

Robust certification of unsharp instruments through sequential quantum advantages in a prepare-measure communication game

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(Received 21 July 2022; accepted 22 December 2022; published 6 January 2023)

Communication games are one of the widely used tools that are designed to demonstrate quantum supremacy over classical resources in which two or more parties collaborate to perform an information processing task to achieve the highest success probability of winning the game. We propose a specific two-party communication game in the prepare-measure scenario that relies on an encoding-decoding task of specific information. We first demonstrate that quantum theory outperforms the classical preparation noncontextual theory and the optimal quantum success probability of such a communication game enables the semi-device-independent certification of qubit states and measurements. Further, we consider the sequential sharing of quantum preparation contextuality and show that, at most, two sequential observers can share the quantum advantage. The suboptimal quantum advantages for two sequential observers form an optimal pair which certifies a unique value of the unsharpness parameter of the first observer. Since the practical implementation inevitably introduces noise, we devised a scheme to demonstrate the robust certification of the states and unsharp measurement instruments of both the sequential observers.

DOI: [10.1103/PhysRevA.107.012411](https://doi.org/10.1103/PhysRevA.107.012411)

I. INTRODUCTION

Arguably, Bell's theorem [1] has left the most prominent mark on the history of quantum foundation research. It states that all the quantum statistics cannot be predicted by a classical model that satisfies locality. This fundamental feature is known as quantum nonlocality, demonstrated through the quantum violation of Bell's inequality [2]. The nonlocal correlation that Bell's theorem certifies is device independent, i.e., the internal functioning of the quantum instruments remains unknown. This is the key reason why nonlocal correlations are used as a resource in various information-theoretic tasks, viz., secure quantum key distribution [3–6], randomness certification [7–10], witnessing Hilbert space dimension [11–18], and communication complexity [19,20].

Another aspect of nonclassicality, the inconsistency between the noncontextual hidden-variable model and the quantum theory, was introduced by Kochen and Specker [21,22]. Unlike Bell's theorem, which requires spatially separated entangled systems to demonstrate nonlocality, the Kochen-Specker (KS) theorem can reveal the contextuality even using a single system. However, the KS theorem has a limited scope of applicability as it only captures the measurement noncontextuality, is not applicable for unsharp measurement, and is restricted to quantum theory. Later, the notion of noncontextuality was generalized in [23] for

arbitrary operational theories and extended the formulation to preparation and transformation. More recently, the preparation contextuality has been studied extensively [24–33], and it is this form of nonclassicality that plays a vital role in the present work. In particular, we show how quantum preparation contextuality powers a sequential communication game and enables the robust certification of states, observables, and unsharpness parameters.

The strongest form of certification is device-independent self-testing [34,35], whereby one can uniquely certify the states and measurements solely from experimental statistics regardless of any knowledge about the functioning of the preparation and measurement instruments. Optimal quantum violation of a Bell inequality enables such self-testing. For instance, the optimal quantum violation of Clauser-Horne-Shimony-Holt [36] inequality self-tests the maximally entangled state and locally anticommuting observables. However, in practice, the loophole-free Bell tests remain a challenging and resource-consuming task. Instinctively, robust certification of states and measurement instruments through the Bell test would be even more challenging. Due to the practical difficulties in experimental testing, the semi-device-independent (SDI) protocols in prepare-measure scenarios [37–45] became appealing to the community. In the SDI certification scheme, the preparation and measurement instruments remain black boxes, but the dimension of the system is bounded from above.

One of the well-studied approaches for showcasing quantum supremacy in information processing tasks is in terms of communication games [19,20,24,25,46]. A typical

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communication game involves two or more independent parties collaborating to perform an information processing task with the maximum probability of success. Apart from structural differences, such games are supplemented with different constraints that have to be fulfilled by the parties. In this work we consider a particular communication game constrained by parity-oblivious communication, which implies that the communication between the parties must not reveal the parity information of the input. The game is technically inspired by but not equivalent to the parity-oblivious communication game [24]. Here we demonstrate that quantum theory predicts the success probability that outperforms the classical theory. Throughout this paper, by classicality we refer to the preparation noncontextuality, as will be introduced shortly.

Recently, the sequential sharing of quantum correlations has attracted considerable attention. Based on the sequential quantum violation of Clauser-Horne-Shimony-Holt inequality, Silva *et al.* [47] first demonstrated that the nonlocality can be shared by a maximum of two sequential observers on one side. Since then, numerous studies [29,38,45,47–52] have been reported that explored how many independent observers can share various forms of quantum correlations, such as steering [48], entanglement [49], nonlocality [50], and preparation contextuality [29]. As long as a communication game is capable of revealing some aspect of quantum supremacy, there is a provision to extend the game to the sequential scenario. We adopt a sequential scenario of the communication game to present the main results, which are explicitly discussed below.

Based on a specific parity-oblivious communication game in the prepare-measure scenario, we first demonstrate the SDI certifications of the prepared qubit states, measurement, and unsharp parameter in an ideal scenario when there is no noise. The sequential quantum advantages over the classical preparation noncontextual bound of the communication game enable us to demonstrate such a certification. We show that, at most, two independent sequential observers can share the quantum advantage in such a game. Specifically, we argue how suboptimal quantum advantages by two sequential observers form an optimal pair which eventually certifies a unique value of the unsharpness parameter of the first observer.

However, the practical implementation of the protocol introduces inevitable losses due to imperfections in experimental procedures. Hence the aforementioned optimal pair of sequential success probabilities may not be achieved. Thus, the exact states and measurements cannot be certified in an actual experiment. On the other hand, in a sequential scenario where more than one observer gets the quantum advantage, the success probabilities must be suboptimal to ensure that the measurements are unsharp. We provide a robust certification of the preparations and measurements and certify a range of the unsharpness parameter of the measurement instrument when the sequential quantum success probabilities do not form an optimal pair.

The paper is organized as follows. In Sec. II we briefly summarize the notion of the ontological model and the assumption of preparation noncontextuality. In Sec. III we introduce the parity-oblivious communication game in the prepare-measure scenario. In Sec. IV, using sequential un-

sharp measurements, we demonstrate the sharing of quantum advantage by multiple independent observers. In Sec. IV A we provide the certification of the unsharpness parameter, which is necessary to violate the preparation noncontextual bound of success probability, and in Sec. IV B we provide a generalized certification statement for the preparations and unsharp measurements in an ideal scenario. In Sec. V we examine the possibility of sharing the sequential quantum advantage with a third observer. In Sec. VI we present the robust certification scheme which robustly certifies Alice's preparations and Bob's and Charlie's measurements of noise for the choice of unsharpness parameter. We summarize our results and in Sec. VII.

II. ONTOLOGICAL MODEL AND PREPARATION NONCONTEXTUALITY

To begin with, we briefly recapitulate the ontological model [53–55] of an operational theory and the assumption of the preparation noncontextuality [23], the notion of classicality that we are considering here. An operational theory essentially involves a set of preparation procedures $\{P\}$ and a set of measurement procedures $\{M\}$, which is designed to predict the operational statistics by means of the probability $p(k|P, M)$ of obtaining a particular outcome k given the experimental arrangements. In operational quantum theory, the preparation is represented by the density matrix ρ and the measurement is in general represented by positive-operator-valued measures (POVMs) $\{E_k\}$ satisfying $\sum_k E_k = \mathbb{I}$. The quantum probability $p(k|P, M)$ is provided by the Born rule as $p(k|P, M) = \text{Tr}(\rho E_k)$.

In general, the ontological model [53–55] of an operational theory provides an objective description of the physical events compatible with the experimental statistics. A preparation procedure P in an ontological model corresponds to a probability distribution $\mu(\lambda|P)$ that satisfies $\int_{\Lambda} \mu(\lambda|P) d\lambda = 1 \forall \lambda \in \Lambda$. The variable $\lambda \in \Lambda$ represents the ontic state of the system. Given a POVM element E_k , the ontic state variable λ assigns a response function $\xi(k|\lambda, E_k)$ with $\sum_k \xi(k|\lambda, E_k) = 1 \forall \lambda$. Any ontological model that is consistent with quantum theory must reproduce the statistical prediction of the theory given by the Born rule as

$$\int_{\Lambda} \mu(\lambda|\rho, P) \xi(k|\lambda, M) d\lambda = \text{Tr}(\rho E_k). \quad (1)$$

For our purposes here, to introduce the notion of preparation noncontextuality in an ontological model, we consider the operationally equivalent experimental procedures. Two different preparation procedures P_0 and P_1 are called operationally equivalent if no measurement can distinguish them. Mathematically, it means that for any two operationally equivalent procedures P_0 and P_1 we must have $p(k|P_0, M) = p(k|P_1, M) \forall M, k$. In quantum theory, a density matrix ρ prepared through two distinct procedures P_0 and P_1 cannot be distinguished by any measurement and thus constitutes such operationally equivalent preparations. As argued in [23], if two operationally equivalent preparations P_0 and P_1 for an operational theory are equivalently represented in the corresponding ontological model by the same ontic state

probability distributions, i.e.,

$$p(k|P_0, M) = p(k|P_1, M) \Rightarrow \mu_{P_0}(\lambda|\rho) = \mu_{P_1}(\lambda|\rho) \forall M, \quad (2)$$

then the model is called preparation noncontextual. This particular form of noncontextuality is taken as the notion of classicality in this paper. To avoid clumsiness, we drop M from the response function for the rest of this paper, as measurement contextuality is not relevant here.

III. PARITY OBLIVIOUS COMMUNICATION GAME

We consider a specific communication game in the prepare-measure scenario, which has a structural resemblance to the games in [28,56]. The game involves two parties, Alice and Bob, who own their respective instruments. Alice receives input $x \in \{1, 2, 3\}$, which is chosen at random from a uniform probability distribution, i.e., $p(x) = \frac{1}{3} \forall x$, based on which her instrument produces an output $a \in \{0, 1\}$. Alice encodes the information about her input x and output a and sends it to Bob. Explicitly, the six different input combinations of Alice are $x^i \in (xa) \equiv \{10, 11, 20, 21, 30, 31\}$, with $i = (1, 2, 3, \dots, 6)$. On the other hand, Bob receives input $y \in \{1, 2, 3\}$, which is also chosen at random, i.e., $p(y) = \frac{1}{3} \forall y$, and produces an output b . The winning condition of the game is that Bob's instrument has to produce the output b such that the condition

$$b = \delta_{x,y} \oplus_2 a \quad (3)$$

is satisfied. The average success probability of the game is then given by

$$\mathcal{A} = \frac{1}{18} \sum_{x,y=1}^3 \sum_{a \in \{0,1\}} p(b = \delta_{x,y} \oplus_2 a | x, y). \quad (4)$$

Such a game can obviously be won with the probability equal to unity, if Alice is allowed to share an unbounded amount of information with Bob. However, in the game, the parity-oblivious constraint limits Alice's communication such that Alice is restricted from revealing the parity information to Bob. To explain this explicitly, let us consider two subsets of Alice's input, the even-parity set $\mathbb{P}_e = \{x \oplus_2 a = 0\}$ and the odd-parity set $\mathbb{P}_o = \{x \oplus_2 a = 1\}$. Then for all y Bob cannot know which parity set the input that belongs to him came from. In an operational theory, the above parity-oblivious restriction can be formally written as

$$\sum_{x^i \in \mathbb{P}_e} p(b|x^i, y) = \sum_{x^i \in \mathbb{P}_o} p(b|x^i, y) \forall y. \quad (5)$$

We derive the maximum classical and quantum success probabilities of the game with the constraints in Eq. (5). Generally, classical strategies are described through the ontological models [23,53–55] wherein the preparation procedures are characterized by a probability distribution $\mu(\lambda)$ corresponding to an ontic state variable λ as discussed earlier. Following [24], we argue that the parity obliviousness at the operational level must also reflect as an equivalent restriction at the level of ontic states if the underlying classical ontological model of the operational theory is preparation noncontextual. Hence, the classical strategy that entails parity obliviousness at the level of ontic states is essentially a preparation noncontextual strategy.

Let us now consider that in operational quantum theory, Alice encodes her input $x^i \in \{x, a\}$ into six qubit states $\{\rho_{xa}\}$. Then the parity obliviousness implies that the following restriction has to be satisfied by Alice's preparation:

$$\sum_{x^i: x \oplus_2 a = 0} \rho_{x^i} = \sum_{x^i: x \oplus_2 a = 1} \rho_{x^i}. \quad (6)$$

Explicitly, the parity-oblivious condition reads $\rho_{11} + \rho_{20} + \rho_{31} = \rho_{10} + \rho_{21} + \rho_{30}$. Hence, if the underlying ontological model of quantum theory is preparation noncontextual, then we must have

$$\begin{aligned} \mu(\lambda|P_{11}) + \mu(\lambda|P_{20}) + \mu(\lambda|P_{31}) \\ = \mu(\lambda|P_{10}) + \mu(\lambda|P_{21}) + \mu(\lambda|P_{30}). \end{aligned} \quad (7)$$

This parity-obliviousness condition at the ontological level dictates that the classical strategy should be such that any message containing information about the parity of the preparation is forbidden from communication. Imposing the above-mentioned constraint, the optimal success probability of the game in a preparation noncontextual model is derived as $(\mathcal{A})_{pnc} = \frac{13}{18}$. This is explicitly proved in the following.

In an ontological model the success probability in Eq. (4) can be written as

$$\mathcal{A} = \frac{1}{18} \sum_{x,y=1}^3 \sum_{a \in \{0,1\}} \sum_{\lambda \in \Lambda} \mu(\lambda|P_{xa}) \xi(b = \delta_{x,y} \oplus_2 a | \lambda, M_y), \quad (8)$$

where $\mu(\lambda|P_{xa})$ is the probability distribution for the preparation of ontic state λ given P_{xa} is prepared in the laboratory, and $\xi(b = \delta_{x,y} \oplus_2 a | \lambda, M_y)$ is the response function corresponding to the correct output of Bob's measurement M_y . By expanding Eq. (8) and further simplifying it we get

$$\begin{aligned} \mathcal{A} = \frac{1}{18} \sum_{\lambda} \{ & [\mu(\lambda|P_{10}) + \mu(\lambda|P_{21}) + \mu(\lambda|P_{31})] \xi(b=1|M_1, \lambda) \\ & + [\mu(\lambda|P_{10}) + \mu(\lambda|P_{21}) + \mu(\lambda|P_{30})] \xi(b=0|M_2, \lambda) \\ & + [\mu(\lambda|P_{10}) + \mu(\lambda|P_{20}) + \mu(\lambda|P_{31})] \xi(b=0|M_3, \lambda) \\ & + [\mu(\lambda|P_{11}) + \mu(\lambda|P_{20}) + \mu(\lambda|P_{30})] \xi(b=0|M_1, \lambda) \\ & + [\mu(\lambda|P_{11}) + \mu(\lambda|P_{20}) + \mu(\lambda|P_{31})] \xi(b=1|M_2, \lambda) \\ & + [\mu(\lambda|P_{11}) + \mu(\lambda|P_{21}) + \mu(\lambda|P_{30})] \xi(b=1|M_3, \lambda) \}. \end{aligned} \quad (9)$$

It has already been mentioned that the game is constrained by the parity-oblivious restriction given by Eq. (7). Applying this constraint relation in Eq. (9) and simplifying we get

$$\begin{aligned} \mathcal{A} = \frac{1}{18} \left(9 + 2 \sum_{\lambda} \{ & [\mu(\lambda|P_{11}) - \mu(\lambda|P_{10})] \xi(b=0|M_1, \lambda) \\ & + [\mu(\lambda|P_{20}) - \mu(\lambda|P_{21})] \xi(b=0|M_2, \lambda) \\ & + [\mu(\lambda|P_{31}) - \mu(\lambda|P_{30})] \xi(b=0|M_3, \lambda) \} \right). \end{aligned} \quad (10)$$

To optimize the above expression we note that the optimization condition requires that the terms having negative signs must be zero, i.e., $\mu(\lambda|P_{10}) = \mu(\lambda|P_{21}) = \mu(\lambda|P_{30}) = 0$.

However, this immediately makes the positive terms zero, which follows directly from parity-oblivious conditions (7) and imposes a trivial bound to the success probability. Thus we can at most set two of the negative terms to zero such that the parity-oblivious condition now takes the form

$$\mu(\lambda|P_{30}) = \mu(\lambda|P_{11}) + \mu(\lambda|P_{20}) + \mu(\lambda|P_{31}). \quad (11)$$

Combining Eq. (11) with Eq. (10), we can write

$$\begin{aligned} \mathcal{A} = \frac{1}{18} & \left(9 + 2 \sum_{\lambda} \{ \mu(\lambda|P_{11}) \xi(b=0|M_1, \lambda) \right. \\ & + \mu(\lambda|P_{20}) \xi(b=0|M_2, \lambda) \\ & \left. - (\mu(\lambda|P_{11}) + \mu(\lambda|P_{20})) \xi(b=0|M_3, \lambda) \right). \quad (12) \end{aligned}$$

It is now straightforward to see from Eq. (12) that a strategy that gives the classical optimal value to the success probability should include the ontic state variable λ that gives $\sum_{\lambda} \mu(\lambda|P_{11}) \xi(b=0|M_1, \lambda) = \sum_{\lambda} \mu(\lambda|P_{20}) \xi(b=0|M_2, \lambda) = 1$ and $\sum_{\lambda} (\mu(\lambda|P_{11}) + \mu(\lambda|P_{21})) \xi(b=0|M_3, \lambda) = 0$. Thus the preparation noncontextual classical bound to the success probability of the game becomes $(\mathcal{A})_{pnc} = \frac{13}{18}$.

A simple strategy can saturate the above bound when Alice makes a trit of communication t to Bob. For example, suppose Alice sends a trit of information $t \in \{1, 2, 3\}$ in a particular run of experiment that corresponds to preparations from the set $\{10, 31\}$, $\{11, 21\}$, or $\{20, 30\}$, respectively. This correspondence is already known to Bob before the game starts. Now the predecided strategy is such that Bob will output $b = 0$, $b = 1$, and $b = 0$ if his inputs are $y = 1$, $y = 2$, and $y = 3$, respectively, when a trit of information $t = 1$ is sent. Similarly, for a trit of information $t = 2$, Bob outputs $b = 0$, $b = 1$, and $b = 1$, respectively, and for a trit of information 3, outputs $b = 1$, $b = 0$, and $b = 0$ in accordance with the inputs $y = 1$, $y = 2$, and $y = 3$, respectively. Thus, in the first and second cases, they will win four times among the six runs in each of the cases, whereas in the third case they win five out of six runs. Thus the success probability becomes $\mathcal{A}_{pnc} = \frac{13}{18}$, as already derived analytically. It is important to note that the discussed strategy ensures that no information about the parity of the preparations is shared due to Alice's communication of the trit of information to Bob.

We use the terms ‘‘classical’’ and ‘‘preparation noncontextual’’ interchangeably throughout the paper. Thus, the classical bound to the success probability will always mean the preparation noncontextual bound derived here.

IV. SEQUENTIAL QUANTUM ADVANTAGE IN THE COMMUNICATION GAME

The sequential sharing of quantum preparation contextuality plays a crucial role in this work in certifying the unsharpness of the measurement instruments. By unsharp measurements we specifically mean the noisy variants of the sharp measurements so that the number of POVMs is the same as the number of projectors. Although the optimal quantum violation of local or noncontextual bounds facilitates the certification of states and measurements, it does not account for the

certification of unsharp POVMs or postmeasurement states. The certifications of postmeasurement states are inevitable to certify unsharp measurements, which can only be done if a joint sequential quantum violation of the classical bound can be achieved. This in turn certifies the states, observables, and unsharpness parameter of the instrument.

As already mentioned, Alice encodes the input $x^i \in (xa)$ into qubit states $\{\rho_{xa}\}$ and sends them to Bob through a unitary channel. Upon receiving the state, Bob performs a measurement $\{E_{b|y}\}$ depending upon his input y and obtains an output b . Now, with the projective measurements, Bob can extract the maximum information from a state, discarding all the information about the initial state. In such circumstances, a second observer (say, Charlie) will not be able to extract any information about the initial state. In essence, a posterior observer can obtain no quantum advantage if the prior observer employs a sharp measurement. Therefore, to extend the quantum advantage to Charlie in Fig. 1, Bob must perform the unsharp measurement [57–64] characterized by POVMs. In that case, the system will be partially disturbed by keeping residual coherence in the postmeasurement state. However, a particular observer can only get the quantum advantage if its unsharpness parameter surpasses a threshold value. This provides the provision of minimal information from the system that is just enough to get the quantum advantage. Hence, whether or not the sequential observer gets the quantum advantage depends on the degree of unsharpness of the preceding observer. Thus, there exists a trade-off between the quantum advantages of sequential observers.

Explicitly, the process runs as follows, as depicted in Fig. 1. Alice prepares qubit state $\{\rho_{xa}\}$ and sends it to Bob. Subsequently, Bob performs an unsharp measurement with his input $y \in \{1, 2, 3\}$ and produces an output b . Thereby, the transformed states are relayed to Charlie and he performs an unsharp measurement with his input $z \in \{1, 2, 3\}$ and produces an output c . This process goes on for a number m of observers ($m \in \{B, C, D, \dots\}$) as long as the m th observer gets the quantum advantage. From the winning condition of the game in Eq. (3) and the success probability in Eq. (4) we can write the quantum success probability for Bob as

$$\mathcal{A}_B = \frac{1}{18} \sum_{x,y=1}^3 \sum_{a \in \{0,1\}} \text{Tr}(\rho_{xa} E_{b|y}), \quad (13)$$

where $E_{b|y}$ is the POVM corresponding to the outcome b of measurement B_y . Now let us take the prepared state to be

$$\rho_{xa} = \frac{\mathbb{I} + \vec{\mathbf{r}}_{xa} \cdot \sigma}{2}, \quad (14)$$

where $\vec{\mathbf{r}}_{xa}$ is the Bloch vector and $E_{b|y} = \frac{\mathbb{I} + (-1)^b \eta_B B_y}{2}$ is the POVM corresponding to the output b of measurement of the operator $B_y = \hat{\mathbf{b}}_y \cdot \sigma$ and η_B is the unsharpness parameter of Bob's measurement instrument, which quantifies the fuzziness in his measurement. The quantum success probability for Bob can then be written as

$$\mathcal{A}_B = \frac{1}{2} + \frac{\eta_B}{36} \sum_{k,y=1}^3 (\delta_{k,y} \vec{\mathbf{n}}_k \cdot \hat{\mathbf{b}}_y), \quad (15)$$

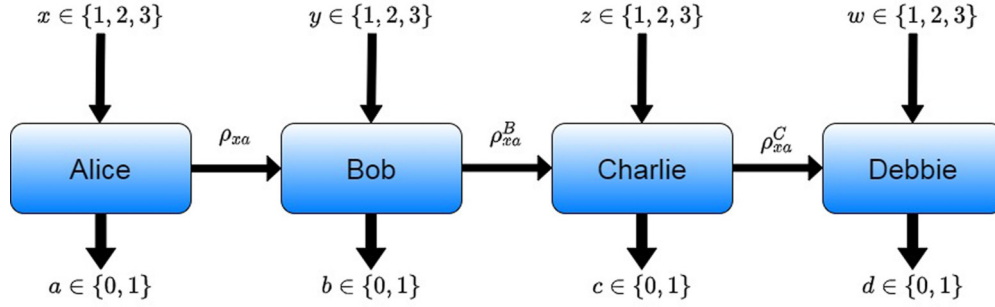


FIG. 1. Sequential setting of the game.

where $\hat{\mathbf{b}}_y$ is the unit vector in the direction of the operator B_y such that $B_y = \hat{\mathbf{b}}_y \cdot \sigma$, and $\bar{\mathbf{n}}_k = \|\bar{\mathbf{n}}_k\| \hat{\mathbf{n}}_k$ ($\hat{\mathbf{n}}_k$ is the unit vector) is the unnormalized Bloch vectors, which are explicitly written as

$$\begin{aligned}\bar{\mathbf{n}}_1 &= -(\bar{\mathbf{r}}_{10} - \bar{\mathbf{r}}_{11}) + (\bar{\mathbf{r}}_{21} - \bar{\mathbf{r}}_{20}) + (\bar{\mathbf{r}}_{30} - \bar{\mathbf{r}}_{31}), \\ \bar{\mathbf{n}}_2 &= (\bar{\mathbf{r}}_{10} - \bar{\mathbf{r}}_{11}) - (\bar{\mathbf{r}}_{21} - \bar{\mathbf{r}}_{20}) + (\bar{\mathbf{r}}_{30} - \bar{\mathbf{r}}_{31}), \\ \bar{\mathbf{n}}_3 &= (\bar{\mathbf{r}}_{10} - \bar{\mathbf{r}}_{11}) + (\bar{\mathbf{r}}_{21} - \bar{\mathbf{r}}_{20}) - (\bar{\mathbf{r}}_{30} - \bar{\mathbf{r}}_{31}).\end{aligned}\quad (16)$$

If Bob's measurement instrument is represented by the set of Kraus operators $\{\mathcal{K}_{b|y}\}$, then the reduced state after his measurement can be written as

$$\rho_{xa}^B = \frac{1}{3} \sum_{y=1}^3 \sum_{b \in \{0,1\}} \mathcal{K}_{b|y} \rho_{xa} \mathcal{K}_{b|y}^\dagger, \quad (17)$$

where the Kraus operators satisfy $\sum_b \mathcal{K}_{b|y} \mathcal{K}_{b|y}^\dagger = \mathbb{I}$, with $\mathcal{K}_{b|y} = \mathcal{U} \sqrt{E_{b|y}}$ for any unitary operator \mathcal{U} . As our results are valid for any unitary transformation, for simplicity, we set $\mathcal{U} = \mathbb{I}$. The Kraus operators are written as $\mathcal{K}_{\pm|y} = \sqrt{\frac{1 \pm \eta_B}{2}} \Pi_{\pm|y} + \sqrt{\frac{1 \mp \eta_B}{2}} \Pi_{\mp|y} = \alpha_B \mathbb{I} \pm \beta_B B_y$, where $\Pi_{\pm|y} = (\mathbb{I} \pm B_y)/2$ are the projectors corresponding to the qubit observable B_y such that

$$\alpha_B = \frac{1}{2} \left(\sqrt{\frac{1 - \eta_B}{2}} + \sqrt{\frac{1 + \eta_B}{2}} \right), \quad (18)$$

$$\beta_B = \frac{1}{2} \left(\sqrt{\frac{1 + \eta_B}{2}} - \sqrt{\frac{1 - \eta_B}{2}} \right), \quad (19)$$

with $\alpha_B^2 + \beta_B^2 = \frac{1}{2}$ and $\eta_B = 4\alpha_B\beta_B$.

Using the above form and simplifying Eq. (17), we find the reduced state after Bob's measurement in its generalized form as

$$\rho_{xa}^B = \frac{1}{2} (\mathbb{I} + \bar{\mathbf{r}}_{xa}^B \cdot \sigma), \quad (20)$$

where

$$\bar{\mathbf{r}}_{xa}^B = 2(\alpha_B^2 - \beta_B^2) \bar{\mathbf{r}}_{xa} + \frac{4\beta_B^2}{3} \sum_{y=1}^3 (\hat{\mathbf{b}}_y \cdot \bar{\mathbf{r}}_{xa}) \hat{\mathbf{b}}_y \quad (21)$$

is the Bloch vector of the reduced state. Therefore, from Eq. (4) the quantum success probability for Charlie can be

written as

$$\begin{aligned}\mathcal{A}_C &= \frac{1}{18} \sum_{x,z=1}^3 \sum_{a \in \{0,1\}} \text{Tr}(\rho_{xa}^B E_{c|z}) \\ &= \frac{1}{2} + \frac{1}{36} \left(2(\alpha_B^2 - \beta_B^2) \sum_{k,z=1}^3 (\delta_{k,z} \bar{\mathbf{n}}_k \cdot \hat{\mathbf{c}}_z) \right. \\ &\quad \left. + \frac{4\beta_B^2}{3} \sum_{k,y,z=1}^3 (\delta_{k,z} \bar{\mathbf{n}}_k \cdot \hat{\mathbf{b}}_y) (\hat{\mathbf{b}}_y \cdot \hat{\mathbf{c}}_z) \right),\end{aligned}\quad (22)$$

where $E_{c|z} = \frac{\mathbb{I} + (-1)^c C_z}{2}$ is the POVM corresponding to the result $c = \delta_{x,z} \oplus_2 a$ of the measurement of $C_z = \hat{\mathbf{c}}_z \cdot \sigma$. The last line comes directly from Eqs. (20) and (21). It is clear from Eq. (22) that, through its dependence on α_B and β_B , the success probability \mathcal{A}_C of Charlie is a function of η_B . Hence, there exists a trade off between \mathcal{A}_B and \mathcal{A}_C . We optimize the success probability of Charlie \mathcal{A}_C as a function of \mathcal{A}_B .

It is seen from Eq. (22) that as $\alpha_B > \beta_B$, for the optimization of \mathcal{A}_C we must have $\bar{\mathbf{n}}_k$ to be in the direction of $\hat{\mathbf{c}}_z$ whenever $k = z$. We further observe that the $\bar{\mathbf{r}}_{xa}$ have to be antipodal pairs to maximize $\bar{\mathbf{n}}_k$ in Eq. (16) such that $\bar{\mathbf{r}}_{10} = -\bar{\mathbf{r}}_{11}$, $\bar{\mathbf{r}}_{21} = -\bar{\mathbf{r}}_{20}$, and $\bar{\mathbf{r}}_{30} = -\bar{\mathbf{r}}_{31}$. The overall optimization also demands that the states $\{\rho_{xa}\}$ are pure, i.e., $\|\bar{\mathbf{r}}_{xa}\| = 1$. From this choice we can rewrite the $\bar{\mathbf{n}}_k$ as $\bar{\mathbf{n}}_1 = 2(-\bar{\mathbf{r}}_{10} + \bar{\mathbf{r}}_{20} + \bar{\mathbf{r}}_{30})$, $\bar{\mathbf{n}}_2 = 2(\bar{\mathbf{r}}_{10} - \bar{\mathbf{r}}_{20} + \bar{\mathbf{r}}_{30})$, and $\bar{\mathbf{n}}_3 = 2(\bar{\mathbf{r}}_{10} + \bar{\mathbf{r}}_{20} - \bar{\mathbf{r}}_{30})$. We can then write

$$\begin{aligned}\mathcal{A}_C &\leq \frac{1}{2} + \frac{1}{36} \left(2(\alpha_B^2 - \beta_B^2) \sum_{k=1}^3 \|\bar{\mathbf{n}}_k\| \right. \\ &\quad \left. + \frac{4\beta_B^2}{3} \sum_{k,y=1}^3 (\bar{\mathbf{n}}_k \cdot \hat{\mathbf{b}}_y) (\hat{\mathbf{b}}_y \cdot \hat{\mathbf{c}}_k) \right).\end{aligned}\quad (23)$$

Using concavity of the square root $\sum_{k=1}^3 \|\bar{\mathbf{n}}_k\| \leq \sqrt{3 \sum_{k=1}^3 \|\bar{\mathbf{n}}_k\|^2}$ and the expressions of $\bar{\mathbf{n}}_k$, we have

$$\sum_{k=1}^3 \|\bar{\mathbf{n}}_k\| \leq 2\sqrt{36 - 3(\bar{\mathbf{r}}_{10} + \bar{\mathbf{r}}_{21} + \bar{\mathbf{r}}_{30})^2}. \quad (24)$$

Evidently, the maximum value of Eq. (24) is given by $\max(\sum_{k=1}^3 \|\bar{\mathbf{n}}_k\|) = 12$ when the condition

$$\bar{\mathbf{r}}_{10} + \bar{\mathbf{r}}_{21} + \bar{\mathbf{r}}_{30} = 0 \quad (25)$$

in Eq. (24) is satisfied. Using Eqs. (16) and (25), it is simple to show that the value of each $\|\hat{\mathbf{n}}_k\|$ is 4 and hence the equality in Eq. (24) holds. The condition in Eq. (25) implies the relations are satisfied by the Bloch vectors as $\hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{20} + \hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{30} = -1$, $\hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{20} + \hat{\mathbf{r}}_{20} \cdot \hat{\mathbf{r}}_{30} = -1$, and $\hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{30} + \hat{\mathbf{r}}_{20} \cdot \hat{\mathbf{r}}_{30} = -1$. The set of equations in turn provides $\hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{20} = \hat{\mathbf{r}}_{10} \cdot \hat{\mathbf{r}}_{30} = \hat{\mathbf{r}}_{20} \cdot \hat{\mathbf{r}}_{30} = -\frac{1}{2}$. Inserting the above conditions into Eq. (16), we immediately get $\hat{\mathbf{n}}_1 = -\hat{\mathbf{r}}_{10}$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{r}}_{20}$, and $\hat{\mathbf{n}}_3 = -\hat{\mathbf{r}}_{30}$.

Without any loss of generality we can then fix a set of vectors that satisfies the above conditions as $\hat{\mathbf{r}}_{10} = \hat{\mathbf{x}}$, $\hat{\mathbf{r}}_{20} = -\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{z}}$, and $\hat{\mathbf{r}}_{30} = -\frac{1}{2}\hat{\mathbf{x}} - \frac{\sqrt{3}}{2}\hat{\mathbf{z}}$. Furthermore, to optimize the quantum success probability of Charlie \mathcal{A}_C of Eq. (22) we also get $\hat{\mathbf{n}}_k = \hat{\mathbf{c}}_z$ for $z = k$. This implies that the maximum value $\max(\mathcal{A}_C) \equiv \Omega_C$ can be obtained when the unit vectors of Charlie are $\hat{\mathbf{c}}_1 = \hat{\mathbf{b}}_1 = \hat{\mathbf{n}}_1$, $\hat{\mathbf{c}}_2 = \hat{\mathbf{b}}_2 = \hat{\mathbf{n}}_2$, and $\hat{\mathbf{c}}_3 = \hat{\mathbf{b}}_3 = \hat{\mathbf{n}}_3$. We then have

$$\mathcal{A}_C \leq \frac{1}{2} + \frac{3\alpha_B^2 - \beta_B^2}{54} \sum_{k=1}^3 \|\hat{\mathbf{n}}_k\|. \quad (26)$$

It can be easily seen that given the values of α_B and β_B , the maximization of \mathcal{A}_C provides the success probability of Bob of the form

$$\mathcal{A}_B = \frac{1}{2} + \frac{\eta_B}{36} \sum_{k=1}^3 \|\hat{\mathbf{n}}_k\|. \quad (27)$$

Note that both \mathcal{A}_B and \mathcal{A}_C are simultaneously optimized when the quantity $\sum_{k=1}^3 \|\hat{\mathbf{n}}_k\|$ is optimized. Let us denote $\max(\mathcal{A}_B)$ by Ω_B . Substituting $\max(\sum_{k=1}^3 \|\hat{\mathbf{n}}_k\|) = 12$, we thus get the optimal pair of success probabilities for Bob and Charlie as

$$\Omega_B = \frac{1}{2} \left(1 + \frac{2\eta_B}{3} \right), \quad (28)$$

$$\Omega_C = \frac{1}{2} \left(1 + \frac{2\eta_C(1 + 2\sqrt{1 - \eta_B^2})}{9} \right). \quad (29)$$

To certify the unsharpness parameter η_B , we assume Charlie performs a sharp measurement (with $\eta_C = 1$). In such a case, both Ω_B and Ω_C become functions of a single parameter η_B .

It is worth pointing out that the set of states which provides the optimal pair of success probabilities Ω_B and Ω_C must also comply with the parity-oblivious restriction (6). In the current scenario this means that the quantum states corresponding to the set $\mathbb{P}_e \in \{10, 20, 30\}$ and the set $\mathbb{P}_o \in \{11, 21, 31\}$ must produce statistics that are equivalent to Bob for all measurements. Explicitly, Alice's input states must satisfy the relation $\rho_{10} + \rho_{20} + \rho_{30} = \rho_{11} + \rho_{21} + \rho_{31}$. One can easily verify that the set of states that are self-tested through the optimal quantum advantage follows the above relational constraint.

A. Certification of the unsharpness parameter

From Eq. (29) we note that Ω_C is a function of Ω_B . The trade-off relation between the optimal pair (Ω_B, Ω_C) of quantum success probabilities for Bob and Charlie can then be written as

$$\Omega_C(\Omega_B) = \frac{1}{2} + \frac{1 + \sqrt{4 - 9(2\Omega_B - 1)^2}}{9}. \quad (30)$$

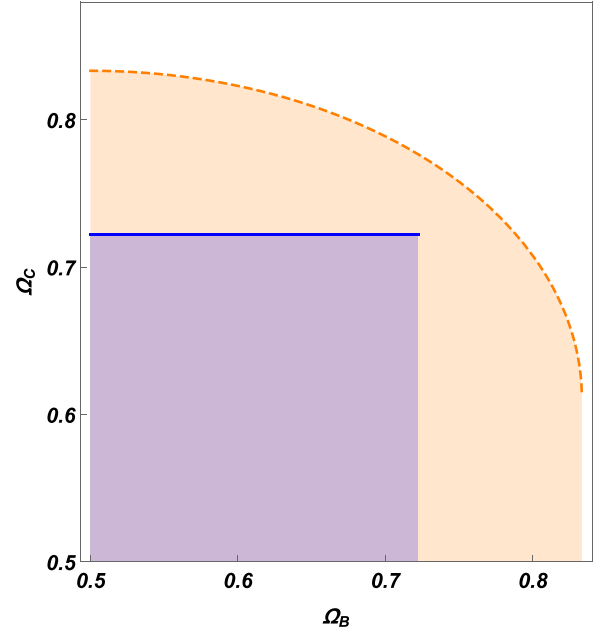


FIG. 2. Trade-off relation between the success probabilities achieved by Bob Ω_B and Charlie Ω_C . The solid blue (dotted orange) line indicates the variation in success probability of Charlie with respect to Bob in classical (quantum) regimes. It is clear that by classical means no trade-off exists as there is no variation in Charlie's success probability with a change in Bob's success probability, whereas in quantum-mechanical prescriptions there is a solid trade-off in success probabilities between Charlie and Bob.

In Fig. 2 we plot the trade-off between the success probabilities Ω_C and Ω_B described by classical and quantum theories. Classically, the measurement does not necessarily disturb a system and there exists no trade-off between the success probabilities, which is represented by the blue line as shown in Fig. 2. On the other hand, due to the information disturbance relations Ω_B and Ω_C follow a trade-off relation in the quantum theory description as expressed in Eq. (30). Such a trade-off between the quantum success probabilities is captured in Fig. 2 by the red curve and each point on it certifies a unique value of unsharpness parameter η_B . For instance, when $\Omega_B = \Omega_C = 0.75457$, the value of $\eta_B = 0.7637$ is certified. A similar argument holds for each point on the red curve in Fig. 2.

It is quite interesting to check the extent to which the value of the unsharpness parameter for observers can be fine-tuned to find the maximum number of independent observers that can sequentially harness the quantum advantage. The sequential observers mutually agree that each of them gets an appropriate amount of quantum advantage in the game by setting up the unsharpness parameters of their measurement instruments. Bob's measurement instrument should have its unsharpness parameter above a certain threshold value. Otherwise, he will not receive any quantum advantage in such a game. By using Eq. (28) the minimum value $(\eta_B)^{\min}$ of the unsharpness parameter required to get the quantum advantage for Bob is determined as a function

of Ω_B as

$$\eta_B \geq (\eta_B)^{\min} = 3(\Omega_B - \frac{1}{2}). \quad (31)$$

Similarly, to upper bound the permissible value for η_B , from Eq. (29) we rewrite η_B in terms of Ω_C as

$$\eta_B \leq (\eta_B)^{\max} = \sqrt{1 - \left\{ \frac{1}{2} \left[\frac{9}{2} (2\Omega_C - 1) - 1 \right] \right\}^2}. \quad (32)$$

From Eqs. (31) and (32), by substituting the classical bound ($\mathcal{A}_{pmc} = \frac{13}{18}$) for Ω_B and Ω_C , we find the quantitative bound to the unsharpness parameter η_B of Bob's measurement instrument as

$$\frac{2}{3} \leq \eta_B \leq \frac{\sqrt{3}}{2}.$$

Certainly, any value of η_B in the above range necessarily provides the quantum advantage to both the observers.

B. SDI certification statements

The optimal pair (Ω_B, Ω_C) of success probabilities for Bob and Charlie provides the following certification statements for the ideal preparations of Alice and the ideal unsharp measurements of Bob.

(i) Among the six states that Alice prepares, three are of a specific type called the trine states and the rest are just their antipodes. For an example, such states are represented by the Bloch vectors $\vec{r}_{10} = \hat{x}$, $\vec{r}_{20} = (\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z})$, and $\vec{r}_{30} = (-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z})$ and their respective antipodal pairs.

(ii) Bob performs unsharp measurements corresponding to the incompatible observables such that $\hat{b}_1 = -\vec{r}_{10}$, $\hat{b}_2 = -\vec{r}_{20}$, and $\hat{b}_3 = -\vec{r}_{30}$. Charlie's measurement settings are the same as Bob's with η_C instead of η_B and are taken to be unity as Charlie performs a sharp measurement.

Furthermore, each point on the red curve in Fig. 2 is a representative of the optimal pair (Ω_B, Ω_C) for a certain η_B . The certification arguments are valid as long as for a certain η_B the observers get optimal pairs. In other words, every point on the red curve in Fig. 2 represents a certified value of η_B . It can be easily seen from Fig. 2 that whether or not Charlie gets the quantum advantage entirely depends upon how much of the same is already extracted from the system by Bob's measurement. They can then agree on a common strategy to achieve the required quantum advantage for a particular purpose.

V. POSSIBLE EXTENSION OF QUANTUM ADVANTAGE TO A THIRD OBSERVER

We have just demonstrated that in a sequential scenario, a sustainable quantum advantage is achieved by Charlie if Bob performs unsharp measurements for a range of values of η_B . A question may immediately arise whether for sufficiently lower values of η_C it is possible to extend the quantum advantage to a third observer (say, Debbie). To analyze this case, we find the maximum success probability for Debbie, who receives input $w \in \{1, 2, 3\}$ and returns output $d \in \{0, 1\}$. By using Eq. (17) we find the reduced states after Charlie's measurement, relayed to Debbie as

$$\rho_{xa}^C = \frac{1}{2}(\mathbb{I} + \vec{r}_{xa}^C \cdot \sigma), \quad (33)$$

where

$$\begin{aligned} \vec{r}_{xa}^C &= 4(\alpha_B^2 - \beta_B^2)(\alpha_C^2 - \beta_C^2)\vec{r}_{xa} \\ &+ \frac{8}{3}\beta_B^2(\alpha_C^2 - \beta_C^2) \sum_{y=1}^3 (\hat{b}_y \cdot \vec{r}_{xa})\hat{b}_y \\ &+ \frac{8}{3}\beta_C^2(\alpha_B^2 - \beta_B^2) \sum_{z=1}^3 (\hat{c}_z \cdot \vec{r}_{xa})\hat{c}_z \\ &+ \frac{16}{9}\beta_B^2\beta_C^2 \sum_{y,z=1}^3 (\hat{b}_y \cdot \vec{r}_{xa})(\hat{b}_y \cdot \hat{c}_z)\hat{c}_z. \end{aligned}$$

Then the quantum success probability for Debbie is given by

$$\begin{aligned} \mathcal{A}_D &= \frac{1}{18} \text{Tr}(\rho_{xa}^C E_{d|w}) \\ &= \frac{1}{2} + \frac{1}{36} \left(4(\alpha_B^2 - \beta_B^2)(\alpha_C^2 - \beta_C^2) \sum_{k,w,z=1}^3 \delta_{k,w} \vec{n}_k \cdot \hat{c}_z \right. \\ &+ \frac{8}{3}\beta_B^2(\alpha_B^2 - \beta_B^2) \sum_{k,y,w=1}^3 (\hat{b}_y \cdot \vec{n}_k)(\hat{b}_y \cdot \hat{d}_w) \\ &+ \frac{8}{3}\beta_C^2(\alpha_B^2 - \beta_B^2) \sum_{k,z,w=1}^3 (\hat{c}_z \cdot \vec{n}_k)(\hat{c}_z \cdot \hat{d}_w) \\ &\left. + \frac{16}{9}\beta_B^2\beta_C^2 \sum_{k,w,y,z=1}^3 (\hat{b}_y \cdot \vec{n}_k)(\hat{b}_y \cdot \hat{c}_z)(\hat{c}_z \cdot \hat{d}_w) \right), \quad (34) \end{aligned}$$

where $E_{d|w} = \frac{\mathbb{I} + (-1)^d \eta_D D_w}{2}$ is the POVM corresponding to the outcome $d = \delta_{x,w} \oplus_2 a$ of measurement $D_w = \hat{d}_w \cdot \sigma$. Using the same optimization technique used in the preceding section from Eq. (34), we obtain

$$\begin{aligned} \mathcal{A}_D &= \frac{1}{2} + \frac{1}{36} \left(4(\alpha_B^2 - \beta_B^2)(\alpha_C^2 - \beta_C^2) \sum_{k=1}^3 \|\vec{n}_k\|^2 \right. \\ &+ \frac{8}{3}\beta_B^2(\alpha_B^2 - \beta_B^2) \sum_{k=1}^3 \|\vec{n}_k\|^2 \\ &\left. + \frac{8}{3}\beta_C^2(\alpha_B^2 - \beta_B^2) \sum_{k=1}^3 \|\vec{n}_k\|^2 + \frac{16}{9}\beta_B^2\beta_C^2 \sum_{k=1}^3 \|\vec{n}_k\|^2 \right). \quad (35) \end{aligned}$$

Further by setting $\sum_{k=1}^3 \|\vec{n}_k\|^2 = 12$, we finally get

$$\Omega_D = \frac{1}{2} \left(1 + \frac{2\eta_D}{27} (1 + 2\sqrt{1 - \eta_B^2})(1 + 2\sqrt{1 - \eta_C^2}) \right). \quad (36)$$

Thus, the unsharpness parameter η_D of Debbie's measurement instrument takes the form

$$\eta_D = \frac{27}{2} \left(\frac{2\Omega_D - 1}{(1 + 2\sqrt{1 - \eta_B^2})(1 + 2\sqrt{1 - \eta_C^2})} \right). \quad (37)$$

From this equation it is clear that, for Debbie to achieve quantum advantage, the values of η_C and η_B should be such that $\eta_D \leq 1$. The range of η_B was specified earlier. Let

the threshold value of the unsharpness parameter of Charlie's measurement instrument be denoted by $(\eta_C)^{\min}$. From Eq. (29) we write

$$(\eta_C)^{\min} = \frac{9}{2} \left(\frac{2\Omega_C - 1}{1 + 2\sqrt{1 - \eta_B^2}} \right). \quad (38)$$

To give a quantum advantage to Charlie, Bob performs his measurement by setting the unsharpness parameter of his instrument slightly above the lower critical value $(\eta_B)^{\min}$. We denote the infinitesimal amount of η_B that is greater than $(\eta_B)^{\min}$ by $\zeta \in [0, 1]$ such that $0.66 \leq \eta_B + \zeta \leq 0.866$.

By setting $\eta_B = \eta_B + \zeta$ in Eq. (38) and upon solving we get $(\eta_C)^{\min} = 0.803 + 0.577\zeta$, with $\zeta \in [0, 0.341]$. Therefore, lowering the degree of unsharpness parameter of Charlie may provide some advantage. To certify that Debbie can harness the quantum advantage, from Eq. (37) we find the minimum value of η_D to violate the noncontextual bound. For that we set $\eta_B = (\eta_B)^{\min}$ and $\eta_C = (\eta_C)^{\min}$ in Eq. (36). This gives $\eta_D = 1.09$, which is not a legitimate value of the unsharpness parameter. This suggests that irrespective of how small the first two observers impose the disturbances, no more than two observers can get simultaneous quantum advantages.

VI. ROBUST CERTIFICATION OF PREPARATIONS AND MEASUREMENTS

In Sec. IV B we discussed the certification of trine states, measurements, and the unsharpness parameter in an ideal scenario. The practical scenario, however, is bound to be imperfect and hence demands a certification scheme tolerant to noise. We now analyze the robustness of preparation and measurement instruments in the presence of noise. For this we adopt a scheme proposed in [35] and subsequently used in the SDI scenario in [37]. In [35] the author derived an operator inequality for the device-independent self-testing bound on the quantum value of the Clauser-Horne-Shimony-Holt expression by analyzing the robustness of the instruments. Following that approach, we construct such operator inequalities to characterize the average fidelity. Relying on the value of average fidelity, the closeness between the target states (measurements) to ideal ones can be determined. Hence obtaining a lower bound to the average fidelity can provide robust certification.

A. Robust certification of the preparations of Alice

Given a set of state preparations $\{\rho_{xa}\}$, the average fidelity for a given success probability \mathcal{A}_B with respect to the set of ideal states $\{\rho_{xa}^{\text{ideal}}\}$ is quantified by

$$\mathcal{S}(\rho_{xa}) = \frac{1}{6} \max_{\Lambda} \sum_{x=1}^3 \sum_{a \in \{0,1\}} F(\rho_{xa}^{\text{ideal}}, \Lambda(\rho_{xa})), \quad (39)$$

where Λ is a completely positive and trace preserving (CPTP) map and fidelity takes the form

$$F(\rho_{xa}^{\text{ideal}}, \Lambda(\rho_{xa})) = \langle \Lambda(\rho_{xa}) | \rho_{xa}^{\text{ideal}} \rangle. \quad (40)$$

There may exist multiple sets of states which are compatible with a given success probability \mathcal{A}_B . Thus, we estimate the

robustness of the preparations by finding the lower bound of the average fidelity for all possible set of states that provides the success probability \mathcal{A}_B such that

$$\mathcal{F}(\mathcal{A}) = \min_{\rho_{xa} \in \mathcal{R}(\mathcal{A}_B)} \mathcal{S}(\rho_{xa}). \quad (41)$$

To ensure the robustness of Alice's preparations, the fidelity must be lower bounded by a function of \mathcal{A}_B in order to satisfy an appropriate operator inequality. By using Eq. (13) we rewrite the success probability of Alice in the form

$$\mathcal{A}_B = \frac{1}{2} + \sum_{x=1}^3 \sum_{a \in \{0,1\}} \text{Tr}(\rho_{xa} W_{xa}), \quad (42)$$

where

$$W_{xa} = \frac{1}{36} \sum_{y=1}^3 \sum_{a \in \{0,1\}} (-1)^{\delta_{x,y} \oplus a} B_y. \quad (43)$$

If the preparations are pure then the fidelity can be written as $F(\rho_{xa}^{\text{ideal}}, \Lambda(\rho_{xa})) = \langle \Lambda(\rho_{xa}) | \rho_{xa}^{\text{ideal}} \rangle$. We can then write $F(\rho_{xa}^{\text{ideal}}, \Lambda(\rho_{xa})) = \text{Tr}(K_{xa} \rho_{xa})$, where

$$K_{xa}(B_1, B_2, B_3) = \Lambda^\dagger(B_1, B_2, B_3)(\rho_{xa}^{\text{ideal}}) \quad (44)$$

represents the action on the set of ideal states, with Λ^\dagger the dual channel to Λ . For any positive quantity s we define an operator in the form $K_{xa} - sW_{xa}$ and let $t_{xa} \in \mathbb{R}$ be its lower bound. Then, in general the operator inequality takes the form

$$K_{xa}(B_1, B_2, B_3) \geq sW_{xa} + t_{xa}(B_1, B_2, B_3)\mathbb{I}. \quad (45)$$

Finding the trace over the inputs of this inequality and applying minimization over B_1, B_2 , and B_3 , the lower bound to the average fidelity turns out to be

$$\frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} \text{Tr}(K_{xa} \rho_{xa}) \geq \frac{s}{6} \text{Tr}(\rho_{xa} W_{xa} + t_{xa} \mathbb{I}) \quad (46)$$

so that

$$\mathcal{S} \geq \frac{s}{6} \left[\left(\mathcal{A}_B - \frac{1}{2} \right) \right] + \frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} t_{xa}. \quad (47)$$

We thus bound the average fidelity from below as

$$\mathcal{F}(\mathcal{A}_B) \geq \frac{s}{6} \left(\mathcal{A}_B - \frac{1}{2} \right) + t, \quad (48)$$

where

$$t = \min_{B_1, B_2, B_3} \frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} t_{xa}(B_1, B_2, B_3)$$

are subject to evaluation.

Here we propose a generalized scheme where robustness can be certified for any choice of the unsharpness parameter η_B in the defined range for Bob's measurement instrument. The detailed derivation is lengthy and hence deferred to Appendix A. We first set $s = 9/\eta_B$, which in turn gives $t = \frac{1}{2}$ such that the lower bound of average fidelity can be determined from Eq. (48) for a given success probability \mathcal{A}_B as

$$\mathcal{F}(\mathcal{A}_B) = \frac{3}{2\eta_B} \left(\mathcal{A}_B - \frac{1}{2} \right) + \frac{1}{2}. \quad (49)$$

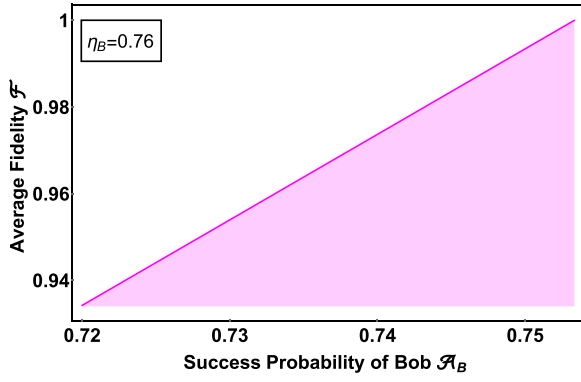


FIG. 3. Variation between the average fidelity \mathcal{F} for prepared states of Alice and the success probability \mathcal{A}_B for a fixed value of the unsharpness parameter η_B .

The trade-off relation between the average fidelity and the success probability is plotted for $\mathcal{A}_{pmc} \leq \mathcal{A}_B \leq \Omega_B$ in Fig. 3 for a particular value of $\eta_B = 0.76$. Due to the dependence of s on η_B , although the nature of the trade-off graph will be the same for different values of η_B , the fidelity for different degrees of unsharpness will be different even for the same value of success probabilities. The lower bound saturates with $F(\mathcal{A}_B) = 1$ for $\mathcal{A}_B = \Omega_B$.

B. Robust certification of measurements

Analogous to robust certification of preparations, we adopt a similar treatment here. The expression of average fidelity, given the set of ideal measurements, is given by

$$\mathcal{S}'(\{E_{b|y}\}) = \frac{1}{6} \max_{\Lambda} \sum_{y=1}^3 \sum_{b \in \{0,1\}} F((E_{b|y})^{\text{ideal}}, \Lambda(E_{b|y})), \quad (50)$$

where Λ is a quantum channel. To estimate the lower bound for the average fidelity, we write

$$\mathcal{F}'(\mathcal{A}) = \min_{E_{b|y} \in \mathcal{R}'(\mathcal{A})} \mathcal{S}'(\{E_{b|y}\}). \quad (51)$$

Measurements are suitably chosen to be compatible with the given value of \mathcal{A} because the instruments are considered to be black boxes. To determine the robustness, the operator inequality is constructed of the form

$$K_{yb}(\{\rho_{yb}\}) \geq sZ_{yb} + t_{yb}\mathbb{I}, \quad (52)$$

where $K_{yb} = \Lambda^\dagger[(E_{b|y})^{\text{ideal}}]$ and thus fidelity can be written as

$$F((E_{b|y})^{\text{ideal}}, \Lambda(E_{b|y})) = \langle \Lambda(E_{b|y}) | (E_{b|y})^{\text{ideal}} \rangle = \text{Tr}[K_{yb}(E_{b|y})]. \quad (53)$$

From the definition of general success probability $\mathcal{A} = \sum_{y=1}^3 \sum_{b \in \{0,1\}} \text{Tr}(E_{b|y}Z_{yb})$, where

$$Z_{yb} = \frac{1}{18} \sum_{x,y=1}^3 \sum_{a \in \{0,1\}} \rho_{xa} \delta_{l,b}, \quad (54)$$

with $l = \delta_{x,y} \oplus a$. Subsequently, we find the expression which quantifies the lower bound of the average fidelity to be

$$\mathcal{F}'(\mathcal{A}) \geq \frac{s}{6} \mathcal{A} + t, \quad (55)$$

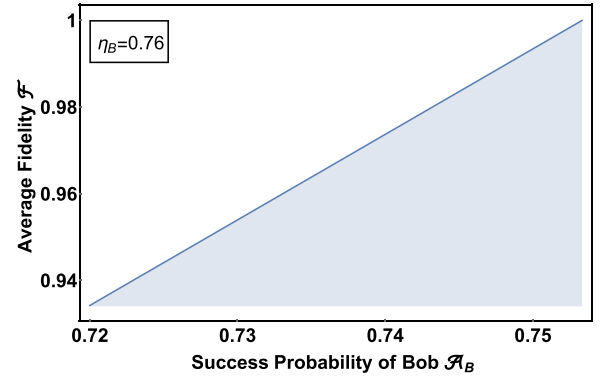


FIG. 4. Variation in the average fidelity of measurements of Bob with respect to success probability \mathcal{A}_B for a fixed value of η_B .

where

$$t = \frac{1}{6} \min_{B_1, B_2, B_3} \sum_{y=1}^3 \sum_{b \in \{0,1\}} t_{yb}(B_1, B_2, B_3). \quad (56)$$

Furthermore, the states that Bob receives are the states sent by Alice and the performed measurements are unsharp. On the other hand, Charlie gets the η_B -dependent reduced states after Bob's measurement and performs sharp measurements. As a result, the robustness analyses for these two observers differ, which are the following.

Bob performs unsharp measurements with unsharpness parameter η_B in the certified range. Thus, essentially the parameter t becomes a function of η_B , which finally end up giving the η_B dependence of the fidelity equation. To lower bound the average fidelity, we choose $s = 9$, and by simple calculation we find $t = \frac{1}{4} - \frac{\eta_B}{2}$. From Eq. (55) we can then write

$$\mathcal{F}'(\mathcal{A}_B) \geq \frac{3}{2} \mathcal{A}_B + \frac{1}{4} - \frac{\eta_B}{2}. \quad (57)$$

For certain η_B we recover the ideal measurements that we aimed to certify, when the success probability \mathcal{A}_B reaches its maximum value, i.e., $\mathcal{F}'(\mathcal{A}_B) = 1$ when $\mathcal{A}_B = \Omega_B$.

Even though Charlie's measurement is sharp, still t is dependent on η_B due to the η_B dependence in the post-measurement state by a factor $\gamma = (1 + \sqrt{1 - \eta_B^2})/2$. Here we set $s = 54/(7\gamma - 1)$, and upon further simplification the value for t turns out to be $t = (9 - 6\gamma)/(2 - 14\gamma)$. Hence, the lower bound to the average fidelity is written from Eq. (55) as

$$\mathcal{F}'(\mathcal{A}_C) = \frac{9}{7\gamma - 1} \mathcal{A}_C + \frac{9 - 6\gamma}{2 - 14\gamma}. \quad (58)$$

For a certain value of η_B in the certified range, the fidelity reaches its maximum value for $\mathcal{A}_C = \Omega_C$.

The variation of fidelity with the success probability for Bob and Charlie is plotted in Figs. 4 and 5, respectively, for a particular choice of η_B . Detailed calculations for robust certification of the measurement instrument for Bob and Charlie are provided in Appendix B.

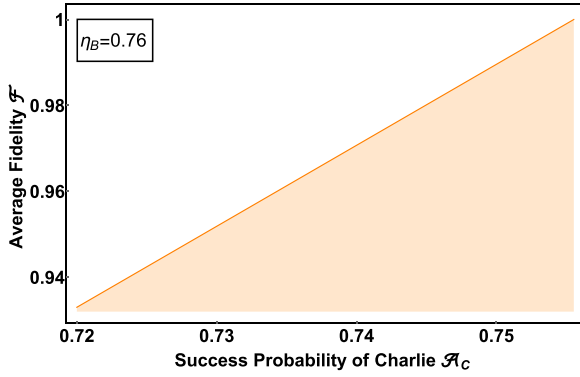


FIG. 5. Variation in the average fidelity of measurement with the variation of the success probability of Charlie.

VII. CONCLUSION

In summary, we have demonstrated robust certification of states, measurements, and the unsharpness parameter based on the sequential quantum advantage through a specific communication game in a semi-device-independent prepare-measure scenario. We explicitly derived the classical preparation noncontextual bound of the proposed game to quantitatively analyze the quantum supremacy in such a task. The viable constraint relations among the states and observable for the optimal quantum success probability were derived. From these constraints, we further argued that the optimal quantum value of the success probability uniquely certifies that the states prepared by Alice are indeed the set of trine states, and the measurement instrument that Bob uses measures along the directions of the trine spin axes.

Next we crafted a scenario that allows multiple independent observers to share sequential quantum advantage when they are allowed to implement their respective measurements with varying degrees of unsharpness. In particular, we have explicitly shown that at most two observers can get the quantum advantage if the degree of unsharpness of the first observer's measurement lies inside a specific range of values. Therefore, when both observers get the quantum advantage sequentially, we demonstrated how to certify the range of the unsharpness parameter of the measurement instrument used by the former observer. Most importantly, it was shown that, even when the success probability does not reach the global optimal value, for a given unsharpness parameter, the optimal pair of success probabilities of the respective observers certifies the prepared states and measurement instruments.

We further extended our investigation to the case when the optimal value of the success probability for a particular unsharpness parameter is not achieved. In such a scenario, we derived the fidelity bounds of the prepared states and implemented measurements to the ideal ones. This enabled us to robustly certify the states and measurements that are subjected to imperfections other than unsharp measurements. In the sequential scenario, as each optimal pair of success probability certifies the states and measurement, the parameters that give the bound to the fidelity possess different values for different optimal pairs. We derived the exact forms of parameters as a function of the degree of unsharpness. Thus, the derived trade-off relation between the fidelity and the success probability

provides the robust certification of the states mentioned above and measurements.

All the works to date considered the robust certification of quantum instruments in a two-party scenario. Nevertheless, as the situation may sometimes demand both the sustainable quantum advantage and certification of the instruments, the robustness analysis in the presence of an unsharp instrument becomes crucial. Our work sheds light in this direction by extending the conventional robustness analysis scheme to the case where the measurements themselves are unsharp along with other experimental imperfections. We hope that future research will find more general methods to handle the unsharpness parameters alongside other experimental imperfections while analyzing the robustness of different certification schemes. Studies along this line would be an interesting avenue for future research.

ACKNOWLEDGMENTS

The authors acknowledge the anonymous referee for valuable suggestions and comments, which helped to improve the quality of the work. A.S.S. and S.M. acknowledge support from the project Department of Science and Technology, Ministry of Science and Technology (DST/ICPS/QuST/Theme-1/2019/4) and A.K.P. acknowledges research grant MTR/2021/000908.

APPENDIX A: ROBUST SELF-TESTING OF ALICE'S PREPARATIONS

We provide a detailed derivation of the robustness of preparations of Alice. The average fidelity of the preparations with the ideal states are quantified by

$$\mathcal{S}(\rho_{xa}) = \frac{1}{6} \max_{\Lambda} \sum_{x=1}^3 \sum_{a \in \{0,1\}} F(\rho_{xa}^{\text{ideal}}, \Lambda(\rho_{xa})), \quad (\text{A1})$$

where Λ is a CPTP map. The average fidelity is estimated by considering all the possible channels. The lower bound is derived by minimizing the average fidelity over the set of preparations compatible with the given value of \mathcal{A}_B , i.e.,

$$\mathcal{F}(\mathcal{A}_B) = \min_{\rho_{xa} \in R(\mathcal{A}_B)} \mathcal{S}(\rho_{xa}), \quad (\text{A2})$$

where $R(\mathcal{A}_B)$ is the set of all preparations compatible with the value of \mathcal{A}_B . As explained in the main text, we write the operator inequality

$$K_{xa}(B_1, B_2, B_3) \leq sW_{xa} + t_{xa}(B_1, B_2, B_3)\mathbb{I}, \quad (\text{A3})$$

where

$$K_{xa}(B_1, B_2, B_3) = \Lambda^\dagger(B_1, B_2, B_3)(\rho_{xa}^{\text{ideal}}), \quad (\text{A4})$$

with Λ^\dagger the dual to a quantum channel Λ , and

$$W_{xa} = \frac{1}{36} \sum_{y=1}^3 (-1)^{\delta_{x,y} \oplus a} B_y. \quad (\text{A5})$$

Since W_{xa} is a function of B_1 , B_2 , and B_3 , the quantities K_{xa} and t_{xa} are chosen to be dependent on B_1 , B_2 , and B_3 in order to be in compliance with W_{xa} . Taking the trace over inputs, we

arrive at

$$\begin{aligned} S &\geq \frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} \text{Tr}(K_{xa} \rho_{xa}) \geq \frac{s}{6} \text{Tr}(\rho_{xa} W_{xa} + t_{xa} \mathbb{I}) \\ &= \frac{s}{6} \left[\left(\mathcal{A}_B - \frac{1}{2} \right) \right] + \frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} t_{xa}. \end{aligned} \quad (\text{A6})$$

Therefore, the lower bound of the average fidelity can be determined by

$$F(\mathcal{A}_B) = \frac{s}{6} \left(\mathcal{A}_B - \frac{1}{2} \right) + t, \quad (\text{A7})$$

where $t = \min_{B_1, B_2, B_3} \frac{1}{6} \sum_{x=1}^3 \sum_{a \in \{0,1\}} t_{xa}(B_1, B_2, B_3)$. Thus, the problem resolves to choose a particular value s , and subsequently the values of t_{xa} are minimized with respect to B_1 , B_2 , and B_3 for the choice of s .

We choose a dephasing channel of the form

$$\Lambda_\theta(\rho) = \frac{1+c(\theta)}{2} \rho + \frac{1-c(\theta)}{2} \Gamma_\theta \rho \Gamma_\theta. \quad (\text{A8})$$

For $\theta \in [0, \frac{\pi}{3}]$, $\Gamma = \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x$; for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$, $\Gamma = -\frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x$. The dephasing function $c(\theta) \in [0, 1]$ needs to be specified. Therefore, using Eqs. (A4) and (A8), we find the action of channel in the intervals: For $\theta \in [0, \frac{\pi}{3}]$,

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(c-1)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right), & K_{11} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(c-1)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right), \\ K_{20} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2} \sigma_z + \frac{1}{2} \sigma_x \right), \\ K_{30} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{1-3c}{4} \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{1-3c}{4} \sigma_x \right); \end{aligned} \quad (\text{A9})$$

similarly, for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right), & K_{11} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right), \\ K_{20} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{1-3c}{4} \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z - \frac{1-3c}{4} \sigma_x \right), \\ K_{30} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{4} \sigma_z - \frac{1}{2} \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{4} \sigma_z + \frac{1}{2} \sigma_x \right). \end{aligned} \quad (\text{A10})$$

Similarly, to find the expressions for W_{xa} we consider that Bob's measurements are projective and are represented by

$$B_1 = -\eta_B \sigma_x, \quad B_2 = \eta_B (\cos \theta \sigma_x - \sin \theta \sigma_z), \quad B_3 = \eta_B (\cos \theta \sigma_x + \sin \theta \sigma_z). \quad (\text{A11})$$

Hence, by Eq. (A5) the W_{xa} can be written as

$$\begin{aligned} W_{10} &= \frac{1}{36} \eta_B (\sigma_x + 2 \cos \theta \sigma_x), & W_{11} &= \frac{1}{36} \eta_B (-\sigma_x - 2 \cos \theta \sigma_x), & W_{20} &= \frac{1}{36} \eta_B (-\sigma_x + 2 \sin \theta \sigma_z), \\ W_{21} &= \frac{1}{36} \eta_B (\sigma_x - 2 \sin \theta \sigma_z), & W_{30} &= \frac{1}{36} \eta_B (-\sigma_x - 2 \sin \theta \sigma_z), & W_{31} &= \frac{1}{36} \eta_B (-\sigma_x - 2 \sin \theta \sigma_z). \end{aligned} \quad (\text{A12})$$

Due to the apparent symmetries of K_{xa} and W_{xa} , the number of inequalities can be further reduced. If we choose $t_{10} = t_{11} \equiv t_1$, $t_{20} = t_{21} \equiv t_2$, and $t_{30} = t_{31} \equiv t_3$, in each interval only three inequalities need to be considered.

Now we write the inequalities in the interval $\theta \in [0, \frac{\pi}{3}]$,

$$\begin{aligned} \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(c-1)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right) - \frac{\eta_B s}{36} (\sigma_x + 2 \cos \theta \sigma_x) - t_1 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x \right) - \frac{\eta_B s}{36} (-\sigma_x + 2 \sin \theta \sigma_z) - t_2 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z - \frac{1(1-3c)}{4} \sigma_x \right) - \frac{\eta_B s}{36} (-\sigma_x - 2 \sin \theta \sigma_z) - t_3 \mathbb{I} &\geq 0, \end{aligned} \quad (\text{A13})$$

and in the interval $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(c-1)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right) - \frac{\eta_B s}{36} (\sigma_x + 2 \cos \theta \sigma_x) - t_1 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z - \frac{(1-3c)}{4} \sigma_x \right) - \frac{\eta_B s}{36} (-\sigma_x + 2 \sin \theta \sigma_z) - t_2 \mathbb{I} &\geq 0, \end{aligned}$$

$$\frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x \right) - \frac{\eta_B s}{36} (-\sigma_x - 2 \sin \theta \sigma_z) - t_3 \mathbb{I} \geq 0. \quad (\text{A14})$$

Solving Eq. (A13), we find the values of t_1 , t_2 , and t_3 for the interval $\theta \in [0, \frac{\pi}{3}]$,

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - \sqrt{81 + 243c^2 - 9s\eta_B - 27cs\eta_B + 3s^2\eta_B^2 - 18s\eta_B \cos \theta - 54cs\eta_B \cos \theta + 4s^2\eta_B^2 \cos \theta + 2s^2\eta_B^2 \cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{81 + 243c^2 - 9s\eta_B - 27cs\eta_B + 3s^2\eta_B^2 - 18s\eta_B \cos \theta - 54cs\eta_B \cos \theta + 4s^2\eta_B^2 \cos \theta + 2s^2\eta_B^2 \cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - \sqrt{324 - 18s\eta_B + 3s^2\eta_B^2 - 2s^2\eta_B^2 \cos 2\theta - 36\sqrt{3}s\eta_B \sin \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{324 - 18s\eta_B + 3s^2\eta_B^2 - 2s^2\eta_B^2 \cos 2\theta - 36\sqrt{3}s\eta_B \sin \theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - \sqrt{81 + 243c^2 + 9s\eta_B - 27cs\eta_B + s^2\eta_B^2 - 18\sqrt{3}s\eta_B \sin \theta - 18\sqrt{3}cs\eta_B \sin \theta + 4s^2\eta_B^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{81 + 243c^2 + 9s\eta_B - 27cs\eta_B + s^2\eta_B^2 - 18\sqrt{3}s\eta_B \sin \theta - 18\sqrt{3}cs\eta_B \sin \theta + 4s^2\eta_B^2 \sin^2 \theta}) \right], \end{aligned} \quad (\text{A15})$$

and for the interval $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - \sqrt{81 + 243c^2 - 9s\eta_B - 27cs\eta_B + 3s^2\eta_B^2 - 18s\eta_B \cos \theta - 54cs\eta_B \cos \theta + 4s^2\eta_B^2 \cos \theta + 2s^2\eta_B^2 \cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{81 + 243c^2 - 9s\eta_B - 27cs\eta_B + 3s^2\eta_B^2 - 18s\eta_B \cos \theta - 54cs\eta_B \cos \theta + 4s^2\eta_B^2 \cos \theta + 2s^2\eta_B^2 \cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - \sqrt{81 + 243c^2 + 9s\eta_B - 27cs\eta_B + s^2\eta_B^2 - 18\sqrt{3}s\eta_B \sin \theta - 18\sqrt{3}cs\eta_B \sin \theta + 4s^2\eta_B^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{81 + 243c^2 + 9s\eta_B - 27cs\eta_B + s^2\eta_B^2 - 18\sqrt{3}s\eta_B \sin \theta - 18\sqrt{3}cs\eta_B \sin \theta + 4s^2\eta_B^2 \sin^2 \theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - \sqrt{324 - 18s\eta_B + 3s^2\eta_B^2 - 2s^2\eta_B^2 \cos 2\theta - 36\sqrt{3}s\eta_B \sin \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 + \sqrt{324 - 18s\eta_B + 3s^2\eta_B^2 - 2s^2\eta_B^2 \cos 2\theta - 36\sqrt{3}s\eta_B \sin \theta}) \right]. \end{aligned} \quad (\text{A16})$$

With the expressions of K_{xa} , W_{xa} , and t , we have constructed the operator inequalities for a given θ in the form (A3). To find lower bound of the average fidelity as given in Eq. (A7), we set the value of s to be $\frac{\eta_B}{6}$. The dephasing function is assumed to be $c(\theta) = \min(1, \frac{\eta_B}{6} \sin \theta)$ for the interval $\theta \in [0, \frac{\pi}{3}]$ and $c(\theta) = \min(1, \frac{\eta_B}{6} \cos \theta)$ for the interval $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$. It is obvious that $c(\theta) \in [0, 1]$ and it is continuous at $\theta = \frac{\pi}{3}$. By the apparent symmetry three out of six inequalities are considered and substituting the values of s and c , we find t to be

$$t = \frac{1}{3} (t_1 + t_2 + t_3) = \frac{1}{2}. \quad (\text{A17})$$

This completely generalizes the robustness analysis such that the general equation for average fidelity will be

$$\mathcal{F}(\mathcal{A}_B) = \frac{3}{2\eta_B} \left(\mathcal{A}_B - \frac{1}{2} \right) + \frac{1}{2}. \quad (\text{A18})$$

Equation (A18) provides the relation between robustness and (non)optimal success probability \mathcal{A}_B . For a fixed value of η_B , the lower bound $\mathcal{F}(\mathcal{A}_B) = 1$ is attained for the optimal success probability $\mathcal{A}_B = \Omega_B$.

APPENDIX B: ROBUST SELF-TESTING OF MEASUREMENTS

Let us now derive the robustness of measurements for the two sequential observers Bob and Charlie by invoking a similar scheme developed for preparations. To begin with, the average fidelity of measurements with respect to ideal ones are quantified as

$$\mathcal{S}'(\{E_{b|y}\}) = \frac{1}{6} \max_{\Lambda} \sum_{y=1}^3 \sum_{b \in \{0,1\}} [F((E_{b|y})^{\text{ideal}}, \Lambda(E_{b|y}))], \quad (\text{B1})$$

where maximization is taken for all possible channels. The lower bound for the average fidelity can be written as

$$\mathcal{F}'(\mathcal{A}) = \min_{E_{b|y} \in \mathcal{R}(\mathcal{A})} \mathcal{S}'(\{E_{b|y}\}), \quad (\text{B2})$$

where \mathcal{F}' is minimized over measurements compatible with the value of \mathcal{A} . Now we write the operator inequality in the form

$$K_{yb}(\rho_{yb}) \geq sZ_{yb} + t_{yb}\mathbb{I}, \quad (\text{B3})$$

where we find that

$$K_{yb} = \Lambda^\dagger[(E_{b|y})^{\text{ideal}}] \quad (\text{B4})$$

and

$$Z_{yb} = \frac{1}{18} \sum_{x=1}^3 \sum_{a \in \{0,1\}} \rho_{xa} \delta_{l,b}, \quad (\text{B5})$$

where $l = \delta_{x,y} \oplus a$. If we take the trace over the input states and minimize t_{xa} , we get

$$\mathcal{F}'(\mathcal{A}) = \frac{s}{6} \mathcal{A} + t, \quad (\text{B6})$$

where $t = \frac{1}{6} \min_{B_1, B_2, B_3} \sum_{y=1}^3 \sum_{b \in \{0,1\}} t_{yb}(B_1, B_2, B_3)$. We adopt the same dephasing channel as in Eq. (A8) and two intervals of θ . For $\theta \in [0, \frac{\pi}{3}]$ and $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$, we recall that $\Gamma = -\frac{\sqrt{3}}{2}\sigma_z + \frac{1}{2}\sigma_x$ and $\Gamma = \frac{\sqrt{3}}{2}\sigma_z + \frac{1}{2}\sigma_x$, respectively.

1. Robust certification of Bob's measurement Bob

As argued in the main text, Bob performs unsharp measurements to extend the quantum advantage to Charlie. It was already derived in the main text that in order to produce optimal success probability Alice's preparation should be pure states. We rewrite the states as

$$\rho_{yb} = \frac{\mathbb{I} + \vec{r}_{yb} \cdot \vec{\sigma}}{2}. \quad (\text{B7})$$

The explicit form of Bloch vectors is

$$\begin{aligned} \vec{r}_{10} &= (1, 0, 0), & \vec{r}_{11} &= (-1, 0, 0), \\ \vec{r}_{20} &= (-\cos \theta, 0, \sin \theta), & \vec{r}_{21} &= (\cos \theta, 0, -\sin \theta), \\ \vec{r}_{30} &= (-\cos \theta, 0, -\sin \theta), & \vec{r}_{31} &= (\cos \theta, 0, \sin \theta). \end{aligned} \quad (\text{B8})$$

We write the ideal POVMs for Bob as

$$\begin{aligned} (M_{b|1})^{\text{ideal}} &= \frac{\mathbb{I} - (-1)^b \eta_B \sigma_x}{2}, & (M_{b|2})^{\text{ideal}} &= \frac{1}{2} \left[\mathbb{I} + (-1)^b \eta_B \left(\frac{1}{2} \sigma_x - \frac{\sqrt{3}}{2} \sigma_z \right) \right], \\ (M_{b|3})^{\text{ideal}} &= \frac{1}{2} \left[\mathbb{I} + (-1)^b \eta_B \left(\frac{1}{2} \sigma_x + \frac{\sqrt{3}}{2} \sigma_z \right) \right]. \end{aligned}$$

Correspondingly, from Eq. (B4) Bob's K_{yb} values in the interval $\theta \in [0, \frac{\pi}{3}]$ can be written as

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)\eta_B}{4} \sigma_z - \frac{(1+3c)\eta_B}{4} \sigma_x \right), & K_{11} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)\eta_B}{4} \sigma_z + \frac{(1+3c)\eta_B}{4} \sigma_x \right), \\ K_{20} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}\eta_B}{2} \sigma_z + \frac{1}{2} \eta_B \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}\eta_B}{2} \sigma_z - \frac{1}{2} \eta_B \sigma_x \right), \\ K_{30} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)\eta_B}{4} \sigma_z + \frac{(3c-1)\eta_B}{4} \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)\eta_B}{4} \sigma_z - \frac{(3c-1)\eta_B}{4} \sigma_x \right) \end{aligned} \quad (\text{B9})$$

and for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)\eta_B}{4} \sigma_z - \frac{(1+3c)\eta_B}{4} \sigma_x \right), & K_{11} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)\eta_B}{4} \sigma_z + \frac{(1+3c)\eta_B}{4} \sigma_x \right), \\ K_{20} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)\eta_B}{4} \sigma_z + \frac{(3c-1)\eta_B}{4} \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)\eta_B}{4} \sigma_z - \frac{(3c-1)\eta_B}{4} \sigma_x \right), \\ K_{30} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}\eta_B}{2} \sigma_z + \frac{1}{2} \eta_B \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}\eta_B}{2} \sigma_z - \frac{1}{2} \eta_B \sigma_x \right). \end{aligned} \quad (\text{B10})$$

By using Eq. (B5) we derive Z_{yb} in the form

$$\begin{aligned} Z_{10} &= \left(\frac{3\mathbb{I}}{2} - \frac{\sigma_x + 2 \cos \theta \sigma_x}{2} \right), & Z_{11} &= \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x + 2 \cos \theta \sigma_x}{2} \right), & Z_{20} &= \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x - 2 \sin \theta \sigma_z}{2} \right), \\ Z_{21} &= \left(\frac{3\mathbb{I}}{2} + \frac{-\sigma_x + 2 \sin \theta \sigma_z}{2} \right), & Z_{30} &= \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x + 2 \sin \theta \sigma_z}{2} \right), & Z_{31} &= \left(\frac{3\mathbb{I}}{2} - \frac{\sigma_x + 2 \sin \theta \sigma_z}{2} \right). \end{aligned} \quad (\text{B11})$$

From the above expressions for K_{yb} and Z_{yb} , we write the inequalities for the respective intervals, i.e., for $\theta \in [0, \frac{\pi}{3}]$,

$$\begin{aligned} \frac{\mathbb{I} - c\eta_B\sigma_x}{2} - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\sigma_x + 2\cos\theta\sigma_x}{2} \right) - t_1\mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2}\eta_B\sigma_z + \frac{1}{2}c\eta_B\sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x - 2\sin\theta\sigma_z}{2} \right) - t_2\mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2}\eta_B\sigma_z + \frac{1}{2}c\eta_B\sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x + 2\sin\theta\sigma_z}{2} \right) - t_3\mathbb{I} &\geq 0, \end{aligned} \quad (\text{B12})$$

and for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} \frac{\mathbb{I} - \eta_B\sigma_x}{2} - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\sigma_x + 2\cos\theta\sigma_x}{2} \right) - t_1\mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2}c\eta_B\sigma_z + \frac{1}{2}\eta_B\sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x - 2\sin\theta\sigma_z}{2} \right) - t_2\mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2}c\eta_B\sigma_z + \frac{1}{2}\eta_B\sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\sigma_x + 2\sin\theta\sigma_z}{2} \right) - t_3\mathbb{I} &\geq 0, \end{aligned} \quad (\text{B13})$$

Solving the inequalities (B12) and (B13), we find the values of t_1 , t_2 , and t_3 for the interval $\theta \in [0, \frac{\pi}{3}]$,

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{3s^2 - 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 + 4s^2\cos\theta - 18s\eta_B\cos\theta - 54cs\eta_B\cos\theta + 2s^2\cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{3s^2 - 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 + 4s^2\cos\theta - 18s\eta_B\cos\theta - 54cs\eta_B\cos\theta + 2s^2\cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{s^2 - 18s\eta_B + 324\eta_B^2 - 36\sqrt{3}s\eta_B\sin\theta + 4s^2\sin^2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{s^2 - 18s\eta_B + 324\eta_B^2 - 36\sqrt{3}s\eta_B\sin\theta + 4s^2\sin^2\theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{s^2 + 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 - 18\sqrt{3}s\eta_B\sin\theta - 18\sqrt{3}cs\eta_B\sin\theta + 4s^2\sin^2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{s^2 + 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 - 18\sqrt{3}s\eta_B\sin\theta - 18\sqrt{3}cs\eta_B\sin\theta + 4s^2\sin^2\theta}) \right], \end{aligned} \quad (\text{B14})$$

and for the interval $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{3s^2 - 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 + 4s^2\cos\theta - 18s\eta_B\cos\theta - 54cs\eta_B\cos\theta + 2s^2\cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{3s^2 - 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 + 4s^2\cos\theta - 18s\eta_B\cos\theta - 54cs\eta_B\cos\theta + 2s^2\cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{s^2 + 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 - 18\sqrt{3}s\eta_B\sin\theta - 18\sqrt{3}cs\eta_B\sin\theta + 4s^2\sin^2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{s^2 + 9s\eta_B - 27cs\eta_B + 81\eta_B^2 + 243c^2\eta_B^2 - 18\sqrt{3}s\eta_B\sin\theta - 18\sqrt{3}cs\eta_B\sin\theta + 4s^2\sin^2\theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{s^2 - 18s\eta_B + 324\eta_B^2 - 36\sqrt{3}s\eta_B\sin\theta + 4s^2\sin^2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{s^2 - 18s\eta_B + 324\eta_B^2 - 36\sqrt{3}s\eta_B\sin\theta + 4s^2\sin^2\theta}) \right]. \end{aligned} \quad (\text{B15})$$

Now, for a given value of θ , with expressions for t , K_{yb} , and Z_{yb} , we have constructed the operator inequalities of the form (B3). To get the lower bound on \mathcal{F}' as given in Eq. (B6) we choose $s = 9$ and minimize t with respect to B_1 , B_2 , and B_3 , by the apparent

symmetry $t = \frac{t_1+t_2+t_3}{3}$. Simple calculation gives $t - \frac{1}{4} - \frac{\eta_B}{2}$. Thus, we write the average fidelity of measurements for Bob as

$$\mathcal{F}'(\mathcal{A}_B) = \frac{3}{2}\mathcal{A}_B + \frac{1}{4} - \frac{\eta_B}{2}, \quad (\text{B16})$$

which is almost the same as Eq. (57). The robustness of Bob's measurement is generalized for the (non)optimal success probability and the choice η_B , quantified in terms of average fidelity. The lower bound of the average fidelity is achieved when $\mathcal{A}_B = \Omega_B$.

2. Robust certification of measurements for Charlie

In the same way, we devised the calculations for robustness for Bob by the same approach; here we derive the same for Charlie as well. We find the values of K_{yb} for Charlie in the interval

$\theta \in [0, \frac{\pi}{3}]$; the values of K_{yb} can be written as

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right), & K_{20} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2} \sigma_z + \frac{1}{2} \sigma_x \right), & K_{30} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{3c-1}{4} \sigma_x \right), \\ K_{11} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z - \frac{3c-1}{4} \sigma_x \right). \end{aligned} \quad (\text{B17})$$

Similarly, for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$, the values of K_{yb} can be written as

$$\begin{aligned} K_{10} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right), & K_{20} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{3c-1}{4} \sigma_x \right), & K_{30} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2} \sigma_z + \frac{1}{2} \sigma_x \right), \\ K_{11} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)}{4} \sigma_z + \frac{1+3c}{4} \sigma_x \right), & K_{21} &= \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z - \frac{3c-1}{4} \sigma_x \right), & K_{31} &= \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2} \sigma_z - \frac{1}{2} \sigma_x \right). \end{aligned} \quad (\text{B18})$$

To find the expression of Z_{yb} , the states provided by Bob are simplified to the form

$$\rho_{yb} = \frac{\mathbb{I} + \gamma \vec{\mathbf{r}}_{yb} \cdot \vec{\sigma}}{2}, \quad (\text{B19})$$

where $\gamma = \frac{1+\sqrt{1-\eta_B^2}}{2}$ and $\vec{\mathbf{r}}_{yb}$ is as defined in Eq. (B8). We find the expressions for Z_{yb} in terms of ρ_{yb} as

$$\begin{aligned} Z_{10} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\gamma}{2} \sigma_x - \gamma \cos \theta \sigma_x \right), & Z_{11} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x + \gamma \cos \theta \sigma_x \right), & Z_{20} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x - \gamma \sin \theta \sigma_z \right), \\ Z_{21} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\gamma}{2} \sigma_x + \gamma \sin \theta \sigma_z \right), & Z_{30} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x + \gamma \sin \theta \sigma_z \right), & Z_{31} &= \frac{1}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\gamma}{2} \sigma_x - \gamma \sin \theta \sigma_z \right). \end{aligned} \quad (\text{B20})$$

The operator inequalities for the interval $\theta \in [0, \frac{\pi}{3}]$ are

$$\begin{aligned} \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1-c)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\gamma}{2} \sigma_x - \gamma \cos \theta \sigma_x \right) - t_1 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}}{2} \sigma_z + \frac{1}{2} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x - \gamma \sin \theta \sigma_z \right) - t_2 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{3c-1}{4} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x + \gamma \sin \theta \sigma_z \right) - t_3 \mathbb{I} &\geq 0. \end{aligned} \quad (\text{B21})$$

For the interval $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$,

$$\begin{aligned} \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1-c)}{4} \sigma_z - \frac{1+3c}{4} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} - \frac{\gamma}{2} \sigma_x - \gamma \cos \theta \sigma_x \right) - t_1 \mathbb{I} &\geq 0, \\ \frac{1}{2} \left(\mathbb{I} - \frac{\sqrt{3}(1+c)}{4} \sigma_z + \frac{3c-1}{4} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x - \gamma \sin \theta \sigma_z \right) - t_2 \mathbb{I} &\geq 0, \end{aligned}$$

$$\frac{1}{2} \left(\mathbb{I} + \frac{\sqrt{3}}{2} \sigma_z + \frac{1}{2} \sigma_x \right) - \frac{s}{18} \left(\frac{3\mathbb{I}}{2} + \frac{\gamma}{2} \sigma_x + \gamma \sin \theta \sigma_z \right) - t_3 \mathbb{I} \geq 0. \quad (\text{B22})$$

Solving the inequalities, the results for t in the interval $\theta \in [0, \frac{\pi}{3}]$ will be

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{81 + 243c^2 - 9s\gamma - 27cs\gamma + 3s^2\gamma^2 - 18s\gamma \cos \theta - 54cs\gamma \cos \theta + 4s^2\gamma^2 \cos \theta + 2s^2\gamma^2 \cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{81 + 243c^2 - 9s\gamma - 27cs\gamma + 3s^2\gamma^2 - 18s\gamma \cos \theta - 54cs\gamma \cos \theta + 4s^2\gamma^2 \cos \theta + 2s^2\gamma^2 \cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{324 - 18s\gamma + s^2\gamma^2 - 36\sqrt{3}s\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{324 - 18s\gamma + s^2\gamma^2 - 36\sqrt{3}s\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{81 + 243c^2 + 9s\gamma - 27cs\gamma + s^2\gamma^2 - 18\sqrt{3}s\gamma \sin \theta - 18\sqrt{3}cs\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{81 + 243c^2 + 9s\gamma - 27cs\gamma + s^2\gamma^2 - 18\sqrt{3}s\gamma \sin \theta - 18\sqrt{3}cs\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}) \right]. \quad (\text{B23}) \end{aligned}$$

Solving the inequalities, the results for $\theta \in (\frac{\pi}{3}, \frac{\pi}{2}]$ will be

$$\begin{aligned} t_1 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{81 + 243c^2 - 9s\gamma - 27cs\gamma + 3s^2\gamma^2 - 18s\gamma \cos \theta - 54cs\gamma \cos \theta + 4s^2\gamma^2 \cos \theta + 2s^2\gamma^2 \cos 2\theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{81 + 243c^2 - 9s\gamma - 27cs\gamma + 3s^2\gamma^2 - 18s\gamma \cos \theta - 54cs\gamma \cos \theta + 4s^2\gamma^2 \cos \theta + 2s^2\gamma^2 \cos 2\theta}) \right], \\ t_2 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{81 + 243c^2 + 9s\gamma - 27cs\gamma + s^2\gamma^2 - 18\sqrt{3}s\gamma \sin \theta - 18\sqrt{3}cs\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{81 + 243c^2 + 9s\gamma - 27cs\gamma + s^2\gamma^2 - 18\sqrt{3}s\gamma \sin \theta - 18\sqrt{3}cs\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}) \right], \\ t_3 &= \min \left[\frac{1}{36} (18 - 3s - \sqrt{324 - 18s\gamma + s^2\gamma^2 - 36\sqrt{3}s\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}), \right. \\ &\quad \left. \frac{1}{36} (18 - 3s + \sqrt{324 - 18s\gamma + s^2\gamma^2 - 36\sqrt{3}s\gamma \sin \theta + 4s^2\gamma^2 \sin^2 \theta}) \right]. \quad (\text{B24}) \end{aligned}$$

Now we have constructed the operator inequality for a given value of θ . To get the lower bound on the average fidelity of the measurement \mathcal{F}' as given in Eq. (B6), we set $s = \frac{54}{7\gamma-1}$ and the subsequent minimization gives $t = \frac{9-6\gamma}{2-14\gamma}$, which provides a complete generalized analysis of robustness based on \mathcal{A}_C . Therefore, the average fidelity of measurements for Charlie is the same as Eq. (58) and describes the variation of the average fidelity with success probability \mathcal{A}_C of Charlie for a particular choice of η_B . The lower bound of the average fidelity $\mathcal{F}'(\mathcal{A}) = 1$ is attained only when $\mathcal{A}_C = \Omega_C$.

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