Incoherent Gaussian equivalence of *m*-mode Gaussian states

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Necessary and sufficient conditions for arbitrary multimode (pure or mixed) Gaussian states to be equivalent under incoherent Gaussian operations are derived. We show that two Gaussian states are incoherent equivalence if and only if they are related by incoherent unitaries. This builds the counterpart of the celebrated result that two pure entangled states are equivalent under local operations and classical communication (LOCC) if and only if they are related by local unitaries. Furthermore, incoherent equivalence of Gaussian states is equivalent to frozen coherence [Phys. Rev. Lett. **114**, 210401 (2015)]. Basing this as foundation, we find all measures of coherence are frozen for an initial Gaussian state under strongly incoherent Gaussian operations if and only if the relative entropy measure of coherence is frozen for the state. This gives an entropy-based dynamical condition in which the coherence of an open quantum system is totally unaffected by noise.

I. INTRODUCTION

Quantum coherence, being at the heart of interference phenomena, stands as one of intrinsic features of quantum mechanics that induces a number of intriguing phenomena in quantum optics [1–4] and quantum information [5]. It constitutes a powerful resource for quantum computing [6], cryptography [7], information processing [8–10], thermodynamics [11], metrology [12], and quantum biology [13].

The first framework for understanding quantum coherence is quantum optics, which requires quantum states in a continuous-variable system. Gaussian states that are processed in most optical experiments have arisen to a privileged position in the context of quantum computation [14–16] over different physical platforms such as optical [17], trapped ions [18,19], atomic ensembles [20,21], and hybrid systems [22]. Recently, there is a growing interest in building the resource theory of coherence of Gaussian states [23–27]. Coherence effects of Gaussian states have been addressed in different branches of quantum information, for example, unitary process in quantum thermodynamics [28], coherence as a resource in charging quantum batteries [29], and dynamical behavior of quantum coherence of a displaced squeezed thermal state [30]. In view of resource theory [31–33], one fundamental issue is the classification of coherent Gaussian states. A natural way of defining equivalence relations in the set of coherent Gaussian states is that equivalent states contain the same amount of coherence. Since the primary tool for analyzing coherent Gaussian states is incoherent Gaussian operations, which are powerful to describe noise and decoherence of optical systems [23,34–36], the monotonicity of coherence under incoherent Gaussian operations allows us to identify any two states that can be transformed from

each other with certainty by incoherent Gaussian operations. Clearly, this criterion is interesting in quantum information theory, since equivalent states are indistinguishable for exactly the same tasks. The aim of this article is to characterize the equivalence of coherent Gaussian states under incoherent Gaussian operations.

For entangled states, a beautiful result is that two pure states can be transformed from each other with certainty by local operations and classical communication (LOCC) if and only if they are related by local unitaries [31,32]. The equivalence of entangled Gaussian states under local unitaries is discussed for bipartite settings [33] and for more parties [37–41]. In [39,40], standard forms of generic *m*-mode pure and mixed states are introduced. For generic pure Gaussian states, it is shown that Gaussian local unitaries equivalence classes are classified by three positive numbers related to local purities. The case of pure 3-mode states has been discussed in detail in [42]. We will classify coherent Gaussian states in terms of incoherent unitaries and the relative entropy measure of coherence.

The paper is organized as follows. In Sec. II we present an explicit description of coherent Gaussian states and incoherent Gaussian operation. In Sec. III we present the results for incoherent Gaussian equivalence of the *m*-mode Gaussian states that we addressed. Section IV is a summary of our findings. The Appendix gives the proof of our results.

II. BACKGROUND AND NOTATION

Here we provide some background on bosonic Gaussian states and Gaussian operations (see [14] for a review). Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with fixed Fock basis $\{|n\rangle\}_{n=0}^{+\infty}$. When we consider the *m*-mode continuous-variable systems $\mathcal{H}^{\otimes m}$, we adopt $(\{|n\rangle\}_{n=0}^{+\infty})^{\otimes m}$ as its reference basis. For a quantum state $\rho \in \mathcal{H}^{\otimes m}$, the characteristic function of ρ

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is defined as

$$\mathcal{X}_\rho(\lambda) = \text{tr}(\rho D(\lambda))$$
$$]D(\lambda) = \otimes_{i=1}^m D(\lambda_i)$$
$$D(\lambda_i) = e^{(\lambda_i \widehat{a}_i^\dagger - \overline{\lambda}_i \widehat{a}_i)},$$

where $\widehat{a}_i$ and $\widehat{a}_i^\dagger$ are the annihilation operator and the creation operator in mode i, $\lambda = (\lambda_1, \ldots, \lambda_m)^t$, $\overline{\lambda}_i$ denotes the complex conjugate of λ_i. Gaussian states are those states for which $\mathcal{X}_\rho(\lambda)$ is a Gaussian function of the phase space, i.e.,

$$\mathcal{X}_\rho(\lambda) = \exp^{-\frac{1}{4}\overrightarrow{r}\Omega V \Omega^t \overrightarrow{r}^t - i(\Omega d)^t \overrightarrow{r}^t},$$

where $\overrightarrow{r} = (\lambda_{1x}, \lambda_{1y}, \ldots, \lambda_{mx}, \lambda_{my})$, $\lambda_{jx}, \lambda_{jy}$ are the real part and imaginary part of λ_j ($j = 1, 2, \ldots, m$), V is a $2m \times 2m$ real Hermitian matrix, which is called the covariance matrix, satisfying the uncertainty relation $V + i\Omega \geqslant 0$, and $d \in \mathbb{R}^{2m}$ is called the mean value, $\Omega = \bigoplus_{k=1}^m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [14]. Note that $\det V \geqslant 1$ and $\det V = 1$ if and only if ρ is pure. It is clear that (V, d) can describe the Gaussian state ρ completely. So ρ can be usually written in $\rho(V, d)$. Every Gaussian operation is a completely positive trace-preserving mapping that takes Gaussian states to Gaussian states. It is described by $(T, N, \overline{d})$, it performs on $\rho(V, d)$, and obtains the Gaussian state with mean value and covariance matrix as follows:

$$d \mapsto Td + \overline{d}, \quad V \mapsto TVT^t + N,$$

where $\overline{d} \in \mathbb{R}^{2m}$, T, N are $2m \times 2m$ real matrices with $N + i\Omega - iT\Omega T^t \geqslant 0$ (complete positivity condition) [14,43].

Inspired by the idea of discrete-variable systems [44], the framework for quantifying coherence of Gaussian states has been built in [23,34]. The incoherent Gaussian states are defined as diagonal Gaussian states in the Fock basis. Every incoherent Gaussian state has the form $\otimes_{i=1}^m \rho_{th}^{A_i}(\overline{n}_i)$; here $\rho_{th}^{A_i}(\overline{n}_i) = \sum_{n=0}^{+\infty} \frac{\overline{n}_i^n}{(\overline{n}_i+1)^{n+1}}|n\rangle\langle n|$ is the incoherent state of the ith-mode A_i. The set of incoherent Gaussian states will be labeled by $\mathcal{I}$. A Gaussian operation is incoherent if it maps incoherent Gaussian states into incoherent Gaussian states. In fact, a Gaussian operation $\Phi(T, N, \overline{d})$ is incoherent if and only if

$$\overline{d} = 0,$$
$$T = \{t_j O_j\}_{j=1}^m \in \mathcal{T}_{2m},$$
$$N = \oplus_{j=1}^m \omega_j I_2,$$
$$\omega_j \geqslant |1 - \sum_{k,r(k)=j} t_k^2 \det O_k|, \quad \forall j,$$

where $t_j, \omega_j \in \mathbb{R}$, O_j is a 2×2 real orthogonal matrix $(O_j O_j^t = I_2)$, $\mathcal{T}_{2m}$ denotes the set of $2m \times 2m$ real matrices such that, for any $T \in \mathcal{T}_{2m}$, the $(2j - 1, 2j)$ two columns of T have just one 2×2 real matrix $t_j O_j$ located in $[2r(j) - 1, 2r(j)]$ rows for $\forall j$, $r(j) \in \{k\}_{k=1}^m$, and other elements are all zero. For Gaussian state $\rho(V, d)$, it performs on $\rho(V, d)$ and obtains a Gaussian state with mean value and covariance matrix as follows:

$$d \mapsto Td, \quad V \mapsto TVT^t + N.$$

Specifically, unitary operators of $\mathcal{T}_{2m}$ are called incoherent unitaries in this paper.

Based on the definition of incoherent Gaussian states and incoherent Gaussian operations (IGOs), any proper coherence measure C is a non-negative function and must satisfy the following conditions [23]:

(C1) $C(\rho) = 0$ for all $\rho \in \mathcal{I}$,

(C2) Monotonicity under all incoherent Gaussian operations (IGOs) Φ:

$$C(\Phi(\rho)) \leqslant C(\rho),$$

(C3) Nonincreasing under mixing of Gaussian states:

$$C\left(\sum_j p_j \rho_j\right) \leqslant \sum_j p_j C(\rho_j),$$

for any set of Gaussian states $\{\rho_j\}$ and any $p_j \geqslant 0$ with $\sum_j p_j = 1$. Note that the set of Gaussian states is not convex; thus ρ_j and $\sum_j p_j \rho_j$ in (C3) should be all Gaussian. Based on the definition of the coherence measure, the relative entropy measure has been provided by

$$C_R(\rho) = \inf_{\delta \in \mathcal{I}} S(\rho||\delta),$$

$S(\rho||\delta) = \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \delta)$ being the relative entropy. Furthermore,

$$C_R(\rho) = -S(\rho) + \sum_{i=1}^m [(\overline{n}_i + 1)\log_2(\overline{n}_i + 1) - n_i \log_2 \overline{n}_i],$$

$S(\rho) = \sum_{i=1}^m [\frac{v_i-1}{2}\log_2 \frac{v_i-1}{2} - \frac{v_i+1}{2}\log_2 \frac{v_i+1}{2}]$, $\overline{n}_i = \frac{1}{4}[\text{tr}(V^{(i)}) + \|d^{(i)}\|^2 - 2]$, where $S(\rho)$ is the von Neumann entropy of ρ, $\{v_i\}_{i=1}^m$ are symplectic eigenvalues of V [45], and $\overline{n}_i$ is determined by ith-mode covariance matrix $V^{(i)}$ and mean value $d^{(i)}$, and $\|d^{(i)}\|$ is the Euclidean norm of $d^{(i)}$ [23].

For coherent Gaussian states $\rho(V, d)$, $\sigma(V', d')$, we say they are IGO equivalent if there exist IGOs Φ and Ψ satisfying $\Phi(\rho(V, d)) = \sigma(V', d')$ and $\Psi(\sigma(V', d')) = \rho(V, d)$. We denote this by $\rho(V, d) \overset{IGO}{\sim} \sigma(V', d')$. We will provide a structural characterization of equivalence for coherent Gaussian states by incoherent unitary operations and the coherence measure based on the relative entropy.

III. MAIN RESULTS

Before giving our main results, we need to introduce the concept of strictly incoherent Gaussian operations which originates from the definition of strictly incoherent operations of discrete-variable systems [46,47]. It plays a key role in classifying the IGO equivalence of Gaussian states.

Definition 3.1. An incoherent Gaussian operation $\Phi(T, N)$ is called strictly incoherent if each $(2i - 1, 2i)$ row and each $(2j - 1, 2j)$ column of T has just one element of $\{t_j O_j\}_{j=1}^m$, $\omega_j \geqslant |1 - t_k^2 \det O_k|$ $(r(k) = j)$. Throughout the paper, we write mean values and covariance matrices of m-mode Gaussian states in terms of two-dimensional sub-blocks as

$$d = (d_1, d_2, \ldots, d_m)^t,$$
$$V = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1m} \\ V_{12}^t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{m-1,m} \\ V_{1m}^t & \cdots & V_{m-1,m}^t & V_{mm} \end{pmatrix}.$$

V_{ii} and d_i are the ith-mode covariance matrix and mean value, respectively. It is known that each subvector d_i and off-diagonal block V_{ij} become 0 for every incoherent Gaussian state [23]. The nonzero elements of off-diagonal blocks of V and subvectors of d reveal coherence of Gaussian states. In order to classify coherent Gaussian states, we assume that every row of V has at least one nonzero off-diagonal block.

Now we are in a position to give our main results.

Theorem 3.2. Assume $m \geqslant 2$ and every row of V and V' have at least one nonzero off-diagonal block, then the following statements are equivalent:

(i) $\rho(V, d) \overset{\text{IGO}}{\sim} \sigma(V', d')$;

(ii) There exists an incoherent Gaussian unitary operator U such that $UVU^t = V'$, $Ud = d'$, where $U = P_\pi \otimes I_2 \left(\begin{smallmatrix} O_1 & & 0 \\ & \ddots & \\ 0 & & O_m \end{smallmatrix} \right)$, P_π is the permutation matrix corresponding to a permutation π of $\{1, 2, \dots, m\}$, O_i ($1 \leqslant i \leqslant m$) are 2×2 real orthogonal matrices with det $O_i = 1$, and I_2 is the 2×2 unit matrix;

(iii) There is a strictly incoherent Gaussian operation Φ satisfying $\Phi(\rho(V, d)) = \sigma(V', d')$ and $C_R(\rho(V, d)) = C_R(\sigma(V', d'))$.

For the one-mode case, our result reads as follows.

Theorem 3.3. Assume $d \neq 0$ or $V \neq \lambda I_2$ [$\rho(V, d)$ is coherent], I_2 is the 2×2 unit matrix, then the following statements are equivalent:

(i) $\rho(V, d) \overset{\text{IGO}}{\sim} \sigma(V', d')$;

(ii) There exists an incoherent Gaussian unitary operator U such that $UVU^t = V'$, $Ud = d'$, here $U = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$ for some $\theta \in \mathbb{R}$;

(iii) There is an incoherent Gaussian operation Φ satisfying $\Phi(\rho(V, d)) = \sigma(V', d')$ and $C_R(\rho(V, d)) = C_R(\sigma(V', d'))$.

Theorems 3.2 and 3.3 show the existence of measure-independent freezing of coherence. Quantum coherence is a useful physical resource, but coherence of a Gaussian state is decreasing under IGOs [$C(\Phi(\rho)) \leqslant C(\rho)$]. The loss of coherence may weaken the abilities of a Gaussian state to fulfill certain quantum information processing tasks. An interesting question is to study when the coherence of an open system is frozen [48], i.e., when $C(\Phi(\rho)) = C(\rho)$ for a coherent Gaussian state ρ and an IGO Φ. However, some coherence measures being frozen do not imply other coherence measures being frozen too, since different coherence measure results in different orderings of coherence in general [49]. Freezing of coherence is dependent on the coherence measures in general. By Theorems 3.2 and 3.3, we find all measures of coherence are frozen for an input Gaussian state if and only if the relative entropy measure of coherence is frozen for the state. A parallel result in discrete-variable systems is that all measures of coherence are frozen for an initial state in a strictly incoherent channel if and only if the relative entropy of coherence is frozen for the state [50].

Theorems 3.2 and 3.3 present frozen phenomenon of coherence and entanglement simultaneously [51]. An important class of two-mode Gaussian states has covariance matrices in standard form

$$V = \begin{pmatrix} aI_2 & C \\ C & bI_2 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

$a \geqslant 1, b \geqslant 1, c, d \in \mathbb{R}$ [14,52,53]. Any Gaussian state can be transformed to the Gaussian state with the covariance matrix in standard form by local linear unitary Bogoliubov operations [52]. Let $d = (d_1, d_2)$, $\rho(V, d)^\sharp$ denote the set of all Gaussian states which are incoherent equivalent with $\rho(V, d)$, by Theorem 3.2:

$$\rho(V, d)^\sharp = \{\sigma[V_1, (O_1 d_1, O_2 d_2)^t]\} \cup \{\delta[V_2, (O_1 d_2, O_2 d_1)^t]\},$$

$V_1 = \left(\begin{smallmatrix} aI_2 & O_1 C O_2^\dagger \\ O_2 C O_1^\dagger & bI_2 \end{smallmatrix} \right)$, $V_2 = \left(\begin{smallmatrix} bI_2 & O_1 C O_2^\dagger \\ O_2 C O_1^\dagger & aI_2 \end{smallmatrix} \right)$, $O_i = \left(\begin{smallmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{smallmatrix} \right)$, $\forall \theta_i \in \mathbb{R}$, $i = 1, 2$. Note that any transformation of

$$V \mapsto \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} V \begin{pmatrix} O_1^t & 0 \\ 0 & O_2^t \end{pmatrix} = V_1$$

is a special kind of local linear unitary Bogoliubov operation [52], which tells us that the amount of coherence of $\{\sigma[V_1, (O_1 d_1, O_2 d_2)^t]\}$ and the amount of entanglement of Gaussian states of $\{\sigma[V_1, (O_1 d_1, O_2 d_2)^t]\}$ are equal, respectively. In addition, by a direct computation, the symplectic spectrum $\{v_+, v_-\}$ of V_i ($i = 1, 2$) is given by

$$v_\pm = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4 \det V}}{2}},$$

$\Delta = a^2 + b^2 + 2 \det C$ (one can also refer to [14]). This implies that the symplectic spectrum of the partial transposed Gaussian states of $\rho(V, d)^\sharp$ are the same [53]. Note that the entanglement measure of formation is a function of the less symplectic spectrum of partial transposed Gaussian states if $a = b$ [14,42], and Gaussian states of $\rho(V, d)^\sharp$ have the same amount of entanglement under Gaussian measure of formation. (The detailed definition of Gaussian entanglement of formation can be found in Appendix A for the convenience of readers.) This shows elements of $\rho(V, d)^\sharp$ have the same amount of entanglement and coherence in the case $a = b$, respectively. That is, coherence and entanglement of $\rho(V, d)^\sharp$ are frozen simultaneously. It hints that there might have been a closed inner link between the measure of coherence and entanglement of formation.

Theorems 3.2 and 3.3 are also key to characterizing the incoherent equivalence class of Gaussian states. It is clear that equivalence class of any pure coherent Gaussian state consists of some pure Gaussian states. The most general pure Gaussian state $|\psi\rangle$ of one mode is a displaced squeezed state obtained by the combined action of the Weyl displacement operator

$$\widehat{D}(\alpha) = e^{\alpha \widehat{a}^\dagger - \overline{\alpha} \widehat{a}}, \quad \alpha \in \mathbb{C},$$

and the squeezing operator

$$\widehat{S}(\beta) = e^{\frac{1}{2}[\beta \widehat{a}^{\dagger 2} - \overline{\beta} \widehat{a}^2]}, \quad \beta \in \mathbb{C},$$

on the vacuum state $|0\rangle$ [54]:

$$|\psi_{\alpha, \beta}\rangle = \widehat{D}(\alpha) \widehat{S}(\beta) |0\rangle.$$

The mean value and covariance matrix of $|\psi_{\alpha,\beta}\rangle$ are

$$2[\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)],$$

$$\begin{pmatrix} \mathrm{ch}(2|\beta|) + \cos\theta\,\mathrm{sh}(2|\beta|) & \sin\theta\,\mathrm{sh}(2|\beta|) \\ \sin\theta\,\mathrm{sh}(2|\beta|) & \mathrm{ch}(2|\beta|) - \cos\theta\,\mathrm{sh}(2|\beta|) \end{pmatrix},$$

where $\beta = |\beta|e^{i\theta}$, $\mathrm{ch}(x) = \frac{e^x + e^{-x}}{2}$, $\mathrm{sh}(x) = \frac{e^x - e^{-x}}{2}$ are hyperbolic functions. Denote

$$\alpha = |\alpha|e^{i\gamma}, \quad \alpha' = |\alpha'|e^{i\gamma'}, \quad \beta' = |\beta'|e^{i\theta'},$$

by Theorem 3.3, a direct computation shows that

$$|\psi_{\alpha,\beta}\rangle \overset{\mathrm{IGO}}{\sim} |\psi_{\alpha',\beta'}\rangle$$
$$\Updownarrow$$
$$|\alpha| = |\alpha'|, |\beta| = |\beta'|, \theta' - \theta = 2(\gamma' - \gamma) - 2k\pi$$

for some integer k. It reveals explicitly the geometric feature of the incoherent equivalence class of displaced squeezed states [55, Fig. 1].

IV. SUMMARY

In this work, necessary and sufficient conditions for arbitrary multimode Gaussian states to be equivalent under incoherent Gaussian operations are obtained. It is shown that two coherent Gaussian states are incoherent equivalence if and only if they are related by incoherent unitaries, if and only if coherence of Gaussian states are frozen [48] under relative entropy measure. Our results first provide an operational description of equivalent Gaussian states and so allow us to formulate a simple criterion to decide whether Gaussian states are equivalent. Second, our results imply that all measures of coherence are frozen for an initial Gaussian state if and only if the relative entropy measure of coherence is frozen for the state. So this provides an entropy-based dynamical condition in which the coherence of an open quantum system is totally unaffected by noise.

Our results raise one interesting question naturally. How about the equivalence of coherent Gaussian states under stochastic incoherent Gaussian operations. The study may produce finitely many kinds of coherence and open a new door for deterministic Gaussian conversion protocols.

APPENDIX A: GAUSSIAN ENTANGLEMENT OF FORMATION

For pure $N \times M$ Gaussian states $|\phi\rangle$, the Gaussian entanglement of formation is defined as the von Neumann entropy of the reduced states $\rho_{A,B} = \mathrm{Tr}_{B,A}(|\phi\rangle\langle\phi|)$, i.e., $E_F(|\phi\rangle) = S(\rho_A) = S(\rho_B)$. The Gaussian entanglement of formation of mixed states is defined as an infimum,

$$E_F(\rho) = \inf\left\{ \left. \sum_k p_k E_F(\phi_k) \right| \rho = \sum_k p_k |\phi_k\rangle\langle\phi_k| \right\},$$

over all possibly convex decompositions (possibly continuous) of the state into pure states [14]. In general, this optimization is difficult to carry out. We only know the solution for two-mode symmetric Gaussian states whose covariance matrix is symmetric under the permutation of the two modes, i.e., two diagonal elements are equal when we write its covariance matrix in the block form, where $E_F(\rho)$ is a function of the less symplectic spectrum of partial transposed Gaussian states.

APPENDIX B: PROOF OF MAIN RESULTS

Proofs of all results in this paper are given in Appendix B.

Proof of Theorem 3.2. By the definition of strictly incoherent Gaussian operations and monotonicity of coherence under all IGOs, it is clear that $(ii) \Rightarrow (iii)$. We will prove $(iii) \Rightarrow (i)$ and $(i) \Rightarrow (ii)$ in the following.

"$(iii) \Rightarrow (i)$": For a fixed orthonormal basis $(\{|n\rangle\}_{n=0}^{\infty})^{\otimes m}$ with positive integer $m > 1$, all Gaussian states are of the form $\rho = \rho(V, d) = \rho^{A_1 A_2 \cdots A_m}$, where A_i denotes the ith mode. Furthermore, if $\rho = (\rho_{n_1 n_2 \cdots n_m, l_1 l_2 \cdots l_m})$ with

$$\rho_{n_1 n_2 \cdots n_m, l_1 l_2 \cdots l_m} = \langle n_1 n_2 \cdots n_m | \rho | l_1 l_2 \cdots l_m \rangle,$$

$$\overline{n}_i = \sum_{n_i}\left(\sum_{n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_m} \rho_{n_1 n_2 \cdots n_m, n_1 n_2 \cdots n_m} \right) n_i, \tag{B1}$$

it is shown that $\mathcal{C}_R(\rho) = S(\rho||\delta)$ with some thermal state $\delta = \otimes_{i=1}^m \rho_{th}^{A_i}(\overline{n}_i)$ [23].

Combining the definition of relative entropy measure of coherence and monotonicity of relative entropy under completely positive and trace-preserving mappings [56], we have

$$\mathcal{C}_R(\Phi(\rho)) \leqslant S(\Phi(\rho)||\Phi(\delta)) \leqslant S(\rho||\delta) = \mathcal{C}_R(\rho).$$

From $\mathcal{C}_R(\Phi(\rho)) = \mathcal{C}_R(\rho)$, it follows that

$$\mathcal{C}_R(\Phi(\rho)) = S(\Phi(\rho)||\Phi(\delta)) = S(\rho||\delta) = \mathcal{C}_R(\rho). \tag{B2}$$

Recall that the strictly incoherent Gaussian operation Φ is described by a pair of operators (T, N). It performs on the Gaussian state $\rho(V, d)$ and gets the Gaussian state with the mean value and the covariance matrix as

$$d \mapsto Td, \quad V \mapsto TVT^t + N.$$

We now demonstrate that there exists an IGO Ψ such that $\Psi(\Phi(\delta)) = \delta$. Noting that $\Phi(\delta)$ is an incoherent Gaussian state, we can assume $\Phi(\delta) = \otimes_{i=1}^m \rho_{th}^{A_i}(\overline{k}_i)$. In order to use a Petz recovery map of Gaussian systems [57], we have to check that $\Phi(\delta)$ is faithful, i.e., $V_{\Phi(\delta)} + i\Omega > 0$, $V_{\Phi(\delta)}$ is the covariance matrix of $\Phi(\delta)$. By [23],

$$V_{\Phi(\delta)} = \bigoplus_{i=1}^m (2\overline{k}_i + 1)I_2.$$

It is easy to check that $V_{\Phi(\delta)} + i\Omega > 0 \Leftrightarrow k_i > 0$ for $i = 1, 2, \ldots, m$. Indeed, by (B1) and (B2), if $\overline{k}_i = 0$ for some

i, then

$$\sum_{k_1 k_2 \cdots k_{i-1} k_{i+1} \cdots k_m} \Phi(\rho)_{k_1 k_2 \cdots k_m, k_1 k_2 \cdots k_m} = 0, \quad \text{if } k_i \neq 0,$$
$$\sum_{k_1 k_2 \cdots k_{i-1} k_{i+1} \cdots k_m} \Phi(\rho)_{k_1 k_2 \cdots k_m, k_1 k_2 \cdots k_m} = 1, \quad \text{if } k_i = 0.$$

Thus $\Phi(\rho)$ has the form $|0^{A_i}\rangle\langle 0^{A_i}| \otimes \Phi(\rho)^{A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_m}$. This deduces the covariance matrix $V_{\Phi(\rho)}$ of $\Phi(\rho)$ has the form $I_2 \oplus V_2'$, which contradicts our assumption. Theorem 1 of [57] shows that the Petz recovery map Ψ is a Gaussian operation with the following action:

$$\Psi : \begin{cases} V \mapsto T_\Psi V T_\psi^t + N_\Psi \\ d \mapsto T_\Psi d + d', \end{cases}$$

where

$$T_\Psi = \sqrt{I + (V_\delta \Omega)^{-2}} V_\delta T^t (\sqrt{I + (\Omega V_{\Phi(\delta)})^{-2}})^{-1} V_{\Phi(\delta)}^{-1},$$

$$N_\Psi = V_\delta - T_\Psi V_{\Phi(\delta)} T_\Psi^t,$$

$$d' = d_\delta - T_\Psi (T d_\delta + 0).$$

From $V_\delta = \oplus_{i=1}^m (2\overline{n}_i + 1) I_2$, $V_{\Phi(\delta)} = \oplus_{i=1}^m (2\overline{k}_i + 1) I_2$, and $d_\delta = 0$, it follows that

$$T_\Psi = \left[\oplus_{i=1}^m \sqrt{(2\overline{n}_i + 1)^2 - 1} I_2 \right]$$
$$\times T^t \left(\oplus_{i=1}^m \frac{I_2}{\sqrt{(2\overline{k}_i + 1)^2 - 1}} \right),$$

$$d' = 0.$$

Note that Φ is a strictly incoherent Gaussian operation, and we may write

$$T = P_\pi \otimes I_2 \begin{pmatrix} t_1 O_1 & & 0 \\ & \ddots & \\ 0 & & t_m O_m \end{pmatrix},$$

where P_π is the permutation matrix corresponding to a permutation π of $\{1, 2, \ldots m\}$. One can check that

$$N_\Psi = V_{\Phi(\delta)} - T_\Psi V_\delta T_\Psi^t = \oplus w_i' I_2,$$

for some scalars w_i'. It is evident that $T_\Psi \in \mathcal{T}_{2m}$. The remaining inequalities in the definition of IGOs are from the complete positivity condition of Ψ. Thus T_Ψ, N_Ψ satisfy the conditions of IGOs, and so Ψ is a IGO with $\Psi(\Phi(\delta)) = \delta$.

Next we claim that $\Psi(\Phi(\rho)) = \rho$. A rotated Petz map Ψ^t, for $t \in \mathbb{R}$, is defined as $\Psi^t(\omega) = \delta^{it} \Psi[\Phi(\delta)^{-it} \omega \Phi(\delta)^{it}] \delta^{-it}$, with $\delta^{it} = \exp(it \log_2 \delta)$ being understood as a unitary evolution according to the Hamiltonian $\log_2 \delta$ [58]. Obviously, $\Psi^t = \Psi$ if $t = 0$. In [59], it is shown that

$$S(\rho || \delta) \geqslant S[\Phi(\rho) || \Phi(\delta)]$$
$$- \int_\mathbb{R} p(t) \log_2 \mathcal{F}(\rho, [\Psi^{\frac{t}{2}}(\Phi(\rho))] dt,$$

where $p(t) = \frac{\pi}{2}[\cosh(\pi t) + 1]^{-1}$ is a probability distribution parametrized by $t \in \mathbb{R}$, and $\mathcal{F}$ denotes the quantum fidelity, defined as $\mathcal{F}(\omega, \tau) := \| \sqrt{\omega} \sqrt{\tau} \|_1^2$ for quantum states ω and τ. From Eq. (B2), it follows that

$$\int_\mathbb{R} p(t) \log_2 \mathcal{F}(\rho, \Psi^{\frac{t}{2}}[\Phi(\rho)]) dt = 0,$$

$$\mathcal{F}(\rho, \Psi^{\frac{t}{2}}[\Phi(\rho)]) = 1,$$

$$\rho = \Psi^{\frac{t}{2}}(\Phi(\rho)), \quad \forall t \in \mathbb{R}$$

$$\rho = \Psi(\Phi(\rho)).$$

"$(i) \Rightarrow (ii)$": From (i), there are matrices $T, S \in \mathcal{T}_{2m}$, and N, N' such that

$$T V T^t + N = V', \quad S V' S^t + N' = V.$$

Hence

$$S T V T^t S^t + S N S^t + N' = V. \tag{B3}$$

By the property of $\mathcal{T}_{2m}$, there are permutations π, π' such that $P_\pi \otimes I_2 T = (T_{ij})$, $P_{\pi'} \otimes I_2 S = (S_{ij})$, where (T_{ij}) and (S_{ij}) are upper triangular blocks with the form

$$T_{ij} = \delta(i, f(j)) t_i O_i, \quad S_{ij} = \delta(i, g(j)) s_i O_i',$$

where f, g are functions from $\{1, \ldots, m\}$ to $\{1, \ldots, m\}$ with $f(j), g(j) \leqslant j$,

$$\delta(i, j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

We need only show that $f(i) = g(i) = i$, $|t_i| = |s_i| = 1$, and $\det O_i = \det O_i' = 1$ for $i = 1, \ldots, m$. Without loss of generality, we assume $T = (T_{ij})$, $S = (S_{ij})$. Since there is at least one nonzero off-diagonal element in every column of V and V', T, S are invertible. Hence $f(i) = g(i) = i$, $i \in \{1, \ldots, m\}$, and so

$$T = \begin{pmatrix} t_1 O_1 & & 0 \\ & \ddots & \\ 0 & & t_m O_m \end{pmatrix}, \quad S = \begin{pmatrix} s_1 O_1' & & 0 \\ & \ddots & \\ 0 & & s_m O_m' \end{pmatrix},$$

$$N = \begin{pmatrix} \omega_1 I_2 & & 0 \\ & \ddots & \\ 0 & & \omega_m I_2 \end{pmatrix}, \quad N' = \begin{pmatrix} \omega_1' I_2 & & 0 \\ & \ddots & \\ 0 & & \omega_m' I_2 \end{pmatrix}.$$

We will show that $w_i = 0$, $\det O_i = 1$, and $|t_i| = 1$.

Computing the diagonal blocks of (B3), we have, for $1 \leqslant i \leqslant m$,

$$s_i^2 t_i^2 O_i' O_i V_{ii} O_i^t O_i'^t + w_i s_i^2 + w_i' = V_{ii}. \tag{B4}$$

From the spectral theorem of positive operators, it follows that $s_i^2 t_i^2 \leqslant 1$. Comparing blocks of (i, j) $(1 \leqslant i < j < m)$ positions in (B3), we have

$$s_i s_j t_i t_j O_i' O_i V_{ij} O_j^t O_j'^t = V_{ij}. \tag{B5}$$

Note that for each i, there is j_i such that $V_{i, j_i} \neq 0$. Computing the trace norm of (B5), we have $|s_i s_{j_i} t_i t_{j_i}| = 1$ and $s_i^2 t_i^2 = 1$. From (B4), we get $\omega_i = \omega_i' = 0$ $(i = 1, 2, \ldots, m)$. Note that $\omega_i \geqslant |1 - t_i^2 \det O_i|$, and hence $\det O_i = 1$, $|t_i| = 1$, $(i = 1, 2, \ldots, m)$, as desired.

Proof of Theorem 3.3. By the definition of IGOs, it is clear that $(ii) \Rightarrow (iii)$. The proof of $(iii) \Rightarrow (i)$ is almost verbatim from $(iii) \Rightarrow (i)$ of Theorem 3.2. We need only show $(i) \Rightarrow (ii)$. We only treat $V \neq \lambda I_2$ and the other case can be treated similarly. Assume that there exists an IGO Φ such that $\Phi[\rho(V_1, d_1)] = \sigma(V_2, d_2)$. By the definition of IGO, we can obtain

$$t^2 O V O^t + \omega I = V', \quad t O d_1 = d_2,$$

where O is a real orthogonal matrix, $\omega, t \in \mathbb{R}$, $\omega \geqslant |1 - t^2 \det O|$. Similarly, there exists a real orthogonal matrix O' and real numbers ω', s with

$$s^2 t^2 O' O V O^t O'^t + \omega s^2 I_2 + \omega' I_2 = V, \tag{B6}$$

$\omega' \geqslant |1 - s^2 \det O'|$. Suppose eigenvalues of V are λ_1 and λ_2, it is clear that $\lambda_1 \neq \lambda_2$. By the spectral mapping theorem of positive operators, from (B6) we have

$$\{s^2 t^2 \lambda_1 + \omega s^2 + \omega', \, s^2 t^2 \lambda_2 + \omega s^2 + \omega'\} = \{\lambda_1, \, \lambda_2\}.$$

If $\begin{cases} s^2 t^2 \lambda_1 + \omega s^2 + \omega' = \lambda_2 \\ s^2 t^2 \lambda_2 + \omega s^2 + \omega' = \lambda_1 \end{cases}$, then $s^2 t^2 (\lambda_1 - \lambda_2) = \lambda_2 - \lambda_1$. Thus $ts = 0$ and so $V = \lambda I_2$ from (B6). This is a contradiction. Therefore

$$\begin{cases} s^2 t^2 \lambda_1 + \omega s^2 + \omega' = \lambda_1 \\ s^2 t^2 \lambda_2 + \omega s^2 + \omega' = \lambda_2 \end{cases}$$

and $|st| = 1$. This implies $\omega = \omega' = 0$. Note that $\omega \geqslant |1 - t^2 \det O|$, and hence $|t| = 1$ and $\det O = 1$ as desired.

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