Sequential generalized measurements: Asymptotics, typicality, and emergent projective measurements

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The relation between projective measurements and generalized quantum measurements is a fundamental problem in quantum physics, and clarifying this issue is also important to quantum technologies. While it has been intuitively known that projective measurements can be constructed from sequential generalized or weak measurements, there is still lack of a proof of this hypothesis in general cases. Here we prove it from the perspective of quantum channels. We show that projective measurements naturally arise from sequential generalized measurements in the asymptotic limit, when the measurement operators are normal and commuting with each other. Specifically, a selective projective measurement arises from a set of typical sequences of selective generalized measurements. We also provide an explicit scheme to construct projective measurements of a quantum system with sequential generalized measurements. Remarkably, a single ancilla qubit is sufficient to mediate sequential generalized measurements for constructing arbitrary projective measurements of a generic system, which can have applications in readout, initialization, and feedback control of a quantum system. As an example, we present a protocol to measure the modular excitation numbers of a bosonic mode with an ancilla qubit.

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Quantum measurements retrieve classical information from quantum states [1,2] and are particularly important to quantum technologies [3]. The traditional description of measurement in quantum mechanics is through projective measurements (PMs) of observables represented by Hermitian operators [4]. Measuring an observable corresponds to statistically projecting the quantum state to one of the orthogonal eigenspaces of this observable. PMs appear most commonly in quantum foundation and quantum information theory and are widely useful for initialization and readout of quantum systems in quantum technologies [5–11].

There exist more general quantum measurements, called generalized measurements (GMs), described by positive-operator-valued measures (POVMs) [12–16]. GMs can outperform PMs in many tasks in quantum technologies, such as quantum tomography [17] and quantum state discrimination or estimation [18,19]. Moreover, continuous or sequential GMs can be exploited for monitoring and maneuvering quantum evolutions [20–29]. In particular, weak measurements can extract partial information without projections and therefore can help realize optimal qubit tomography [30], reconcile measurement incompatibility [31,32], and extract arbitrary bath correlations [33–35].

Substantial efforts have been devoted to illustrating the relation between PMs and GMs. A celebrated result is Naimark's theorem [4], implying that any GM can be implemented as a PM on an enlarged Hilbert space. The measurement statistics of GMs can also be simulated by PMs with classical randomness or postselection [36–38]. In the opposite direction, it has been argued that sequential GMs can generate PMs by analyzing the gradual state collapse [39–42], the statistics of measurement results [43–45], and saturation of knowledge [46]. However, to our knowledge the general relation between PMs and sequential GMs still remains elusive.

In this paper we prove that PMs can emerge from sequential GMs in the asymptotic limit, when the measurement operators are normal and commuting with each other. The proof is based on the observation that projections are fixed points of the quantum channels for such GMs. Moreover, from the theory of classical typicality we find that different selective PMs arise from different sets of typical sequences of selective GMs. These results completely characterize the structures of sequential GMs with normal and commuting measurement operators. We further present a general scheme to realize such GMs with a single qubit ancilla and show that sequential GMs can simulate arbitrary PMs for arbitrary finite-dimensional quantum systems. The scheme will be useful for initialization, readout, and feedback control of a quantum system. As

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an example, we provide a protocol to measure the modular excitation numbers of an infinite-dimensional bosonic mode with an ancilla qubit, which are the error syndromes of several bosonic quantum error correction codes.

I. GMs AND QUANTUM CHANNELS

For a d-level quantum system, an r-outcome POVM is a set of positive semidefinite operators acting on its Hilbert space that sum to the identity, $\sum_{\alpha=1}^{r} M_\alpha^\dagger M_\alpha = \mathbb{I}$, with $\dagger$ denoting the Hermitian conjugation. The αth outcome is obtained with probability $\mathrm{Tr}(M_\alpha^\dagger M_\alpha \rho)$, with ρ being the density matrix. A GM is characterized by a POVM and the set of measurement operators $\{M_\alpha\}_{\alpha=1}^{r}$. The state change induced by a GM is described by a completely positive and trace-preserving (CPTP) map or a quantum channel [12,47],

$$\Phi(\rho) = \sum_{\alpha=1}^{r} \mathcal{M}_\alpha \rho = \sum_{\alpha=1}^{r} M_\alpha \rho M_\alpha^\dagger, \tag{1}$$

where $\mathcal{M}_\alpha = M_\alpha(\cdot)M_\alpha^\dagger$ is a superoperator acting on the operator space of the quantum system, representing a trace-nonincreasing and completely positive (CP) map corresponding to the αth outcome. The set of superoperators $\{\mathcal{M}_\alpha\}_{\alpha=1}^{r}$ forms a quantum instrument [48,49], which belongs to a class of quantum channels that can include both classical and quantum outputs. Hereafter, we define a nonselective GM as the channel $\Phi = \sum_{\alpha=1}^{r} \mathcal{M}_\alpha$ and a selective GM as a specific CP map $\mathcal{M}_\alpha$.

Quantum channels have natural matrix representations acting on the Hilbert-Schmidt (HS) space of the quantum system [50] (see Appendix A). While the density matrices are operators on the Hilbert space with an orthonormal basis $\{|a\rangle\}_{a=1}^{d}$, they are turned into vectors in the HS space, i.e., $\rho = \sum_{a,b=1}^{d} \rho_{ab}|a\rangle\langle b| \leftrightarrow |\rho\rangle\rangle = \sum_{a,b=1}^{d} \rho_{ab}|ab\rangle\rangle$, such that $X\rho Y \leftrightarrow X \otimes Y^T|\rho\rangle\rangle$, with X, Y being operators on the Hilbert space and Y^T being the transpose of Y. The inner product in the HS space is defined as $\langle\langle\sigma|\rho\rangle\rangle = \mathrm{Tr}[\sigma^\dagger\rho]$ with σ being another operator on the Hilbert space. The quantum channel is a linear operator on the HS space,

$$\hat{\Phi}|\rho\rangle\rangle = \sum_{\alpha=1}^{r} \hat{\mathcal{M}}_\alpha|\rho\rangle\rangle = \sum_{\alpha=1}^{r} M_\alpha \otimes M_\alpha^*|\rho\rangle\rangle, \tag{2}$$

where M_α^* is the complex conjugate of M_α. Note that we add hats for operators acting on the HS space to distinguish them from the corresponding superoperators acting on the operator space of the quantum system. With the HS space, the probability to get the αth outcome is $\langle\langle\mathbb{I}|\hat{\mathcal{M}}_\alpha|\rho\rangle\rangle = \mathrm{Tr}(M_\alpha\rho M_\alpha^\dagger)$.

II. GMs WITH NORMAL AND COMMUTING MEASUREMENT OPERATORS

We assume that the set of measurement operators $\{M_\alpha\}_{\alpha=1}^{r}$ are normal and commuting with each other, i.e., $[M_\alpha, M_\alpha^\dagger] = [M_\alpha, M_\beta] = 0$ for all integers $\alpha, \beta \in [1, r]$, such that $\{M_\alpha\}_{\alpha=1}^{r}$ can be simultaneously diagonalized in an orthonormal eigenbasis $\{|i\rangle\}_{i=1}^{d}$ of the quantum system [51,52]:

$$\begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1d} \\ \vdots & \vdots & \vdots \\ c_{r1} & \cdots & c_{rd} \end{bmatrix} \begin{bmatrix} |1\rangle\langle 1| \\ \vdots \\ |d\rangle\langle d| \end{bmatrix}. \tag{3}$$

This can be simply denoted as $\mathbf{M} = \mathbf{CP}$, where $\mathbf{M} = [M_1, \ldots, M_r]^T$, $\mathbf{P} = [|1\rangle\langle 1|, \ldots, |d\rangle\langle d|]^T$, and $\mathbf{C}$ is a $r \times d$ complex matrix (r and d are generally different). We partition $\mathbf{C}$ according to its columns as $[\mathbf{c}_1, \ldots, \mathbf{c}_d]$, then $\|\mathbf{c}_j\|^2 = \mathbf{c}_j^\dagger\mathbf{c}_j = 1$ for any integer $j \in [1, d]$ due to $\mathbf{M}^\dagger\mathbf{M} = \sum_{\alpha=1}^{r} M_\alpha^\dagger M_\alpha = \mathbb{I}$, and $\{\mathbf{c}_j\}_{j=1}^{d}$ is a set of unit vectors in a r-dimensional complex vector space, with j corresponding to the basis state $|j\rangle$. Note that these unit vectors are not necessarily orthogonal to each other (see Appendix B). For a specific GM, the measurement operators are not unique, since we can define a new set of measurement operators by $\mathbf{M}' = \mathbf{TM}$ with $\mathbf{T}$ being a $r \times r$ unitary matrix, which satisfy $\mathbf{M}'^\dagger\mathbf{M}' = \mathbb{I}$ and also characterize the same quantum channel.

The quantum channel is then a diagonal operator on the HS space,

$$\hat{\Phi} = \sum_{i,j=1}^{d} \mathbf{c}_j^\dagger\mathbf{c}_i|ij\rangle\rangle\langle\langle ij|, \tag{4}$$

where $\{|ij\rangle\rangle\}_{i,j=1}^{d}$ are the eigenvectors (eigenmatrices on the Hilbert space) of $\hat{\Phi}$ with the corresponding eigenvalues $\{\mathbf{c}_j^\dagger\mathbf{c}_i\}_{i,j=1}^{d}$. Since $|\mathbf{c}_j^\dagger\mathbf{c}_i| \leqslant 1$ (due to the Cauchy-Schwarz inequality) with equality if and only if $\mathbf{c}_i = e^{i\varphi}\mathbf{c}_j$ for some real φ, all the eigenvalues of $\hat{\Phi}$ lie within the unit disk of the complex plane (actually this nonexpansive property holds for arbitrary quantum channels [51]). The eigenvectors with eigenvalue 1 are called *fixed points* [51,53], and those with eigenvalues $e^{i\varphi}$ with $\varphi \neq 0$ are *rotating points* [54]. The HS subspace spanned by the fixed points and rotating points are called *asymptotic subspace* (also known as peripheral or attractor subspace). For $\hat{\Phi}$ in Eq. (4), the fixed points must include $\{|jj\rangle\rangle\}_{j=1}^{d}$, and the rotating points are $\{|ij\rangle\rangle | \forall i, j \in [1, d], \mathbf{c}_j^\dagger\mathbf{c}_i = e^{i\varphi} \neq 1\}$.

As a simple example, consider $\{\mathbf{c}_j\}_{j=1}^{d}$ as a set of orthonormal vectors, then the channel is $\hat{\Phi} = \sum_{j=1}^{d} |jj\rangle\rangle\langle\langle jj|$, representing a nonselective PM with rank-1 projectors (von Neumann measurements), $\Phi(\cdot) = \sum_{j=1}^{d} |j\rangle\langle j|(\cdot)|j\rangle\langle j|$. This channel has only fixed points but no rotating points. As another example, consider $\{\mathbf{c}_j\}_{j=1}^{d} = \{\widetilde{\mathbf{c}}e^{i\varphi_j}\}_{j=1}^{d}$ with $\widetilde{\mathbf{c}}$ being also a unit vector, then $\hat{\Phi} = \sum_{j=1}^{d} e^{i(\varphi_i - \varphi_j)}|ij\rangle\rangle\langle\langle ij|$ is a unitary channel $\Phi(\cdot) = U(\cdot)U^\dagger$ with $U = \sum_{j=1}^{d} e^{i\varphi_j}|j\rangle\langle j|$. For the unitary channel, $|ij\rangle\rangle$ is a fixed point if $i = j$ or $\varphi_i = \varphi_j$, and a rotating point if $\varphi_i \neq \varphi_j$.

For general cases, we divide the index set $A = \{1, \ldots, d\}$ into $s(\leqslant d)$ disjoint subsets $A_1, \ldots, A_s$, with the corresponding cardinalities (number of elements) being $d_1, \ldots, d_s$, satisfying $\sum_{i=1}^{s} d_i = d$. Then divide the set of unit vectors $C = \{\mathbf{c}_j\}_{j=1}^{d}$ into s disjoint subsets $C_1, \ldots, C_s$ with $C_k = \{\mathbf{c}_j | j \in A_k\}$. This division should ensure that the unit vectors in each subset are the same up to some phase factors but are different from any other unit vectors in other subsets, i.e., $C_k = \{\widetilde{\mathbf{c}}_k e^{i\varphi_j} | j \in A_k\}$ but $\widetilde{\mathbf{c}}_p \neq \widetilde{\mathbf{c}}_q e^{i\varphi}$ for any φ and

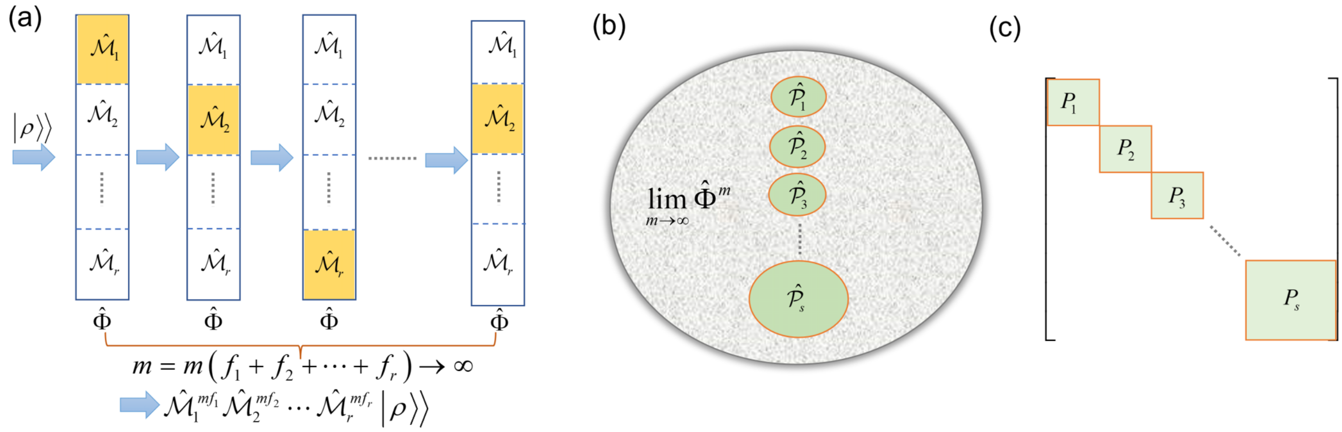

FIG. 1. (a) Schematic of sequential nonselective GMs and sequences of selective GMs in the asymptotic limit. (b) Emergent PMs arising from summation over the sets of typical sequences of selective GMs. (c) The emergent projections in the operator space of the quantum system.

$p, q \in [1, s]$. This implies that $|ij\rangle\rangle$ with $i, j \in A_k$ is either a fixed point ($\varphi_i = \varphi_j$) or a rotating point ($\varphi_i \neq \varphi_j$).

The division of the index set also partitions the Hilbert space $\mathcal{H}$ of the quantum system into the direct sum of s subspaces, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_s$, where $\mathcal{H}_k = \mathrm{Span}\{|j\rangle | j \in A_k\}$ with rank-d_k projection $P_k = \sum_{j\in A_k} |j\rangle\langle j|$. Thus the measurement operators in Eq. (3) can be written in a compact matrix form, $\mathbf{M} = \widetilde{\mathbf{C}}\mathbf{P}$, where $\widetilde{\mathbf{C}} = [\widetilde{\mathbf{c}}_1, \ldots, \widetilde{\mathbf{c}}_s]$ and $\mathbf{P} = [\widetilde{P}_1, \ldots, \widetilde{P}_s]^T$ with $\widetilde{P}_k = \sum_{j\in A_k} e^{i\varphi_j} |j\rangle\langle j|$. Note that $\widetilde{P}_k$ is either a projection operator or a unitary operator in $\mathcal{H}_k$, satisfying $\widetilde{P}_k^\dagger \widetilde{P}_{k'} = \delta_{kk'} P_k$ and $\sum_{k=1}^s \widetilde{P}_k^\dagger \widetilde{P}_k = \mathbb{I}$, with $\delta_{kk'}$ being the Kronecker delta. Such a compact form of $\mathbf{M}$ allows us to extend the above formulation to infinite-dimensional quantum systems (see Appendix B), if we divide the identity operator into a finite set of orthogonal projections.

III. ASYMPTOTICS OF SEQUENTIAL GMs

Sequential nonselective GMs correspond to sequential applications of the quantum channel $\hat{\Phi}$ [Fig. 1(a)]. Previous works have studied the asymptotic behaviors of sequential general quantum channels [54–57], mostly trying to find which information from an initial state can be preserved during the process.

For the channel in Eq. (4), as the number of applications m increases, the projections of an initial state to eigenvectors with eigenvalues lying in the interior of the unit disk ($|\mathbf{c}_j^\dagger \mathbf{c}_i| < 1$) gradually vanish, while the projections to eigenvectors with eigenvalues on the unit circle ($|\mathbf{c}_j^\dagger \mathbf{c}_i| = 1$) remain unchanged or change by some phase factors (see Appendix C). So sequential nonselective GMs tend to preserve the quantum coherence within subspaces $\{\mathcal{H}_k\}_{k=1}^s$ but diminish the coherence between different subspaces. First assume that the channel has only fixed points, i.e., elements in each C_k are all the same or $\varphi_j = 0$ for all $j \in [1, d]$, then in the asymptotic limit of large m,

$$\lim_{m\to\infty} \hat{\Phi}^m = \sum_{k=1}^s \sum_{i,j\in A_k} |ij\rangle\rangle\langle\langle ij| = \sum_{k=1}^s \hat{\mathcal{P}}_k, \qquad (5)$$

corresponding to $\lim_{m\to\infty} \Phi^m(\cdot) = \sum_{k=1}^s P_k(\cdot)P_k$ [Figs. 1(b) and 1(c)], which represents nonselective PMs. Then consider the channel with also rotating points, i.e., there are different phase factors in $C_k = \{\widetilde{\mathbf{c}}_k e^{i\varphi_j} | j \in A_k\}$, each application of $\hat{\Phi}$ produces a unitary operation in the Hilbert subspace $\mathcal{H}_k$, i.e., $P_k(\cdot)P_k$ in the former case should be replaced by $\widetilde{P}_k^m(\cdot)(\widetilde{P}_k^\dagger)^m$. For example, if $C_k = \{\mathbf{c}_i, \mathbf{c}_j\} = \{\widetilde{\mathbf{c}}_k e^{i\varphi_i}, \widetilde{\mathbf{c}}_k e^{i\varphi_j}\}$, then $\widetilde{P}_k = e^{i\varphi_i}|i\rangle\langle i| + e^{i\varphi_j}|j\rangle\langle j|$. Then the asymptotic limit for $\hat{\Phi}^m$ may not exist, but the typicality theory below for finite m still applies in these cases.

IV. TYPICALITY OF SEQUENTIAL GMs

Now that sequential nonselective GMs produce projections (or oscillatory unitary operations in the projected subspaces) in the asymptotic limit, we further ask which sequences of sequential selective GMs produce a specific projection. This problem can be perfectly solved by the theory of classical typicality [58–62]. Classical typicality mainly cares about the following problem: if a random variable takes r different values with the probability distribution $(p_1, \ldots, p_r)$, generate m independent realizations of this variable and find the statistical distributions of the event sequences with $(m_1/m, \ldots, m_r/m)$, where m_i is the number of the occurrences of the ith value. For infinitely large m, the event sequences that are overwhelmingly likely to occur are the set of *typical sequences* with $(p_1, \ldots, p_r)$.

A nonselective GM is a quantum instrument, which has r outcomes with an analogous "probability distribution" $(\hat{\mathcal{M}}_1, \ldots, \hat{\mathcal{M}}_r)$ (note that $\{\hat{\mathcal{M}}_r\}_{\alpha=1}^r$ are all diagonal matrices, and their diagonal entries on the space of each fixed point define a probability distribution). For sequential nonselective GMs, we can define sequences of selective GMs [Fig. 1(a)]. Below we show that the asymptotic projections are induced by the sets of typical sequences of selective GMs.

Since $\hat{\Phi} = \sum_{\alpha=1}^r \hat{\mathcal{M}}_\alpha$ and $[\hat{\mathcal{M}}_\alpha, \hat{\mathcal{M}}_\beta] = 0$ for $\alpha, \beta \in [1, r]$, we can expand $\hat{\Phi}^m$ according to the multinomial theorem,

$$\hat{\Phi}^m = \sum_{\{F\}} \frac{m!}{(mf_1)! \cdots (mf_r!)} \hat{\mathcal{M}}_1^{mf_1} \cdots \hat{\mathcal{M}}_r^{mf_r}, \qquad (6)$$

where $F = (f_1, \ldots, f_r)$, with $f_i \in [0, 1]$ (also a rational number with denominator m) satisfying $\sum_{i=1}^r f_i = 1$, and the summation is over all distributions $\{F\}$ in a $(r-1)$-dimensional probability space. For large m, $\hat{\Phi}^m$ can be approximated by its projected operation on the asymptotic subspace (see Appendix D),

$$\hat{\Phi}^m \approx \sum_{k=1}^{s} \hat{\mathcal{P}}_k \hat{\Phi}^m \hat{\mathcal{P}}_k \approx \sum_{k=1}^{s} \sum_{\{F\}} e^{-mS(F\|F_k)} \hat{\mathcal{P}}_k, \qquad (7)$$

where $F_k = (f_{k1}, \dots, f_{kr}) = (|\widetilde{c}_{1k}|^2, \dots, |\widetilde{c}_{rk}|^2)$, with $\widetilde{c}_{1k}, \dots, \widetilde{c}_{rk}$ being entries of $\widetilde{\mathbf{c}}_k$ satisfying $\sum_{i=1}^r |\widetilde{c}_{ik}|^2 = 1$, and $S(F\|F_k) = \sum_{i=1}^r f_i \ln(f_i/f_{ki})$ is the relative entropy between F and F_k (the derivation above uses Stirling's formula $\ln m! \approx m \ln m - m$ for large m). $S(F\|F_k)$ takes the minimum when $F = F_k$, so for infinitely large m, $\{F_k\}_{k=1}^s$ represents s sets of ideal typical sequences of selective GMs, leading to the projections $\{\hat{\mathcal{P}}_k\}_{k=1}^s$ correspondingly [Fig. 1(b)].

For large but finite m, the distributions of selective GM sequences for $\hat{\mathcal{P}}_k$ are concentrated around F_k, so $S(F\|F_k) \approx \sum_{i=1}^r (f_i - f_{ki})^2/(2f_{ki})$. Then Eq. (7) represents the summation of s Gaussians around $F_1, \dots, F_s$, with integration of the kth Gaussian over the whole probability space giving rise to $\hat{\mathcal{P}}_k$. For any two Gaussians around F_j and F_k, they are well separated if the distance between F_j and F_k is larger than the sum of the respective Gaussian half-widths. This requires $m > 2|\ln\eta|[(\sum_{i=1}^r (f_{ji} - f_{ki})^2/f_{ji})^{-1/2} + (\sum_{i=1}^r (f_{ji} - f_{ki})^2/f_{ki})^{-1/2}]^2$ [63] (see Appendix D for the derivation), where η is the ratio of the minimum height to the maximum height within the Gaussian width. If all the Gaussians are well separated, integration of the selective GM sequences within a small neighborhood around F_k can approximate $\hat{\mathcal{P}}_k$ up to arbitrary small error as m increases (see Appendix D for the error rates with finite m).

It may happen that two Gaussians coincide around $F_j = F_k$ but $\widetilde{\mathbf{c}}_j \neq \widetilde{\mathbf{c}}_k$, i.e., only partial elements of $\widetilde{\mathbf{c}}_j$ and $\widetilde{\mathbf{c}}_k$ differ by some phase factors. Since $|\widetilde{\mathbf{c}}_j^\dagger \widetilde{\mathbf{c}}_k| < 1$, the coinciding Gaussians actually correspond to different projections, and the selective GM sequences around F_j approximately produce $\hat{\mathcal{P}}_j + \hat{\mathcal{P}}_k$. To realize selective projections, we can get a new set of measurement operators by a unitary transformation, thus creating different typical sequences of selective GMs for $\hat{\mathcal{P}}_j$ and $\hat{\mathcal{P}}_k$.

V. PHYSICAL REALIZATION

We present a general physical model to perform PMs on a d-level target system with sequential GMs. Without loss of generality, we assume that the GMs are realized by PMs of an ancilla qubit. The coupling Hamiltonian of the composite system (including the ancilla and target systems) is in the pure-dephasing form [64]

$$H(t) = \sigma_z \otimes B(t), \qquad (8)$$

where σ_i is the Pauli-i operator of the ancilla qubit ($i = x, y, z$), and $B(t)$ is a time-dependent Hermitian operator of the target system (the time dependence of $B(t)$ is due to being in some interaction picture or external drivings).

The dynamics of the composite system induces a general class of quantum channels on the target system, which can be

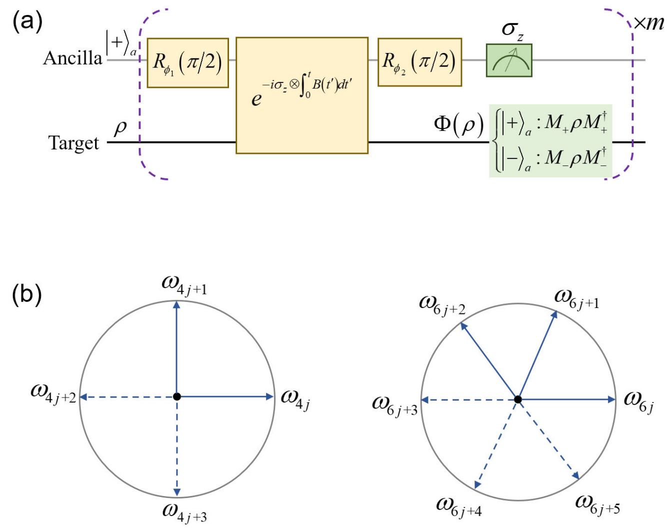

FIG. 2. (a) Quantum circuit diagram to realize sequential GMs on the target system with PMs of an ancilla qubit. (b) Distributions of eigenvalues of $U_\pm(t) = e^{\pm i\chi a^\dagger a t/2} = \sum_{j=0}^{\infty}\sum_{l=0}^{2N-1} e^{\pm i\omega_{2jN+l}}|2jN + l\rangle\langle 2jN + l|$ in the complex unit circle to detect the k mod N excitation numbers of a bosonic mode with $t = 2\pi/(N\chi)$ and $N = 2, 3$.

written in the Stinespring representation as [65]

$$\Phi(\rho) = \mathrm{Tr}_a[U(t)(\rho_a \otimes \rho)U^\dagger(t)], \qquad (9)$$

where $U(t) = \mathcal{T}e^{-i\sigma_z \otimes \int_0^t B(t')dt'}$, with $\mathcal{T}$ being the time-ordering operator, $\rho_a = |\psi\rangle_a\langle\psi|$ is the initial state of the ancilla, ρ denotes the density matrix of the target system, and Tr_a denotes the partial trace over the ancilla. With an orthonormal ancilla basis $\{|v_+\rangle_a, |v_-\rangle_a\}$, we obtain the Kraus representation of the quantum channels, $\Phi(\rho) = \sum_{\alpha\in\{+,-\}} M_\alpha \rho M_\alpha^\dagger$ with $M_\alpha = \langle v_\alpha|U(t)|\psi\rangle_a$. (Note that we add subscripts to the kets only when representing matrix elements or inner products with respect to the ancilla states.) With another orthonormal basis $\{T^\dagger|v_+\rangle_a, T^\dagger|v_-\rangle_a\}$, with T being a unitary operator for the ancilla, the measurement operators become $\{M'_\alpha\}$ with $M'_\alpha = \sum_{\beta\in\{+,-\}} T_{\alpha\beta}M_\beta$ and $T_{\alpha\beta} = \langle v_\alpha|T|v_\beta\rangle_a$, while the quantum channels remain unchanged.

We expand $U(t)$ in the ancilla eigenbasis $\{|+\rangle_a, |-\rangle_a\}$ of σ_z as $U(t) = |+\rangle_a\langle +| \otimes U_+(t) + |-\rangle_a\langle -| \otimes U_-(t)$, where $U_\pm(t) = \mathcal{T}e^{\mp i\int_0^t B(t')dt'}$. If $U_\pm(t)$ is exactly equal to or well approximated by its first-order Magnus expansion [66], i.e., $U_\pm(t) = e^{\mp i\int_0^t B(t')dt'}$, then $U_+(t) = U_-^\dagger(t)$ and $[U_+(t), U_-(t)] = 0$, and $U_+(t)$ and $U_-(t)$ can be simultaneously diagonalized as $U_\pm(t) = \sum_{j=1}^d e^{\pm i\omega_j}|j\rangle\langle j|$. So the measurement operators are $M_\pm = \sum_{j=1}^d (\langle v_\pm|\psi\rangle_a \cos\omega_j + i\langle v_\pm|\sigma_z|\psi\rangle_a \sin\omega_j)|j\rangle\langle j|$. As a special case, take $|\psi\rangle_a = R_{\phi_1}(\frac{\pi}{2})|+\rangle_a$ and $|v_\pm\rangle_a = R_{\phi_2}(-\frac{\pi}{2})|\pm\rangle_a$ with $R_\phi(\theta) = e^{-i(\cos\phi\sigma_x + \sin\phi\sigma_y)\theta/2}$, then

$$\begin{bmatrix} M_+ \\ M_- \end{bmatrix} = \begin{bmatrix} e^{i\omega_1} - e^{i(\Delta\phi-\omega_1)} & \cdots & e^{i\omega_d} - e^{i(\Delta\phi-\omega_d)} \\ e^{i\omega_1} + e^{i(\Delta\phi-\omega_1)} & \cdots & e^{i\omega_d} + e^{i(\Delta\phi-\omega_d)} \end{bmatrix} \frac{\mathbf{P}}{2}, \qquad (10)$$

where $\Delta\phi = \phi_1 - \phi_2$. Each round of such GMs corresponds to a three-step physical process [Fig. 2(a)]: (1) the ancilla starts from $|+\rangle_a$ and is rotated by $R_{\phi_1}(\frac{\pi}{2})$; (2) let the ancilla and target systems evolve under $H(t)$ for time t; (3) finally rotate the ancilla by $R_{\phi_2}(\frac{\pi}{2})$ and make a PM of the ancilla in the basis $\{|+\rangle_a, |-\rangle_a\}$. Similar schemes have been designed to realize single-shot readouts of nuclear spins-1/2 in diamond [44], but here we show this scheme can be extended to perform PMs of a generic system.

Since the GMs have only two outcomes, the measurement results are solely determined by the measurement polarization $\Delta f = (m_- - m_+)/m$ [43], with m_+/m_- being the number of outcomes $+/-$ in m sequential measurements. For the spectra $\{e^{\pm i\omega_j}\}$ of $U_\pm(t)$, calculate $\Delta f_j = \cos(2\omega_j - \Delta\phi)$ for all $j \in [1, d]$. Weak measurement corresponds to the regime $|\Delta f_j| \ll 1$. If $\Delta f_j \neq \Delta f_k$ for any $j, k \in [1, d]$ and $j \neq k$, sequential GMs produce von Neumann measurements of the target system, with the rank-1 projection $P_j = |j\rangle\langle j|$ corresponding to typical selective GM sequences with Δf_j. If $\Delta f_j = \Delta f_k$, then either (I) $\omega_j + \omega_k = \Delta\phi + n\pi$ or (II) $\omega_j - \omega_k = n\pi$ with n being integers. In case I, the typical selective GM sequences for P_j and P_k are the same, but selective projections can still be achieved by choosing a different $\Delta\phi'$. In case II, the typical selective GM sequences with Δf_j induce the operation $P_j + (-1)^n P_k$.

A. Example: Modular excitation number measurements of bosonic modes

As an example, we present a protocol to measure the modular excitation numbers of a bosonic mode with an ancilla qubit. The ancilla is dispersively coupled to a bosonic mode with the Hamiltonian $H = -\chi\sigma_z a^\dagger a/2$, where a ($a^\dagger$) is the annihilation (creation) operator of the bosonic mode and χ is the dispersive coupling strength. The dispersive coupling arises naturally from the Jaynes-Cummings coupling in cavity quantum electrodynamics (QED) [67] and circuit QED [68] when the detuning between the ancilla and the bosonic mode is much larger than the coupling strength.

We construct the projectors into the sets of bosonic Fock states with modular excitation number $l \bmod 2N$, $P_{2N}^l = \sum_{j=0}^{\infty} |2jN + l\rangle\langle 2jN + l|$, with $l \in \{0, 1, \ldots, 2N-1\}$ and N being any positive integer. With the scheme below Eq. (10) and the evolution time $t = 2\pi/(N\chi)$, $U_\pm(t) = e^{\pm i\chi a^\dagger a t/2} = \sum_{k=0}^{N-1} e^{\pm ik\pi/N}(P_{2N}^k - P_{2N}^{k+N})$, i.e., the eigenvalues of $U_\pm(t)$ divide the complex unit circle into $2N$ equal pieces [Fig. 2(b)]. The measurement operators are $M_\pm = \sum_{k=0}^{N-1}(e^{ik\pi/N} \mp e^{i(\Delta\phi - k\pi/N)})(P_{2N}^k - P_{2N}^{k+N})$, and the measurement polarization $\Delta f_k = \cos(2k\pi/N - \Delta\phi)$. We can tune $\Delta\phi$ so that Δf_k is maximally distinguishable for different $k \in [0, N-1]$. For $N = 2$, $\Delta\phi = 0$ is optimal as $\Delta f_0 = -\Delta f_1 = 1$; while for $N \geqslant 3$, we can choose $\Delta\phi = \pi/(2N)$ so that $\Delta f_k = \cos[(2k - 1/2)\pi/N]$. Then for large and even m, sequential GMs approximately induce the $k \bmod N$ excitation number measurement of the bosonic mode. The modular excitation numbers are the error syndromes of rotation-symmetric error correction codes of bosonic modes [69], such as cat codes [70–73] and binomial codes [74]. So this protocol is useful for quantum nondemolition measurements in bosonic quantum information processing [75–78], especially for tracking the error syndromes of higher-order bosonic error correction codes [79–81].

VI. SUMMARY AND OUTLOOK

We have revealed the elegant structures of sequential GMs by studying their asymptotic behaviors and typical sequences. We prove that nonselective PMs can emerge from sequential nonselective GMs when the measurement operators are normal and commuting with each other. Each selective PM comes from a set of typical sequences of selective GMs, which is determined solely by the structures of the measurement operators. While the GMs here are restricted to have normal and commuting measurement operators, they describe a large class of quantum channels on a quantum system induced by a pure-dephasing coupling between this system and an ancilla system. For future works it will be interesting to relax this restriction and study the asymptotics and typicality of sequential GMs with general measurement operators.

Note added. We recently became aware of a related but different work [82]. In the work of Linden and Skrzypczyk, they find that with many copies of available GMs in parallel (aided by entangling gates) one can simulate target GMs in the asymptotic limit.

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APPENDIX A: INTRODUCTIONS TO HS SPACE

Here we briefly introduce the HS space. For a d-dimensional quantum system, the space of operators acting on its Hilbert space $\mathcal{H}$ forms a linear vector space. This is easily seen if the $d \times d$ complex matrix of an operator X in an orthonormal eigenbasis $\{|i\rangle\}_{i=1}^d$ is reshaped into a $d^2 \times 1$ column vector,

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \vdots & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_d \end{bmatrix} \tag{A1a}$$

$$\Updownarrow$$

$$|X\rangle\rangle = \begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_d^T \end{bmatrix}, \tag{A1b}$$

where $\mathbf{x}_i$ is the ith row of X with $i \in [1, d]$, and $\mathbf{x}_i^T$ is the transpose of $\mathbf{x}_i$. With Dirac notations, the matrix reshaping can be simply represented by $X = \sum_{i,j=1}^d x_{ij}|i\rangle\langle j| \leftrightarrow |X\rangle\rangle = \sum_{i,j=1}^d x_{ij}|ij\rangle\rangle$. Then the Euclidean inner product between $|X\rangle\rangle$ and $|Y\rangle\rangle$ defines an inner product between X and Y,

$$\langle\langle Y|X\rangle\rangle = \sum_{i=1}^d \mathbf{y}_i^* \mathbf{x}_i^T = \sum_{i,j=1}^d y_{ij}^* x_{ij} = \mathrm{Tr}(Y^\dagger X), \tag{A2}$$

which is the so-called HS inner product. The HS space is the space of vectorized operators equipped with the HS inner product.

The density matrices of the quantum system, as the class of positive semidefinite operators with trace 1, are also vectors in the HS space. In the HS space, the trace 1 constraint of a density matrix ρ is equivalent to $\langle\langle \mathbb{I}|\rho\rangle\rangle = \mathrm{Tr}(\rho) = 1$, with $\mathbb{I}$ being the identity operator. Left and right multiplications of ρ by operators X and Y correspond to multiplying $|\rho\rangle\rangle$ with a $d^2 \times d^2$ matrix,

$$X\rho Y = \sum_{i,j=1}^{d} x_{ik} y_{lj} \rho_{kl} |i\rangle\langle j|, \tag{A3a}$$

$$\Updownarrow$$

$$X \otimes Y^T |\rho\rangle\rangle. \tag{A3b}$$

So the operation $X(\cdot)Y$ as a superoperator is equivalent to a linear operator $X \otimes Y^T$ on the HS space.

For the quantum channel of a nonselective GM with the measurement operators $\{M_\alpha\}_{\alpha=1}^r$, the transformation from its Kraus representation to the natural matrix representation on the HS space is given by

$$\Phi(\rho) = \sum_{\alpha=1}^{r} \mathcal{M}_\alpha(\rho) = \sum_{\alpha=1}^{r} M_\alpha \rho M_\alpha^\dagger, \tag{A4a}$$

$$\Updownarrow$$

$$\hat{\Phi}|\rho\rangle\rangle = \sum_{\alpha=1}^{r} \hat{\mathcal{M}}_\alpha |\rho\rangle\rangle = \sum_{\alpha=1}^{r} M_\alpha \otimes M_\alpha^* |\rho\rangle\rangle, \tag{A4b}$$

where $\mathcal{M}_\alpha = M_\alpha(\cdot)M_\alpha^\dagger$ is a superoperator representing a selective GM with the αth outcome, $\sum_{\alpha=1}^r M_\alpha^\dagger M_\alpha = \mathbb{I}$ ensures the trace-preserving property, $\hat{\mathcal{M}}_\alpha = M_\alpha \otimes M_\alpha^*$ is an operator on the HS space corresponding to $\mathcal{M}_\alpha$, and M_α^* is the complex conjugate of M_α. For a selective GM with the αth outcome, the density matrix undergoes the following evolution:

$$\rho_\alpha = \frac{\mathcal{M}_\alpha(\rho)}{p_\alpha} = \frac{M_\alpha \rho M_\alpha^\dagger}{p_\alpha}, \tag{A5a}$$

$$\Updownarrow$$

$$|\rho_\alpha\rangle\rangle = \frac{\hat{\mathcal{M}}_\alpha |\rho\rangle\rangle}{p_\alpha} = \frac{M_\alpha \otimes M_\alpha^* |\rho\rangle\rangle}{p_\alpha}, \tag{A5b}$$

where $p_\alpha = \mathrm{Tr}(M_\alpha \rho M_\alpha^\dagger) = \langle\langle \mathbb{I}|\hat{\mathcal{M}}_\alpha|\rho\rangle\rangle$ is the probability to get the αth outcome, satisfying $\sum_{\alpha=1}^r p_\alpha = 1$.

APPENDIX B: STRUCTURES OF GMs WITH NORMAL AND COMMUTING MEASUREMENT OPERATORS

In this section we provide a systematic description of GMs with normal and commuting measurement operators $\{M_\alpha\}_{\alpha=1}^r$. Since $[M_\alpha, M_\alpha^\dagger] = [M_\alpha, M_\beta] = 0$ for all integers, $\alpha, \beta \in [1, r]$, $\{M_\alpha\}_{\alpha=1}^r$ can be simultaneously diagonalized in an orthonormal eigenbasis $\{|i\rangle\}_{i=1}^d$ of the quantum system [52],

$$\begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1d} \\ \vdots & \vdots & \vdots \\ c_{r1} & \cdots & c_{rd} \end{bmatrix} \begin{bmatrix} |1\rangle\langle 1| \\ \vdots \\ |d\rangle\langle d| \end{bmatrix}, \tag{B1}$$

which can be written in a matrix form as $\mathbf{M} = \mathbf{CP}$, with the definitions below:

$$\mathbf{M} = \begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} |1\rangle\langle 1| \\ \vdots \\ |d\rangle\langle d| \end{bmatrix}, \tag{B2}$$

$$\mathbf{C} = [\mathbf{c}_1, \cdots, \mathbf{c}_d] = \begin{bmatrix} c_{11} & \cdots & c_{1d} \\ \vdots & \vdots & \vdots \\ c_{r1} & \cdots & c_{rd} \end{bmatrix}, \tag{B3}$$

where $\mathbf{M}$ is a $r \times 1$ column vector of operators, $\mathbf{C}$ is a $r \times d$ complex matrix with $\mathbf{c}_i$ being its ith column, and $\mathbf{P}$ is a $d \times 1$ column vector of operators. If we further define $\mathbf{M}^\dagger = [M_1^\dagger, \ldots, M_r^\dagger]$ and $\mathbf{P}^\dagger = \mathbf{P}^T = [|1\rangle\langle 1|, \ldots, |d\rangle\langle d|]$, then $\mathbf{M}^\dagger\mathbf{M} = \sum_{i=1}^r M_i^\dagger M_i = \mathbf{P}^\dagger\mathbf{P} = \sum_{i=1}^d |i\rangle\langle i| = \mathbb{I}$. This condition restricts the form of $\mathbf{C}$, as can be seen by

$$\begin{aligned}
\mathbf{M}^\dagger\mathbf{M} &= \mathbf{P}^\dagger\mathbf{C}^\dagger\mathbf{C}\mathbf{P} \\
&= [|1\rangle\langle 1|, \ldots, |d\rangle\langle d|] \begin{bmatrix} \mathbf{c}_1^\dagger\mathbf{c}_1 & \cdots & \mathbf{c}_1^\dagger\mathbf{c}_d \\ \vdots & \vdots & \vdots \\ \mathbf{c}_d^\dagger\mathbf{c}_1 & \cdots & \mathbf{c}_d^\dagger\mathbf{c}_d \end{bmatrix} \\
&\quad \times \begin{bmatrix} |1\rangle\langle 1| \\ \vdots \\ |d\rangle\langle d| \end{bmatrix} \\
&= \sum_{i,j=1}^{d} \mathbf{c}_i^\dagger\mathbf{c}_j |i\rangle\langle i|j\rangle\langle j| = \sum_{i=1}^{d} \mathbf{c}_i^\dagger\mathbf{c}_i |i\rangle\langle i|,
\end{aligned} \tag{B4}$$

which clearly shows $\mathbf{c}_i^\dagger\mathbf{c}_i = \sum_{\alpha=1}^r |c_{\alpha i}|^2 = 1$ for any $i \in [1, d]$, i.e., all the columns $\{\mathbf{c}_j\}_{j=1}^d$ of $\mathbf{C}$ are unit vectors in a r-dimensional complex vector space. But these unit vectors are not necessarily orthogonal to each other. The reason is that entries of $\mathbf{P}$ are not real or complex numbers but rank-1 projectors $\{|i\rangle\langle i|\}_{i=1}^d$, satisfying $|i\rangle\langle i|j\rangle\langle j| = \delta_{ij}|i\rangle\langle i|$.

Now we take a closer look at the structures of matrix $\mathbf{C}$. Define the set of its column vectors as $C = \{\mathbf{c}_j\}_{j=1}^d$ with an index set $A = \{1, \ldots, d\}$. Then divide C into s disjoint subsets $C_1, \ldots, C_s$ with the corresponding index subsets $A_1, \ldots, A_s$, where $C_k = \{\mathbf{c}_j | j \in A_k\}$ for any integer $k \in [1, s]$. The cardinality of C_k and A_k is d_k, satisfying $\sum_{i=1}^s d_i = d$ and $d_i \geqslant 1$. This division should ensure that the unit vectors in each subset are the same up to some phase factors but are different from any other unit vectors in other subsets, i.e., $C_k = \{\widetilde{\mathbf{c}}_k e^{i\varphi_j} | j \in A_k\}$, but $\widetilde{\mathbf{c}}_p \neq \widetilde{\mathbf{c}}_q e^{i\varphi}$ for any real φ and $p, q \in [1, s]$. This means that we can always simultaneously reorder the columns of C and the entries of $\mathbf{P}$, and then relabel the eigenbasis $\{|i\rangle\}_{i=1}^d$, so that $\mathbf{C}$ is in the following canonical form:

$$\mathbf{C} = [\overbrace{\widetilde{\mathbf{c}}_1 e^{i\varphi_1}, \ldots, \widetilde{\mathbf{c}}_1 e^{i\varphi_{d_1}}}^{d_1}, \;\ldots,\; \overbrace{\widetilde{\mathbf{c}}_s e^{i\varphi_{d-d_s+1}}, \ldots, \widetilde{\mathbf{c}}_s e^{i\varphi_{d-1}}}^{d_s}], \tag{B5}$$

and $\mathbf{P}$ remains unchanged. Then Eq. (B1) can be written in a more compact matrix form, $\mathbf{M} = \widetilde{\mathbf{C}}\widetilde{\mathbf{P}}$, where $\widetilde{\mathbf{C}} = [\widetilde{\mathbf{c}}_1, \ldots, \widetilde{\mathbf{c}}_s]$ and $\widetilde{\mathbf{P}} = [\widetilde{P}_1, \ldots, \widetilde{P}_s]^T$ with $\widetilde{P}_k = \sum_{j\in A_k} e^{i\varphi_j}|j\rangle\langle j|$. More explicitly,

$$\begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix} = \begin{bmatrix} \widetilde{c}_{11} & \dots & \widetilde{c}_{1s} \\ \vdots & \vdots & \vdots \\ \widetilde{c}_{r1} & \dots & \widetilde{c}_{rs} \end{bmatrix} \begin{bmatrix} \widetilde{P}_1 \\ \vdots \\ \widetilde{P}_s \end{bmatrix}. \tag{B6}$$

While Eq. (B1) mainly concerns finite-dimensional quantum systems, Eq. (B6) can describe both finite- and infinite-dimensional systems. The key point is to first partition the identity operator $\mathbb{I}$ of a generic system into a set of orthogonal projections $\{P_k\}_{k=1}^s$, which corresponds to partitioning the Hilbert space $\mathcal{H}$ of the system into the direct sum of s subspaces, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_s$. Then $\widetilde{P}_k$ is a unitary operator in subspace $\mathcal{H}_k$ (with projection P_k as a special case), satisfying $\widetilde{P}_k^\dagger \widetilde{P}_{k'} = \delta_{kk'} P_k$ and $\widetilde{\mathbf{P}}^\dagger \widetilde{\mathbf{P}} = \sum_{k=1}^s \widetilde{P}_k^\dagger \widetilde{P}_k = \mathbb{I}$. Obviously, projective measurements and unitary evolutions are both special cases of Eq. (B6).

Moreover, for both Eqs. (B1) and (B6), we can define a new set of measurement operators by $\mathbf{M}' = \mathbf{T}\mathbf{M}$, with $\mathbf{T} = [T_{\alpha\beta}]$ being a $r \times r$ unitary matrix, which satisfies $\mathbf{M}'^\dagger \mathbf{M}' = \mathbf{M}\mathbf{T}^\dagger \mathbf{T}\mathbf{M} = \mathbb{I}$. $\mathbf{M}'$ and $\mathbf{M}$ also characterize the same CPTP map, since

$$\begin{aligned} \sum_{\alpha=1}^r M'_\alpha(\cdot) M'^\dagger_\alpha &= \sum_{\alpha,\beta,\gamma=1}^r T^*_{\alpha\gamma} T_{\alpha\beta} M_\beta(\cdot) M^\dagger_\gamma \\ &= \sum_{\beta,\gamma=1}^r \delta_{\gamma\beta} M_\beta(\cdot) M^\dagger_\gamma = \sum_{\beta=1}^r M_\beta(\cdot) M^\dagger_\beta, \end{aligned} \tag{B7}$$

where we have used $\sum_{\alpha=1}^r T^*_{\alpha\gamma} T_{\alpha\beta} = \delta_{\gamma\beta}$ since $\mathbf{T}$ is a unitary matrix.

APPENDIX C: DETAILS ABOUT ASYMPTOTICS OF SEQUENTIAL GMs

For the measurement operators in Eq. (B1), the matrix representation of the channel acting on the HS space is

$$\begin{aligned} \hat{\Phi} &= \sum_{\alpha=1}^r \sum_{i,j=1}^d (c_{\alpha i}|i\rangle\langle i|) \otimes (c^*_{\alpha j}|j\rangle\langle j|) \\ &= \sum_{i,j=1}^d \left(\sum_{\alpha=1}^r c^*_{\alpha j} c_{\alpha i} \right) |ij\rangle\rangle\langle\langle ij| = \sum_{i,j=1}^d \mathbf{c}_j^\dagger \mathbf{c}_i |ij\rangle\rangle\langle\langle ij|, \end{aligned} \tag{C1}$$

while for the more general case in Eq. (B6), we can similarly obtain

$$\begin{aligned} \hat{\Phi} &= \sum_{\alpha=1}^r \sum_{k,l=1}^s (\widetilde{c}_{\alpha k}\widetilde{P}_k) \otimes (\widetilde{c}^*_{\alpha l}\widetilde{P}^*_l) = \sum_{k,l=1}^s \widetilde{\mathbf{c}}_l^\dagger \widetilde{\mathbf{c}}_k (\widetilde{P}_k \otimes \widetilde{P}^*_l) \\ &= \sum_{k,l=1}^s \widetilde{\mathbf{c}}_l^\dagger \widetilde{\mathbf{c}}_k \left(\sum_{i\in A_k} \sum_{j\in A_l} e^{i(\varphi_i-\varphi_j)} |ij\rangle\rangle\langle\langle ij| \right), \end{aligned} \tag{C2}$$

where $\{\widetilde{P}_k \otimes \widetilde{P}^*_l\}_{k,l=1}^s$ is a set of s^2 diagonal matrices in HS space that has orthogonal supports, i.e., $(\widetilde{P}_k \otimes \widetilde{P}^*_l)(\widetilde{P}_{k'} \otimes \widetilde{P}^*_{l'}) = \delta_{kk'}\delta_{ll'}(\widetilde{P}_k \otimes \widetilde{P}^*_l)^2$. From the Cauchy-Schwarz inequality, $|\widetilde{\mathbf{c}}_l^\dagger \widetilde{\mathbf{c}}_k| < 1$ if $k \neq l$, since $\widetilde{\mathbf{c}}_k \neq \widetilde{\mathbf{c}}_l e^{i\varphi}$ for any real φ. So with many applications of the channel,

$$\begin{aligned} \hat{\Phi}^m &= \sum_{k,l=1}^s (\widetilde{\mathbf{c}}_l^\dagger \widetilde{\mathbf{c}}_k)^m (\widetilde{P}_k \otimes \widetilde{P}^*_l)^m \approx \sum_{k=1}^s (\widetilde{P}_k \otimes \widetilde{P}^*_k)^m \\ &= \sum_{k=1}^s \sum_{i,j\in A_k} e^{im(\varphi_i-\varphi_j)} |ij\rangle\rangle\langle\langle ij|. \end{aligned} \tag{C3}$$

If $\varphi_j = 0$ for any $j \in [1, d]$, i.e., $\widetilde{P}_k = P_k$, then

$$\hat{\Phi}^m \approx \sum_{k=1}^s P_k \otimes P_k = \sum_{k=1}^s \hat{\mathcal{P}}_k = \sum_{k=1}^s \sum_{i,j\in A_k} |ij\rangle\rangle\langle\langle ij|, \tag{C4}$$

which is just Eq. (5).

APPENDIX D: DETAILS ABOUT TYPICALITY OF SEQUENTIAL GMs

Since $[M_\alpha, M_\beta] = 0$ for $\alpha, \beta \in [1, r]$, we can easily prove that $[\hat{\mathcal{M}}_\alpha, \hat{\mathcal{M}}_\beta] = 0$. So $\hat{\Phi}^m$ can be expanded according to the following multinomial theorem:

$$\begin{aligned} \hat{\Phi}^m &= \left(\sum_{\alpha=1}^r \hat{\mathcal{M}}_\alpha \right)^m = \sum_{\alpha_1,\dots,\alpha_m=1}^r \hat{\mathcal{M}}_{\alpha_1} \cdots \hat{\mathcal{M}}_{\alpha_m} \\ &= \sum_{\substack{m_1,\dots,m_r \geqslant 0 \\ m_1+\cdots+m_r=m}} \frac{m!}{m_1! m_2! \cdots m_r!} \hat{\mathcal{M}}_1^{m_1} \hat{\mathcal{M}}_2^{m_2} \cdots \hat{\mathcal{M}}_r^{m_r}. \end{aligned} \tag{D1}$$

We define a distribution $F = (f_1, \dots, f_r) = (m_1/m, \dots, m_r/m)$ to represent the frequencies of each element in $\{\hat{\mathcal{M}}_\alpha\}_{\alpha=1}^r$ to appear in the selective GM sequence $\hat{\mathcal{M}}_{\alpha_1} \cdots \hat{\mathcal{M}}_{\alpha_m}$, where $\sum_{i=1}^r f_i = 1$. Then $\hat{\Phi}^m$ can be rewritten as

$$\hat{\Phi}^m = \sum_{\{F\}} \frac{m!}{(mf_1)! \cdots (mf_r!)} \hat{\mathcal{M}}_1^{mf_1} \cdots \hat{\mathcal{M}}_r^{mf_r}, \tag{D2}$$

where the summation is over all distributions $\{F\}$ in a $(r-1)$-dimensional probability space. Substituting $\hat{\mathcal{M}}_\alpha = \sum_{k,l=1}^s \widetilde{c}_{\alpha k} \widetilde{c}^*_{\alpha l} (\widetilde{P}_k \otimes \widetilde{P}^*_l)$ into Eq. (D2) gives

$$\begin{aligned} \hat{\Phi}^m &= \sum_{k,l=1}^s \sum_{\{F\}} \frac{m!}{(mf_1)! \cdots (mf_r!)} (\widetilde{c}_{1k}\widetilde{c}^*_{1l})^{mf_1} \cdots (\widetilde{c}_{rk}\widetilde{c}^*_{rl})^{mf_r} \\ &\quad \times (\widetilde{P}_k \otimes \widetilde{P}^*_l)^m \\ &\approx \sum_{k=1}^s \sum_{\{F\}} \frac{m!}{(mf_1)! \cdots (mf_r!)} |\widetilde{c}_{1k}|^{2mf_1} \cdots |\widetilde{c}_{rk}|^{2mf_r} \\ &\quad \times (\widetilde{P}_k \otimes \widetilde{P}^*_k)^m, \end{aligned} \tag{D3}$$

where we have used $(\widetilde{P}_k \otimes \widetilde{P}^*_l)(\widetilde{P}_{k'} \otimes \widetilde{P}^*_{l'}) = \delta_{kk'}\delta_{ll'}(\widetilde{P}_k \otimes \widetilde{P}^*_l)^2$ and approximated $\hat{\Phi}^m$ by its projected operation on the asymptotic subspace. To further simplify Eq. (D3), we can use Stirling's formula $\ln m! \approx m \ln m - m$ for large m to obtain

$$\begin{aligned} &\ln \left(\frac{m!}{(mf_1)! \cdots (mf_r!)} |\widetilde{c}_{1k}|^{2mf_1} \cdots |\widetilde{c}_{rk}|^{2mf_r} \right) \\ &\approx -m \sum_{i=1}^r f_i \ln \frac{f_i}{|\widetilde{c}_{ik}|^2} = -mS(F \| F_k), \end{aligned} \tag{D4}$$

where we define $F_k = (f_{k1}, \ldots, f_{kr}) = (|\widetilde{c}_{1k}|^2, \ldots, |\widetilde{c}_{rk}|^2)$, with $\widetilde{c}_{1k}, \ldots, \widetilde{c}_{rk}$ being entries of $\widetilde{\mathbf{c}}_k$ in Eq. (B5) satisfying $\sum_{i=1}^r |\widetilde{c}_{ik}|^2 = 1$, and $S(F\|F_k) = \sum_{i=1}^r f_i \ln(f_i/f_{ki})$ is the relative entropy between F and F_k. Then Eq. (D3) is reduced to

$$\hat{\Phi}^m \approx \sum_{k=1}^{s} \sum_{\{F\}} e^{-mS(F\|F_k)} (\widetilde{P}_k \otimes \widetilde{P}_k^*)^m. \tag{D5}$$

Moreover, for large m, the distribution $e^{-mS(F\|F_k)}$ is concentrated within a small neighborhood around F_k, so $S(F\|F_k) \approx \sum_{i=1}^r (f_i - f_{ki})^2/(2f_{ki})$, and Eq. (D3) can be further approximated as

$$\hat{\Phi}^m \approx \sum_{k=1}^{s} \sum_{\{F\}} e^{-\frac{m}{2}\sum_{i=1}^r \frac{(f_i - f_{ki})^2}{f_{ki}}} (\widetilde{P}_k \otimes \widetilde{P}_k^*)^m. \tag{D6}$$

For the special case $\widetilde{P}_k = P_k$, Eqs. (D5) and (D7) become

$$\begin{aligned}\hat{\Phi}^m &\approx \sum_{k=1}^{s} \sum_{\{F\}} e^{-mS(F\|F_k)} \hat{\mathcal{P}}_k \\ &\approx \sum_{k=1}^{s} \sum_{\{F\}} e^{-\frac{m}{2}\sum_{i=1}^r \frac{(f_i - f_{ki})^2}{f_{ki}}} \hat{\mathcal{P}}_k,\end{aligned} \tag{D7}$$

which represents summations of s Gaussians around $F_1, \ldots, F_s$, with integration of the kth Gaussian over the whole probability space giving rise to $\hat{\mathcal{P}}_k$. Note that we neglect the normalization constants of the Gaussians in this paper as we use a simplified version of Stirling's formula, but this does not affect the typicality analyses below.

For any two Gaussians around F_j and F_k, they are well separated if the distance between F_j and F_k is larger than the sum of the respective Gaussian half-widths. In the $(r-1)$-dimensional probability space, the straight line connecting F_j and F_k is

$$\begin{aligned}F_{jk}(y) &= (1-y)F_j + yF_k \\ &= [(1-y)f_{j1} + yf_{k1}, \ldots, (1-y)f_{jr} + yf_{kr}],\end{aligned} \tag{D8}$$

where y is a real number within [0,1]. Define η as the ratio of the minimum hight to the maximum hight within the Gaussian width. Then the half-widths Δy_j and Δy_k of the two Gaussians along the line $F_{jk}(y)$ can be derived as

$$\begin{aligned}&e^{-\frac{m}{2}\sum_{i=1}^r \frac{(\Delta y_j)^2 (f_{ji} - f_{ki})^2}{f_{ji}}} = \eta, \implies \\ &\Delta y_j = \sqrt{\frac{2|\ln \eta|}{m}} \left(\sum_{i=1}^r \frac{(f_{ji} - f_{ki})^2}{f_{ji}} \right)^{-\frac{1}{2}},\end{aligned} \tag{D9a}$$

$$\begin{aligned}&e^{-\frac{m}{2}\sum_{i=1}^r \frac{(\Delta y_k)^2 (f_{ji} - f_{ki})^2}{f_{ki}}} = \eta, \implies \\ &\Delta y_k = \sqrt{\frac{2|\ln \eta|}{m}} \left(\sum_{i=1}^r \frac{(f_{ji} - f_{ki})^2}{f_{ki}} \right)^{-\frac{1}{2}}.\end{aligned} \tag{D9b}$$

The two Gaussians around F_j and F_k are well separated if $\Delta y_j + \Delta y_k < 1$, so we have $m > 2|\ln \eta| [(\sum_{i=1}^r (f_{ji} - f_{ki})^2/f_{ji})^{-1/2} + (\sum_{i=1}^r (f_{ji} - f_{ki})^2/f_{ki})^{-1/2}]^2$. For all the Gaussians to be well separated, the lower bound for the number of measurements is obtained by taking the maximum of the above bound over all pairs of $j, k \in [1, s]$.

For the jth Gaussian, we define a closed neighborhood $\mathcal{F}_j^\delta$ around F_j in the probability space such that summation of all the selective GM sequences within $\mathcal{F}_j^\delta$ well approximates $\hat{\mathcal{P}}_j$. Explicitly, summation of all the selective GM sequences within $\mathcal{F}_j^\delta$ gives

$$\hat{\mathcal{P}}_j^\delta \approx \sum_{k=1}^{s} \sum_{F \in \mathcal{F}_j^\delta} e^{-mS(F\|F_k)} \hat{\mathcal{P}}_k = \sum_{k=1}^{s} w_{jk} \hat{\mathcal{P}}_k, \tag{D10}$$

where $w_{jk} = \sum_{F \in \mathcal{F}_j^\delta} e^{-mS(F\|F_k)}$, satisfying $\sum_{j=1}^s w_{jk} < 1$ and $w_{jk} \geqslant 0$. Define F_j^* as a point on the boundary of $\mathcal{F}_j^\delta$ where $S(F_j^*\|F_j)$ takes the minimum on the boundary. Then from classical typicality theory [59,83] we have

$$1 - w_{jj} \leqslant \frac{(m+r-1)!}{m!(r-1)!} e^{-mS(F_j^*\|F_j)}. \tag{D11}$$

As $\hat{\mathcal{P}}_j^\delta$ and $\hat{\mathcal{P}}_j$ are both diagonal operators on the HS space, we can use the l_1 norm for matrices [84] to estimate an upper bound of the error rate in approximating $\hat{\mathcal{P}}_j$ with $\hat{\mathcal{P}}_j^\delta$,

$$\begin{aligned}\left|\left|\hat{\mathcal{P}}_j^\delta - \hat{\mathcal{P}}_j\right|\right|_1 &\approx 1 - w_{jj} + \sum_{k \neq j} w_{jk} \\ &< \sum_{k=1}^{s} (1 - w_{kk}) \leqslant \sum_{k=1}^{s} \frac{(m+r-1)!}{m!(r-1)!} e^{-mS(F_k^*\|F_k)},\end{aligned} \tag{D12}$$

which can be made arbitrarily small for large enough m. Here we do not account for the errors arising from the projected operation of $\hat{\mathcal{P}}_j^\delta$ in the orthogonal complement of the asymptotic subspace, which are exponentially small as m increases.

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