entanglement cost* [8] is an alternative and natural way to quantify the cost of entanglement dilution, being defined as the smallest asymptotic rate R at which $\Phi_2^{\otimes nR}$ is required to reproduce $\rho_{AB}^{\otimes n}$ exactly. The exact entanglement cost under PPT-preserving operations (PPT entanglement cost) was introduced and solved for a large class of quantum states in Ref. [8], but it has hitherto remained unknown for general quantum states until the recent solutions in Refs. [11,12] (note that Ref. [12] is a companion paper of the original announcement in Ref. [11]).

The above resource-theoretic problems can alternatively be phrased as simulation problems: How many copies of Φ_2 are needed to simulate n copies of a given bipartite state ρ_{AB}? As discussed above, the simulation can be either approximate, such that a verifier has little chance of distinguishing the

*wangxin73@baidu.com
†wilde@cornell.edu

simulation from the ideal case, while it can also be exact, such that a verifier has no chance at all for distinguishing the simulation from the ideal case.

With this perspective, it is also natural to consider the simulation of a quantum channel when allowing some set of operations for free and metering the entanglement cost of the simulation. The authors of Ref. [13] defined the entanglement cost of a channel to be the smallest rate R at which $\Phi_2^{\otimes nR}$ is needed, along with the free assistance of LOCC, to simulate the channel $\mathcal{N}^{\otimes n}$ in such a way that a verifier would have little chance of distinguishing the simulation from the ideal case of $\mathcal{N}^{\otimes n}$. In Ref. [13], it was shown that the regularized entanglement of formation of the channel is equal to its entanglement cost, thus extending the result of Ref. [5] in a natural way.

In a recent work [14], it was observed that the channel simulation task defined in Ref. [13] is actually a particular kind of simulation, called a parallel channel simulation. The paper [14] then defined an alternative notion of channel simulation, called sequential channel simulation, in which the goal is to simulate n uses of the channel $\mathcal{N}$ in such a way that the most general verification strategy would have little chance of distinguishing the simulation from the ideal n uses of the channel. Although a general formula for the entanglement cost in this scenario was not found, it was determined for several key channel models, including erasure, dephasing, three-dimensional Holevo–Werner, and single-mode pure-loss and pure-amplifier bosonic Gaussian channels.

B. Summary of results

In this paper, we solve significant questions in the resource theory of entanglement, one of which has remained open since the inception of entanglement theory over two decades ago. Namely, we prove that the exact PPT-entanglement cost for quantum channels has an efficiently computable, single-letter formula, reflecting the fundamental entanglement structure of bipartite quantum states and channels. Along with this claim, we prove that the exact parallel and sequential entanglement costs of quantum channels are given by the same efficiently computable, single-letter formula.

We note here that all our results apply to the resource theory of NPT (nonpositive partial transpose) entanglement, introduced in Refs. [6,7] and considered in Ref. [8], rather than to the more standard resource theory of entanglement, as introduced in Ref. [4]. The key difference is that the free operations allowed here are completely PPT-preserving (C-PPT-P) operations, whereas the free operations allowed in the standard resource theory are LOCC. Since LOCC is contained in the set of C-PPT-P operations, the operational quantities considered here provide bounds on operational quantities in the standard resource theory.

Our paper is structured as follows. We first introduce the κ-entanglement measure of a bipartite state and review its desirable properties [15], including monotonicity under completely PPT-preserving channels, additivity, normalization, faithfulness, nonconvexity, and nonmonogamy. For finite-dimensional states, it is also efficiently computable by means of a semidefinite program. In particular, the κ entanglement is equal to the exact entanglement cost of a quantum state. We further evaluate the κ entanglement (and the exact entanglement cost) for several bipartite states of interest (cf. Sec. II B), including isotropic states, Werner states, maximally correlated states, some states supported on the 3×3 antisymmetric subspace, and all bosonic Gaussian states.

In Sec. III, we extend the κ-entanglement measure from bipartite states to point-to-point quantum channels. We prove that it also satisfies several desirable properties, including nonincrease under amortization, monotonicity under a class of PPT superchannels, additivity, normalization, faithfulness, and nonconvexity. For finite-dimensional channels, it is also efficiently computable by means of a semidefinite program.

In Sec. IV, we prove that the κ entanglement of channels has a direct operational meaning as the entanglement cost of both parallel and sequential channel simulation. Thus, the theory of channel simulation significantly simplifies for the setting in which completely PPT-preserving channels are allowed for free. In addition to all the properties that it satisfies, this operational interpretation solidifies the κ entanglement of a channel as a foundational measure of the entanglement of a quantum channel.

As a last contribution of this paper (cf. Secs. V and VI), we evaluate the κ entanglement (and exact entanglement cost) of several important channel models, including erasure, depolarizing, dephasing, and amplitude-damping channels. We also leverage recent results in the literature [16] regarding the teleportation simulation of bosonic Gaussian channels to evaluate the κ entanglement and exact entanglement cost for several fundamental bosonic Gaussian channels. We remark that these latter results provide a direct operational interpretation of the Holevo-Werner quantity [17] for these channels.

Finally, we conclude with a summary and some open questions.

II. κ-ENTANGLEMENT MEASURE AND EXACT ENTANGLEMENT COST OF QUANTUM STATES

A. κ-entanglement measure and its operational meaning

We first recall an entanglement measure called the κ-entanglement measure for a bipartite state, which was introduced and analyzed in the original arXiv version of this paper in 2018 [11] and published in the companion paper [12]. Here, we review the important properties of this entanglement measure and its operational meaning as the exact entanglement cost.

Definition 1 (κ-entanglement measure [12]). Let ρ_{AB} be a bipartite state acting on a separable Hilbert space. The κ-entanglement measure is defined as follows:

$$E_\kappa(\rho_{AB}) := \inf_{S_{AB}\geqslant 0} \{\log_2 \operatorname{Tr} S_{AB} : -S_{AB}^{T_B} \leqslant \rho_{AB}^{T_B} \leqslant S_{AB}^{T_B}\}. \quad (1)$$

In the case that the state ρ_{AB} acts on a finite-dimensional Hilbert space, then $E_\kappa(\rho_{AB})$ is calculable by a semidefinite program, and it is thus efficiently computable with respect to the dimension of the Hilbert space. Throughout this paper, we consider completely PPT-preserving operations [6,7], defined as a bipartite operation $\mathcal{P}_{AB\to A'B'}$ (completely positive map) such that the map $T_{B'} \circ \mathcal{P}_{AB\to A'B'} \circ T_B$ is also completely positive, where T_B and $T_{B'}$ denote the partial transpose map acting on the input system B and the output system B', respectively. If $\mathcal{P}_{AB\to A'B'}$ is also trace preserving, such that it is a quantum channel, and $T_{B'} \circ \mathcal{P}_{AB\to A'B'} \circ T_B$ is also completely positive, then we say that $\mathcal{P}_{AB\to A'B'}$ is a completely PPT-preserving channel.

Monotonicity under completely PPT-preserving channels. The most important property of the κ-entanglement measure is that it does not increase under the action of a completely PPT-preserving channel. Note that an LOCC channel [4,18], as considered in entanglement theory, is a special kind of completely PPT-preserving channel, as observed in Refs. [6,7].

Theorem 1 (Monotonicity [12]). Let ρ_{AB} be a quantum state acting on a separable Hilbert space, and let $\{\mathcal{P}^x_{AB\to A'B'}\}_x$ be a set of completely positive, trace nonincreasing maps that are each completely PPT preserving, such that the sum map $\sum_x \mathcal{P}^x_{AB\to A'B'}$ is a quantum channel. Then the following entanglement monotonicity inequality holds:

$$E_\kappa(\rho_{AB}) \geqslant \sum_{x\,:\,p(x)>0} p(x) E_\kappa\left(\frac{\mathcal{P}^x_{AB\to A'B'}(\rho_{AB})}{p(x)}\right), \tag{2}$$

where $p(x) := \operatorname{Tr} \mathcal{P}^x_{AB\to A'B'}(\rho_{AB})$. In particular, for a completely PPT-preserving quantum channel $\mathcal{P}_{AB\to A'B'}$, the following inequality holds:

$$E_\kappa(\rho_{AB}) \geqslant E_\kappa(\mathcal{P}_{AB\to A'B'}(\rho_{AB})). \tag{3}$$

Dual representation and additivity. The optimization problem dual to $E_\kappa(\rho_{AB})$ in Definition 1 is as follows:

$$E^{\text{dual}}_\kappa(\rho_{AB}) := \sup_{V^{T_B}_{AB},\, W^{T_B}_{AB} \geqslant 0} \{\log_2 \operatorname{Tr} \rho_{AB}(V_{AB} - W_{AB}) : V_{AB} + W_{AB} \leqslant \mathbb{1}_{AB}\}, \tag{4}$$

which can be found by the Lagrange multiplier method (see, e.g., Ref. [19], Sec. 1.2.2]). By weak duality [19], Sec. 1.2.2], we have for every bipartite state ρ_{AB} acting on a separable Hilbert space that

$$E^{\text{dual}}_\kappa(\rho_{AB}) \leqslant E_\kappa(\rho_{AB}). \tag{5}$$

For all finite-dimensional states ρ_{AB}, strong duality holds, so

$$E_\kappa(\rho_{AB}) = E^{\text{dual}}_\kappa(\rho_{AB}). \tag{6}$$

This follows as a consequence of Slater's theorem. By employing the strong duality equality in (6) for the finite-dimensional case, along with the approach from Ref. [20], we conclude that the following equality holds for all bipartite states ρ_{AB} acting on a separable Hilbert space:

$$E_\kappa(\rho_{AB}) = E^{\text{dual}}_\kappa(\rho_{AB}). \tag{7}$$

We provide an explicit proof of (7) in Appendix. Both the primal and dual SDPs for E_κ are important, as the combination of them allows for proving the following additivity of E_κ with respect to tensor-product states.

Proposition 2. (Additivity [12]) For all bipartite states ρ_{AB} and $\omega_{A'B'}$ acting on separable Hilbert spaces, the following additivity identity holds:

$$E_\kappa(\rho_{AB} \otimes \omega_{A'B'}) = E_\kappa(\rho_{AB}) + E_\kappa(\omega_{A'B'}). \tag{8}$$

Relation to logarithmic negativity. There is an inequality relating E_κ to the logarithmic negativity [21,22], defined as

$$E_N(\rho_{AB}) := \log_2 \left\|\rho^{T_B}_{AB}\right\|_1. \tag{9}$$

Let ρ_{AB} be a bipartite state acting on a separable Hilbert space. Then

$$E_\kappa(\rho_{AB}) \geqslant E_N(\rho_{AB}). \tag{10}$$

If ρ_{AB} satisfies the binegativity condition

$$\left|\rho^{T_B}_{AB}\right|^{T_B} \geqslant 0, \tag{11}$$

then

$$E_\kappa(\rho_{AB}) = E_N(\rho_{AB}). \tag{12}$$

Normalization. E_κ is normalized on maximally entangled states, and for finite-dimensional states, it achieves its largest value on maximally entangled states.

Proposition 3 (Normalization [12]). Let Φ^M_{AB} be a maximally entangled state of Schmidt rank M. Then,

$$E_\kappa(\Phi^M_{AB}) = \log_2 M. \tag{13}$$

Furthermore, for every bipartite state ρ_{AB}, the following bound holds:

$$E_\kappa(\rho_{AB}) \leqslant \log_2 \min\{d_A, d_B\}, \tag{14}$$

where d_A and d_B denote the dimensions of systems A and B, respectively.

Faithfulness. E_κ is faithful in the sense that it is nonnegative and equal to zero if and only if the state is a PPT state. To be specific, the following proposition holds.

Proposition 4 (Faithfulness [12]). For a state ρ_{AB} acting on a separable Hilbert space, we have that $E_\kappa(\rho_{AB}) \geqslant 0$ and $E_\kappa(\rho_{AB}) = 0$ if and only if $\rho^{T_B}_{AB} \geqslant 0$.

No convexity. The κ-entanglement measure is not generally convex. Due to (12) and the fact that the binegativity condition in (11) holds for every two-qubit state [23], the nonconvexity of E_κ boils down to finding a two-qubit example for which the logarithmic negativity is not convex. In particular, let us choose the two-qubit states

$$\rho_1 = \Phi_2, \quad \rho_2 = \tfrac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \tag{15}$$

and their average $\rho = \frac{1}{2}(\rho_1 + \rho_2)$. By direct calculation, we have

$$E_\kappa(\rho) > \tfrac{1}{2}(E_\kappa(\rho_1) + E_\kappa(\rho_2)), \tag{16}$$

which implies that the κ entanglement is not convex.

No monogamy. If an entanglement measure E is monogamous [24–26], then the following inequality should be satisfied for every tripartite state ρ_{ABC}:

$$E(\rho_{AB}) + E(\rho_{AC}) \leqslant E(\rho_{A(BC)}), \tag{17}$$

where the entanglement in $E(\rho_{A(BC)})$ is understood to be with respect to the bipartite cut between systems A and BC. It is known that some entanglement measures satisfy the monogamy inequality above [24,26]. However, the κ-entanglement measure is not generally monogamous. Consider a state $|\psi\rangle\langle\psi|_{ABC}$ of three qubits, where $|\psi\rangle_{ABC} = \frac{1}{2}(|000\rangle_{ABC} + |011\rangle_{ABC} + \sqrt{2}|110\rangle_{ABC})$. Due the fact that $|\psi\rangle_{ABC}$ can be written as

$$|\psi\rangle_{ABC} = [|0\rangle_A \otimes |\Phi\rangle_{BC} + |1\rangle_A \otimes |10\rangle_{BC}]/\sqrt{2}, \tag{18}$$

where $|\Phi\rangle_{BC} = [|00\rangle_{BC} + |11\rangle_{BC}]/\sqrt{2}$, this state is locally equivalent to $|\Phi\rangle_{AB} \otimes |0\rangle_C$ with respect to the bipartite cut $A|BC$. One then finds that $E_\kappa(\psi_{A(BC)}) = E_\kappa(\Phi_{AB}) = E_N(\Phi_{AB}) = 1$. Furthermore, we have that $E_\kappa(\psi_{AB}) = E_N(\psi_{AB}) = \log_2 \frac{3}{2}$, and $E_\kappa(\psi_{AC}) = E_N(\psi_{AC}) = \log_2 \frac{3}{2}$, which implies that

$$E_\kappa(\psi_{AB}) + E_\kappa(\psi_{AC}) > E_\kappa(\psi_{A(BC)}). \tag{19}$$

κ-entanglement measure is equal to the exact PPT-entanglement cost. The κ entanglement of a bipartite state is equal to its exact entanglement cost, when completely PPT-preserving channels are allowed for free. Let Ω represent a set of free channels, which can be either LOCC or PPT. The one-shot exact entanglement cost of a state ρ_{AB}, under the Ω channels, is defined as

$$E_\Omega^{(1)}(\rho_{AB}) = \inf_{\Lambda\in\Omega} \left\{\log_2 d : \rho_{AB} = \Lambda_{\hat{A}\hat{B}\to AB}\left(\Phi^d_{\hat{A}\hat{B}}\right)\right\}, \tag{20}$$

where $\Phi^d_{\hat{A}\hat{B}} = [1/d]\sum_{i,j=1}^d |ii\rangle\langle jj|_{\hat{A}\hat{B}}$ represents the standard maximally entangled state of Schmidt rank d. The exact entanglement cost of a bipartite state ρ_{AB}, under the Ω channels, is defined as

$$E_\Omega(\rho_{AB}) = \limsup_{n\to\infty} \frac{1}{n} E_\Omega^{(1)}\left(\rho_{AB}^{\otimes n}\right). \tag{21}$$

The exact entanglement cost under LOCC channels was previously considered in Refs. [10,27–29], while the exact entanglement cost under PPT channels was considered in Refs. [8,30]. In Ref. [8], the following bounds were given for E_{PPT}:

$$E_N(\rho_{AB}) \leqslant E_{\mathrm{PPT}}(\rho_{AB}) \leqslant \log_2 Z(\rho_{AB}), \tag{22}$$

the lower bound being the logarithmic negativity recalled in (9), and the upper bound defined as

$$Z(\rho_{AB}) := \mathrm{Tr}\left|\rho_{AB}^{T_B}\right| + \dim(\rho_{AB}) \max\left\{0, -\lambda_{\min}\left(\left|\rho_{AB}^{T_B}\right|^{T_B}\right)\right\}. \tag{23}$$

Due to the presence of the dimension factor $\dim(\rho_{AB})$, the upper bound in (22) clearly only applies in the case that ρ_{AB} is finite-dimensional.

In what follows, we first recast $E_{\mathrm{PPT}}^{(1)}(\rho_{AB})$ as an optimization problem by building on previous developments in Refs. [8,30]. After that, we bound $E_{\mathrm{PPT}}^{(1)}(\rho_{AB})$ in terms of E_κ by observing that E_κ is a relaxation of the optimization problem for $E_{\mathrm{PPT}}^{(1)}(\rho_{AB})$. We then finally prove that $E_{\mathrm{PPT}}(\rho_{AB})$ is equal to E_κ.

Theorem 5 ([12]). Let ρ_{AB} be a bipartite state acting on a separable Hilbert space. Then the one-shot exact PPT-entanglement cost $E_{\mathrm{PPT}}^{(1)}(\rho_{AB})$ is given by the following optimization:

$$\begin{aligned} E_{\mathrm{PPT}}^{(1)}(\rho_{AB}) = \inf\{\log_2 m :&\\ -(m-1)G_{AB}^{T_B} \leqslant \rho_{AB}^{T_B} &\leqslant (m+1)G_{AB}^{T_B},\\ G_{AB} \geqslant 0,&\ \mathrm{Tr}\, G_{AB} = 1\}. \end{aligned} \tag{24}$$

Theorem 6 (Operational meaning [12]). Let ρ_{AB} be a bipartite state acting on a separable Hilbert space. Then the exact PPT-entanglement cost of ρ_{AB} is given by

$$E_{\mathrm{PPT}}(\rho_{AB}) = E_\kappa(\rho_{AB}). \tag{25}$$

Note that Theorem 6 constitutes a significant development for entanglement theory, showing that an entanglement measure is not only efficiently computable but also possesses a direct operational meaning. In the work of Refs. [31,32], it was conjectured that the regularized relative entropy of entanglement is equal to the entanglement cost and distillable entanglement of a bipartite quantum state, with the set of free operations being asymptotically nonentangling maps. However, in spite of the fact that the work of Refs. [31,32] conjectured a direct operational meaning to the regularized relative entropy of entanglement, this entanglement measure arguably has limited applications beyond being a formal expression, due to the fact that there is no known efficient procedure for computing it. See [33] for recent developments and discussions.

Furthermore, in prior work, most discussions about the structure and properties of entanglement are based on entanglement measures. However, none of these measures, with the exception of the regularized relative entropy of entanglement, possesses a direct operational meaning. Thus, the connection made by Theorem 6 allows for the study of the structure of entanglement via an entanglement measure possessing a direct operational meaning. Given that $E_\kappa = E_{\mathrm{PPT}}$ is neither convex nor monogamous, this raises questions of whether these properties should really be required or necessary for measures of entanglement, in contrast to the discussions put forward in Refs. [1,25] based on intuition. Furthermore, E_κ is additive (Proposition 2), so Theorem 6 implies that E_{PPT} is additive as well:

$$E_{\mathrm{PPT}}(\rho_{AB}\otimes\omega_{A'B'}) = E_{\mathrm{PPT}}(\rho_{AB}) + E_{\mathrm{PPT}}(\omega_{A'B'}). \tag{26}$$

Thus, E_{PPT} is the only known example of an operational quantity in entanglement theory for which the optimal rate is additive as a function of general quantum states.

B. Exact entanglement cost of particular bipartite states

To have a better understanding of exact entanglement cost, we evaluate the exact entanglement cost for particular bipartite states of interest, including isotropic states [34], Werner states [35], maximally correlated states [6,7], some states supported on the 3×3 antisymmetric subspace, and bosonic Gaussian states [36]. For isotropic and Werner states, the exact PPT-entanglement cost was already determined [8,10], and so we recall these developments here.

Let A and B be quantum systems, each of dimension d. For $t\in[0,1]$ and $d\geqslant 2$, an isotropic state is defined as follows [34]:

$$\rho_{AB}^{(t,d)} := t\Phi_{AB}^d + (1-t)\frac{\mathbb{1}_{AB} - \Phi_{AB}^d}{d^2-1}. \tag{27}$$

An isotropic state is PPT if and only if $t \leqslant 1/d$. It was shown in Ref. [10], Exercise 8.73] that $\rho_{AB}^{(t,d)}$ satisfies the binegativity condition: $|(\rho_{AB}^{(t,d)})^{T_B}|^{T_B} \geqslant 0$. By applying (22), this implies that

$$E_{\mathrm{PPT}}\left(\rho_{AB}^{(t,d)}\right) = E_N\left(\rho_{AB}^{(t,d)}\right) \tag{28}$$

$$= \begin{cases} \log_2 dt & \text{if } t > \frac{1}{d} \\ 0 & \text{if } t \leqslant \frac{1}{d}, \end{cases} \tag{29}$$

with the second equality shown in Refs. [10,37].

Let A and B be quantum systems, each of dimension d. A Werner state is defined for $p \in [0, 1]$ as [35]

$$W_{AB}^{(p,d)} := (1-p)\frac{2}{d(d+1)}\Pi_{AB}^{\mathcal{S}} + p\frac{2}{d(d-1)}\Pi_{AB}^{\mathcal{A}}, \quad (30)$$

where $\Pi_{AB}^{\mathcal{S}} := (\mathbb{1}_{AB} + F_{AB})/2$ and $\Pi_{AB}^{\mathcal{A}} := (\mathbb{1}_{AB} - F_{AB})/2$ are the projections onto the symmetric and antisymmetric subspaces of A and B, respectively, with F_{AB} denoting the swap operator. A Werner state is PPT if and only if $p \leqslant 1/2$. It was shown in Ref. [8] that $W_{AB}^{(p,d)}$ satisfies the binegativity condition: $|(W_{AB}^{(p,d)})^{T_B}|^{T_B} \geqslant 0$. By applying (22), this implies that [8]

$$E_{\mathrm{PPT}}\left(W_{AB}^{(p,d)}\right) = E_N\left(W_{AB}^{(p,d)}\right) \quad (31)$$

$$= \begin{cases} \log_2\left[\frac{2}{d}(2p-1)+1\right] & \text{if } p > 1/2 \\ 0 & \text{if } p \leqslant 1/2, \end{cases} \quad (32)$$

with the second equality shown in Refs. [10,37].

A maximally correlated state is defined as [6,7]

$$\rho_{AB}^{\mathbf{c}} := \sum_{i,j=0}^{d-1} c_{ij}|ii\rangle\langle jj|, \quad (33)$$

with the complex coefficients $\mathbf{c} := \{c_{ij}\}_{i,j}$ being chosen such that $\sum_{i,j=0}^{d-1} c_{ij}|i\rangle\langle j|$ is a legitimate quantum state. Noting that $(\rho_{AB}^{\mathbf{c}})^{T_B} = \sum_{i,j=0}^{d-1} c_{ij}|ij\rangle\langle ji|$, a direct calculation reveals that

$$\left|\left(\rho_{AB}^{\mathbf{c}}\right)^{T_B}\right| = \sum_{i,j=0}^{d-1} |c_{ij}||ij\rangle\langle ij|. \quad (34)$$

Considering that $|(\rho_{AB}^{\mathbf{c}})^{T_B}|^{T_B} = |(\rho_{AB}^{\mathbf{c}})^{T_B}| \geqslant 0$, we have that

$$E_{\mathrm{PPT}}\left(\rho_{AB}^{\mathbf{c}}\right) = E_N\left(\rho_{AB}^{\mathbf{c}}\right) = \log_2\left(\sum_{i,j}|c_{ij}|\right). \quad (35)$$

The maximally correlated state $\widehat{\omega}_\alpha$ was considered recently in Ref. [29],

$$\widehat{\omega}_{AB}^{\alpha} := \alpha\Phi_{AB}^2 + \frac{1-\alpha}{2}(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}) \quad (36)$$

$$= \frac{\alpha}{2}|00\rangle\langle 11|_{AB} + \frac{\alpha}{2}|11\rangle\langle 00|_{AB} + \frac{1}{2}|00\rangle\langle 00|_{AB} + \frac{1}{2}|11\rangle\langle 11|_{AB}, \quad (37)$$

where $\alpha \in [0, 1]$. The authors of Ref. [29] showed that the exact entanglement cost under LOCC is bounded as

$$\left\lfloor \frac{1}{\log_2(\alpha+1)} \right\rfloor^{-1} \geqslant E_{\mathrm{LOCC}}\left(\widehat{\omega}_{AB}^{\alpha}\right) \geqslant \log_2(\alpha+1) \quad (38)$$

for $0 < \alpha < \sqrt{2} - 1$. However, under PPT-preserving operations, by (35) it holds that

$$E_{\mathrm{PPT}}\left(\widehat{\omega}_{AB}^{\alpha}\right) = \log_2(\alpha+1) \quad (39)$$

for $\alpha \in [0, 1]$. This demonstrates that the lower bound in (38) can be understood as arising from the fact that the inequality $E_{\mathrm{LOCC}} \geqslant E_{\mathrm{PPT}}$ generally holds for an arbitrary bipartite state.

The next example indicates the irreversibility of exact PPT entanglement manipulation, and it also implies that E_{PPT} is generally not equal to the logarithmic negativity E_N. Consider the following rank-two state supported on the 3×3 antisymmetric subspace [38]:

$$\rho_v = \tfrac{1}{2}(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|), \quad (40)$$

with $|v_1\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and $|v_2\rangle = (|02\rangle - |20\rangle)/\sqrt{2}$. For the state ρ_v, it holds that

$$R_{\max}(\rho_v) = E_N(\rho_v) = \log_2\left(1 + \frac{1}{\sqrt{2}}\right) < E_{\mathrm{PPT}}(\rho_v) = 1$$

$$< \log_2 Z(\rho) = \log_2\left(1 + \frac{13}{4\sqrt{2}}\right), \quad (41)$$

where $R_{\max}(\rho_v)$ denotes the max-Rains relative entropy [39]. The strict inequalities in (41) also imply that both the lower and upper bounds from (22), i.e., from Ref. [8], are generally not tight.

The last examples that we consider are bosonic Gaussian states [36]. As shown in Ref. [8], all bosonic Gaussian states ρ_{AB}^G satisfy the binegativity condition $|(\rho_{AB}^G)^{T_B}|^{T_B} \geqslant 0$. Thus, as a consequence of Theorem 6 and Eq. (12), we conclude that

$$E_{\mathrm{PPT}}\left(\rho_{AB}^G\right) = E_N\left(\rho_{AB}^G\right) \quad (42)$$

for every bosonic Gaussian state ρ_{AB}^G. Note that an explicit expression for the logarithmic negativity of a bosonic Gaussian state is available in Ref. [40], Eq. (15)]. We stress again that it is not clear whether the equality in (42) follows from the upper bound in (22), given that the dimension of a bosonic Gaussian state is generally equal to infinity.

III. κ-ENTANGLEMENT MEASURE FOR QUANTUM CHANNELS

Quantum channels underlie the dynamics of quantum systems and they enable the manipulation of quantum states. To better effectively exploit quantum resources, it is important to understand the resource cost of quantum channels. In this section, we extend the κ-entanglement measure from bipartite states to point-to-point quantum channels. We establish several properties of the κ-entanglement of quantum channels, including the fact that it does not increase under amortization, that it is monotone under the action of a class of PPT superchannels, that it is additive, normalized, faithful, and that it is generally not convex. The fact that it is monotone under the action of a class of PPT superchannels is a basic property that we would expect to hold for a good measure of the entanglement of a quantum channel.

In what follows, we consider a channel $\mathcal{N}_{A\to B}$ that takes density operators acting on a separable Hilbert space $\mathcal{H}_A$ to those acting on a separable Hilbert space $\mathcal{H}_B$. We refer to such channels simply as quantum channels, regardless of whether $\mathcal{H}_A$ or $\mathcal{H}_B$ is finite-dimensional. If the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are both finite-dimensional, then we specifically refer to $\mathcal{N}_{A\to B}$ as a finite-dimensional channel.

We also make use of the Choi operator $J_{RB}^{\mathcal{N}}$ [41,42] of the channel $\mathcal{N}_{A\to B}$, defined as

$$J_{RB}^{\mathcal{N}} := \mathcal{N}_{A\to B}(\Gamma_{RA}) := \sum_{i,j} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}(|i\rangle\langle j|_A), \tag{43}$$

where R is isomorphic to the channel input A, we employ the shorthand $\Gamma_{RA} \equiv |\Gamma\rangle\langle\Gamma|_{RA}$, and $|\Gamma\rangle_{RA}$ denotes the unnormalized maximally entangled vector,

$$|\Gamma\rangle_{RA} := \sum_i |i\rangle_R \otimes |i\rangle_A, \tag{44}$$

where $\{|i\rangle_R\}_i$ and $\{|i\rangle_A\}_i$ are orthonormal bases for the Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_A$.

Definition 2 (κ-entanglement of a channel). Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then the κ-entanglement of the channel $\mathcal{N}_{A\to B}$ is defined as

$$E_\kappa(\mathcal{N}_{A\to B}) := \inf_{Q_{AB}\geqslant 0} \{\log_2 \|\mathrm{Tr}_B[Q_{AB}]\|_\infty : -Q_{AB}^{T_B} \leqslant (J_{AB}^{\mathcal{N}})^{T_B} \leqslant Q_{AB}^{T_B}\}. \tag{45}$$

Proposition 7. Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})), \tag{46}$$

where the supremum is with respect to all states ρ_{RA} with system R arbitrary.

Proof. Due to Proposition 1, i.e., the fact that E_κ for states is monotone nonincreasing with respect to completely PPT-preserving channels (with one such channel being a local partial trace), it follows from purification, the Schmidt decomposition, and this local data processing, that it suffices to optimize with respect to pure states ρ_{RA} with system R isomorphic to system A. Thus, we conclude that

$$\sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) = \sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA})), \tag{47}$$

where ϕ_{RA} is pure and $R \simeq A$.

By definition, and using the fact that every pure state ϕ_{RA} of the form mentioned above can be represented as $X_R\Gamma_{RA}X_R^\dagger$ with $\|X_R\|_2 = 1$, we have that

$$\begin{aligned}
&\sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA}))\\
&= \log_2 \sup_{X_R:\|X_R\|_2=1,|X_R|>0} \inf_{S_{RB}\geqslant 0} \{\mathrm{Tr}\, S_{RB} : -S_{RB}^{T_B} \leqslant X_R[J_{RB}^{\mathcal{N}}]^{T_B} X_R^\dagger \leqslant S_{RB}^{T_B}\},
\end{aligned} \tag{48}$$

where the equality follows because the set of operators X_R satisfying $\|X_R\|_2 = 1$, and $|X_R| > 0$ is dense in the set of all operators satisfying $\|X_R\|_2 = 1$. Now defining Q_{RB} in terms of $S_{RB} = X_R Q_{RB} X_R^\dagger$, and using the facts that

$$-S_{RB}^{T_B} \leqslant X_R[J_{RB}^{\mathcal{N}}]^{T_B} X_R^\dagger \leqslant S_{RB}^{T_B} \Leftrightarrow -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}, \tag{49}$$

$$S_{RB} \geqslant 0 \Leftrightarrow Q_{RB} \geqslant 0, \tag{50}$$

for operators X_R satisfying $|X_R| > 0$, we find that

$$\begin{aligned}
&\sup_{X_R:\|X_R\|_2=1,|X_R|>0} \inf_{S_{RB}\geqslant 0} \{\mathrm{Tr}\, S_{RB} : -S_{RB}^{T_B} \leqslant X_R[J_{RB}^{\mathcal{N}}]^{T_B} X_R^\dagger \leqslant S_{RB}^{T_B}\}\\
&= \sup_{X_R:\|X_R\|_2=1,|X_R|>0} \inf_{Q_{RB}\geqslant 0} \{\mathrm{Tr}\, X_R Q_{RB} X_R^\dagger : -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}\}\\
&= \sup_{\rho_R:\mathrm{Tr}\,\rho_R=1,\rho_R>0} \inf_{Q_{RB}\geqslant 0} \{\mathrm{Tr}[\rho_R \mathrm{Tr}_B[Q_{RB}]] : -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}\}\\
&= \sup_{\rho_R:\mathrm{Tr}\,\rho_R=1,\rho_R\geqslant 0} \inf_{Q_{RB}\geqslant 0} \{\mathrm{Tr}[\rho_R \mathrm{Tr}_B[Q_{RB}]] : -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}\}\\
&= \inf_{Q_{RB}\geqslant 0} \left[\sup_{\rho_R:\mathrm{Tr}\,\rho_R=1,\rho_R>0} \{\mathrm{Tr}[\rho_R \mathrm{Tr}_B[Q_{RB}]] : -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}\}\right]\\
&= \inf_{Q_{RB}\geqslant 0} \{\|\mathrm{Tr}_B[Q_{RB}]\|_\infty : -Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B} \leqslant Q_{RB}^{T_B}\}.
\end{aligned} \tag{51}$$

The fourth equality follows from an application of the Sion minimax theorem [43], given that the set of operators satisfying $\mathrm{Tr}\,\rho_R = 1$ and $\rho_R \geqslant 0$ is compact and both sets over which we are optimizing are convex. Putting everything together, we conclude (46). ■

A. Amortization collapse and monotonicity under a class of PPT superchannels

In this subsection, we prove that the κ entanglement of a quantum channel does not increase under amortization, which is a property that holds for the squashed entanglement of a channel [44,45], a channel's max-relative entropy of entanglement [46], and the max-Rains information of a channel [47]. We additionally prove that this property implies that the κ entanglement of a quantum channel does not increase under the action of a class of PPT superchannels. A PPT superchannel Θ^{PPT} is a physical transformation of a quantum channel. The class of PPT superchannels that we consider realizes the following transformation of a channel $\mathcal{M}_{\hat{A}\to\hat{B}}$ to a channel $\mathcal{N}_{A\to B}$ in terms of completely PPT-preserving channels $\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}$ and $\mathcal{P}^{\mathrm{post}}_{A_M\hat{B}B_M}$:

$$\begin{aligned}
\mathcal{N}_{A\to B} &= \Theta^{\mathrm{PPT}}(\mathcal{M}_{\hat{A}\to\hat{B}})\\
&:= \mathcal{P}^{\mathrm{post}}_{A_M\hat{B}B_M} \circ \mathcal{M}_{\hat{A}\to\hat{B}} \circ \mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}.
\end{aligned} \tag{52}$$

We also state that the same property holds for the max-Rains information of a quantum channel due to the main result of Ref. [47], while a channel's squashed entanglement and max-relative entropy of entanglement do not increase under the action of an LOCC superchannel.

We begin our development with the following amortization inequality:

Proposition 8 (Amortization inequality). Let $\rho_{A'AB'}$ be a quantum state acting on a separable Hilbert space and let $\mathcal{N}_{A\to B}$ be a quantum channel. Then the following amortization inequality holds:

$$E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'})) - E_\kappa(\rho_{A'AB'}) \leqslant E_\kappa(\mathcal{N}_{A\to B}). \tag{53}$$

Proof. A proof for this inequality follows similarly to the proof of Ref. [47], Proposition 1]. We first rewrite the desired inequality as

$$E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'})) \leqslant E_\kappa(\mathcal{N}_{A\to B}) + E_\kappa(\rho_{A'AB'}), \tag{54}$$

and then once again as

$$2^{E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'}))} \leqslant 2^{E_\kappa(\mathcal{N}_{A\to B})} \cdot 2^{E_\kappa(\rho_{A'AB'})}. \tag{55}$$

Consider that

$$2^{E_\kappa(\rho_{A'AB'})} = \inf\big\{\operatorname{Tr} S_{A'AB'} : \\ -S_{A'AB'}^{T_{B'}} \leqslant \rho_{A'AB'}^{T_{B'}} \leqslant S_{A'AB'}^{T_{B'}},\ S_{A'AB'} \geqslant 0\big\}, \tag{56}$$

$$2^{E_\kappa(\mathcal{N}_{A\to B})} = \inf\{\|\operatorname{Tr}_B Q_{RB}\|_\infty : \\ -Q_{RB}^{T_B} \leqslant \big[J_{RB}^{\mathcal{N}}\big]^{T_B} \leqslant Q_{RB}^{T_B},\ Q_{RB} \geqslant 0\big\}. \tag{57}$$

Let $S_{A'AB'}$ be an arbitrary operator satisfying

$$-S_{A'AB'}^{T_{B'}} \leqslant \rho_{A'AB'} \leqslant S_{A'AB'}^{T_{B'}},\ S_{A'AB'} \geqslant 0, \tag{58}$$

and let Q_{RB} be an arbitrary operator satisfying

$$-Q_{RB}^{T_B} \leqslant J_{RB}^{\mathcal{N}} \leqslant Q_{RB}^{T_B},\ Q_{RB} \geqslant 0. \tag{59}$$

Then let

$$F_{A'BB'} = \langle\Gamma|_{RA}(S_{A'AB'} \otimes Q_{RB})|\Gamma\rangle_{RA}, \tag{60}$$

where $|\Gamma\rangle_{RA}$ denotes the unnormalized maximally entangled vector. It follows that $F_{A'BB'} \geqslant 0$ because $S_{A'AB'} \geqslant 0$ and $Q_{RB} \geqslant 0$. Furthermore, we have from (58) and (59) that

$$F_{A'BB'}^{T_{BB'}} = [\langle\Gamma|_{RA}(S_{A'AB'} \otimes Q_{RB})|\Gamma\rangle_{RA}]^{T_{BB'}} \tag{61}$$
$$= \langle\Gamma|_{RA}\big(S_{A'AB'}^{T_{B'}} \otimes Q_{RB}^{T_B}\big)|\Gamma\rangle_{RA} \tag{62}$$
$$\geqslant \langle\Gamma|_{RA}\big(\rho_{A'AB'}^{T_{B'}} \otimes \big[J_{RB}^{\mathcal{N}}\big]^{T_B}\big)|\Gamma\rangle_{RA} \tag{63}$$
$$= \big[\langle\Gamma|_{RA}\big(\rho_{A'AB'} \otimes J_{RB}^{\mathcal{N}}\big)|\Gamma\rangle_{RA}\big]^{T_{BB'}} \tag{64}$$
$$= [\mathcal{N}_{A\to B}(\rho_{A'AB'})]^{T_{BB'}}. \tag{65}$$

Similarly, we have that

$$-F_{A'BB'}^{T_{BB'}} \leqslant [\mathcal{N}_{A\to B}(\rho_{A'AB'})]^{T_{BB'}} \tag{66}$$

by using $-S_{A'AB'}^{T_{B'}} \leqslant \rho_{A'AB'}^{T_{B'}}$ and $-Q_{RB}^{T_B} \leqslant [J_{RB}^{\mathcal{N}}]^{T_B}$. Thus, $F_{A'BB'}$ is feasible for $2^{E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'}))}$.

Finally, consider that

$$2^{E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'}))} \leqslant \operatorname{Tr} F_{A'BB'} \tag{67}$$
$$= \operatorname{Tr}\langle\Gamma|_{RA}(S_{A'AB'} \otimes Q_{RB})|\Gamma\rangle_{RA} \tag{68}$$
$$= \operatorname{Tr} S_{A'AB'} Q_{AB}^{T_A} \tag{69}$$
$$= \operatorname{Tr}\big[S_{A'AB'} \operatorname{Tr}_B Q_{AB}^{T_A}\big] \tag{70}$$
$$\leqslant \operatorname{Tr} S_{A'AB'} \big\|\operatorname{Tr}_B Q_{AB}^{T_A}\big\|_\infty \tag{71}$$
$$= \operatorname{Tr} S_{A'AB'} \|\operatorname{Tr}_B Q_{AB}\|_\infty. \tag{72}$$

The inequality above follows from Hölder's inequality. The last equality follows because the spectrum of an operator remains invariant under the action of a transpose. Since the inequality above holds for all $S_{A'AB'}$ and Q_{RB} satisfying (58) and (59), respectively, we conclude the inequality in (55). ■

Definition 3 (Amortized κ-entanglement of a channel). Following Ref. [48], we define the amortized κ entanglement of a quantum channel $\mathcal{N}_{A\to B}$ as

$$E_\kappa^{\mathcal{A}}(\mathcal{N}_{A\to B}) := \sup_{\rho_{A'AB'}} [E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'})) - E_\kappa(\rho_{A'AB'})]. \tag{73}$$

where the supremum is with respect to every state $\rho_{A'AB'}$, with the A' and B' systems arbitrary.

In spite of the possibility that amortization might increase E_κ, a consequence of Proposition 8 is that in fact it does not.

Proposition 9. Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then the κ entanglement of a channel does not increase under amortization:

$$E_\kappa^{\mathcal{A}}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}). \tag{74}$$

Proof. The inequality $E_\kappa^{\mathcal{A}}(\mathcal{N}_{A\to B}) \geqslant E_\kappa(\mathcal{N}_{A\to B})$ follows from Proposition 7 by identifying A' with R, setting B' to be a trivial system, and noting that $E_\kappa(\rho_{A'AB'})$ vanishes for this choice. The opposite inequality is a direct consequence of Proposition 8. ■

Theorem 10 (Monotonicity). Let $\mathcal{M}_{\hat{A}\to\hat{B}}$ be a quantum channel and Θ^{PPT} a completely PPT-preserving superchannel of the form in (52). The channel measure E_κ is monotone under the action of the superchannel Θ^{PPT} in the sense that

$$E_\kappa(\mathcal{M}_{\hat{A}\to\hat{B}}) \geqslant E_\kappa(\Theta^{\mathrm{PPT}}(\mathcal{M}_{\hat{A}\to\hat{B}})). \tag{75}$$

Proof. The proof is similar to that of Ref. [49], Proposition 6]. Let $\rho_{A'AB'}$ be an arbitrary input state. Then we have that

$$E_\kappa(\mathcal{N}_{A\to B}(\rho_{A'AB'})) - E_\kappa(\rho_{A'AB'})$$
$$= E_\kappa\big(\big(\mathcal{P}^{\mathrm{post}}_{A_M\hat{B}B_M} \circ \mathcal{M}_{\hat{A}\to\hat{B}} \circ \mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}\big)(\rho_{A'AB'})\big) \\ - E_\kappa(\rho_{A'AB'}) \tag{76}$$
$$\leqslant E_\kappa\big(\big(\mathcal{P}^{\mathrm{post}}_{A_M\hat{B}B_M} \circ \mathcal{M}_{\hat{A}\to\hat{B}} \circ \mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}\big)(\rho_{A'AB'})\big) \\ - E_\kappa\big(\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}(\rho_{A'AB'})\big) \tag{77}$$
$$\leqslant E_\kappa\big(\big(\mathcal{M}_{\hat{A}\to\hat{B}} \circ \mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}\big)(\rho_{A'AB'})\big) \\ - E_\kappa\big(\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}(\rho_{A'AB'})\big) \tag{78}$$
$$\leqslant E_\kappa^{\mathcal{A}}(\mathcal{M}_{\hat{A}\to\hat{B}}) \tag{79}$$
$$= E_\kappa(\mathcal{M}_{\hat{A}\to\hat{B}}). \tag{80}$$

The first inequality follows because $E_\kappa(\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}(\rho_{A'AB'})) \leqslant E_\kappa(\rho_{A'AB'})$, given that E_κ does not increase under the action of the completely PPT-preserving channel $\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}$ (Proposition 1). The second inequality follows from a similar reasoning, but with respect to the completely PPT-preserving channel $\mathcal{P}^{\mathrm{post}}_{A_M\hat{B}B_M}$. The last inequality follows because $\mathcal{P}^{\mathrm{pre}}_{A\to\hat{A}A_MB_M}(\rho_{A'AB'})$ is a particular bipartite state to consider at the input of the channel $\mathcal{M}_{\hat{A}\to\hat{B}}$, but the quantity $E_\kappa^{\mathcal{A}}$ involves an optimization over all such states. The final equality is a consequence of Proposition 9. ■

Remark 1. We remark here that the same inequality holds for the max-Rains information of a channel $R_{\max}(\mathcal{N})$, defined in Refs. [50,51] and considered further in Ref. [47] (see also Ref. [52]). That is, for $\mathcal{M}_{\hat{A}\to\hat{B}}$, a quantum channel, and $\Theta^{\rm PPT}$ a completely PPT-preserving superchannel of the form in (52), the following inequality holds:

$$R_{\max}(\mathcal{M}_{\hat{A}\to\hat{B}}) \geqslant R_{\max}(\Theta^{\rm PPT}(\mathcal{M}_{\hat{A}\to\hat{B}})). \tag{81}$$

This follows because $R_{\max}$ does not increase under amortization, as shown in Ref. [47], and because the max-Rains relative entropy does not increase under the action of a completely PPT-preserving channel [39].

Furthermore, a similar inequality holds for the squashed entanglement $E_{\rm sq}$ of a channel and for a channel's max-relative entropy of entanglement $E_{\max}$. In particular, let $\Theta^{\rm LOCC}$ denote an LOCC superchannel, which realizes the following transformation of a channel $\mathcal{M}_{\hat{A}\to\hat{B}}$ to a channel $\mathcal{N}_{A\to B}$ in terms of LOCC channels $\mathcal{L}^{\rm pre}_{A\to\hat{A}A_MB_M}$ and $\mathcal{L}^{\rm post}_{A_M\hat{B}B_M}$:

$$\mathcal{N}_{A\to B} = \Theta^{\rm LOCC}(\mathcal{M}_{\hat{A}\to\hat{B}}) \tag{82}$$

$$:= \mathcal{L}^{\rm post}_{A_M\hat{B}B_M}\circ\mathcal{M}_{\hat{A}\to\hat{B}}\circ\mathcal{L}^{\rm pre}_{A\to\hat{A}A_MB_M}. \tag{83}$$

Then the following inequalities hold:

$$E_{\rm sq}(\mathcal{M}_{\hat{A}\to\hat{B}}) \geqslant E_{\rm sq}(\Theta^{\rm LOCC}(\mathcal{M}_{\hat{A}\to\hat{B}})), \tag{84}$$

$$E_{\max}(\mathcal{M}_{\hat{A}\to\hat{B}}) \geqslant E_{\max}(\Theta^{\rm LOCC}(\mathcal{M}_{\hat{A}\to\hat{B}})), \tag{85}$$

with both inequalities following because these measures do not increase under amortization, as shown in Refs. [44–46], respectively, and the squashed entanglement [53] and max-relative entropy of entanglement of states [54,55] do not increase under LOCC channels.

B. Dual representation and additivity

The optimization that is dual to (45) is as follows:

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) := \sup_{V_{AB}^{T_B},\,W_{AB}^{T_B},\,\rho_A\geqslant 0}\big\{\log_2 \operatorname{Tr} J^{\mathcal{N}}_{AB}(V_{AB}-W_{AB})$$
$$: V_{AB}+W_{AB}\leqslant \rho_A\otimes \mathbb{1}_B,\ \operatorname{Tr}\rho_A = 1\big\}. \tag{86}$$

This follows from applying the Lagrange multiplier method. By weak duality, we have that

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) \leqslant E_\kappa(\mathcal{N}_{A\to B}). \tag{87}$$

If the channel $\mathcal{N}_{A\to B}$ is finite-dimensional, then strong duality holds, so

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}). \tag{88}$$

Furthermore, by employing the fact that $E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) = \sup_{\rho_{RA}} E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}(\rho_{RA}))$, Proposition 7, and (7), we conclude that the following equality holds for a quantum channel $\mathcal{N}_{A\to B}$:

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}). \tag{89}$$

The additivity of E_κ with respect to tensor-product channels follows from both the primal and dual representations of $E_\kappa(\mathcal{N})$.

Proposition 11 (Additivity). Given two quantum channels $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A'\to B'}$, it holds that

$$E_\kappa(\mathcal{N}_{A\to B}\otimes\mathcal{M}_{A'\to B'}) = E_\kappa(\mathcal{N}_{A\to B}) + E_\kappa(\mathcal{M}_{A'\to B'}). \tag{90}$$

Proof. The proof is similar to that of Proposition 2. To be self-contained, we show the details as follows. First, by definition, we can write $E_\kappa(\mathcal{N}_{A\to B})$ as

$$E_\kappa(\mathcal{N}_{A\to B}) = \inf_{Q_{AB}\geqslant 0}\big\{\log_2\|\operatorname{Tr}_B Q_{AB}\|_\infty$$
$$: -Q_{AB}^{T_B} \leqslant \big(J^{\mathcal{N}}_{AB}\big)^{T_B} \leqslant Q_{AB}^{T_B}\big\}. \tag{91}$$

Let Q_{AB} be an arbitrary operator satisfying $-Q_{AB}^{T_B}\leqslant (J^{\mathcal{N}}_{AB})^{T_B}\leqslant Q_{AB}^{T_B}$, $Q_{AB}\geqslant 0$ and let $P_{A'B'}$ be an arbitrary operator satisfying $-P_{A'B'}^{T_{B'}}\leqslant (J^{\mathcal{M}}_{A'B'})^{T_{B'}}\leqslant P_{A'B'}^{T_{B'}}$, $P_{A'B'}\geqslant 0$. Then $Q_{AB}\otimes P_{A'B'}$ satisfies

$$-(Q_{AB}\otimes P_{A'B'})^{T_{BB'}} \leqslant \big(J^{\mathcal{N}}_{AB}\otimes J^{\mathcal{M}}_{A'B'}\big)^{T_{BB'}} \leqslant (Q_{AB}\otimes P_{A'B'})^{T_{BB'}}, \tag{92}$$

$$Q_{AB}\otimes P_{A'B'} \geqslant 0, \tag{93}$$

so

$$E_\kappa(\mathcal{N}_{A\to B}\otimes\mathcal{M}_{A'\to B'})$$
$$\leqslant \log_2\|\operatorname{Tr}_{BB'} Q_{AB}\otimes P_{A'B'}\|_\infty \tag{94}$$
$$= \log_2\|\operatorname{Tr}_B Q_{AB}\|_\infty + \log_2\|\operatorname{Tr}_{B'} P_{A'B'}\|_\infty. \tag{95}$$

Since the inequality holds for all Q_{AB} and $P_{A'B'}$ satisfying the above conditions, we conclude that

$$E_\kappa(\mathcal{N}\otimes\mathcal{M}) \leqslant E_\kappa(\mathcal{N}) + E_\kappa(\mathcal{M}). \tag{96}$$

To see the superadditivity of E_κ for quantum channels, let us suppose that $\{V^1_{AB}, W^1_{AB}, \rho^1_A\}$ and $\{V^2_{A'B'}, W^2_{A'B'}, \rho^2_{A'}\}$ are arbitrary operators satisfying the conditions in (86) for $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A'\to B'}$, respectively. Now we choose

$$R_{ABA'B'} = V^1_{AB}\otimes V^2_{A'B'} + W^1_{AB}\otimes W^2_{A'B'}, \tag{97}$$

$$S_{ABA'B'} = V^1_{AB}\otimes W^2_{A'B'} + W^1_{AB}\otimes V^2_{A'B'}. \tag{98}$$

One can verify from (86) that

$$R_{ABA'B'}^{T_{BB'}},\ S_{ABA'B'}^{T_{BB'}} \geqslant 0, \tag{99}$$

$$R_{ABA'B'} + S_{ABA'B'} = \big(V^1_{AB}+W^1_{AB}\big)\otimes\big(V^2_{A'B'}+W^2_{A'B'}\big)$$
$$\leqslant \rho^1_A\otimes\rho^2_{A'}\otimes\mathbb{1}_{BB'}, \tag{100}$$

which implies that $\{R_{ABA'B'}, S_{ABA'B'}, \rho^1_A\otimes\rho^2_{A'}\}$ is feasible for $E_\kappa(\mathcal{N}_{A\to B}\otimes\mathcal{M}_{A'\to B'})$ in (86). Thus, we have that

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}\otimes\mathcal{M}_{A'\to B'})$$
$$\geqslant \log_2 \operatorname{Tr}\big(J^{\mathcal{N}}_{AB}\otimes J^{\mathcal{M}}_{A'B'}\big)(R_{ABA'B'}-S_{ABA'B'}) \tag{101}$$
$$= \log_2\big[\operatorname{Tr} J^{\mathcal{N}}_{AB}\big(V^1_{AB}-W^1_{AB}\big)\cdot \operatorname{Tr} J^{\mathcal{M}}_{A'B'}\big(V^2_{A'B'}-W^2_{A'B'}\big)\big] \tag{102}$$
$$= \log_2\big(\operatorname{Tr} J^{\mathcal{N}}_{AB}\big(V^1_{AB}-W^1_{AB}\big)\big)$$
$$+ \log_2\big(\operatorname{Tr} J^{\mathcal{M}}_{A'B'}\big(V^2_{A'B'}-W^2_{A'B'}\big)\big). \tag{103}$$

Since the inequality has been shown for arbitrary $\{V^1_{AB}, W^1_{AB}, \rho^1_A\}$ and $\{V^2_{A'B'}, W^2_{A'B'}, \rho^2_{A'}\}$ satisfying the conditions in (86) for $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A'\to B'}$, respectively, we conclude that

$$E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}\otimes\mathcal{M}_{A'\to B'}) \geqslant E_\kappa^{\rm dual}(\mathcal{N}_{A\to B}) + E_\kappa^{\rm dual}(\mathcal{M}_{A'\to B'}). \tag{104}$$

The proof is concluded by combining (96), (104), and (89). ■

C. Normalization, faithfulness, and no convexity

In this subsection, we prove that the κ entanglement of a quantum channel is normalized, faithful, and generally not convex.

Proposition 12 (Normalization). Let $\mathrm{id}^M_{A\to B}$ be a noiseless quantum channel with dimension $d_A = d_B = M$. Then

$$E_\kappa(\mathrm{id}^M) = \log_2 M. \tag{105}$$

Moreover, for every finite-dimensional quantum channel $\mathcal{N}_{A\to B}$,

$$E_\kappa(\mathcal{N}_{A\to B}) \leqslant \min\{\log_2 d_A, \log_2 d_B\}. \tag{106}$$

Proof. By Propositions 3 and 7, we have

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) \tag{107}$$

$$= \sup_{\psi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\psi_{RA})) \tag{108}$$

$$\leqslant \log_2 \min\{d_A, d_B\}, \tag{109}$$

where, in the second equality, the optimization is with respect to pure states with system R isomorphic to the channel input system A.

This implies that $E_\kappa(\mathrm{id}^M) \leqslant \log_2 M$. Furthermore,

$$E_\kappa(\mathrm{id}^M) \geqslant E_\kappa\left(\mathrm{id}_{A\to B}\left(\Phi^M_{RA}\right)\right) = \log_2 M, \tag{110}$$

where Φ^M_{RA} denotes a maximally entangled state of Schmidt rank M and the second equality follows from Proposition 3. ■

Proposition 13 (Faithfulness). Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then $E_\kappa(\mathcal{N}_{A\to B}) \geqslant 0$ and $E_\kappa(\mathcal{N}_{A\to B}) = 0$ if and only if $\mathcal{N}_{A\to B}$ is a PPT entanglement binding channel [56].

Proof. To see that $E_\kappa(\mathcal{N}_{A\to B}) \geqslant 0$, we could utilize the dual representation in (86) and the equality in (89) or, alternatively, employ Propositions 4 and 7 to find that

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) \geqslant 0. \tag{111}$$

Now if $\mathcal{N}_{A\to B}$ is a PPT entanglement binding channel (as defined in Ref. [56]), then the state $\mathcal{N}_{A\to B}(\rho_{RA})$ is PPT for every input state ρ_{RA}. Thus, $E_\kappa(\mathcal{N}_{A\to B}) = 0$. On the other hand, if $E_\kappa(\mathcal{N}_{A\to B}) = 0$, then for every ρ_{RA} it holds that $E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) = 0$. By Proposition 4, we conclude that $\mathcal{N}_{A\to B}(\rho_{RA})$ is PPT for every state ρ_{RA}, and thus $\mathcal{N}_{A\to B}$ is a PPT entanglement binding channel. ■

Proposition 14 (No convexity). The κ entanglement of a quantum channel is not generally convex.

Proof. To see this, we construct channels with Choi states given by the examples in Eq. (15). Let us choose the following qubit channels:

$$\mathcal{N}_1(\rho) = \rho, \tag{112}$$

$$\mathcal{N}_2(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|. \tag{113}$$

Since $\mathcal{N}_1$ is a qubit noiseless channel, Proposition 12 implies that $E_\kappa(\mathcal{N}_1) = 1$. Noting that $\mathcal{N}_2$ is a PPT entanglement binding channel, Proposition 13 implies that $E_\kappa(\mathcal{N}_2) = 0$.

Let $\mathcal{N} = \frac{1}{2}(\mathcal{N}_1 + \mathcal{N}_2)$ denote the uniform mixture of the two channels. The mixed channel $\mathcal{N}$ is actually a dephasing channel with dephasing parameter 1/2. Then we have that $E_\kappa(\mathcal{N}) \geqslant \log_2 \frac{3}{2}$, which follows by inputting one share of the maximally entangled state. Thus, we find that

$$E_\kappa(\mathcal{N}) > \tfrac{1}{2}(E_\kappa(\mathcal{N}_1) + E_\kappa(\mathcal{N}_1)). \tag{114}$$

This concludes the proof. ■

IV. EXACT ENTANGLEMENT COST OF QUANTUM CHANNELS

In this section, we introduce two channel simulation tasks. First, we consider the exact parallel simulation of a quantum channel when completely PPT-preserving channels are allowed for free and the goal is to meter the entanglement cost. We also consider the exact sequential simulation of a quantum channel. In both cases, the entanglement cost is equal to the κ entanglement of the channel, thus endowing it with a direct operational meaning. After these results are established, we focus on PPT-simulable [48] and resource-seizable [14] channels, demonstrating that the theory significantly simplifies for these kinds of channels.

A. Exact parallel simulation of quantum channels

Another fundamental problem is to quantify the entanglement required for an exact simulation of an arbitrary quantum channel via free channels (LOCC or PPT) and by making use of an entangled resource state. Recall that Ω represents the set of free channels. Also, two quantum channels $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A\to B}$ are equal if, for orthonormal bases $\{|i\rangle_A\}_i$ and $\{|k\rangle_B\}_k$, the following equalities hold for all $i, j, k, l \in \mathbb{N}$:

$$\langle k|_B \mathcal{N}_{A\to B}(|i\rangle_A\langle j|_A)|l\rangle_B = \langle k|_B \mathcal{M}_{A\to B}(|i\rangle_A\langle j|_A)|l\rangle_B. \tag{115}$$

This is equivalent to the Choi operators of the channels being equal:

$$\mathcal{N}_{A\to B}(\Gamma_{RA}) = \mathcal{M}_{A\to B}(\Gamma_{RA}). \tag{116}$$

Furthermore, the following identity holds for an arbitrary state ρ_{CS} with $S \simeq R \simeq A$:

$$\langle\Gamma|_{SR}[\rho_{CS} \otimes \mathcal{N}_{A\to B}(\Gamma_{RA})]|\Gamma\rangle_{SR} = \mathcal{N}_{A\to B}(\rho_{CA}), \tag{117}$$

understood intuitively as a postselected variant [57,58] of quantum teleportation [59]. From the identity in (117), we conclude that if two channels are equal in the sense of (115) and (116), then there is no physical procedure that can distinguish them.

We define the one-shot exact entanglement cost of a quantum channel $\mathcal{N}_{A\to B}$, under the Ω channels, as

$$E^{(1)}_\Omega(\mathcal{N}_{A\to B}) = \inf_{\Lambda\in\Omega}\left\{\log_2 d : \mathcal{N}_{A\to B}(\Gamma_{RA}) = \Lambda_{\hat{A}\hat{B}A\to B}\left(\Gamma_{RA}\otimes\Phi^d_{\hat{A}\hat{B}}\right)\right\}. \tag{118}$$

Figure 1 depicts this simulation task. The exact parallel entanglement cost of quantum channel $\mathcal{N}_{A\to B}$, under the Ω channels, is defined as

$$E^{(p)}_\Omega(\mathcal{N}_{A\to B}) = \limsup_{n\to\infty} \frac{1}{n} E^{(1)}_\Omega\left(\mathcal{N}^{\otimes n}_{A\to B}\right). \tag{119}$$

Theorem 15. The one-shot exact PPT-entanglement cost $E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B})$ of a quantum channel $\mathcal{N}_{A\to B}$ is given by the

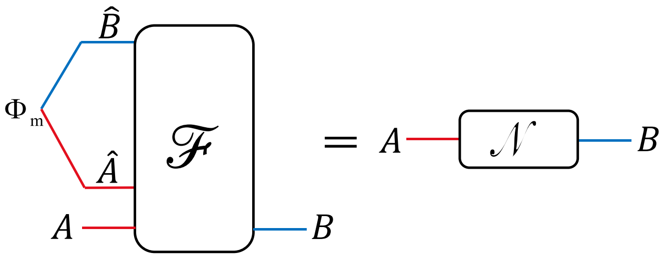

FIG. 1. Simulating the quantum channel $\mathcal{N}$ via a free channel $\mathcal{F}_{A\hat{A}\hat{B}\to B}$ and a maximally entangled state Φ_m

following optimization:

$$E_{\mathrm{PPT}}^{(1)}(\mathcal{N}_{A\to B}) = \inf_{m\in\mathbb{Z}^+, Q_{AB}\geqslant 0}\big\{\log_2 m : \operatorname{Tr}_B Q_{AB} = \mathbb{1}_A, \\ -(m-1)Q_{AB}^{T_B} \leqslant \big(J_{AB}^{\mathcal{N}}\big)^{T_B} \leqslant (m+1)Q_{AB}^{T_B}\big\}. \tag{120}$$

Proof. The proof is somewhat similar to the proof of Theorem 5, which is available in [12]. The achievability part features a construction of a completely PPT-preserving channel $\mathcal{P}_{\hat{A}\hat{B}\to AB}$ such that $\mathcal{P}_{A\hat{A}\hat{B}\to B}(X_A\otimes\Phi_{\hat{A}\hat{B}}^m) = \mathcal{N}_{A\to B}(X_A)$ for every input operator X_A (including density operators), and then the converse part demonstrates that the constructed channel is essentially the only form that is needed to consider for the one-shot exact PPT-entanglement cost task.

First, to have an exact simulation of a channel, it is only necessary to check the simulation on a single input, the maximally entangled vector $|\Gamma\rangle_{RA}$. So, we require that

$$\mathcal{P}_{A\hat{A}\hat{B}\to B}\big(\Gamma_{RA}\otimes\Phi_{\hat{A}\hat{B}}^m\big) = \mathcal{N}_{A\to B}(\Gamma_{RA}), \tag{121}$$

where Γ_{RA} is the unnormalized maximally entangled operator.

We now prove the achievability part. Let $m\geqslant 1$ be a positive integer and Q_{AB} a Choi operator for a quantum channel (i.e., $Q_{AB}\geqslant 0$, $\operatorname{Tr}_B Q_{AB} = \mathbb{1}_A$) such that the following inequalities hold:

$$-(m-1)Q_{AB}^{T_B} \leqslant \big(J_{AB}^{\mathcal{N}}\big)^{T_B} \leqslant (m+1)Q_{AB}^{T_B}. \tag{122}$$

Then we take the completely-PPT-preserving channel $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ to have a Choi operator given by

$$J_{A\hat{A}\hat{B}B}^{\mathcal{P}} = J_{AB}^{\mathcal{N}}\otimes\Phi_{\hat{A}\hat{B}}^m + Q_{AB}\otimes\big(\mathbb{1}_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m\big). \tag{123}$$

Observe that $J_{A\hat{A}\hat{B}B}^{\mathcal{P}}\geqslant 0$. Furthermore, we have that

$$\operatorname{Tr}_B J_{A\hat{A}\hat{B}B}^{\mathcal{P}} = \operatorname{Tr}_B J_{AB}^{\mathcal{N}}\otimes\Phi_{\hat{A}\hat{B}}^m + \operatorname{Tr}_B Q_{AB}\otimes\big(\mathbb{1}_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m\big) \tag{124}$$

$$= \mathbb{1}_A\otimes\Phi_{\hat{A}\hat{B}}^m + \mathbb{1}_A\otimes\big(\mathbb{1}_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m\big) \tag{125}$$

$$= \mathbb{1}_{A\hat{A}\hat{B}}. \tag{126}$$

Thus, $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ is a quantum channel. Setting $|\Gamma\rangle_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'} := |\Gamma\rangle_{AA'}\otimes|\Gamma\rangle_{\hat{A}\hat{A}'}\otimes|\Gamma\rangle_{\hat{B}\hat{B}'}$, its action on the input $\Gamma_{RA}\otimes\Phi_{\hat{A}\hat{B}}^m$ is given by

$$\langle\Gamma|_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'}\big(\Gamma_{RA}\otimes\Phi_{\hat{A}\hat{B}}^m\otimes J_{A'\hat{A}'\hat{B}'B}^{\mathcal{P}}\big)|\Gamma\rangle_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'} = \langle\Gamma|_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'}\big(\Gamma_{RA}\otimes\Phi_{\hat{A}\hat{B}}^m\otimes J_{A'B}^{\mathcal{N}}\otimes\Phi_{\hat{A}'\hat{B}'}^m\big)|\Gamma\rangle_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'} \\ + \langle\Gamma|_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'}\big(\Gamma_{RA}\otimes\Phi_{\hat{A}\hat{B}}^m\otimes Q_{A'B}\otimes\big(\mathbb{1}_{\hat{A}'\hat{B}'} - \Phi_{\hat{A}'\hat{B}'}^m\big)\big)|\Gamma\rangle_{AA'\hat{A}\hat{A}'\hat{B}\hat{B}'} \tag{127}$$

$$= \langle\Gamma|_{AA'}\big(\Gamma_{RA}\otimes J_{A'B}^{\mathcal{N}}\big)|\Gamma\rangle_{AA'} \tag{128}$$

$$= \mathcal{N}_{A\to B}(\Gamma_{RA}). \tag{129}$$

The second equality follows because

$$(\langle\Gamma|_{\hat{A}\hat{A}'}\otimes\langle\Gamma|_{\hat{B}\hat{B}'})\big(\Phi_{\hat{A}\hat{B}}^m\otimes\Phi_{\hat{A}'\hat{B}'}^m\big)(|\Gamma\rangle_{\hat{A}\hat{A}'}\otimes|\Gamma\rangle_{\hat{B}\hat{B}'}) = \operatorname{Tr}\Phi_{\hat{A}\hat{B}}^m\Phi_{\hat{A}\hat{B}}^m = 1, \tag{130}$$

$$(\langle\Gamma|_{\hat{A}\hat{A}'}\otimes\langle\Gamma|_{\hat{B}\hat{B}'})\big(\Phi_{\hat{A}\hat{B}}^m\otimes\mathbb{1}_{\hat{A}'\hat{B}'}\big)(|\Gamma\rangle_{\hat{A}\hat{A}'}\otimes|\Gamma\rangle_{\hat{B}\hat{B}'}) = \operatorname{Tr}\Phi_{\hat{A}\hat{B}}^m = 1. \tag{131}$$

Thus, for the constructed channel, we have that (121) holds. Finally, we need to show that the constructed channel $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ is completely PPT preserving:

$$\big(J_{A\hat{A}\hat{B}B}^{\mathcal{P}}\big)^{T_{\hat{B}B}} \geqslant 0. \tag{132}$$

Consider that

$$\big(J_{A\hat{A}\hat{B}B}^{\mathcal{P}}\big)^{T_{\hat{B}B}} = \big(J_{AB}^{\mathcal{N}}\big)^{T_B}\otimes\big(\Phi_{\hat{A}\hat{B}}^m\big)^{T_{\hat{B}}} + Q_{AB}^{T_B}\otimes\big(\mathbb{1}_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m\big)^{T_{\hat{B}}} \tag{133}$$

$$= \frac{1}{m}\big(J_{AB}^{\mathcal{N}}\big)^{T_B}\otimes(F_{\hat{A}\hat{B}}) + Q_{AB}^{T_B}\otimes\left(\mathbb{1}_{\hat{A}\hat{B}} - \frac{1}{m}F_{\hat{A}\hat{B}}\right) \tag{134}$$

$$= \frac{1}{m}\big(J_{AB}^{\mathcal{N}}\big)^{T_B}\otimes\big(\Pi_{\hat{A}\hat{B}}^{\mathcal{S}} - \Pi_{\hat{A}\hat{B}}^{\mathcal{A}}\big) + Q_{AB}^{T_B}\otimes\left(\Pi_{\hat{A}\hat{B}}^{\mathcal{S}} + \Pi_{\hat{A}\hat{B}}^{\mathcal{A}} - \frac{1}{m}\big[\Pi_{\hat{A}\hat{B}}^{\mathcal{S}} - \Pi_{\hat{A}\hat{B}}^{\mathcal{A}}\big]\right) \tag{135}$$

$$= \left[\frac{1}{m}\big(J_{AB}^{\mathcal{N}}\big)^{T_B} + \left(1-\frac{1}{m}\right)Q_{AB}^{T_B}\right]\otimes\Pi_{\hat{A}\hat{B}}^{\mathcal{S}} + \left[\left(1+\frac{1}{m}\right)Q_{AB}^{T_B} - \frac{1}{m}\big(J_{AB}^{\mathcal{N}}\big)^{T_B}\right]\otimes\Pi_{\hat{A}\hat{B}}^{\mathcal{A}} \tag{136}$$

$$= \frac{1}{m}\big[\big(J_{AB}^{\mathcal{N}}\big)^{T_B} + (m-1)Q_{AB}^{T_B}\big]\otimes\Pi_{\hat{A}\hat{B}}^{\mathcal{S}} + \frac{1}{m}\big[(m+1)Q_{AB}^{T_B} - \big(J_{AB}^{\mathcal{N}}\big)^{T_B}\big]\otimes\Pi_{\hat{A}\hat{B}}^{\mathcal{A}}. \tag{137}$$

Applying the condition in (122), we conclude (132). Thus, we have shown that for all m and Q_{AB} satisfying (122) and $Q_{AB} \geqslant 0$, $\mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A$, there exists a completely PPT-preserving channel $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ such that (121) holds. Now taking an infimum over all such m and Q_{AB}, we conclude that the right-hand side of (120) is greater than or equal to $E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B})$.

To see the opposite inequality, let $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ be a completely PPT-preserving channel such that (121) holds. Then preceding $\mathcal{P}_{A\hat{A}\hat{B}\to B}$ by the isotropic twirling channel $\mathcal{T}_{\hat{A}\hat{B}}$ results in a completely PPT-preserving channel $\mathcal{P}'_{A\hat{A}\hat{B}\to B} = \mathcal{P}_{A\hat{A}\hat{B}\to B}\circ\mathcal{T}_{\hat{A}\hat{B}}$ achieving the same simulation task, and so it suffices to focus on the channel $\mathcal{P}'_{A\hat{A}\hat{B}\to B}$ to establish an expression for the one-shot exact PPT-entanglement cost. Consider that

$$\begin{aligned} J^{\mathcal{P}'}_{R\hat{A}'\hat{B}'B} &= \mathcal{P}'_{A\hat{A}\hat{B}\to B}(\Gamma_{RA}\otimes\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}}) \\ &= (\mathcal{P}_{A\hat{A}\hat{B}\to B}\circ\mathcal{T}_{\hat{A}\hat{B}})(\Gamma_{RA}\otimes\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}}). \end{aligned} \tag{138}$$

Considering that

$$\begin{aligned} &\mathcal{T}_{\hat{A}\hat{B}}(\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}}) \\ &\quad = \Phi^m_{\hat{A}\hat{B}}\otimes\mathrm{Tr}_{\hat{A}\hat{B}}\left[\Phi^m_{\hat{A}\hat{B}}(\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}})\right] \\ &\qquad + \frac{\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}}{m^2-1}\mathrm{Tr}_{\hat{A}\hat{B}}\left[\left(\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}\right)(\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}})\right] \end{aligned} \tag{139}$$

$$= \Phi^m_{\hat{A}\hat{B}}\otimes\Phi^m_{\hat{A}'\hat{B}'} + \frac{\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}}{m^2-1}\otimes\left(\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}\right), \tag{140}$$

with the equalities understood in terms of entanglement swapping [59], we conclude that

$$\begin{aligned} &(\mathcal{P}_{A\hat{A}\hat{B}\to B}\circ\mathcal{T}_{\hat{A}\hat{B}})(\Gamma_{RA}\otimes\Gamma_{\hat{A}'\hat{A}}\otimes\Gamma_{\hat{B}'\hat{B}}) \\ &\quad = \mathcal{P}_{A\hat{A}\hat{B}\to B}\left(\Gamma_{RA}\otimes\Phi^m_{\hat{A}\hat{B}}\right)\otimes\Phi^m_{\hat{A}'\hat{B}'} \\ &\qquad + \mathcal{P}_{A\hat{A}\hat{B}\to B}\left(\Gamma_{RA}\otimes\frac{\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}}{m^2-1}\right)\otimes\left(\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}\right) \end{aligned} \tag{141}$$

$$\begin{aligned} &= \mathcal{N}_{A\to B}(\Gamma_{RA})\otimes\Phi^m_{\hat{A}'\hat{B}'} \\ &\quad + \mathcal{P}_{A\hat{A}\hat{B}\to B}\left(\Gamma_{RA}\otimes\frac{\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}}{m^2-1}\right)\otimes\left(\mathbb{1}_{\hat{A}'\hat{B}'}-\Phi^m_{\hat{A}'\hat{B}'}\right) \end{aligned} \tag{142}$$

$$= J^{\mathcal{N}}_{RB}\otimes\Phi^m_{\hat{A}'\hat{B}'} + Q_{RB}\otimes\left(\mathbb{1}_{\hat{A}'\hat{B}'}-\Phi^m_{\hat{A}'\hat{B}'}\right). \tag{143}$$

where we have used the assumption that (121) holds and set

$$Q_{RB} = \mathcal{P}_{A\hat{A}\hat{B}\to B}\left(\Gamma_{RA}\otimes\frac{\mathbb{1}_{\hat{A}\hat{B}}-\Phi^m_{\hat{A}\hat{B}}}{m^2-1}\right), \tag{144}$$

from which it follows that $Q_{RB}\geqslant 0$ and $\mathrm{Tr}_B\, Q_{RB} = \mathbb{1}_R$. For the channel $\mathcal{P}'_{A\hat{A}\hat{B}\to B}$ to be completely PPT-preserving, it is necessary that

$$\left(J^{\mathcal{P}'}_{R\hat{A}'\hat{B}'B}\right)^{T_{\hat{B}'B}} \geqslant 0. \tag{145}$$

Writing this out and using calculations given above, we find that it is necessary that the following operator is positive semidefinite:

$$\begin{aligned} &\frac{1}{m}\left[\left(J^{\mathcal{N}}_{AB}\right)^{T_B} + (m-1)Q^{T_B}_{AB}\right]\otimes\Pi^{\mathcal{S}}_{\hat{A}\hat{B}} \\ &\quad + \frac{1}{m}\left[(m+1)Q^{T_B}_{AB} - \left(J^{\mathcal{N}}_{AB}\right)^{T_B}\right]\otimes\Pi^{\mathcal{A}}_{\hat{A}\hat{B}}. \end{aligned} \tag{146}$$

Since $\Pi^{\mathcal{S}}_{\hat{A}\hat{B}}$ and $\Pi^{\mathcal{A}}_{\hat{A}\hat{B}}$ project onto orthogonal subspaces, we find that the condition (122) is necessary. Thus, it follows that the quantity on the right-hand side of (120) is less than or equal to $E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B})$. ■

Proposition 16. Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then

$$\log_2(2^{E_\kappa(\mathcal{N})}-1) \leqslant E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \leqslant \log_2(2^{E_\kappa(\mathcal{N})}+2). \tag{147}$$

Proof. The idea of the proof is to use the technique of SDP relaxation. Consider that

$$\begin{aligned} E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) &= \inf\left\{\log_2 m : -(m-1)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (m+1)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A\right\} \\ &\geqslant \inf\left\{\log_2 m : -(m+1)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (m+1)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A\right\} \\ &= \inf\left\{\log_2 m : -R^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant R^{T_B}_{AB},\ R_{AB}\geqslant 0,\ \mathrm{Tr}_B\, R_{AB} = (m+1)\mathbb{1}_A\right\} \\ &= \inf\left\{\log_2(\|\mathrm{Tr}_B\, R_{AB}\|_\infty - 1) : -R^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant R^{T_B}_{AB},\ R_{AB}\geqslant 0\right\} \\ &= \log_2(2^{E_\kappa(\mathcal{N})}-1). \end{aligned} \tag{148}$$

The first inequality follows by relaxing the constraint $-(m-1)Q^{T_B}_{AB} \leqslant (J^{\mathcal{N}}_{AB})^{T_B}$ to $-(m+1)Q^{T_B}_{AB} \leqslant (J^{\mathcal{N}}_{AB})^{T_B}$. The second equality follows by absorbing m into Q_{AB} and setting $R_{AB} = (m+1)Q_{AB}$. The last equality follows from the definition of $E_\kappa(\mathcal{N})$.

Similarly, we have that $E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \leqslant \log_2(2^{E_\kappa(\mathcal{N})}+2)$, following the chain of inequalities:

$$\begin{aligned} E^{(1)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) &= \inf\left\{\log_2 m : -(m-1)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (m+1)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A, m\in\mathbb{N}, m\geqslant 2\right\} \\ &\leqslant \inf\left\{\log_2 m : -(m-1)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (m-1)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A, m\in\mathbb{N}, m\geqslant 2\right\} \\ &= \inf\left\{\log_2\lfloor\mu\rfloor : -(\lfloor\mu\rfloor-1)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (\lfloor\mu\rfloor-1)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A, \mu\geqslant 2\right\} \\ &\leqslant \inf\left\{\log_2\lfloor\mu\rfloor : -(\mu-2)Q^{T_B}_{AB} \leqslant \left(J^{\mathcal{N}}_{AB}\right)^{T_B} \leqslant (\mu-2)Q^{T_B}_{AB},\ Q_{AB}\geqslant 0,\ \mathrm{Tr}_B\, Q_{AB} = \mathbb{1}_A, \mu\geqslant 2\right\} \end{aligned}$$

$$\begin{aligned}&\leqslant \inf\left\{\log_2\mu : -(\mu-2)Q_{AB}^{T_B} \leqslant \left(J_{AB}^{\mathcal{N}}\right)^{T_B} \leqslant (\mu-2)Q_{AB}^{T_B},\ Q_{AB}\geqslant 0,\ \operatorname{Tr}_B Q_{AB} = \mathbb{1}_A,\ \mu\geqslant 2\right\}\\ &= \inf\left\{\log_2(\|\operatorname{Tr}_B R_{AB}\|_\infty + 2) : -R_{AB}^{T_B} \leqslant \left(J_{AB}^{\mathcal{N}}\right)^{T_B} \leqslant R_{AB}^{T_B},\ R_{AB}\geqslant 0\right\}\\ &= \log_2(2^{E_\kappa(\mathcal{N})}+2). \end{aligned} \tag{149}$$

The first inequality follows since we choose more restricted condition $(m-1)Q_{AB}^{T_B} \leqslant (J_{AB}^{\mathcal{N}})^{T_B} \leqslant (m-1)Q_{AB}^{T_B}$. The second inequality follows since $-(\lfloor\mu\rfloor-1) \leqslant -(\mu-2)$ and $\mu-2 \leqslant \lfloor\mu\rfloor-1$. In this case, the set over which we are optimizing becomes smaller. The third inequality follows since $\lfloor\mu\rfloor \leqslant \mu$ in the loss function. We also take $R_{AB} = (\mu-2)Q_{AB}$ to simplify the optimization and then arrive at the final equality following the definition of $E_\kappa(\mathcal{N})$. ∎

Theorem 17 (Exact parallel cost). Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then the exact parallel entanglement cost of $\mathcal{N}_{A\to B}$ is equal to its κ entanglement:

$$E_{\mathrm{PPT}}^{(p)}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}). \tag{150}$$

Proof. The main idea behind the proof is to employ the one-shot bound in Proposition 16 and then the additivity relation from Proposition 11. Consider that

$$E_{\mathrm{PPT}}^{(p)}(\mathcal{N}_{A\to B}) = \limsup_{n\to\infty}\frac{1}{n}E_{\mathrm{PPT}}^{(1)}\left(\mathcal{N}_{A\to B}^{\otimes n}\right) \tag{151}$$

$$\leqslant \limsup_{n\to\infty}\frac{1}{n}\log_2(2^{E_\kappa(\mathcal{N}^{\otimes n})}+2) \tag{152}$$

$$= \limsup_{n\to\infty}\frac{1}{n}\log_2(2^{nE_\kappa(\mathcal{N})}+2) \tag{153}$$

$$= E_\kappa(\mathcal{N}_{A\to B}). \tag{154}$$

Similarly, $E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \geqslant E_\kappa(\mathcal{N}_{A\to B})$. ∎

B. Exact sequential simulation of quantum channels

A more general notion of channel simulation, called sequential channel simulation, was recently proposed and studied in Ref. [14]. In this section, we define and characterize exact sequential channel simulation, as opposed to the approximate sequential channel simulation focused on in Ref. [14]. For concreteness, we set the free channels Ω to be completely PPT-preserving channels. The main idea behind sequential channel simulation is to simulate n uses of the channel $\mathcal{N}_{A\to B}$ in such a way that they can be called in an arbitrary order, i.e., on demand when they are needed. An (n, M) exact sequential channel simulation code consists of a maximally entangled resource state $\Phi_{\bar{A}_0\bar{B}_0}^M$ of Schmidt rank M and a set

$$\left\{\mathcal{P}^{(i)}_{A_i\bar{A}_{i-1}\bar{B}_{i-1}\to B_i\bar{A}_i\bar{B}_i}\right\}_{i=1}^n \tag{155}$$

of completely PPT-preserving channels. Note that the systems $\bar{A}_n\bar{B}_n$ of the final completely PPT-preserving channel $\mathcal{P}^{(n)}_{A_n\bar{A}_{n-1}\bar{B}_{n-1}\to B_n\bar{A}_n\bar{B}_n}$ can be taken trivial without loss of generality. As before, Alice has access to all systems labeled by A, Bob has access to all systems labeled by B, and they are in distant laboratories. The structure of this simulation protocol is intended to be compatible with a discrimination strategy that can test the actual n channels versus the above simulation in a sequential way, along the lines discussed in Refs. [60–62].

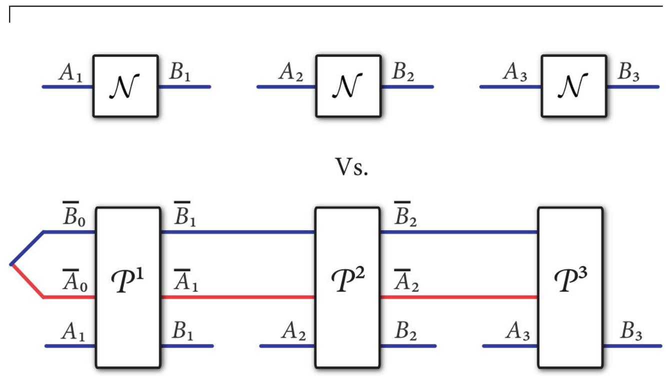

FIG. 2. The top part of the figure depicts the $n = 3$ sequential uses of the channel $\mathcal{N}_{A\to B}$ that should be simulated. The bottom part of the figure depicts the simulation. The simulation is considered to be exact, as written in (156), if, after inputting the operator $|i_r\rangle\langle j_r|_{A_r}$ to the input system A_r and contracting the output system B_r in terms of $\langle k_r|_{B_r}(\cdot)|l_r\rangle_{B_r}$, the resulting numbers are the same for both the original channels and their simulation, for all possible $|i_r\rangle_{A_r}$, $|j_r\rangle_{A_r}$, $|k_r\rangle_{B_r}$, and $|l_r\rangle_{B_r}$ and for $r\in\{1,\dots,n\}$.

We define the simulation to be exact if the following equalities hold for orthonormal bases $\{|i\rangle_A\}_A$ and $\{|k\rangle_B\}_k$ and for all $i_1, j_1, k_1, l_1, \dots, i_n, j_n, k_n, l_n \in \mathbb{N}$:

$$p^{\{i_r,j_r,k_r,l_r\}_{r=1}^n} = \prod_{r=1}^n \langle k_r|_{B_r}\mathcal{N}_{A_r\to B_r}\left(|i_r\rangle\langle j_r|_{A_r}\right)|l_r\rangle_{B_r}, \tag{156}$$

where

$$P_{\bar{A}_1\bar{B}_1}^{i_1,j_1,k_1,l_1} := \langle k_1|_{B_1}\left[\mathcal{P}^{(1)}_{A_1\bar{A}_0\bar{B}_0\to B_1\bar{A}_1\bar{B}_1}\left(|i_1\rangle\langle j_1|_{A_1}\otimes\Phi^M_{\bar{A}_0\bar{B}_0}\right)\right]|l_1\rangle_{B_1}, \tag{157}$$

$$P_{\bar{A}_2\bar{B}_2}^{i_2,j_2,k_2,l_2,i_1,j_1,k_1,l_1} := \langle k_2|_{B_2}\left[\mathcal{P}^{(2)}_{A_2\bar{A}_1\bar{B}_1\to B_2\bar{A}_2\bar{B}_2}\left(|i_2\rangle\langle j_2|_{A_2}\otimes P^{i_1,j_1,k_1,l_1}_{\bar{A}_1\bar{B}_1}\right)\right]|l_2\rangle_{B_2}, \tag{158}$$

$$\vdots$$

$$\begin{aligned}P_{\bar{A}_{n-1}\bar{B}_{n-1}}^{\{i_r,j_r,k_r,l_r\}_{r=1}^{n-1}} &:= \langle k_{n-1}|_{B_{n-1}}\left[\mathcal{P}^{(n-1)}_{A_{n-1}\bar{A}_{n-2}\bar{B}_{n-2}\to B_{n-1}\bar{A}_{n-1}\bar{B}_{n-1}}\left(|i_{n-1}\rangle\langle j_{n-1}|_{A_{n-1}}\right.\right.\\ &\quad\left.\left.\otimes P^{\{i_r,j_r,k_r,l_r\}_{r=1}^{n-2}}_{\bar{A}_{n-2}\bar{B}_{n-2}}\right)\right]|l_{n-1}\rangle_{B_{n-1}},\end{aligned} \tag{159}$$

$$p^{\{i_r,j_r,k_r,l_r\}_{r=1}^n} := \langle k_n|_{B_n}\left[\mathcal{P}^{(n)}_{A_n\bar{A}_{n-1}\bar{B}_{n-1}\to B_n}\left(|i_n\rangle\langle j_n|_{A_n}\otimes P^{\{i_r,j_r,k_r,l_r\}_{r=1}^{n-1}}_{\bar{A}_{n-1}\bar{B}_{n-1}}\right)\right]|l_n\rangle_{B_n}. \tag{160}$$

Figure 2 depicts the channel simulation and the exact simulation condition in (156).

By defining the completely PPT-preserving quantum channel $\mathcal{P}_{A^n\bar{A}_0\bar{B}_0\to B^n}$ as the serial composition of the individual

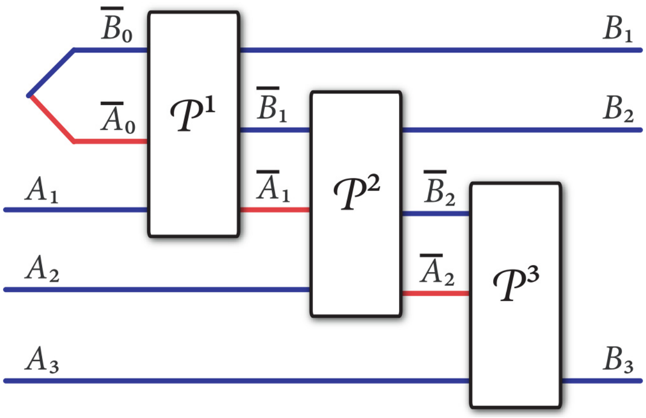

FIG. 3. The channel in (161), defined as the serial composition of the completely PPT-preserving channels in the simulation.

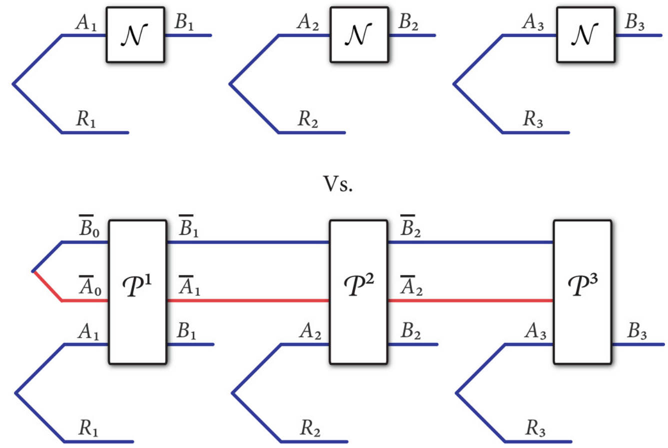

FIG. 4. The exact channel simulation condition in (156) is equivalent to the condition that the Choi operators as depicted above are equal, as written in (162).

channels in (155) (depicted in Fig. 3)

$$\begin{aligned}\mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n} := \big(&\mathcal{P}^{(n)}_{A_n\overline{A}_{n-1}\overline{B}_{n-1}\to B_n}\\ &\circ \mathcal{P}^{(n-1)}_{A_{n-1}\overline{A}_{n-2}\overline{B}_{n-2}\to B_{n-1}\overline{A}_{n-1}\overline{B}_{n-1}}\circ\cdots\\ &\circ \mathcal{P}^{(2)}_{A_2\overline{A}_1\overline{B}_1\to B_2\overline{A}_2\overline{B}_2}\circ \mathcal{P}^{(1)}_{A_1\overline{A}_0\overline{B}_0\to B_1\overline{A}_1\overline{B}_1}\big),\end{aligned} \tag{161}$$

we conclude that the condition in (156) is equivalent to the following condition:

$$(\mathcal{N}_{A\to B})^{\otimes n}(\Gamma_{R^nA^n}) = \mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}\big(\Gamma_{R^nA^n}\otimes \Phi^M_{\overline{A}_0\overline{B}_0}\big), \tag{162}$$

where $\Gamma_{R^nA^n} := \bigotimes_{i=1}^n \Gamma_{R_iA_i}$. This latter condition is depicted in Fig. 4.

The n-shot exact sequential simulation cost of the channel $\mathcal{N}_{A\to B}$ is then defined as

$$\begin{aligned}E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}, n) := \inf\big\{\log_2 M : (&\mathcal{N}_{A\to B})^{\otimes n}(\Gamma_{R^nA^n})\\ &= \mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}\big(\Gamma_{R^nA^n}\otimes\Phi^M_{\overline{A}_0\overline{B}_0}\big)\big\},\end{aligned} \tag{163}$$

where the optimization is with respect to sequential protocols of the form in (155) and the channel $\mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}$ is defined as in (161). The exact (sequential) simulation cost of the channel $\mathcal{N}_{A\to B}$ is defined as

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) := \limsup_{n\to\infty}\frac{1}{n}E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}, n). \tag{164}$$

The condition in (162) illustrates that a sequential simulation is a particular kind of parallel simulation, but with more constraints. That is, in a parallel simulation, the channel $\mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}$ can be arbitrary, whereas in a sequential simulation, it is constrained to have the form in (155). For this reason, we can immediately conclude the following bound for every integer $n \geqslant 1$:

$$E^{(1)}_{\mathrm{PPT}}((\mathcal{N}_{A\to B})^{\otimes n}) \leqslant E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}, n), \tag{165}$$

which in turn implies that

$$E^{(p)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \leqslant E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}). \tag{166}$$

C. Physical justification for definition of exact sequential channel simulation

The most general method for distinguishing the n channel uses from its simulation is with an adaptive discrimination strategy. Such a strategy was described in Ref. [14] and consists of an initial state $\rho_{R_1A_1}$, a set $\{\mathcal{A}^{(i)}_{R_iB_i\to R_{i+1}A_{i+1}}\}_{i=1}^{n-1}$ of adaptive channels, and a quantum measurement $\{Q_{R_nB_n}, \mathbb{1}_{R_nB_n} - Q_{R_nB_n}\}$. Let us employ the shorthand $\{\rho, \mathcal{A}, Q\}$ to abbreviate such a discrimination strategy. Note that, in performing a discrimination strategy, the discriminator has a full description of the channel $\mathcal{N}_{A\to B}$ and the simulation protocol, which consists of $\Phi_{\overline{A}_0\overline{B}_0}$ and the set in (155). If this discrimination strategy is performed on the n uses of the actual channel $\mathcal{N}_{A\to B}$, the relevant states involved are

$$\rho_{R_{i+1}A_{i+1}} := \mathcal{A}^{(i)}_{R_iB_i\to R_{i+1}A_{i+1}}\big(\rho_{R_iB_i}\big) \tag{167}$$

for $i \in \{1, \dots, n-1\}$ and

$$\rho_{R_iB_i} := \mathcal{N}_{A_i\to B_i}\big(\rho_{R_iA_i}\big) \tag{168}$$

for $i \in \{1, \dots, n\}$. If this discrimination strategy is performed on the simulation protocol discussed above, then the relevant states involved are

$$\begin{aligned}\tau_{R_1B_1\overline{A}_1\overline{B}_1} &:= \mathcal{P}^{(1)}_{A_1\overline{A}_0\overline{B}_0\to B_1\overline{A}_1\overline{B}_1}\big(\tau_{R_1A_1}\otimes\Phi_{\overline{A}_0\overline{B}_0}\big),\\ \tau_{R_{i+1}A_{i+1}\overline{A}_i\overline{B}_i} &:= \mathcal{A}^{(i)}_{R_iB_i\to R_{i+1}A_{i+1}}\big(\tau_{R_iB_i\overline{A}_i\overline{B}_i}\big),\end{aligned} \tag{169}$$

for $i \in \{1, \dots, n-1\}$, where $\tau_{R_1A_1} = \rho_{R_1A_1}$, and

$$\tau_{R_iB_i\overline{A}_i\overline{B}_i} := \mathcal{P}^{(i)}_{A_i\overline{A}_{i-1}\overline{B}_{i-1}\to B_i\overline{A}_i\overline{B}_i}\big(\tau_{R_iA_i\overline{A}_{i-1}\overline{B}_{i-1}}\big) \tag{170}$$

for $i \in \{2, \dots, n\}$. The discriminator then performs the measurement $\{Q_{R_nB_n}, \mathbb{1}_{R_nB_n} - Q_{R_nB_n}\}$ and guesses "actual channel" if the outcome is $Q_{R_nB_n}$ and "simulation" if the outcome is $\mathbb{1}_{R_nB_n} - Q_{R_nB_n}$. Figure 5 depicts the discrimination strategy in the case that the actual channel is called $n = 3$ times and in the case that the simulation is performed.

From the physical point of view, the n channel uses of $\mathcal{N}_{A\to B}$ are perfectly indistinguishable from the simulation if every possible discrimination strategy as described above leads to the exact same final decision probabilities. That is,

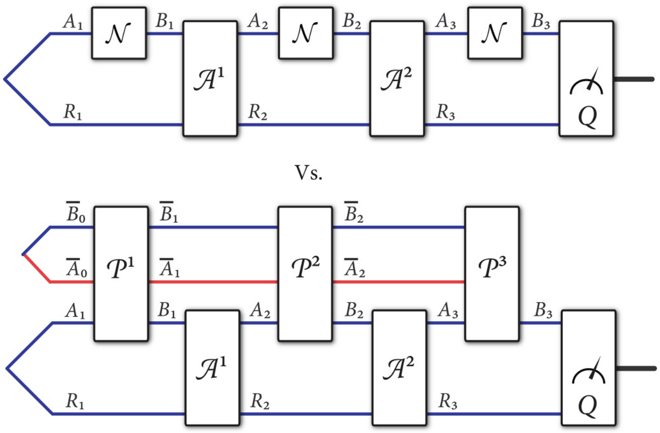

FIG. 5. An adaptive protocol for discriminating the original channels (top) from their simulation (bottom).

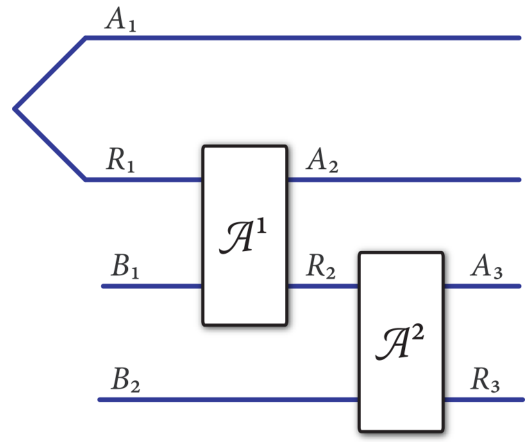

FIG. 6. The discrimination strategy $\rho_{R_1A_1}$ and $\{\mathcal{A}^{(i)}_{R_iB_i\to R_{i+1}A_{i+1}}\}_{i=1}^{n-1}$ represented as a single channel $\mathcal{A}_{B^n\to A^nR_n}$, as written in (162).

for all possible discrimination strategies, the original channels and their simulations are indistinguishable if the following equality holds:

$$\mathrm{Tr}\, Q_{R_nB_n}\rho_{R_nB_n} = \mathrm{Tr}\, Q_{R_nB_n}\tau_{R_nB_n}. \tag{171}$$

We now prove that this physical notion of exact channel simulation is equivalent to the more mathematical notion of exact channel simulation described in the previous section. First, suppose that the physical notion of exact channel simulation holds; i.e., the equality in (171) holds for all possible discrimination strategies. Then this means that $\rho_{R_nB_n} = \tau_{R_nB_n}$ for all possible discrimination strategies. One possible strategy could be to pick the input state for each system A_i as one of the following states:

$$\rho_A^{x,y} = \begin{cases} |x\rangle\langle x|_A & \text{if } x = y \\ \frac{1}{2}(|x\rangle_A + |y\rangle_A)(\langle x|_A + \langle y|_A) & \text{if } x < y \\ \frac{1}{2}(|x\rangle_A + i|y\rangle_A)(\langle x|_A - i\langle y|_A) & \text{if } x > y, \end{cases} \tag{172}$$

and the output system B_i could be measured in the same way, but with respect to an orthonormal basis for the output system. Then all input state choices and measurement outcomes could be stored in auxiliary classical registers. Consider that for all x, y such that $x < y$, the following holds:

$$|x\rangle\langle y|_A = \left(\rho_A^{x,y} - \tfrac{1}{2}\rho_A^{x,x} - \tfrac{1}{2}\rho_A^{y,y}\right) - i\left(\rho_A^{y,x} - \tfrac{1}{2}\rho_A^{x,x} - \tfrac{1}{2}\rho_A^{y,y}\right), \tag{173}$$

$$|y\rangle\langle x|_A = \left(\rho_A^{x,y} - \tfrac{1}{2}\rho_A^{x,x} - \tfrac{1}{2}\rho_A^{y,y}\right) + i\left(\rho_A^{y,x} - \tfrac{1}{2}\rho_A^{x,x} - \tfrac{1}{2}\rho_A^{y,y}\right), \tag{174}$$

so linear combinations of all the outcomes realize the operator basis discussed in the mathematical definition of equivalence. Since the equivalence holds for all possible discrimination strategies, we can collect the data from them in the auxiliary registers, and then finally conclude that the condition in (156) holds.

To see that the mathematical notion of exact sequential simulation implies the physical one, we use the method of postselected teleportation, essentially the same idea as what was used in the proof of Ref. [63], Theorem 4]. Consider the channel defined by the serial composition of the channels in the discrimination strategy $\{\rho, \mathcal{A}, Q\}$:

$$\mathcal{A}_{B^n\to A^nR_n} = \mathcal{A}^{(n-1)}_{R_{n-1}B_{n-1}\to R_nA_n} \circ \cdots \circ \mathcal{A}^{(2)}_{R_2B_2\to R_3A_3} \circ \mathcal{A}^{(1)}_{R_1B_1\to R_2A_2} \circ \rho_{R_1A_1}, \tag{175}$$

where the notation $\rho_{R_1A_1}$ indicates a preparation channel that tensors in the state $\rho_{R_1A_1}$. Figure 6 depicts this channel. By acting on both sides of the exact simulation condition with the channel and then the projection onto $|\Gamma\rangle\langle\Gamma|_{A^nS^n}$, with $S \simeq R$, we find that

$$\langle\Gamma|_{A^nS^n}\left[\mathcal{A}_{B^n\to A^nR_n} \circ (\mathcal{N}_{A\to B})^{\otimes n}(\Gamma_{S^nA^n})\right]|\Gamma\rangle_{A^nS^n} = \langle\Gamma|_{A^nS^n}\left[\mathcal{A}_{B^n\to A^nR_n} \circ \mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}\left(\Gamma_{S^nA^n} \otimes \Phi^M_{\overline{A}_0\overline{B}_0}\right)\right]|\Gamma\rangle_{A^nS^n}, \tag{176}$$

where

$$|\Gamma\rangle_{A^nS^n} = |\Gamma\rangle_{A_1S_1} \otimes |\Gamma\rangle_{A_2S_2} \otimes \cdots \otimes |\Gamma\rangle_{A_nS_n}. \tag{177}$$

From the method of postselected teleportation, we conclude that

$$\langle\Gamma|_{A^nS^n}\left[\mathcal{A}_{B^n\to A^nR_n} \circ (\mathcal{N}_{A\to B})^{\otimes n}(\Gamma_{S^nA^n})\right]|\Gamma\rangle_{A^nS^n} = \rho_{R_nB_n}, \tag{178}$$

$$\langle\Gamma|_{A^nS^n}\left[\mathcal{A}_{B^n\to A^nR_n} \circ \mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}\left(\Gamma_{S^nA^n} \otimes \Phi^M_{\overline{A}_0\overline{B}_0}\right)\right]|\Gamma\rangle_{A^nS^n} = \tau_{R_nB_n}. \tag{179}$$

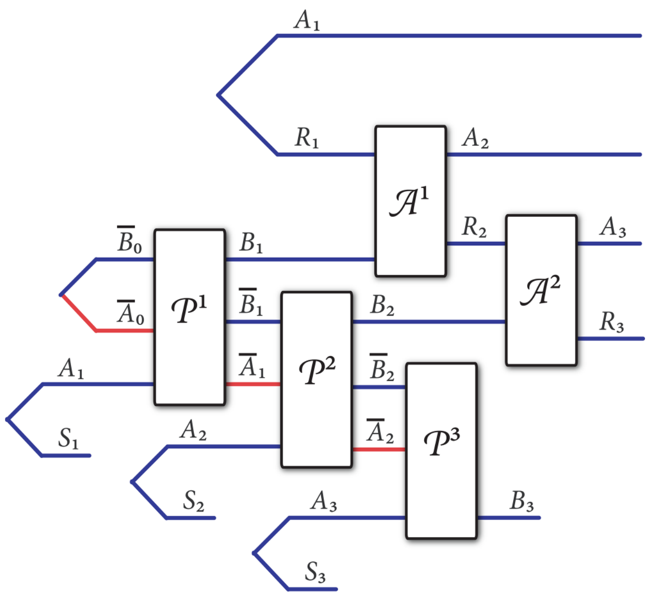

FIG. 7. This figure depicts the operator $\mathcal{A}_{B^n\to A^nR_n}\circ\mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}(\Gamma_{S^nA^n}\otimes\Phi^M_{\overline{A}_0\overline{B}_0})$ in order to help visualize the argument in (176)–(180). By projecting the systems S_1A_1 onto $\langle\Gamma|_{S_1A_1}$, S_2A_2 onto $\langle\Gamma|_{S_2A_2}$, and S_3A_3 onto $\langle\Gamma|_{S_3A_3}$, the method of postselected teleportation guarantees that the remaining state is $\tau_{R_3B_3}$, which is the final state of the bottom part of Fig. 5.

Putting these together, we finally conclude that

$$\rho_{R_nB_n}=\tau_{R_nB_n}. \tag{180}$$

Thus, no physical discrimination strategy can distinguish the original channels from their simulation if the exact simulation condition in (162) holds. Figure 7 depicts the operator $\mathcal{A}_{B^n\to A^nR_n}\circ\mathcal{P}_{A^n\overline{A}_0\overline{B}_0\to B^n}(\Gamma_{S^nA^n}\otimes\Phi^M_{\overline{A}_0\overline{B}_0})$ to help visualize the above argument.

D. Exact sequential channel simulation cost

We first establish the following bounds on the n-shot exact sequential simulation cost:

Proposition 18. Let $\mathcal{N}_{A\to B}$ be a quantum channel such that $E_\kappa(\mathcal{N})>0$. Then the n-shot exact sequential simulation cost is bounded as

$$\log_2[2^{nE_\kappa(\mathcal{N})}-1]\leqslant E_{\mathrm{PPT}}(\mathcal{N}_{A\to B},n) \tag{181}$$

$$\leqslant\log_2\left[\frac{2^{(n+1)E_\kappa(\mathcal{N})}-1}{2^{E_\kappa(\mathcal{N})}-1}\right]. \tag{182}$$

If $E_\kappa(\mathcal{N})=0$, then $E_{\mathrm{PPT}}(\mathcal{N}_{A\to B},n)=0$.

Proof. Suppose that $E_\kappa(\mathcal{N})>0$. The inequality

$$\log_2[2^{nE_\kappa(\mathcal{N})}-1]\leqslant E_{\mathrm{PPT}}(\mathcal{N}_{A\to B},n) \tag{183}$$

is a direct consequence of (165), Proposition 16, and Proposition 11.

So we now prove the other inequality. The main idea behind the construction is for the ith completely PPT-preserving channel to perform the following exact simulation:

$$\mathcal{P}^{(i)}_{A_i\overline{A}_{i-1}\overline{B}_{i-1}\to B_i\overline{A}_i\overline{B}_i}\left(\rho_{A_i}\otimes\Phi^{M_{i-1}}_{\overline{A}_{i-1}\overline{B}_{i-1}}\right)=\mathcal{N}_{A\to B}\left(\rho_{A_i}\right)\otimes\Phi^{M_i}_{\overline{A}_i\overline{B}_i} \tag{184}$$

for $i\in\{1,\ldots,n-1\}$ and for the nth completely PPT-preserving channel to perform the following exact simulation:

$$\mathcal{P}^{(n)}_{A_n\overline{A}_{n-1}\overline{B}_{n-1}\to B_n}\left(\rho_{A_n}\otimes\Phi^{M_{n-1}}_{\overline{A}_{n-1}\overline{B}_{n-1}}\right)=\mathcal{N}_{A\to B}\left(\rho_{A_n}\right). \tag{185}$$

Note that, to perform the simulation in (184), we could actually simulate the channel $\mathcal{N}_{A\to B}\otimes\mathrm{id}^{M_i}$, and then send one share of the maximally entangled state $\Phi^{M_i}_{\overline{A}_i\overline{B}_i}$ through the exactly simulated identity channel id^{M_i} to produce the output in (184).

Thus, we should now determine an upper bound on the simulation cost when using this construction. The most effective way to do so is to start from the final (nth) simulation. By the one-shot bound from Proposition 16, its cost $\log_2 M_{n-1}$ is bounded as

$$\log_2 M_{n-1}\leqslant\log_2[2^{E_\kappa(\mathcal{N})}+1]. \tag{186}$$

The cost $\log_2 M_{n-2}$ of the $n-1$ simulation is then bounded as

$$\log_2 M_{n-2}\leqslant\log_2\left[2^{E_\kappa(\mathcal{N}\otimes\mathrm{id}^{M_{n-1}})}+1\right] \tag{187}$$

$$\leqslant\log_2[2^{E_\kappa(\mathcal{N})+\log_2 M_{n-1}}+1] \tag{188}$$

$$=\log_2[2^{E_\kappa(\mathcal{N})}M_{n-1}+1] \tag{189}$$

$$\leqslant\log_2[2^{E_\kappa(\mathcal{N})}(2^{E_\kappa(\mathcal{N})}+1)+1] \tag{190}$$

$$=\log_2\left[\sum_{\ell=0}^{2}2^{\ell E_\kappa(\mathcal{N})}\right], \tag{191}$$

where we made use of the subadditivity inequality from Proposition 11. Performing this kind of reasoning iteratively, going backward until the first simulation, we find the following bound:

$$\log_2 M_0\leqslant\log_2\left[\sum_{\ell=0}^{n}2^{\ell E_\kappa(\mathcal{N})}\right]=\log_2\left[\frac{2^{(n+1)E_\kappa(\mathcal{N})}-1}{2^{E_\kappa(\mathcal{N})}-1}\right]. \tag{192}$$

If $E_\kappa(\mathcal{N})=0$, then the channel $\mathcal{N}$ is PPT entanglement binding by Proposition 13 and thus can be simulated at no cost, so $E_{\mathrm{PPT}}(\mathcal{N}_{A\to B},n)=0$. This concludes the proof. ■

Theorem 19 (Exact sequential cost). Let $\mathcal{N}_{A\to B}$ be a quantum channel. Then the exact sequential channel simulation cost of $\mathcal{N}_{A\to B}$ is equal to its κ entanglement:

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B})=E_\kappa(\mathcal{N}_{A\to B}). \tag{193}$$

Proof. First, suppose that $E_\kappa(\mathcal{N})>0$. The lower bound follows from Proposition 18 and Theorem 17. The upper bound follows from Proposition 18:

$$\limsup_{n\to\infty}\frac{1}{n}E_{\mathrm{PPT}}(\mathcal{N}_{A\to B},n)$$

$$\leqslant\limsup_{n\to\infty}\frac{1}{n}\log_2\left[\frac{2^{(n+1)E_\kappa(\mathcal{N})}-1}{2^{E_\kappa(\mathcal{N})}-1}\right] \tag{194}$$

$$=\limsup_{n\to\infty}\frac{1}{n}\log_2\left[\frac{2^{nE_\kappa(\mathcal{N})}-2^{-E_\kappa(\mathcal{N})}}{1-2^{-E_\kappa(\mathcal{N})}}\right] \tag{195}$$

$$=E_\kappa(\mathcal{N}). \tag{196}$$

If $E_\kappa(\mathcal{N})=0$, then the channel $\mathcal{N}$ is PPT entanglement binding by Proposition 13 and thus can be simulated at no cost. This concludes the proof. ■

By combining Theorems 17 and 19, we reach the conclusion that the exact entanglement cost of parallel and sequential simulation of quantum channels are in fact equal and given by the κ entanglement of the channel. Thus, the κ entanglement is a fundamental measure of the entanglement of a quantum channel. Not only is it efficiently computable by means of a semidefinite program (for finite-dimensional channels), but it also possesses a direct operational meaning in terms of these channel simulation tasks. It is the only known channel entanglement measure possessing these properties, and from this perspective, it can be helpful in understanding the fundamental structure of entanglement of quantum channels.

E. PPT-simulable channels

Although the theory of exact simulation of quantum channels under PPT operations simplifies significantly due to Theorems 17 and 19, there is a class of channels for which the theory is even simpler. These channels were defined in Ref. [48] and are known as PPT-simulable channels. In this section, we recall their definition and show how the theory of exact entanglement cost is quite simple for certain PPT-simulable channels.

Definition 4. (PPT-simulable channel [48]) A channel $\mathcal{N}_{A\to B}$ is PPT simulable with associated resource state $\omega_{A'B'}$ if there exists a completely PPT-preserving channel $\mathcal{P}_{AA'B'\to B}$ such that, for every input state ρ_A:

$$\mathcal{N}_{A\to B}(\rho_A) = \mathcal{P}_{AA'B'\to B}(\rho_A \otimes \omega_{A'B'}). \tag{197}$$

A particular kind of PPT-simulable channel is one that is resource-seizable, as defined in Ref. [14], Sec. VI]:

Definition 5 (Resource-seizable [14]). Let $\mathcal{N}_{A\to B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. The channel $\mathcal{N}_{A\to B}$ is resource-seizable if there exists a PPT state $\tau_{A_M AB_M}$ and a completely PPT-preserving postprocessing channel $\mathcal{D}_{A_M BB_M\to A'B'}$ such that

$$\mathcal{D}_{A_M BB_M\to A'B'}\big(\mathcal{N}_{A\to B}\big(\tau_{A_M AB_M}\big)\big) = \omega_{A'B'}. \tag{198}$$

For PPT-simulable channels, it follows that the exact entanglement cost of sequential channel simulation is bounded from above by the exact entanglement cost of the underlying resource state.

Theorem 20. Let $\mathcal{N}_{A\to B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. Then the PPT-assisted entanglement cost of a channel is bounded from above as

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \leqslant E_{\mathrm{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}). \tag{199}$$

Proof. The proof for this inequality follows the same reasoning given in Ref. [14], Corollary 1]. First, simulate a large number of copies of the resource state $\omega_{A'B'}$ and then use the PPT-preserving channel $\mathcal{P}_{AA'B'\to B}$ from (197) to simulate the channel $\mathcal{N}_{A\to B}$. The equality follows from Proposition 6. ■

If a PPT-simulable channel is additionally resource-seizable, then its exact entanglement cost is given by the κ entanglement of the underlying resource state:

Theorem 21. Let $\mathcal{N}_{A\to B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. Suppose furthermore that it is resource-seizable, as given in Definition 5. Then

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) = E^{(p)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}) \tag{200}$$

$$= E_{\mathrm{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}). \tag{201}$$

Proof. The following inequality:

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \leqslant E_{\mathrm{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}) \tag{202}$$

is a consequence of Theorem 20. To establish the opposite inequality, consider that we always have that

$$E_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) \geqslant E^{(p)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}), \tag{203}$$

where $E^{(p)}_{\mathrm{PPT}}$ denotes the exact parallel simulation entanglement cost. From Theorem 17, we have that

$$E^{(p)}_{\mathrm{PPT}}(\mathcal{N}_{A\to B}) = E_\kappa(\mathcal{N}_{A\to B}). \tag{204}$$

So, it suffices to prove that

$$E_\kappa(\mathcal{N}_{A\to B}) = E_\kappa(\omega_{A'B'}). \tag{205}$$

Letting ρ_{RA} be an arbitrary input state, we have that

$$E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) = E_\kappa(\mathcal{P}_{AA'B'\to B}(\rho_{RA}\otimes\omega_{A'B'})) \tag{206}$$

$$\leqslant E_\kappa(\rho_{RA}\otimes\omega_{A'B'}) \tag{207}$$

$$= E_\kappa(\omega_{A'B'}), \tag{208}$$

where the inequality follows from the monotonicity of E_κ under PPT-preserving channels and the final equality follows because the bipartite cut is taken as $RAA'|B'$. Since this holds for an arbitrary input state ρ_{RA}, we conclude that

$$E_\kappa(\omega_{A'B'}) \geqslant E_\kappa(\mathcal{N}_{A\to B}). \tag{209}$$

Now we prove the opposite inequality by using the fact that $\mathcal{N}_{A\to B}$ is resource-seizable. Let $\tau_{A_M AB_M}$ be the input PPT state from Definition 5. Consider that

$$E_\kappa(\omega_{A'B'}) = E_\kappa\big[\mathcal{D}_{A_M BB_M\to A'B'}\big(\mathcal{N}_{A\to B}\big(\tau_{A_M AB_M}\big)\big)\big] \tag{210}$$

$$\leqslant E_\kappa\big(\mathcal{N}_{A\to B}(\tau_{A_M AB_M})\big) \tag{211}$$

$$= E_\kappa\big(\mathcal{N}_{A\to B}\big(\tau_{A_M AB_M}\big)\big) - E_\kappa\big(\tau_{A_M AB_M}\big) \tag{212}$$

$$\leqslant E_\kappa(\mathcal{N}_{A\to B}). \tag{213}$$

The first inequality follows because E_κ does not increase under the action of the completely PPT-preserving channel $\mathcal{D}_{A_M BB_M\to A'B'}$ (Theorem 1). The second equality follows because $\tau_{A_M AB_M}$ is a PPT state, so $E_\kappa(\tau_{A_M AB_M}) = 0$. The final inequality is a consequence of the amortization inequality in Proposition 8. ■

F. Relationship to other quantities

A previously known efficiently computable upper bound for quantum capacity is the partial transposition bound [17],

$$Q_\Theta(\mathcal{N}) := \log_2 \|T_{B\to B}\circ\mathcal{N}_{A\to B}\|_\diamond, \tag{214}$$

where $T_{B\to B}$ is the transpose map and $\|\cdot\|_\diamond$ is the completely bounded trace norm or diamond norm. Note that $\|\cdot\|_\diamond$ for finite-dimensional channels is efficiently computable via semidefinite programming [64].

Proposition 22. For every quantum channel $\mathcal{N}_{A\to B}$, we have that

$$Q_\Theta(\mathcal{N}_{A\to B}) \leqslant E_\kappa(\mathcal{N}_{A\to B}). \tag{215}$$

Proof. Given an arbitrary quantum channel $\mathcal{N}_{A\to B}$, it holds that

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA})) \tag{216}$$
$$\geqslant \sup_{\phi_{RA}} E_N(\mathcal{N}_{A\to B}(\phi_{RA})) \tag{217}$$
$$= \sup_{\phi_{RA}} \log_2 \|\mathcal{N}_{A\to B}(\phi_{RA})^{T_B}\|_1 \tag{218}$$
$$= \log_2 \|T_{B\to B}\circ \mathcal{N}_{A\to B}\|_\diamond. \tag{219}$$

The equality in (216) follows from Proposition 7. The inequality in (217) follows from the property of E_κ in Eq. (12). The last equality follows due to the definition of the completely bounded trace norm. ■

Remark 2. For qubit-input qubit-output channels, we have that

$$E_\kappa(\mathcal{N}_{A\to B}) = Q_\Theta(\mathcal{N}_{A\to B}). \tag{220}$$

This follows because it suffices to optimize $E_\kappa(\mathcal{N}_{A\to B})$ with respect to two-qubit input states ϕ_{RA}, and then the output state consists of two qubits, so the result of Ref. [23] applies. That is, for this case:

$$E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA})) \tag{221}$$
$$= \sup_{\phi_{RA}} E_N(\mathcal{N}_{A\to B}(\phi_{RA})) \tag{222}$$
$$= Q_\Theta(\mathcal{N}_{A\to B}). \tag{223}$$

V. EXACT ENTANGLEMENT COST OF FUNDAMENTAL CHANNELS

Theorem 21 provides a formula for the exact PPT-entanglement cost of an arbitrary resource-seizable, PPT-simulable channel, given in terms of the entanglement cost of the underlying resource state $\omega_{A'B'}$. We detail some simple examples here for which this simplified formula applies. We also consider amplitude damping channels for which it is necessary to invoke Theorems 17 and 19 to determine their exact entanglement costs.

Let us begin by recalling the notion of a covariant channel $\mathcal{N}_{A\to B}$ [65]. For a group G with unitary channel representations $\{\mathcal{U}_A^g\}_{g\in G}$ and $\{\mathcal{V}_B^g\}_{g\in G}$ acting on the input system A and output system B of channel $\mathcal{N}_{A\to B}$, channel $\mathcal{N}_{A\to B}$ is covariant with respect to group G if the following equality holds for all $g\in G$:

$$\mathcal{N}_{A\to B}\circ \mathcal{U}_A^g = \mathcal{V}_B^g \circ \mathcal{N}_{A\to B}. \tag{224}$$

If the averaging channel is such that $\frac{1}{|G|}\sum_g \mathcal{U}_A^g(X) = \mathrm{Tr}[X]I/|A|$, then we simply say that the channel $\mathcal{N}_{A\to B}$ is covariant.

Then, from Ref. [66], Sec. 7], we conclude that a covariant channel is PPT simulable with associated resource state given by the Choi state of the channel, i.e., $\omega_{A'B'} = \mathcal{N}_{A\to B}(\Phi_{A'A})$. As such, covariant channels are resource-seizable, so the equality in Theorem 21 applies to all covariant channels. Thus, the exact entanglement cost of a covariant channel is equal to the exact entanglement cost of its Choi state.

A. Erasure channel

The quantum erasure channel is denoted by

$$\mathcal{E}_p(\rho) = (1-p)\rho + p|e\rangle\langle e|, \tag{225}$$

where ρ is a d-dimensional input state, $p\in[0,1]$ is the erasure probability, and $|e\rangle\langle e|$ is a pure erasure state orthogonal to every input state, so the output state has $d+1$ dimensions. This channel is covariant.

The Choi matrix of $\mathcal{E}_p$ is given by

$$J_{\mathcal{E}_p} = (1-p)\sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + p\sum_{i=0}^{d-1}|i\rangle\langle i|\otimes|e\rangle\langle e|. \tag{226}$$

By direct calculation, we find that

$$E_{\mathrm{PPT}}(\mathcal{E}_p) = E_{\mathrm{PPT}}\big(J_{\mathcal{E}_p}/d\big) \tag{227}$$
$$= E_N\big(J_{\mathcal{E}_p}/d\big) \tag{228}$$
$$= \log_2(d[1-p]+p). \tag{229}$$

B. Depolarizing channel

Consider the qudit depolarizing channel,

$$\mathcal{N}_{D,p}(\rho) = (1-p)\rho + \frac{p}{d^2-1}\sum_{\substack{0\leqslant i,j\leqslant d-1\\ (i,j)\neq(0,0)}} X^iZ^j\rho(X^iZ^j)^\dagger, \tag{230}$$

where $p\in[0,1]$ and X, Z are the generalized Pauli operators. This channel is covariant.

The Choi matrix of $\mathcal{N}_{D,p}$ is

$$J_{\mathcal{N}_{D,p}} = d\bigg[(1-p)\Phi_{AB} + \frac{p}{d^2-1}(\mathbb{1}_{AB}-\Phi_{AB})\bigg], \tag{231}$$

where $\Phi = \frac{1}{d}\sum_{i,j=0}^{d-1}|ii\rangle\langle jj|$. Observe that the state $\frac{J_{\mathcal{N}_{D,p}}}{d}$ is an isotropic state. Applying the previous result from (29), we conclude that

$$E_{\mathrm{PPT}}(\mathcal{N}_{D,p}) = \begin{cases} \log_2 d(1-p) & \text{if } 1-p\geqslant \frac{1}{d}\\ 0 & \text{if } 1-p<\frac{1}{d}.\end{cases} \tag{232}$$

C. Dephasing channel

The qubit dephasing channel is given as

$$\mathcal{D}_q(\rho) = (1-q)\rho + qZ\rho Z. \tag{233}$$

Note that this channel is covariant with respect to the Heisenberg-Weyl group of unitaries. The Choi matrix of $\mathcal{D}_q$ is as follows:

$$J_{\mathcal{D}_q} = 2[(1-q)\psi_1 + q\psi_2], \tag{234}$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle). \tag{235}$$

By direct calculation, we find that

$$E_{\mathrm{PPT}}(\mathcal{D}_q) = E_{\mathrm{PPT}}\big(J_{\mathcal{D}_q}/2\big) \tag{236}$$
$$= E_N\big(J_{\mathcal{D}_q}/2\big) \tag{237}$$
$$= \log_2(1+2|q-1/2|). \tag{238}$$

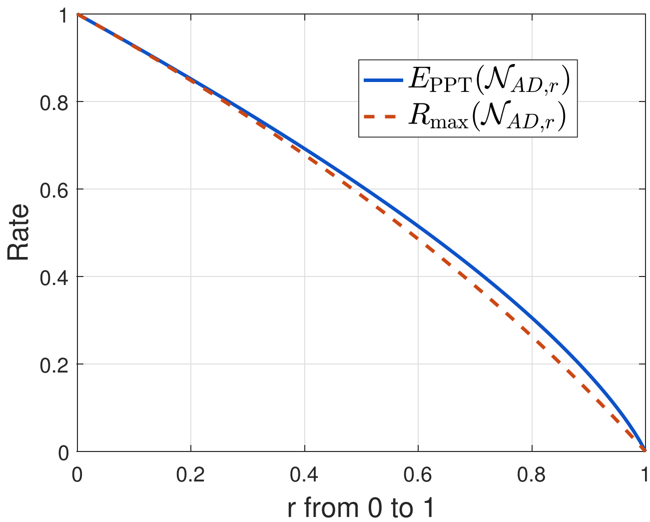

FIG. 8. This plot demonstrates the difference between $E_{\rm PPT}(\mathcal{N}_{{\rm AD},r})$ and $R_{\max}(\mathcal{N}_{{\rm AD},r})$, where $\mathcal{N}_{{\rm AD},r}$ is the amplitude damping channel in Sec. V D. The solid line depicts $E_{\rm PPT}(\mathcal{N}_{{\rm AD},r})$ while the dashed line depicts $R_{\max}(\mathcal{N}_{{\rm AD},r})$. The parameter r ranges from 0 to 1, and the units of the rate (vertical axis) are ebits per channels use.

We note that this approach also works for a d-dimensional dephasing channel.

D. Amplitude damping channel

An amplitude damping channel corresponds to the process of asymmetric relaxation in a quantum system, which is a key noise process in quantum information science. The qubit amplitude damping channel is given as $\mathcal{N}_{{\rm AD},r} = \sum_{i=0}^{1} E_i \cdot E_i^\dagger$ with

$$E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1|, \quad E_1 = \sqrt{r}|0\rangle\langle 1|, \tag{239}$$

and where $r \in [0, 1]$ is the damping parameter. This channel is covariant with respect to $\{I, Z\}$, but not with respect to a one-design. So, Theorem 21 does not apply, and we instead need to evaluate the exact entanglement cost of this channel by applying Theorems 17 and 19.

We plot $E_{\rm PPT}(\mathcal{N}_{{\rm AD},r})$ in Fig. 8 and compare it with the max-Rains information of [50,51]. The fact that there is a gap between these two quantities demonstrates that the resource theory of entanglement (exact PPT case) is irreversible, given that the max-Rains information is an upper bound on the exact distillable entanglement of an arbitrary channel [47].

VI. EXACT ENTANGLEMENT COST OF QUANTUM GAUSSIAN CHANNELS

In this subsection, we determine formulas for the exact entanglement cost of particular quantum Gaussian channels, which include all single-mode bosonic Gaussian channels with the exception of the pure-loss and pure-amplifier channels. In this sense, the results found here are complementary to those found recently in Ref. [14], Theorem 2]. The presentation and background given in this section largely follows that given recently in Ref. [14].

A. Preliminary observations about the exact entanglement cost of single-mode bosonic Gaussian channels

The starting point for our analysis of single-mode bosonic Gaussian channels is the Holevo classification from Ref. [67], in which canonical forms for all single-mode bosonic Gaussian channels have been given, classifying them up to local Gaussian unitaries acting on the input and output of the channel. It then suffices for us to focus our attention on the canonical forms, as it is self-evident from definitions that local unitaries do not alter the exact entanglement cost of a quantum channel. The thermal and amplifier channels form the class C discussed in Ref. [67], and the additive-noise channels form the class B_2 discussed in the same work. The classes that remain are labeled A, B_1, and D in Ref. [67]. The channels in A and D are entanglement breaking [68] and are thus entanglement binding, and as a consequence of Proposition 13 and Theorems 17 and 19, they have zero exact entanglement cost. Channels in class B_1 are perhaps not interesting for practical applications and, as it turns out, they have infinite quantum capacity [67]. Thus, their exact entanglement cost is also infinite because a channel's quantum capacity is a lower bound on its distillable entanglement, which is in turn a lower bound on its partial transposition bound. The partial transposition bound is finally a lower bound on its κ entanglement, as shown in Proposition 22. For the same reason, the exact entanglement cost of the bosonic identity channel is also infinite.

B. Thermal, amplifier, and additive-noise bosonic Gaussian channels

In light of the previous discussion, for the remainder of this section, let us focus our attention on the thermal, amplifier, and additive-noise channels. Each of these are defined, respectively, by the following Heisenberg input-output relations:

$$\hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}, \tag{240}$$

$$\hat{b} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^\dagger, \tag{241}$$

$$\hat{b} = \hat{a} + (x + ip)/\sqrt{2}, \tag{242}$$

where $\hat{a}$, $\hat{b}$, and $\hat{e}$ are the field-mode annihilation operators for the sender's input, the receiver's output, and the environment's input of these channels, respectively.

The channel in (240) is a thermalizing channel, in which the environmental mode is prepared in a thermal state $\theta(N_B)$ of mean photon number $N_B \geqslant 0$, defined as

$$\theta(N_B) := \frac{1}{N_B+1} \sum_{n=0}^{\infty} \left(\frac{N_B}{N_B+1}\right)^n |n\rangle\langle n|, \tag{243}$$

where $\{|n\rangle\}_{n=0}^{\infty}$ is the orthonormal, photonic number-state basis. When $N_B = 0$, the state $\theta(N_B)$ reduces to the vacuum state, in which case the resulting channel in (240) is called the pure-loss channel—it is said to be quantum limited in this case because the environment is injecting the minimum amount of noise allowed by quantum mechanics. The parameter $\eta \in (0, 1)$ is the transmissivity of the channel, representing the average fraction of photons making it from the input to the output of the channel. Let $\mathcal{L}_{\eta,N_B}$ denote this channel, and we make the further abbreviation $\mathcal{L}_\eta \equiv \mathcal{L}_{\eta,N_B=0}$ when it is the pure-loss channel. The channel in (240) is entanglement breaking when $(1-\eta)N_B \geqslant \eta$ [68] and is thus entanglement binding in this case, and as a consequence of Proposition 13 and Theorems 17 and 19, it has zero exact entanglement cost for these values.

The channel in (241) is an amplifier channel and the parameter $G > 1$ is its gain. For this channel, the environment is prepared in the thermal state $\theta(N_B)$. If $N_B = 0$, the amplifier channel is called the pure-amplifier channel—it is said to be quantum limited for a similar reason as stated above. Let $\mathcal{A}_{G,N_B}$ denote this channel, and we make the further abbreviation $\mathcal{A}_G \equiv \mathcal{A}_{G,N_B=0}$ when it is the quantum-limited amplifier channel. The channel in (241) is entanglement breaking when $(G-1)N_B \geqslant 1$ [68] and is thus entanglement binding, and as a consequence of Proposition 13 and Theorems 17 and 19, it has zero exact entanglement cost for these values.

Finally, the channel in (242) is an additive-noise channel, representing a quantum generalization of the classical additive white Gaussian noise channel. In (242), x and p are zero-mean, independent Gaussian random variables each having variance $\xi \geqslant 0$. Let $\mathcal{T}_\xi$ denote this channel. The channel in (242) is entanglement breaking when $\xi \geqslant 1$ [68] and is thus entanglement binding, and as a consequence of Proposition 13 and Theorems 17 and 19, it has zero exact entanglement cost for these values.

Kraus representations for the channels in (240)–(242) are available in Ref. [69], which can be helpful for further understanding their action on input quantum states.

Due to the entanglement-breaking regions discussed above, we are left with a limited range of single-mode bosonic Gaussian channels to consider, which is delineated by the white strip in Fig. 1 of Ref. [70].

C. Exact entanglement cost of thermal, amplifier, and additive-noise bosonic Gaussian channels

We can now state our main result for this section, which applies to all thermal, amplifier, and additive-noise channels that are neither entanglement breaking nor quantum limited:

Theorem 23. For a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in (0,1)$ and thermal photon number $N_B \in (0, \eta/[1-\eta])$, an amplifier channel $\mathcal{A}_{G,N_B}$ with gain $G > 1$ and thermal photon number $N_B \in (0, 1/[G-1])$, and an additive-noise channel $\mathcal{T}_\xi$ with noise variance $\xi \in (0,1]$, the following formulas characterize the exact entanglement costs of these channels:

$$E_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) = E^{(p)}_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) = \log_2\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right), \tag{244}$$

$$E_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) = E^{(p)}_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) = \log_2\left(\frac{G+1}{(G-1)(2N_B+1)}\right), \tag{245}$$

$$E_{\mathrm{PPT}}(\mathcal{T}_\xi) = E^{(p)}_{\mathrm{PPT}}(\mathcal{T}_\xi) = \log_2(1/\xi). \tag{246}$$

Proof. To arrive at the following inequalities:

$$E_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) \leqslant \log_2\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right), \tag{247}$$

$$E_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) \leqslant \log_2\left(\frac{G+1}{(G-1)(2N_B+1)}\right), \tag{248}$$

$$E_{\mathrm{PPT}}(\mathcal{T}_\xi) \leqslant \log_2(1/\xi), \tag{249}$$

we apply Proposition 20, along with some recent developments, to the single-mode thermal, amplifier, and additive-noise channels that are neither entanglement breaking nor quantum limited. Some recent papers [16,71,72] have shown how to simulate each of these channels by using a bosonic Gaussian resource state along with variations of the continuous-variable quantum teleportation protocol [73]. Of these works, the one most relevant for us is the original one [16], because these authors proved that the logarithmic negativity of the underlying resource state is equal to the logarithmic negativity that results from transmitting through the channel one share of a two-mode squeezed vacuum state with arbitrarily large squeezing strength. That is, let $\mathcal{N}_{A\to B}$ denote a single-mode thermal, amplifier, or additive-noise channel. Then one of the main results of Ref. [16] is that, associated to this channel, there is a bosonic Gaussian resource state $\omega_{A'B'}$ and a Gaussian LOCC channel $\mathcal{G}_{AA'B'\to B}$ such that

$$E_N(\omega_{A'B'}) = \sup_{N_S \geqslant 0} E_N(\sigma^{N_S}_{RB}) \tag{250}$$

$$= \lim_{N_S\to\infty} E_N(\sigma^{N_S}_{RB}), \tag{251}$$

where

$$\sigma^{N_S}_{RB} := \mathcal{N}_{A\to B}(\phi^{N_S}_{RA}), \tag{252}$$

$$\phi^{N_S}_{RA} := |\phi^{N_S}\rangle\langle\phi^{N_S}|_{RA}, \tag{253}$$

$$|\phi^{N_S}\rangle_{RA} := \frac{1}{\sqrt{N_S+1}}\sum_{n=0}^{\infty}\sqrt{\left(\frac{N_S}{N_S+1}\right)^n}|n\rangle_R|n\rangle_A, \tag{254}$$

and for every input state ρ_A,

$$\mathcal{N}_{A\to B}(\rho_A) = \mathcal{G}_{AA'B'\to B}(\rho_A \otimes \omega_{A'B'}). \tag{255}$$

In the above, $\phi^{N_S}_{RA}$ is the two-mode squeezed vacuum state [36]. Note that the equality in (251) holds because one can always produce $\phi^{N_S}_{RA}$ from $\phi^{N'_S}_{RA}$ such that $N'_S \geqslant N_S$ by using Gaussian LOCC and the local displacements involved in the Gaussian LOCC commute with the channel $\mathcal{N}_{A\to B}$ [74] (whether it be thermal, amplifier, or additive noise). Furthermore, the logarithmic negativity does not increase under the action of an LOCC channel.

Thus, applying the above observations and Proposition 20, it follows that there exist bosonic Gaussian resource states $\omega^{\eta,N_B}_{A'B'}$, $\omega^{G,N_B}_{A'B'}$, and $\omega^{\xi}_{A'B'}$ associated to the respective thermal, amplifier, and additive-noise channels in (240)–(242) such that the following inequalities hold:

$$E_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) \leqslant E_\kappa(\omega^{\eta,N_B}_{A'B'}) = E_N(\omega^{\eta,N_B}_{A'B'}) = \log_2\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right), \tag{256}$$

$$
\begin{aligned}
E_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) &\leqslant E_\kappa(\omega^{G,N_B}_{A'B'}) = E_N(\omega^{G,N_B}_{A'B'}) \\
&= \log_2\left(\frac{G+1}{(G-1)(2N_B+1)}\right), \qquad (257)\\
E_{\mathrm{PPT}}(\mathcal{T}_\xi) &\leqslant E_\kappa(\omega^{\xi}_{A'B'}) = E_N(\omega^{\xi}_{A'B'}) \\
&= \log_2(1/\xi), \qquad (258)
\end{aligned}
$$

where the first equalities in each line follow because $E_\kappa = E_N$ for bosonic Gaussian states (see (12) and Ref. [8]), and the explicit formulas on the right-hand side are found in Refs. [16,17].

On the other hand, Theorems 17 and 19 imply that

$$
\begin{aligned}
E_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) &= E^{(p)}_{\mathrm{PPT}}(\mathcal{L}_{\eta,N_B}) \qquad (259)\\
&\leqslant \lim_{N_S\to\infty} E_N(\sigma^{\eta,N_B}(N_S)_{RB}) \qquad (260)\\
&= \log_2\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right), \qquad (261)\\
E_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) &= E^{(p)}_{\mathrm{PPT}}(\mathcal{A}_{G,N_B}) \qquad (262)\\
&\leqslant \lim_{N_S\to\infty} E_N(\sigma^{G,N_B}(N_S)_{RB}) \qquad (263)\\
&= \log_2\left(\frac{G+1}{(G-1)(2N_B+1)}\right), \qquad (264)\\
E_{\mathrm{PPT}}(\mathcal{T}_\xi) &= E^{(p)}_{\mathrm{PPT}}(\mathcal{T}_\xi) \qquad (265)\\
&\leqslant \lim_{N_S\to\infty} E_N(\sigma^{\xi}(N_S)_{RB}) \qquad (266)\\
&= \log_2(1/\xi). \qquad (267)
\end{aligned}
$$

Combining the inequalities above, we conclude the statement of the theorem. ■

The significance of Theorem 23 above is that it establishes a clear operational meaning of the Holevo-Werner quantity [17] (partial transposition bound) for the basic bosonic channels that are not quantum limited. This quantity has been used for a variety of purposes in prior work, as an upper bound on unassisted quantum capacity [17], as an upper bound on LOCC-assisted quantum capacity [75], as a tool in arriving at a no-go theorem for Gaussian quantum error correction [76], and as a tool in the teleportation simulation of bosonic Gaussian channels [16]. Finally, Theorem 23 solves the long-standing open problem of giving the Holevo-Werner quantity a direct operational meaning for the basic bosonic channels, in terms of exact entanglement cost of parallel and sequential channel simulation.

In light of the results stated in Theorem 23, it is quite natural to conjecture that the following formulas hold for the pure-loss and pure-amplifier channels with $\eta \in (0,1)$ and $G > 1$, respectively:

$$
E_{\mathrm{PPT}}(\mathcal{L}_\eta) = E^{(p)}_{\mathrm{PPT}}(\mathcal{L}_\eta) \stackrel{?}{=} \log_2\left(\frac{1+\eta}{1-\eta}\right), \qquad (268)
$$

$$
E_{\mathrm{PPT}}(\mathcal{A}_G) = E^{(p)}_{\mathrm{PPT}}(\mathcal{A}_G) \stackrel{?}{=} \log_2\left(\frac{G+1}{G-1}\right). \qquad (269)
$$

Theorems 17 and 19 imply that the following inequalities hold:

$$
E_{\mathrm{PPT}}(\mathcal{L}_\eta) = E^{(p)}_{\mathrm{PPT}}(\mathcal{L}_\eta) \geqslant \log_2\left(\frac{1+\eta}{1-\eta}\right), \qquad (270)
$$

$$
E_{\mathrm{PPT}}(\mathcal{A}_G) = E^{(p)}_{\mathrm{PPT}}(\mathcal{A}_G) \geqslant \log_2\left(\frac{G+1}{G-1}\right). \qquad (271)
$$

However, what excludes us from making a rigorous statement about the opposite inequalities is the lack of a legitimate quantum state that can be used to simulate these channels exactly, as was the case for the channels considered in Theorem 23. For example, it is not clear that we could simply plug in the EPR state (i.e., the limiting object $\lim_{N_S\to\infty}\phi^{N_S}_{RA}$) and use the teleportation simulation argument as before. There are several issues: the limiting object is not actually a state and any finite squeezing leads to a slight error or inexact simulation. In spite of these obstacles, we think that it is highly plausible that the equalities in (268) and (269) hold. More generally, based on the results of Ref. [76], we suspect that the following equality holds for an arbitrary Gaussian channel $\mathcal{N}$ described by a scaling matrix X and a noise matrix Y [36]:

$$
E_{\mathrm{PPT}}(\mathcal{N}) \stackrel{?}{=} Q_\Theta(\mathcal{N}) \stackrel{?}{=} \frac{1}{2}\log_2 \min\left\{\frac{(1+\det X)^2}{\det Y}, 1\right\}. \qquad (272)
$$

VII. CONCLUDING REMARKS

In the zoo of entanglment measures [1,2,77], the κ entanglement of a bipartite state is the first entanglement measure that is efficiently computable while possessing a direct operational meaning for general bipartite states. This unique feature of E_κ may help us better understand the structure and power of quantum entanglement. As a generalization of this notion, the κ entanglement of a quantum channel is also efficiently computable while possessing a direct operational meaning as the entanglement cost for exact parallel and sequential simulation of a quantum channel.

Moving forward, the most pressing open question is to determine whether the formula in (272) holds for the exact entanglement cost of quantum Gaussian channels. One could potentially require new methods beyond the scope of this paper to establish (272).

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APPENDIX: EQUALITY OF E_κ AND E_κ^{dual} FOR STATES ACTING ON SEPARABLE HILBERT SPACES

In this Appendix, we prove that

$$
E_\kappa(\rho_{AB}) = E_\kappa^{\mathrm{dual}}(\rho_{AB}) \qquad (A1)
$$

for a state ρ_{AB} acting on a separable Hilbert space. To begin, let us recall that the following inequality always holds from weak duality:

$$E_\kappa(\rho_{AB}) \geqslant E_\kappa^{\text{dual}}(\rho_{AB}). \tag{A2}$$

So, our goal is to prove the opposite inequality. We suppose throughout that $E_\kappa^{\text{dual}}(\rho_{AB}) < \infty$. Otherwise, the desired equality in (A1) is trivially true. We also suppose that ρ_{AB} has full support. Otherwise, it is finite-dimensional and the desired equality in (A1) is trivially true.

To this end, consider sequences $\{\Pi_A^k\}_k$ and $\{\Pi_B^k\}_k$ of projectors weakly converging to the identities $\mathbb{1}_A$ and $\mathbb{1}_B$ and such that $\Pi_A^k \leqslant \Pi_A^{k'}$ and $\Pi_B^k \leqslant \Pi_B^{k'}$ for $k' \geqslant k$. Furthermore, we suppose that $[\Pi_B^k]^{T_B} = \Pi_B^k$ for all k. Then define

$$\rho_{AB}^k := (\Pi_A^k \otimes \Pi_B^k)\rho_{AB}(\Pi_A^k \otimes \Pi_B^k). \tag{A3}$$

It follows that [78]

$$\lim_{k\to\infty} \left\|\rho_{AB} - \rho_{AB}^k\right\|_1 = 0. \tag{A4}$$

We now prove that

$$E_\kappa^{\text{dual}}(\rho_{AB}) \geqslant E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right) \tag{A5}$$

for all k. Let A^k and B^k denote the subspaces onto which Π_A^k and Π_B^k project. Let $V_{A^kB^k}^k$ and $W_{A^kB^k}^k$ be arbitrary operators satisfying $V_{AB}^k + W_{AB}^k \leqslant \mathbb{1}_{A^kB^k} = (\Pi_A^k \otimes \Pi_B^k)$, $[V_{A^kB^k}^k]^{T_B}, [W_{A^kB^k}^k]^{T_B} \geqslant 0$. Set

$$\overline{V}_{AB}^k := (\Pi_A^k \otimes \Pi_B^k)V_{A^kB^k}^k(\Pi_A^k \otimes \Pi_B^k), \tag{A6}$$

$$\overline{W}_{AB}^k := (\Pi_A^k \otimes \Pi_B^k)W_{A^kB^k}^k(\Pi_A^k \otimes \Pi_B^k), \tag{A7}$$

and note that

$$\overline{V}_{AB}^k + \overline{W}_{AB}^k \leqslant \mathbb{1}_{AB}, \tag{A8}$$

$$\left[\overline{V}_{AB}^k\right]^{T_B}, \left[\overline{W}_{AB}^k\right]^{T_B} \geqslant 0. \tag{A9}$$

Then

$$\begin{aligned}
&\operatorname{Tr} \rho_{AB}^k\left(V_{A^kB^k}^k - W_{A^kB^k}^k\right) \\
&= \operatorname{Tr}\left(\Pi_A^k \otimes \Pi_B^k\right)\rho_{AB}\left(\Pi_A^k \otimes \Pi_B^k\right)\left(V_{A^kB^k}^k - W_{A^kB^k}^k\right) &\text{(A10)}\\
&= \operatorname{Tr} \rho_{AB}\left(\Pi_A^k \otimes \Pi_B^k\right)\left(V_{A^kB^k}^k - W_{A^kB^k}^k\right)\left(\Pi_A^k \otimes \Pi_B^k\right) &\text{(A11)}\\
&= \operatorname{Tr} \rho_{AB}\left(\overline{V}_{AB}^k - \overline{W}_{AB}^k\right) &\text{(A12)}\\
&\leqslant E_\kappa^{\text{dual}}(\rho_{AB}). &\text{(A13)}
\end{aligned}$$

Since the inequality holds for arbitrary $V_{A^kB^k}^k$ and $W_{A^kB^k}^k$ satisfying the conditions above, we conclude the inequality in (A5).

Thus, we conclude that

$$E_\kappa^{\text{dual}}(\rho_{AB}) \geqslant \limsup_{k\to\infty} E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right). \tag{A14}$$

Now let us suppose that $E_\kappa^{\text{dual}}(\rho_{AB}) < \infty$. Then for all V_{AB} and W_{AB} satisfying $V_{AB} + W_{AB} \leqslant \mathbb{1}_{AB}$, $[V_{AB}]^{T_B}, [W_{AB}]^{T_B} \geqslant 0$, as well as $\operatorname{Tr} \rho_{AB}(V_{AB} - W_{AB}) \geqslant 0$, we have that

$$\operatorname{Tr} \rho_{AB}(V_{AB} - W_{AB}) < \infty. \tag{A15}$$

Since ρ_{AB} has full support, this means that

$$\|V_{AB} - W_{AB}\|_\infty < \infty. \tag{A16}$$

Considering that from Hölder's inequality

$$\begin{aligned}
&\left|\operatorname{Tr}\left(\rho_{AB} - \rho_{AB}^k\right)(V_{AB} - W_{AB})\right| \\
&\quad \leqslant \left\|\rho_{AB} - \rho_{AB}^k\right\|_1 \|V_{AB} - W_{AB}\|_\infty, &\text{(A17)}
\end{aligned}$$

and setting

$$V_{AB}^k := \left(\Pi_A^k \otimes \Pi_B^k\right)V_{AB}\left(\Pi_A^k \otimes \Pi_B^k\right), \tag{A18}$$

$$W_{AB}^k := \left(\Pi_A^k \otimes \Pi_B^k\right)W_{AB}\left(\Pi_A^k \otimes \Pi_B^k\right), \tag{A19}$$

we conclude that

$$\begin{aligned}
&\operatorname{Tr} \rho_{AB}(V_{AB} - W_{AB}) \\
&\leqslant \liminf_{k\to\infty} \operatorname{Tr} \rho_{AB}^k(V_{AB} - W_{AB}) &\text{(A20)}\\
&= \liminf_{k\to\infty} \operatorname{Tr} \rho_{AB}^k\left(V_{AB}^k - W_{AB}^k\right) &\text{(A21)}\\
&\leqslant \liminf_{k\to\infty} \sup_{V^k, W^k} \operatorname{Tr} \rho_{AB}^k\left(V_{AB}^k - W_{AB}^k\right) &\text{(A22)}\\
&= \liminf_{k\to\infty} E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right). &\text{(A23)}
\end{aligned}$$

Since the inequality holds for arbitrary V_{AB} and W_{AB} satisfying the above conditions, we conclude that

$$E_\kappa^{\text{dual}}(\rho_{AB}) \leqslant \liminf_{k\to\infty} E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right). \tag{A24}$$

Putting together (A14) and (A24), we conclude that

$$E_\kappa^{\text{dual}}(\rho_{AB}) = \lim_{k\to\infty} E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right). \tag{A25}$$

From strong duality for the finite-dimensional case, we have for all k that

$$E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right) = E_\kappa\left(\rho_{AB}^k\right), \tag{A26}$$

and thus that

$$\lim_{k\to\infty} E_\kappa^{\text{dual}}\left(\rho_{AB}^k\right) = \lim_{k\to\infty} E_\kappa\left(\rho_{AB}^k\right). \tag{A27}$$

It thus remains to prove that

$$\lim_{k\to\infty} E_\kappa\left(\rho_{AB}^k\right) = E_\kappa(\rho_{AB}). \tag{A28}$$

We first prove that

$$E_\kappa(\rho_{AB}) \geqslant \limsup_{k\to\infty} E_\kappa\left(\rho_{AB}^k\right). \tag{A29}$$

Let S_{AB} be an arbitrary operator satisfying

$$S_{AB} \geqslant 0, \quad -S_{AB}^{T_B} \leqslant \rho_{AB}^{T_B} \leqslant S_{AB}^{T_B}. \tag{A30}$$

Then, defining $S_{AB}^k = (\Pi_A^k \otimes \Pi_B^k)S_{AB}(\Pi_A^k \otimes \Pi_B^k)$, we have that

$$S_{AB}^k \geqslant 0, \quad -\left[S_{AB}^k\right]^{T_B} \leqslant \left[\rho_{AB}^k\right]^{T_B} \leqslant \left[S_{AB}^k\right]^{T_B}. \tag{A31}$$

Then

$$\log_2 \operatorname{Tr} S_{AB} \geqslant \log_2 \operatorname{Tr} S_{AB}^k \geqslant E_\kappa\left(\rho_{AB}^k\right). \tag{A32}$$

Since the inequality holds for all S_{AB} satisfying (A30), we conclude that

$$E_\kappa(\rho_{AB}) \geqslant E_\kappa\left(\rho_{AB}^k\right) \tag{A33}$$

for all k, and thus (A29) holds.

The rest of the proof follows Ref. [20] closely. Since the condition $\Pi_A^k \leqslant \Pi_A^{k'}$ and $\Pi_B^k \leqslant \Pi_B^{k'}$ for $k' \geqslant k$ holds, in fact the same sequence of steps as above allows for concluding that

$$E_\kappa\left(\rho_{AB}^{k'}\right) \geqslant E_\kappa\left(\rho_{AB}^{k}\right), \tag{A34}$$

meaning that the sequence is monotone non-decreasing with k. Thus, we can define

$$\mu := \lim_{k\to\infty} E_\kappa\left(\rho_{AB}^{k}\right) \in \mathbb{R}^+, \tag{A35}$$

and note from the above that

$$\mu \leqslant E_\kappa(\rho_{AB}). \tag{A36}$$

For each k, let S_{AB}^k denote an optimal operator such that $E_\kappa(\rho_{AB}^k) = \log_2 \operatorname{Tr} S_{AB}^k$. From the fact that $S_{AB}^k \geqslant 0$ and $\operatorname{Tr} S_{AB}^k \leqslant 2^\mu$, we conclude that $\{S_{AB}^k\}_k$ is a bounded sequence in the trace class operators. Since the trace class operators form the dual space of the compact operators $\mathcal{K}(\mathcal{H}_{AB})$ [79], we can apply the Banach–Alaoglu theorem [79] to find a subsequence $\{S_{AB}^k\}_{k\in\Gamma}$ with a weak* limit $\widetilde{S}_{AB}$ in the trace class operators such that $\widetilde{S}_{AB} \geqslant 0$ and $\operatorname{Tr}[\widetilde{S}_{AB}] \leqslant 2^\mu$. Furthermore, the sequences $[\rho_{AB}^k]^{T_B} + [S_{AB}^k]^{T_B}$ and $[S_{AB}^k]^{T_B} - [\rho_{AB}^k]^{T_B}$ converge in the weak operator topology to $\rho_{AB}^{T_B} + \widetilde{S}_{AB}^{T_B}$ and $\widetilde{S}_{AB}^{T_B} - \rho_{AB}^{T_B}$, respectively, and we can then conclude that $\rho_{AB}^{T_B} + \widetilde{S}_{AB}^{T_B}, \widetilde{S}_{AB}^{T_B} - \rho_{AB}^{T_B} \geqslant 0$. But this means that

$$E_\kappa(\rho_{AB}) \leqslant \log_2 \operatorname{Tr} \widetilde{S}_{AB} \leqslant \mu, \tag{A37}$$

which implies that

$$E_\kappa(\rho_{AB}) \leqslant \liminf_{k\to\infty} E_\kappa\left(\rho_{AB}^{k}\right). \tag{A38}$$

Putting together (A29) and (A38), we conclude that

$$E_\kappa(\rho_{AB}) = \lim_{k\to\infty} E_\kappa\left(\rho_{AB}^{k}\right). \tag{A39}$$

Finally, putting together (A25), (A27), and (A39), we conclude (A1).

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