



Testing of quantum nonlocal correlations under constrained free will and imperfect detectorsAbhishek Sadhu ^{*}*Raman Research Institute, Bengaluru, Karnataka 560080, India*Siddhartha Das [†]*Center for Security, Theory and Algorithmic Research, Centre of Quantum Science and Technology, International Institute of Information Technology, Hyderabad, Gachibowli, Telangana 500032, India*

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In this work, we deal with the relaxation of two central assumptions in standard locally realistic hidden variable (LRHV) inequalities: free will in choosing measurement settings, and the presence of perfect detectors at the measurement devices. Quantum correlations violating LRHV inequalities are called quantum nonlocal correlations. In principle, in an adversarial situation, there could be a hidden variable introducing bias in the selection of measurement settings, but observers with no access to that hidden variable could be unaware of the bias. In practice, however, detectors do not have perfect efficiency. A main focus of this paper is the introduction of the framework in which given a quantum state with nonlocal behavior under constrained free will, we can determine the threshold values of detector parameters (detector inefficiency and dark counts) such that the detectors are robust enough to certify nonlocality. We also introduce a class of LRHV inequalities with constrained free will, and we discuss their implications in the testing of quantum nonlocal correlations.

DOI: [10.1103/PhysRevA.107.012212](https://doi.org/10.1103/PhysRevA.107.012212)**I. INTRODUCTION**

Randomness acts as a primitive resource for various desirable information processing and computational tasks, e.g., cryptography [1–6], statistical sampling [7,8], probabilistic algorithms [9–11], genetic algorithms [12–14], simulated annealing algorithms [15–17], neural networks [18–20], etc. Generation and certification of private randomness also constitute a major goal of cryptographic protocols [4–6,21–23]. The quantum theory allows for the generation and certification of true randomness [5,6,22,24,25], which otherwise seems to be impossible within classical theory [26].

In a seminal paper [24], Bell showed that the statistical predictions of quantum mechanics cannot be explained by local realistic hidden variable (LRHV) theories. Several LRHV inequalities, also called Bell-type inequalities, which are based on two physical assumptions (the existence of local realism and the no-signalling criterion [26]), have been derived since then (see, e.g., [27,28] and references therein). Quantum systems violating these LRHV inequalities [27] are said to have quantum nonlocal correlations. Quantum nonlocal correlations are not explainable by any LRHV theories. Hence, the randomness generated from the behavior of quantum systems with nonlocal correlations can be deemed to exhibit true randomness. The presence of these nonlocal correlations allows for the generation and certification of randomness and a secret key in a device-independent way [6,25,29–33].

It is known that the experimental verification of Bell's inequality requires additional assumptions that could lead to incurring loopholes such as locality loophole [34], freedom-of-choice (measurement independence or free will) loophole [35,36], and fair-sampling loophole (detection loophole) [35,37]. In recent major breakthroughs [38–41], the incompatibility of quantum mechanics with LRHV theories has been demonstrated by considerably loophole-free experiments showing a violation of Bell's inequality by quantum states with quantum nonlocal correlations. In an experimental setup, the locality assumption requires spacelike separation between the measurement events [24,26], which leads to the locality loophole [42,43]. In a Bell experiment with a photonic setup, the detection events of the two parties are identified as belonging to the same pair by observing whether the difference in the time of detection is small [44]. This makes it possible for an adversary to fake the observed quantum correlations by exploiting the time difference between the detection events of the parties [45] giving rise to the detection loophole, which can be closed by using predefined coincidence detection windows [45].

The free will assumption [46] in the Bell-type inequality states that the parties (users) can choose the measurement settings freely or use uncorrelated random number generators. In an experimental demonstration presented in [41], human random decisions were used to choose the measurement settings in an attempt to close this loophole. In practice, the detectors are imperfect, which introduces dark counts [47] and detection inefficiency [47]. In this paper, we discuss the implications of bias in the freedom-of-choice and nonideal detectors on the LRHV tests of quantum nonlocal correlations. To the best of our knowledge, the assumption of free will was relaxed

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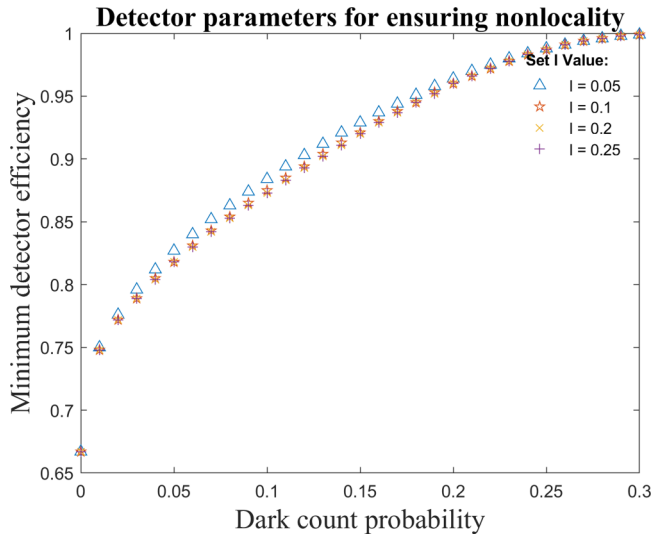


FIG. 1. In this figure, we plot the minimum detection efficiency η as a function of the dark count probability δ obtained using Algorithm 1. For each pair of (η, δ) in this figure, there exists a quantum behavior $\{p(a^{id}, b^{id}|xy)\}$ that violates Eq. (8). From such a behavior, $\{p(a^{ob}, b^{ob}|xy)\}$ is obtained by using the values of the pair (η, δ) . The behavior $\{p(a^{ob}, b^{ob}|xy)\}$ violates Eq. (32). See Sec. IV.

in [48] using a distance-measure-based quantification of the measurement dependence. It was shown that the Bell-CHSH inequality can be violated by sacrificing an equal amount of free will for both parties. This result was extended to the scenario of parties having different amounts of free will in [49] and for one of the parties in [50]. In an alternate approach, measurement dependence was quantified in [51,52] by bounding the probability of choosing the measurement settings to be in a given range. Following this approach, tests for nonlocality have been constructed [51,53]. These inequalities have been applied to randomness amplification protocols [53,54]. However, consideration of imperfect detectors in the implication of these measurement-dependent LRHV inequalities is still lacking, as the above-mentioned works assumed perfect detection while allowing for relaxation in measurement dependence.

A main focus of this work is to study the implications of imperfect detectors and constrained free will on the test of quantum nonlocal correlations. We adapt the approach discussed in [55] to model imperfect detectors for the Bell experiment as a sequential application of a perfectly working inner box followed by a lossy outer box. The inner box contains a quantum source whose behavior is nonlocal under constrained free will, i.e., it violates certain measurement-dependent LRHV inequality. The outer box separately introduces detector inefficiency and dark counts for each party. Using this model, we determine the threshold values of the detector parameters that make detectors robust for testing quantum nonlocality under constrained free will (e.g., see Fig. 1 with details in Sec. IV). Next for the scenario of perfect detectors, we compare the implications of two different approaches presented in [51,48] to quantify measurement dependence (a) by bounding the probability of choosing the measurement settings x (for Alice's side) and y (for Bob's side) conditioned on a hidden variable λ to be in

the range $[l, 1 - 3l]$ [51], and (b) by using a distance measure M to quantify measurement setting distinguishability [48]. This comparison is made in the two (party)-two (measurement settings per party)-two (outcome per measurement) scenario and their effects on the certification of the nonlocality. We also introduce a new set of measurement-dependent LRHV (MDL) inequalities by introducing a distance-based measurement-dependent quantity in the adapted AMP tilted Bell inequality [21], and we discuss implications and tradeoff between measurement dependence parameters and tilted parameters for the certification of quantum nonlocal correlations.

Throughout this paper, we limit our discussions to LRHV for bipartite quantum systems. The structure of this paper is as follows. In Sec. II, we introduce the framework of locality and measurement dependence, which is a constraint that limits the free will of the user. We present a model in which the user is tricked by an adversary into thinking that they have freedom-of-choice for the measurement. In Sec. III, we consider and compare two different approaches to quantify the measurement dependence in terms of parameters l and M mentioned earlier. We determine the critical values of these parameters necessary for the certification of nonlocality in the measurement dependence settings, and we compare them with the amount of violation obtained for the Bell-CHSH inequality, tilted Hardy relations, and the tilted Bell inequalities. In Sec. IV, we determine the threshold values of the detector parameters, namely inefficiency and dark count probability, such that the detectors are robust enough to certify nonlocality in the presence of constrained free will. In Sec. V, we introduce a new set of LRHV inequalities adapted from the AMP tilted Bell inequality using distance-measure-based measurement dependence quantities. We use these inequalities to observe the effect of relaxing the free will assumption for either party on the certification of quantum nonlocal correlations.

II. THE ADVERSARIAL ROLE IN THE CHOICE OF MEASUREMENT SETTINGS

Formally a quantum behavior of a device with bipartite state ρ_{AB} is given by its quantum representation $\{p(ab|xy)\}$, where $p(ab|xy) = \text{Tr}[\Gamma_a^x \otimes \Gamma_b^y \rho_{AB}]$, with $\{\Gamma_a^x\}_a$ and $\{\Gamma_b^y\}_b$ being positive operator valued measures (POVMs) for each input (x, y) to the device, for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. If a state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ is pure, then we may simply represent it as a ket $|\psi\rangle_{AB}$ instead of density operator ρ_{AB} . Henceforth, the abbreviation MDL inequalities would stand for measurement-dependent LRHV inequalities.

We consider the Bell scenario in which two parties, Alice and Bob, share a bipartite quantum state ρ_{AB} . Each party can choose to perform one of two measurements available to them, i.e., $|\mathcal{X}| = 2 = |\mathcal{Y}|$. We attribute POVM $\{\Lambda_a^x\}_x$ to Alice and $\{\Lambda_b^y\}_y$ to Bob. Each of these measurements can have two outcomes. We denote measurement outcomes for Alice and Bob by a and b , respectively. The statistics of the measurement outcomes in the experiment can then be described by the probability distribution $\mathbf{P} = \{p(ab|xy)\}$, which is also termed as behavior. In this framework, let there exist a hidden variable λ belonging to some hidden-variable space, Λ . The probability distribution of the outputs conditioned on the inputs can then

TABLE I. Probability distribution for choice of settings by Alice and Bob based on the hidden variable λ . An adversary can use this distribution, and by suitably choosing the parameters of the table the adversary can trick Alice and Bob into thinking they have free will in choosing the measurement settings.

| Probability distributions of Eve | | |
|----------------------------------|--|--|
| Joint setting | Distribution 1 (λ_1) | Distribution 2 (λ_2) |
| $p(0, 0 \lambda)$ | $\cos^2(\phi_{s_1})$ | $\cos^2(\phi_{s_2})$ |
| $p(0, 1 \lambda)$ | $\sin^2(\theta_{s_1}) \sin^2(\phi_{s_1})$ | $\sin^2(\theta_{s_2}) \sin^2(\phi_{s_2})$ |
| $p(1, 0 \lambda)$ | $\cos^2(\delta_{s_1}) \cos^2(\theta_{s_1})$ | $\cos^2(\delta_{s_2}) \cos^2(\theta_{s_2})$ |
| $p(1, 1 \lambda)$ | $\sin^2(\phi_{s_1}) \sin^2(\delta_{s_1}) \cos^2(\theta_{s_1})$ | $\sin^2(\phi_{s_2}) \sin^2(\delta_{s_2}) \cos^2(\theta_{s_2})$ |

be expressed as

$$p(ab|xy) = \sum_{\lambda \in \Lambda} p(ab|xy\lambda)p(\lambda|xy). \quad (1)$$

The hidden variable λ [distribution according to $p(\lambda)$] can provide an explanation of the observed experimental (measurement) statistics. In each run of the experiment, there exists a fixed λ that describes the outcome of the experimental trial following the distribution $p(ab|xy\lambda)$. After multiple runs of the experiment, the output statistics are described by sampling from the distribution $p(\lambda|xy)$.

In an adversarial scenario, Alice and Bob can believe that they are choosing all the settings with equal probability, i.e., $p(xy) = 1/4$ for each pair (x, y) while an adversary biases their choice in the scale λ . The adversary can distribute the settings chosen by Alice and Bob according to

$$p(xy) = \sum_{\lambda \in \Lambda} p(xy|\lambda)p(\lambda). \quad (2)$$

In Eq. (2), let λ take two values, λ_1 and λ_2 , whose probability distributions are given as $p(\lambda_1) = \sin^2(\theta_\lambda)$ and $p(\lambda_2) = \cos^2(\theta_\lambda)$, respectively. In the simplest scenario, x and y can each take values 0 or 1 and there are four possible ways in which the measurement settings can be chosen by Alice and Bob, i.e., $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Let the probability of choosing the measurement setting (x, y) conditioned on the hidden variable be distributed according to Table I. To observe the effect of the conditional probability distribution for the choice of x and y from Table I on $p(xy)$, consider the following examples:

(i) For the case of $\theta_{s_2} = 0.6847$, $\phi_{s_2} = 1.2491$, $\phi_{s_1} = 0.8861$, $\delta_{s_2} = 1.2491$, $\theta_\lambda = 0.785398$, $\theta_{s_1} = 0.50413$, and $\delta_{s_1} = 0.175353$. It can be seen that this choice of parameters set $p(xy) = 0.25 \forall (x, y)$.

(ii) For the case of $\theta_{s_2} = 0.5796$, $\phi_{s_2} = 1.2491$, $\phi_{s_1} = 0.6847$, $\delta_{s_2} = 0.75$, $\theta_\lambda = 0.57964$, $\theta_{s_1} = 0.793732$, and $\delta_{s_1} = 1.06395$. It can be seen that this choice of parameters set $p(xy) = 0.25 \forall (x, y)$.

The above examples show that by proper choice of parameters in Table I, the adversary can trick Alice and Bob into thinking that they have free will in choosing the measurement setting. In the scenario in which Alice and Bob choose between the measurement settings with unequal probabilities,

i.e., $p(xy) \neq 1/4$ for each pair (x, y) , the adversary can adjust the parameters of Table I accordingly. In the presence of a bias in the choice of measurement settings in the λ scale of which Alice and Bob are unaware, the following constraints [56] can be imposed on the conditional joint probability distribution:

(a) The signal locality, i.e., no-signaling, assumption imposes the factorizability constraint on the conditional joint probability distribution,

$$p(ab|xy\lambda) = p(a|x\lambda)p(b|y\lambda). \quad (3)$$

(b) The measurement independence, i.e., freedom of choice or free will, assumption requires that λ does not contain any information about x and y , which is equivalent to stating

$$p(\lambda|xy) = p(\lambda)$$

$$\text{or equivalently, } p(xy|\lambda) = p(xy). \quad (4)$$

III. QUANTIFYING MEASUREMENT DEPENDENCE

In this section, we first review two different approaches considered in [51,53] and [48–50] to quantify the measurement dependence. We then compare these two approaches and observe their effects on the tests of quantum nonlocal behaviors.

A. Review of MDL inequalities from prior works

We review the approach discussed in [51,52] to quantify measurement dependence by bounding the probability of choice of measurement settings conditioned on a hidden variable to be in a specific range (Sec. III A 1). Then we review the approach discussed in [48–50] to quantify measurement dependence using a distance measure (Sec. III A 2).

1. Bound on the probability of choosing the measurement settings

In Refs. [51,52], we observe that the probability of Alice and Bob to choose measurement settings x and y conditioned on λ can be bounded as

$$l \leq p(xy|\lambda) \leq h, \quad (5)$$

where $0 \leq l \leq p(xy|\lambda) \leq h \leq 1$. If Alice and Bob each choose from two possible values of measurement settings, then $l = h = 0.25$ corresponds to the complete measurement independence; other values of l and h represent bias in the choice of measurement settings.

In the two (user)–two (measurement settings per user)–two (outcome per measurement) scenario with $a, b \in \{+, -\}$ and $x, y \in \{0, 1\}$, it was shown in [51,52] that all the measurement-dependent local correlations satisfy the PRBLG MDL inequality

$$lp(++00) - h[p(++01) + p(-+10) + p(++11)] \leq 0. \quad (6)$$

A two-dimensional slice in the nonsignaling space is shown in Fig. 2 (figure from [51]). In the figure, the quantum set is bounded by the green line, and the set of nonsignaling correlations lie within the black triangle. In Fig. 2, the red dotted line corresponds to Eq. (6) with $h = 1 - 3l$. If we

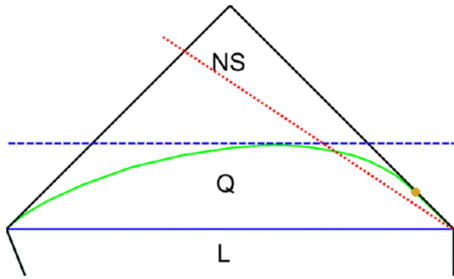


FIG. 2. A two-dimensional slice of the no-signaling space with the MDL correlations discussed in [51]. The blue line encloses the set of Bell-CHSH local correlations. The green line encloses the quantum set. The black triangle encloses the no-signaling distributions. For the case of $h = 1 - 3l$, the inequality (6) shifts from the Bell-CHSH boundary to the no-signaling boundary via the red dotted line.

set $h = 1 - 3l$, the PRBLG MDL inequality tilts from the Bell-CHSH inequality ($l = 0.25$) to the nonsignaling border ($l = 0$). For $h = 1 - 3l$, Eq. (6) is expressed as

$$lp(++00) - (1 - 3l)[p(+ - 01) + p(- + 10) + p(++11)] \leq 0. \quad (7)$$

We note that if Alice and Bob believe they have complete measurement dependence, i.e., $p(xy) = 0.25 \forall (x, y)$, then Eq. (7) reduces to

$$lp(++|00) - (1 - 3l)[p(+ - |01) + p(- + |10) + p(++|11)] \leq 0. \quad (8)$$

It follows from [53] that invoking the PRBLG MDL inequality (7) in the tilted Hardy test [53], we obtain the ZRLH MDL inequality. The ZRLH MDL inequality is expressed as

$$l[p(++|00) + wp(--|00) - \max\{0, w\}] - (1 - 3l)[p(+ - |01) + p(- + |10) + p(++|11)] \leq 0, \quad (9)$$

where w is the tilting parameter taking real numbers in the range, $w \in (-0.25, 1)$. We call the quantum behaviors that violate Eqs. (8) and (9) as quantum nonlocal in the presence of measurement dependence.

2. Distance measure to quantify measurement distinguishability

We discussed in Sec. II that the experimental statistics described by the joint probability distribution $p(ab|xy)$ can be

$$M_1 \equiv \max \left\{ \int d\lambda |p(\lambda|x_1, y_1) - p(\lambda|x_2, y_1)|, \int d\lambda |p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)| \right\}, \quad (15)$$

$$M_2 \equiv \max \left\{ \int d\lambda |p(\lambda|x_1, y_1) - p(\lambda|x_1, y_2)|, \int d\lambda |p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2)| \right\}. \quad (16)$$

M_1 quantifies the measurement dependence for Alice's settings keeping Bob's settings fixed, and similarly the other way round for M_2 . The above parameters will be useful in deriving the bounds on the AMP tilted Bell inequalities [21] in Sec. V.

explained by $\lambda \in \Lambda$ in the following form:

$$p(ab|xy) = \int d\lambda p(ab|xy\lambda)p(\lambda|xy). \quad (10)$$

The assumption of the measurement independence constrains the probability distribution of measurement settings via

$$p(\lambda|xy) = p(\lambda). \quad (11)$$

Equation (11) implies that no extra information about λ can be obtained from the knowledge of x and y . This is equivalent to saying

$$p(xy|\lambda) = \frac{p(\lambda|xy)p(xy)}{p(\lambda)} = p(xy). \quad (12)$$

Equation (12) implies that Alice and Bob have complete freedom in choosing the measurement settings x and y , respectively. If $x \in U \equiv \{x_1, x_2\}$ and $y \in V \equiv \{y_1, y_2\}$, measurement dependence implies $p(\lambda|x_1, y_1) \neq p(\lambda|x_2, y_2)$. Distinguishability between $p(\lambda|x_1, y_1)$ and $p(\lambda|x_2, y_2)$ can be quantified using a distance measure defined in [48],

$$M = \int d\lambda |p(\lambda|x_1, y_1) - p(\lambda|x_2, y_2)|. \quad (13)$$

We can express the probability to successfully distinguish $p(\lambda|x_1, y_1)$ and $p(\lambda|x_2, y_2)$ based on the knowledge of λ as

$$F = \frac{1}{2} \left(1 + \frac{M}{2} \right). \quad (14)$$

When $M = 0$, we have $F = \frac{1}{2}$, which suggests that no additional information about the hidden variable λ can be obtained from knowing the choice of measurement settings.

This observation is consistent with maximum free will that Alice and Bob have while choosing the measurement settings, whereas for $M = 2$ we have $F = 1$, which suggests that the complete information about the hidden variable λ can be obtained from knowing the choice of the measurement settings. This observation is consistent with no free will for Alice and Bob while choosing the measurement settings.

The local degrees of measurement dependence for Alice and Bob as introduced in [48] are given by

B. Comparison and discussion on the implications of different measurement-dependent LRHV inequalities

In this section, we check for the allowed values of measurement dependence parameter l to ensure nonlocality of

TABLE II. This table presents the expectation values of operators from [21] that violate the AMP tilted Bell inequality given in Eq. (34). The parameter $\mu := \tan^{-1}(\sin(2\phi)/\alpha)$, where ϕ is the state parameter and α is the tilting parameter in Eq. (34) as defined in [21].

| Expectation values of the operators | |
|---|--|
| $\langle x_1 \rangle = \cos(2\phi);$ | $\langle x_2 \rangle = 0;$ |
| $\langle y_1 \rangle = \cos(2\phi) \cos(\mu);$ | $\langle y_2 \rangle = \cos(2\phi) \cos(\mu);$ |
| $\langle x_1, y_1 \rangle = \cos(\mu);$ | $\langle x_1, y_2 \rangle = \cos(\mu);$ |
| $\langle x_2, y_1 \rangle = \sin(2\phi) \sin(\mu);$ | $\langle x_2, y_2 \rangle = -\sin(2\phi) \sin(\mu).$ |

quantum behaviors that are used to obtain randomness. Consider the quantum behavior in Table II that violates the AMP tilted Bell inequality (34). In the limit of $\alpha \rightarrow \infty$, close to two bits of randomness can be obtained from such a behavior by violating the AMP tilted Bell inequality [21]. For the quantum behavior in Table II, the PRBLG MDL inequality (8) reduces to

$$l \leq \frac{3\alpha t + \alpha \cos(2\phi)t - 2\sqrt{2}\alpha \sin^2(\phi) + \sqrt{2}[\cos(4\phi) - 1]}{10\alpha t + 4\alpha \cos(2\phi)(t + \sqrt{2}) - 2\sqrt{2}[\alpha + 3 \sin^2(2\phi)]}, \tag{17}$$

where $t = \sqrt{-\frac{\cos(4\phi)}{\alpha^2} + \frac{1}{\alpha^2} + 2}$. In the limit of $\alpha \rightarrow \infty$, from Eq. (17) we have $l \leq 0.25$. We see that in the limit of $\alpha \rightarrow \infty$ the quantum behavior in Table II does not violate any PRBLG MDL inequality. For $\alpha = 1$ (which is equivalent to the Bell-CHSH inequality) and $\phi = \frac{\pi}{4}$ (the maximum violation of the Bell-CHSH inequality,) we see from Eq. (17) that the quantum behavior in Table II violates the family of PRBLG MDL inequalities for $l > 0.2023$.

We observe in Fig. 3 that for a fixed value of α , the range of the allowed values of ϕ from Table II that violates the PRBLG MDL inequality given by Eq. (8) increases with the increase in l . Also as α increases, for a particular value of l , there is a decrease in the range of the allowed values of ϕ from Table II that violates the AMP tilted Bell inequality. It was shown in [53] that the state $\rho_g = |\psi_g\rangle\langle\psi_g|$ (18) and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$ can be used to obtain close to 1.6806 bits of global randomness (at $\theta \approx 1.13557$), where

$$|\psi_g\rangle = \cos\left(\frac{\theta}{2}\right)|00\rangle - \sin\left(\frac{\theta}{2}\right)|11\rangle, \tag{18}$$

$$A_0^g = B_0^g = \frac{-\sqrt{2} \sin(\theta) \sqrt{\sin(\theta)}}{[2 - \sin(\theta)] \sqrt{\sin(\theta) + 1}} \sigma_x + \frac{-[\sin(\theta) + 2] \sqrt{1 - \sin(\theta)}}{[2 - \sin(\theta)] \sqrt{\sin(\theta) + 1}} \sigma_z, \tag{19}$$

$$A_1^g = B_1^g = \frac{\sqrt{2} \sqrt{\sin(\theta)}}{\sqrt{\sin(\theta) + 1}} \sigma_x - \frac{\sqrt{1 - \sin(\theta)}}{\sqrt{\sin(\theta) + 1}} \sigma_z, \tag{20}$$

such that $\theta = \sin^{-1}(3 - \sqrt{4w + 5})$ and A_x^g, B_y^g for $x, y \in \{0, 1\}$ denote measurement settings for Alice and Bob, respectively.

Observation 1. Let us consider the quantum behavior given by the state ρ_g and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$. For

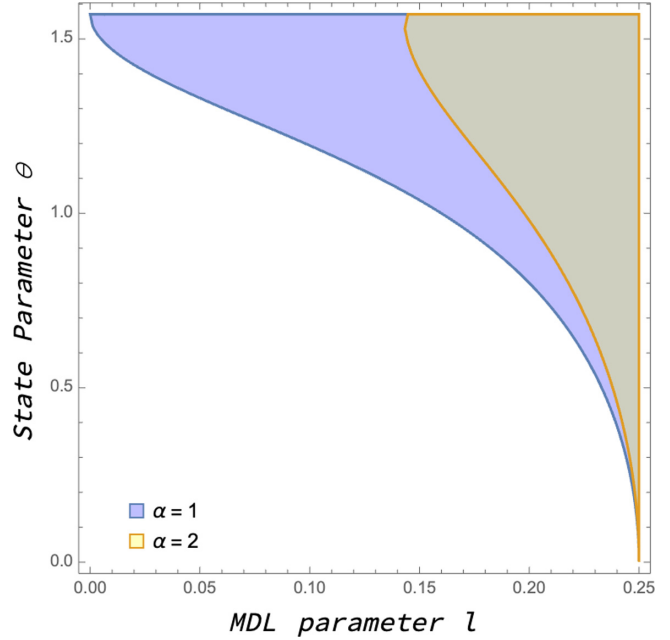


FIG. 3. For $\alpha \in \{1, 2\}$, we plot the range of the state parameter ϕ from Table II [$\phi = \frac{1}{2} \sin^{-1}(\alpha \tan \mu)$] that violates the PRBLG MDL inequality [given by Eq. (8)] for different values of the measurement dependence parameter l .

such a behavior, the PRBLG MDL inequality given by Eq. (8) reduces to

$$\frac{2l}{(\xi - 1)^2} [31\xi + w(4\xi - 34) - 69] + \frac{2l(\xi - 5)\sqrt{\xi - 2}}{\sqrt{4 - \xi}(\xi - 1)} \sqrt{-4w + 6\xi - 13} - \frac{2(1 - 3l)}{1 - \xi} \left(8w - 10\xi + \frac{\sqrt{\xi - 2} \sin(2 \sin^{-1}(3 - \xi))}{\sqrt{4 - \xi}} \right) + 22 \leq 0, \tag{21}$$

where $\xi = \sqrt{4w + 5}$. On inspection we see that for all $w \in (-0.25, 1)$, Eq. (21) reduces to $l \leq 0$. The quantum behavior specified by the state ρ_g and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$ is nonlocal for $l > 0$.

For the given quantum behavior, the ZRLH MDL inequality (9) reduces to

$$\frac{1}{2(\xi - 1)} \left(2l(-4w + 7\xi - 15) + \frac{(1 - 3l)\sqrt{\xi - 2} \sin(2 \sin^{-1}(3 - \xi))}{\sqrt{4 - \xi}} + 8w - 10\xi + 22 \right) - l \max(0, w) \leq 0. \tag{22}$$

On inspection we see that for all $w \in (-0.25, 1)$, Eq. (22) reduces to $l \leq 0$.

That is, the quantum behavior given by state ρ_g and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$ violates both PRBLG and ZRLH MDL inequalities, and its quantum nonlocality can be certified for all possible $l > 0$.

IV. IMPERFECT DETECTOR AND CONSTRAINED FREE WILL

We model the detection units of Alice and Bob using a two-box approach following Ref. [55]. There is an inner box containing a quantum source generating bipartite quantum states whose behavior is nonlocal under constrained free will but assuming that detectors are perfect. Nonlocality of the quantum behavior in the inner box is tested by violation of a given MDL inequality. The output of the inner box is quantum nonlocal behavior that violates a given MDL inequality. An outer box introduces the detector imperfections, namely the detection inefficiency and dark counts. The quantum nonlocal behavior obtained from the inner box gets mapped at the outer box to the output behavior with detector imperfection parameters. The output behavior then undergoes an LRHV test based on which we are able to determine the threshold values of detection inefficiency and dark counts such that the given quantum nonlocal behavior can still be certified to be nonlocal with imperfect detectors. A deviation from a two-box approach in [55] is the introduction of the measurement dependence assumption to the working of the inner box. Alice and Bob have access only to the input settings and the outputs of the outer box. We assume that either party (Alice and Bob) has access to two identical detectors that can distinguish between the orthogonal outputs.

The measurement outcomes for the inner box are labeled as a^{id} and b^{id} , respectively. We note that a^{id} and b^{id} can each take values from the set $\{+, -\}$. Introducing nonunit detection efficiency, $0 \leq \eta \leq 1$, and nonzero dark count probability, $0 \leq \delta \leq 1$, in the outer box, the ideal two-outcome scenario becomes a four-outcome scenario with the addition of no-detection event Φ and the dark-count event χ . These events are defined in the following way:

Φ : One particle is sent to the party, and none of the two detectors of that party click.

χ : One particle is sent to the party, and both the detectors of the party click.

We label the measurement outcomes for Alice and Bob obtained from the outer box as a^{ob} and b^{ob} , respectively. We note that a^{ob} and b^{ob} can each take values from the set $\{+, -, \Phi, \chi\}$. Furthermore, we assume that Alice and Bob's detection units have identical values for η and δ . The conditional probability of observing the outcome t^{ob} from the outer box conditioned on observing t^{id} in the ideal scale is given by $p(t^{\text{ob}}|t^{\text{id}})$ with $t^{\text{ob}} \in \{a^{\text{ob}}, b^{\text{ob}}\}$ and $t^{\text{id}} \in \{a^{\text{id}}, b^{\text{id}}\}$. The observed joint probabilities can then be expressed as [55]

$$p(a^{\text{ob}}b^{\text{ob}}|xy) = \sum_{a^{\text{id}}, b^{\text{id}}} p(a^{\text{ob}}|a^{\text{id}})p(b^{\text{ob}}|b^{\text{id}})p(a^{\text{id}}b^{\text{id}}|xy). \quad (23)$$

We relax the free-will assumption in the inner box by introducing the hidden variable $\lambda \in \Lambda$. Considering this

assumption, $p(a^{\text{id}}b^{\text{id}}|xy)$ is expressed as

$$p(a^{\text{id}}b^{\text{id}}|xy) = \sum_{\lambda \in \Lambda} p(a^{\text{id}}b^{\text{id}}|xy\lambda)p(\lambda|xy). \quad (24)$$

The hidden variable λ [distributed according to $p(\lambda)$] provides an explanation of the observed experimental statistics of the inner box. The distribution of settings that are chosen by Alice and Bob depends on λ via the following relation:

$$p(xy) = \sum_{\lambda \in \Lambda} p(xy|\lambda)p(\lambda). \quad (25)$$

If we impose the locality condition from Eq. (3) on the experimental statistics of the inner box, we arrive at the following factorizability constraint:

$$p(a^{\text{id}}b^{\text{id}}|xy\lambda) = p(a^{\text{id}}|x\lambda)p(b^{\text{id}}|y\lambda). \quad (26)$$

Also, if we impose the measurement independence assumption from Eq. (4), we arrive at the following constraint:

$$p(xy|\lambda) = p(xy) \text{ or equivalently } p(\lambda|xy) = p(\lambda). \quad (27)$$

The output statistics of the outer box for imperfect detectors can depend on the output of the inner box in the following four ways [55]:

(i) No particle is detected on either of the detectors, and no dark count detection event takes place. We can then write the following:

$$p(a^{\text{ob}}|a^{\text{id}}) = (1 - \eta)(1 - \delta)^2. \quad (28)$$

(ii) No particle is detected by the detector that should have detected it, and a dark count takes place in the other detector. We can then write the following:

$$p(a^{\text{ob}}|a^{\text{id}}) = (1 - \eta)(1 - \delta)\delta. \quad (29)$$

(iii) Either the particle is detected by one of the detectors and a dark count takes place in the other detector, or the particle is not detected and dark counts take place in both of the detectors. We can then write the following:

$$\begin{aligned} p(a^{\text{ob}}|a^{\text{id}}) &= \eta\delta + (1 - \eta)\delta^2 \\ &= \delta[1 - (1 - \eta)(1 - \delta)]. \end{aligned} \quad (30)$$

(iv) Either the particle is detected and no dark count takes place, or the particle is not detected and a dark count takes place in the detector in which the particle should have been registered. We can then write the following:

$$\begin{aligned} p(a^{\text{ob}}|a^{\text{id}}) &= \eta(1 - \delta) + (1 - \eta)\delta(1 - \delta) \\ &= (1 - \delta)[1 - (1 - \eta)(1 - \delta)]. \end{aligned} \quad (31)$$

The quantum nonlocal behavior obtained from the inner box after getting mapped to the output behavior with

Algorithm 1

```

1: Initialize:
    $l \leftarrow$  parameter of Eq. (8)
2: for  $\delta$  in range (0, 1) do
3:   for  $\eta$  in range (0, 1) do
4:      $P(a^{\text{id}}, b^{\text{id}}|xy) \leftarrow$  inner box quantum behavior
5:      $P(a^{\text{ob}}, b^{\text{ob}}|xy) \leftarrow f(P(a^{\text{id}}, b^{\text{id}}|xy), \delta, \eta)$ 
6:     obj  $\leftarrow$  LHS of Eq. (32) for  $P(a^{\text{ob}}, b^{\text{ob}}|xy)$ 
7:     MDL  $\leftarrow$  LHS of Eq. (8) for  $P(a^{\text{id}}, b^{\text{id}}|xy)$ 
8:     maximize: obj
9:     such that: MDL > 0
10:    opt  $\leftarrow$  max value of obj
11:    if opt > 0 then
12:      print:  $\delta, \eta$ 
13:    Break

```

Algorithm 2

```

1: Initialize:
    $w \leftarrow$  parameter of Eq. (9)
    $l \leftarrow$  parameter of Eq. (9)
2: for  $\delta$  in range (0, 1) do
3:   for  $\eta$  in range (0, 1) do
4:      $P(a^{\text{id}}, b^{\text{id}}|xy) \leftarrow$  inner box quantum behavior
5:      $P(a^{\text{ob}}, b^{\text{ob}}|xy) \leftarrow f(P(a^{\text{id}}, b^{\text{id}}|xy), \delta, \eta)$ 
6:     obj  $\leftarrow$  LHS of Eq. (32)
7:     MDL  $\leftarrow$  LHS of Eq. (9)
8:     maximize: obj
9:     such that: MDL > 0.
10:    opt  $\leftarrow$  max value of obj
11:    if opt > 0 then
12:      print:  $\delta, \eta$ 
13:    Break

```

detector imperfection parameters remains nonlocal if the behavior obtained from the outer box violates the inequality [55]

$$p(+ + |00) + p(+ + |01) + p(+ + |10) - p(+ + |11) - p_A(+|0) - p_B(+|0) \leq 0, \tag{32}$$

where $p_A(o|s)$ and $p_B(o|s)$ are the probabilities of Alice and Bob obtaining the outcome o on measuring s .

At first, let us assume there is a quantum source in the inner box that is generating a bipartite quantum state whose behavior $\{p(a^{\text{id}}, b^{\text{id}}|xy)\}$ violates the PRBLG MDL inequality given by Eq. (8) assuming that detectors are perfect. The quantum behavior $\{p(a^{\text{id}}, b^{\text{id}}|xy)\}$ obtained from the inner box gets mapped at the outer box to the behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ with the introduction of the detector inefficiency η and dark count probability δ . The behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ is then inserted in Eq. (32) to obtain the critical detector parameters using Algorithm 1. For a fixed value of δ we obtain the minimum value of η that violates Eq. (32) using Algorithm 1. We abbreviate left-hand side as LHS.

We plot the critical values of η and δ obtained using Algorithm 1 in Fig. 1.

Observation 2. From Fig. 1, we see that the minimum value of η for a given δ takes the highest value for $l = 0$ and decreases as l increases. For a fixed value of l , the minimum value of η increases monotonically with the increase in dark count probability. We note that for $\delta = 0$, we have $\eta \approx 0.667$ for all the values of l .

We next assume there is a quantum source in the inner box that is generating a bipartite quantum state whose behavior $\{p(a^{\text{id}}, b^{\text{id}}|xy)\}$ violates the ZRLH MDL inequality given by Eq. (9) assuming that detectors are perfect. The quantum behavior $\{p(a^{\text{id}}, b^{\text{id}}|xy)\}$ obtained from the inner box gets mapped at the outer box to the behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ with the introduction of the detector inefficiency η and dark count probability δ . The behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ is then inserted in Eq. (32) to obtain the critical detector parameters using Algorithm 2. For a fixed value of δ we obtain the minimum

value of η that violates Eq. (32) using Algorithm 2. We abbreviate left-hand side as LHS.

We plot in Fig. 1, the critical values of η and δ from Algorithm 2 for $w = 0$.

Observation 3. The ZRLH MDL inequality given by Eq. (9) and the PRBLG MDL inequality given by Eq. (8) reduce to the same inequality for $l = 0$, and they are independent of w . For $l = 0$, Eqs. (9) and (8) cannot be violated [57]. For $w = 0$, Eqs. (9) and (8) reduce to the same inequality, and their violation can happen only when $l > 0$ [58]. For the case $w = 0$, we observe the same dependence for the threshold detector parameters as can be seen from Fig. 1. We observe from Fig. 1 that for a fixed value of l , the minimum detection efficiency increases monotonically with the dark count probability.

We comment here that using Algorithm 2, one can calculate the detector requirements for other values of (w, l) not mentioned in this section. Next, we consider the state ρ_g and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$ with $\theta \approx 1.13557$. This choice of state and measurement settings was shown to ensure 1.6806 bits of global randomness in [53].

Observation 4. Consider that the quantum source in the inner box generates a bipartite quantum state ρ_g . Let Alice and Bob choose the measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$, with $\theta \approx 1.13557$. With this choice of state and measurement settings, we obtain the quantum behavior $\{p(a^{\text{id}}, b^{\text{id}}|xy)\}$ assuming that the detectors are perfect. The output behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ obtained from the outer box is evaluated using Eq. (23). The behavior $\{p(a^{\text{ob}}, b^{\text{ob}}|xy)\}$ is nonlocal if we have a violation of the inequality

$$\begin{aligned} & \delta\eta[\delta(9.50424\delta - 4\delta^2 - 9.00848) + 4.25636] \\ & + \eta^2[\delta(2\delta^3 - 5.50424\delta^2 + 5.82473\delta - 3.13675) \\ & + 0.816258] + 2\delta(\delta - 1)(\delta^2 - \delta + 1) \\ & - 0.752119\eta \leq 0. \end{aligned} \tag{33}$$

The allowed values of the pair (η, δ) for which $p(a^{\text{ob}}, b^{\text{ob}}|xy)$ is quantum nonlocal shown in Fig. 4. In Fig. 4 we observe that for $\delta = 0$ the minimum detection efficiency $\eta_{\text{crit}} \approx 0.92$ is required to ensure that $p(a^{\text{ob}}, b^{\text{ob}}|xy)$ is quantum nonlocal.

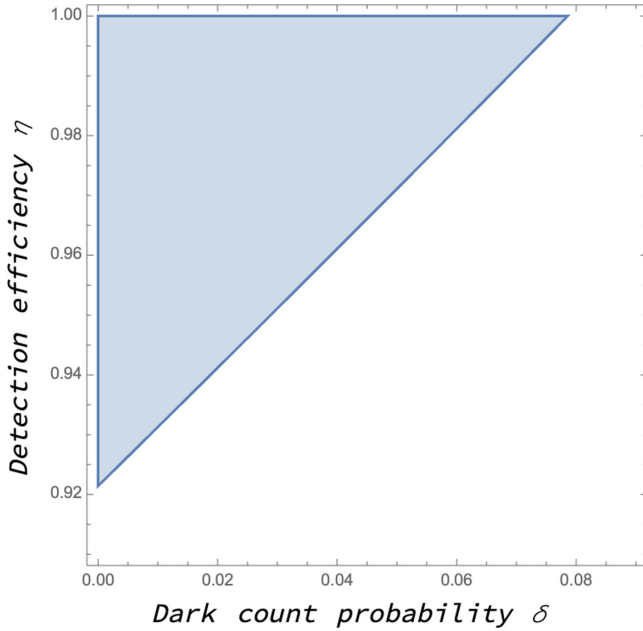


FIG. 4. In this figure we plot the detector parameters η and δ for which the behavior $\{p(a^{ob}, b^{ob}|xy)\}$ produced in the outer box is quantum nonlocal when the quantum behavior $\{p(a^{id}, b^{id}|xy)\}$ is produced in the inner box using the state ρ_g and measurement settings $\{A_0^g, A_1^g, B_0^g, B_1^g\}$ for $\theta \approx 1.13557$.

We also observe that η_{crit} increases monotonically with an increase in δ .

V. THE TILTED BELL INEQUALITY

The AMP tilted Bell inequality is given as [21]

$$I_\alpha^\beta := \beta \langle x_1 \rangle + \alpha \langle x_1 y_1 \rangle + \alpha \langle x_1 y_2 \rangle + \langle x_2 y_1 \rangle - \langle x_2 y_2 \rangle \leq \beta + 2\alpha. \quad (34)$$

It plays an important role in (a) demonstrating the inequivalence between the amount of certified randomness and the amount of nonlocality [21], (b) self-testing of all bipartite pure entangled states [59], (c) protocol for device-independent quantum random number generation with a sublinear amount of quantum communication [60], and (d) unbounded randomness certification from a single pair of entangled qubits with sequential measurements [61].

In the following proposition, we obtain the bound on I_α^β when the measurement independence assumption is relaxed (see Appendix A for proof).

Proposition 1. The AMP tilted Bell expression I_α^β in the presence of locality and the relaxed measurement independence is bounded by

$$I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\}, \quad (35)$$

where M_1 and M_2 are the measurement dependence parameters for Alice and Bob.

A. Comparison of one-sided and two-sided measurement dependence to ensure quantum representation

Let Alice and Bob have free will in choosing the measurement settings. Then the maximum violation of I_α^β obtained by

quantum nonlocal behaviors is given by [21]

$$I_\alpha^\beta \leq 2\sqrt{(1 + \alpha^2)\left(1 + \frac{\beta^2}{4}\right)}. \quad (36)$$

As direct consequences of Proposition 1, we have following corollaries.

Corollary 1. When $M_1 = M_2 = M$, the quantum nonlocal behaviors that maximally violate Eq. (34) with the amount of violation given by the right-hand side (RHS) of Eq. (36) remain nonlocal for

$$M < \frac{-2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}}{1 + 2\alpha}. \quad (37)$$

For $\alpha = 1$ and $\beta = 0$ (the Bell-CHSH inequality), Eq. (37) reduces to $M < \frac{2}{3}(\sqrt{2} - 1) \approx 0.276$ and is consistent with the observation in [49].

Corollary 2. When $M_1 = 0$ and $M_2 = M$, the quantum nonlocal behaviors that maximally violate Eq. (34), with the amount of violation given by the RHS of Eq. (36), remain nonlocal for

$$M < -2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}. \quad (38)$$

For $\beta = 0$ and $\alpha = 1$ (the Bell-CHSH inequality), Eq. (38) reduces to $M < 2(\sqrt{2} - 1) \approx 0.828$ and is consistent with the observation in [49].

Corollary 3. When $M_1 = M$ and $M_2 = 0$, the quantum nonlocal behaviors that maximally violate Eq. (34), with the amount of violation given by the RHS of Eq. (36), remain nonlocal for

$$M < \frac{1}{\alpha}[-2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}]. \quad (39)$$

For $\beta = 0$ and $\alpha = 1$ (the Bell-CHSH inequality), Eq. (39) reduces to $M < 2(\sqrt{2} - 1) \approx 0.828$ and is consistent with the observation in [49].

In Fig. 5, we plot an upper bound on M as a function of α for MDL violating behaviors when $\beta = 0$ and $\alpha \geq 1$. The values of M and α from Fig. 5 ensure that the quantum nonlocal behaviors that maximally violate Eq. (34) remain nonlocal in the presence of measurement dependence. We observe in Fig. 5 that for $\beta = 0$ and $\alpha \geq 1$, the introduction of one-sided measurement dependence allows for higher values of M (implying a lower degree of freedom-of-choice) as compared to introducing both-sided measurement dependence. Also, for the case of one-sided measurement dependence, Alice can have higher values of M as compared to Bob.

B. Bounds on the measurement dependence for testing nonlocality

We observe in Proposition 1 that in the presence of relaxed measurement dependence, the behaviors $\{p(ab|xy)\}$ that violates Eq. (40) are nonlocal,

$$I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\}. \quad (40)$$

In the following, we discuss some of the cases when Eq. (40) can be violated.

(a) We consider a situation when both Alice and Bob have the same measurement dependence. We have $M_1 = M_2 = M$.

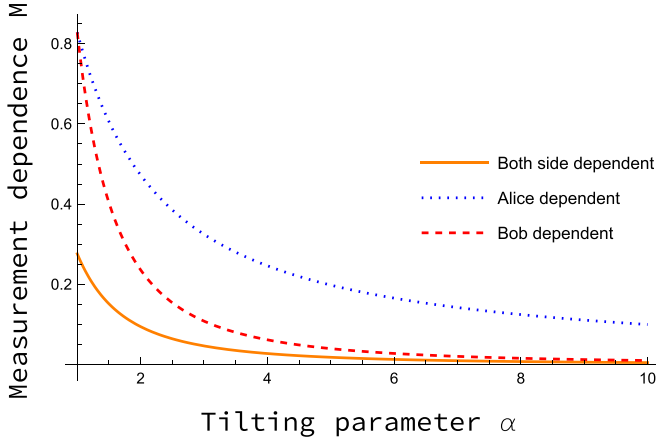


FIG. 5. In this figure, we consider $\beta = 0$ and plot the maximum values of the measurement dependence parameter M as a function of the tilting parameter α in Eq. (34). The values of M and α in this figure ensure that the quantum nonlocal behaviors that maximally violate Eq. (34) remain nonlocal for the cases of (a) both-sided measurement dependence (in orange), (b) only Alice has measurement dependence (in dotted blue line), and (c) only Bob has measurement dependence (in dashed red line).

For this case, violating Eq. (40) requires

$$I_{\alpha}^{\beta} > \beta + 2\alpha + \min\{(2\alpha + 1)M, 2\}. \quad (41)$$

If $(2\alpha + 1)M \geq 2$, I_{α}^{β} reaches the no-signaling boundary, whereas if $(2\alpha + 1)M < 2$, then Eq. (41) reduces to the inequality

$$M < \frac{I_{\alpha}^{\beta} - \beta - 2\alpha}{2\alpha + 1}. \quad (42)$$

(b) We consider the situation when only Bob has measurement dependence, and we have $M_1 = 0, M_2 = M$. For this case, violating Eq. (40) requires

$$I_{\alpha}^{\beta} > \beta + 2\alpha + \min\{M, 2\}. \quad (43)$$

If $M = 2$, I_{α}^{β} reaches the no-signaling boundary, whereas, if $M < 2$, then Eq. (43) reduces to the inequality

$$M < I_{\alpha}^{\beta} - \beta - 2\alpha. \quad (44)$$

(c) We consider the situation when only Alice has measurement dependence, and we have $M_1 = M, M_2 = 0$. For this case, violating Eq. (40) requires

$$I_{\alpha}^{\beta} > \beta + 2\alpha + \min\{\alpha M, 2\}. \quad (45)$$

If $\alpha M \geq 2$, I_{α}^{β} reaches the no-signaling boundary, whereas if $\alpha M < 2$, then Eq. (45) reduces to the inequality

$$M < \frac{I_{\alpha}^{\beta} - \beta - 2\alpha}{\alpha}. \quad (46)$$

We note that for $\alpha = 1$ and $\beta = 0$, i.e., for the Bell-CHSH inequality, the values of M_1 and M_2 that violate Eq. (40) are in agreement with that obtained in [49].

Consider that there exists some behavior $\mathbf{P}' = \{p'(ab|xy)\}$ that violates Eq. (34) with the amount of violation given by $I_{\alpha}^{\beta} = I'(\alpha, \beta, \mathbf{P}')$. For $\alpha = 1, \beta = 8$, we show in Fig. 6 the possible values of measurement dependence parameters M_1

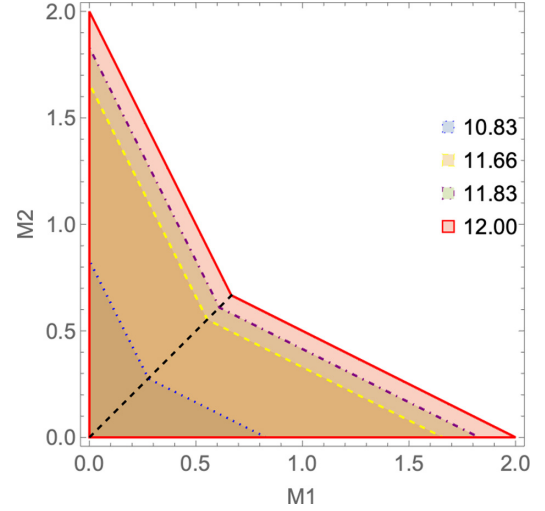


FIG. 6. In this figure, we plot the values of the measurement dependence parameters M_1 and M_2 for which the violation of Eq. (34) given by $I_{\alpha}^{\beta} = I'(\alpha = 1, \beta = 8, \mathbf{P}') \in \{10.83, 11.66, 11.83, 12.00\}$ cannot be described by a deterministic MDL model. The correlations violating Eq. (34) with (a) $I_{\alpha}^{\beta} = 12.00$ belong to the no-signaling boundary (shown enclosed by a red line), (b) $I_{\alpha}^{\beta} = 11.66$ belong to the quantum boundary (shown enclosed by a dashed yellow line), (c) $I_{\alpha}^{\beta} = 10.83$ belong to the quantum set (shown enclosed by a dotted blue line), and (d) $I_{\alpha}^{\beta} = 11.83$ belong to the no-signaling set (shown enclosed by a dot-dashed purple line). The black line in the figure denotes equal values of M_1 and M_2 in the regions (a), (b), (c), and (d).

and M_2 for which the amount of violation of Eq. (34) given by I' cannot be described by a deterministic measurement-dependent local model. The plots for the other combinations of (α, β) are given in Appendix B.

VI. DISCUSSION

In this paper, two different approaches are presented in quantifying measurement dependence via parameters l and (M_1, M_2) from standard literature, and a bound on these parameters for certifying the nonlocality of different behaviors is obtained. It is observed that the quantum behavior certifying close to two bits of randomness remains measurement-dependent nonlocal only in the limit of complete measurement independence, i.e., in the limit of $l \rightarrow 0.25$. The quantum behavior that provides 1.6806 bits of global randomness from the violation of the tilted Hardy relations is measurement-dependent nonlocal for arbitrarily small values of l . This motivates further study on obtaining an inequality that provides close to two bits of global randomness in the limit of arbitrarily low measurement dependence, a problem we leave for future study. Deviating from the conventional approach of assuming perfect detectors, we present a framework to determine the threshold values of the detector parameters that are robust enough to certify nonlocality of given quantum behaviors. This is an important step towards experimentally obtaining nonlocality in the presence of relaxed measurement independence. For an illustration, we presented the critical requirements for generating 1.6806 bits of tilted-Hardy

certified global randomness. The detector parameters obtained from this study are expected to have important applications in experimentally implementing various information processing tasks that rely on quantum nonlocality as a resource.

The modified analytical bound on the AMP tilted Bell inequality in terms of M_1 and M_2 has been obtained. Using the analytical bound, the bounds on M_1 and M_2 to ensure quantumness and nonlocality are observed. It is observed that one-sided measurement dependence is more advantageous from the point of view of the user as compared to two-sided measurement dependence. The analytical bound obtained is expected to have applications in self-testing of quantum states and other device-independent information processing protocols like randomness generation and secure communication.

For future work, it would be interesting to see the implications of measurement dependence in multipartite Bell-type inequalities [27,28] for the certification of multipartite nonlocality and device-independent conference keys [62–64].

Note added. After completion of this work, we noticed a related work by Šupić, Bancal, and Brunner [65].

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATING THE MODIFIED LOCAL BOUND FOR THE AMP TILTED BELL INEQUALITY

We consider the AMP tilted-Bell inequality introduced in [21] and given by

$$I_\alpha^\beta = \beta \langle x_1 \rangle + \alpha \langle x_1 y_1 \rangle + \alpha \langle x_1 y_2 \rangle + \langle x_2 y_1 \rangle - \langle x_2 y_2 \rangle. \quad (\text{A1})$$

The determinism assumption states that the measurement outcomes are deterministic functions of the choice of settings and the hidden variable λ , i.e., $a = A(x, \lambda)$ and $b = B(x, \lambda)$ with $p(a|x\lambda) = \delta_{a,A(x,\lambda)}$ and $p(b|y\lambda) = \delta_{b,B(y,\lambda)}$. If we assume that the choice of the measurement settings on the side of Alice and Bob can depend on some hidden variable, λ , the correlations of Alice and Bob can be expressed as

$$\langle xy \rangle = \int d\lambda p(\lambda|x, y) A(x, \lambda) B(y, \lambda). \quad (\text{A2})$$

The correlation function of Alice can also be written as

$$\langle x \rangle = \int d\lambda p(\lambda|x) A(x, \lambda). \quad (\text{A3})$$

Applying Eqs. (A2) and (A3) to Eq. (A1), we have

$$I_\alpha^\beta = \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) + \alpha \int d\lambda p(\lambda|x_1, y_1) A(x_1, \lambda) B(y_1, \lambda)$$

$$+ \alpha \int d\lambda p(\lambda|x_1, y_2) A(x_1, \lambda) B(y_2, \lambda) + \int d\lambda p(\lambda|x_2, y_1) A(x_2, \lambda) B(y_1, \lambda) - \int d\lambda p(\lambda|x_2, y_2) A(x_2, \lambda) B(y_2, \lambda). \quad (\text{A4})$$

Adding and subtracting the terms

$$\alpha \int d\lambda p(\lambda|x_1, y_2) A(x_1, \lambda) B(y_1, \lambda) \quad \text{and} \\ \int d\lambda p(\lambda|x_2, y_2) A(x_2, \lambda) B(y_1, \lambda)$$

to Eq. (A4) we obtain

$$I_\alpha^\beta = \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) [p(\lambda|x_1, y_1) - p(\lambda|x_1, y_2)] + \int d\lambda A(x_2, \lambda) B(y_1, \lambda) [p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2)] + \alpha \int d\lambda p(\lambda|x_1, y_2) [A(x_1, \lambda) B(y_2, \lambda) + A(x_1, \lambda) B(y_1, \lambda)] - \int d\lambda p(\lambda|x_2, y_2) \times [A(x_2, \lambda) B(y_2, \lambda) - A(x_2, \lambda) B(y_1, \lambda)]. \quad (\text{A5})$$

We observe that Eq. (A5) is bounded by $I_\alpha^\beta \leq \max[T_1] + \max[T_2] + \max[T_3]$. We also note that as $p(\lambda|x_1)$ is a normalized probability distribution, $\int d\lambda p(\lambda|x_1) = 1$. This implies that the maximum value of the quantity $\beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda)$ is given by β when $A(x_1, \lambda)$ is set to 1. With these observations, T_1 can be simplified as

$$T_1 = \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) [p(\lambda|x_1, y_1) - p(\lambda|x_1, y_2)] \leq \beta + \alpha M_2, \quad (\text{A6})$$

where we have set $B(y_1, \lambda) = 1$. To evaluate the maximum value of T_2 , we set $A(x_2, \lambda) = 1$ and obtain

$$T_2 = \int d\lambda A(x_2, \lambda) B(y_1, \lambda) [p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2)] \leq M_2. \quad (\text{A7})$$

We evaluate T_3 as follows:

$$T_3 = \alpha \int d\lambda p(\lambda|x_1, y_2) [A(x_1, \lambda) B(y_2, \lambda) + A(x_1, \lambda) B(y_1, \lambda)] - \int d\lambda p(\lambda|x_2, y_2) [A(x_2, \lambda) B(y_2, \lambda) - A(x_2, \lambda) B(y_1, \lambda)] = \int d\lambda A(x_1, \lambda) B(y_2, \lambda) \left[\alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \frac{A(x_2, \lambda)}{A(x_1, \lambda)} \right]$$

$$\begin{aligned}
 & + \int d\lambda A(x_1, \lambda)B(y_1, \lambda) \left[\alpha p(\lambda|x_1, y_2) \right. \\
 & \left. + p(\lambda|x_2, y_2) \frac{A(x_2, \lambda)}{A(x_1, \lambda)} \right]. \tag{A8}
 \end{aligned}$$

To get the maximum value of T_3 , we set the values of $A(x_1, \lambda), B(y_1, \lambda), A(x_2, \lambda), B(y_2, \lambda)$ to 1. This then implies,

$$\begin{aligned}
 T_3 & = \int d\lambda [\alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \\
 & + \int d\lambda [\alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2)]. \tag{A9}
 \end{aligned}$$

We evaluate the first term of Eq. (A9) as

$$\begin{aligned}
 & \int d\lambda [\alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \\
 & = \int d\lambda [\alpha p(\lambda|x_1, y_2) - \alpha p(\lambda|x_2, y_2) + \alpha p(\lambda|x_2, y_2) \\
 & - p(\lambda|x_2, y_2)] \tag{A10}
 \end{aligned}$$

$$\begin{aligned}
 & = \int d\lambda [\alpha(p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)) \\
 & + (\alpha - 1)p(\lambda|x_2, y_2)] \tag{A11}
 \end{aligned}$$

$$\begin{aligned}
 & = \alpha \int d\lambda [p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \\
 & + (\alpha - 1) \int d\lambda p(\lambda|x_2, y_2). \tag{A12}
 \end{aligned}$$

In Eq. (A12) we note that

$$\alpha \int d\lambda (p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)) \leq \alpha M_1.$$

We note that as $p(\lambda|x_2, y_2)$ is a normalized probability distribution, $\int d\lambda p(\lambda|x_2, y_2) = 1$. This then implies

$$\int d\lambda [\alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \leq \alpha M_1 + \alpha - 1, \tag{A13}$$

which on simplification reduces to

$$\int d\lambda [\alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \leq \alpha(M_1 + 1) - 1. \tag{A14}$$

We proceed in the same way for the second term in Eq. (A9) as follows:

$$\begin{aligned}
 & \int d\lambda [\alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2)] \\
 & = \int d\lambda [\alpha p(\lambda|x_1, y_2) + \alpha p(\lambda|x_2, y_2) - \alpha p(\lambda|x_2, y_2) \\
 & + p(\lambda|x_2, y_2)] \tag{A15}
 \end{aligned}$$

$$\begin{aligned}
 & = \int d\lambda \alpha [p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2)] \\
 & - (\alpha - 1) \int d\lambda p(\lambda|x_2, y_2) \tag{A16}
 \end{aligned}$$

$$\begin{aligned}
 & = \alpha \int d\lambda [p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2)] \\
 & + (1 - \alpha) \int d\lambda p(\lambda|x_2, y_2). \tag{A17}
 \end{aligned}$$

We note that as $p(\lambda|x_1, y_2), p(\lambda|x_2, y_2)$ are normalized probability distributions, $\int d\lambda p(\lambda|x_1, y_2) = 1$ and $\int d\lambda p(\lambda|x_2, y_2) = 1$. This then implies

$$\int d\lambda [\alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2)] = \alpha + 1. \tag{A18}$$

If we insert Eq. (A14) and Eq. (A18) in Eq. (A9), we obtain

$$T_3 \leq \alpha(M_1 + 2). \tag{A19}$$

Now combining Eqs. (A6), (A7), and (A19), we have the bound on I_α^β as

$$I_\alpha^\beta \leq T_1 + T_2 + T_3 \tag{A20}$$

$$\leq \beta + \alpha M_2 + M_2 + \alpha(M_1 + 2) \tag{A21}$$

$$\leq \beta + 2\alpha + \alpha M_1 + (\alpha + 1)M_2 \tag{A22}$$

$$\leq \beta + 2\alpha + \alpha(M_1 + M_2) + M_2. \tag{A23}$$

We again start by considering Eq. (A4) as

$$\begin{aligned}
 I_\alpha^\beta & = \beta \int d\lambda p(\lambda|x_1)A(x_1, \lambda) \\
 & + \alpha \int d\lambda p(\lambda|x_1, y_1)A(x_1, \lambda)B(y_1, \lambda) \\
 & + \alpha \int d\lambda p(\lambda|x_1, y_2)A(x_1, \lambda)B(y_2, \lambda) \\
 & + \int d\lambda p(\lambda|x_2, y_1)A(x_2, \lambda)B(y_1, \lambda) \\
 & - \int d\lambda p(\lambda|x_2, y_2)A(x_2, \lambda)B(y_2, \lambda). \tag{A24}
 \end{aligned}$$

Adding and subtracting the terms $\alpha \int d\lambda p(\lambda|x_2, y_1)A(x_1, \lambda)B(y_1, \lambda)$ and $\alpha \int d\lambda p(\lambda|x_2, y_2)A(x_1, \lambda)B(y_2, \lambda)$ to Eq. (A4), we obtain

$$\begin{aligned}
 I_\alpha^\beta & = \beta \int d\lambda p(\lambda|x_1)A(x_1, \lambda) \\
 & + \alpha \int d\lambda A(x_1, \lambda)B(y_1, \lambda)[p(\lambda|x_1, y_1) - p(\lambda|x_2, y_1)] \\
 & + \alpha \int d\lambda A(x_1, \lambda)B(y_2, \lambda)[p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \\
 & + \int d\lambda A(x_2, \lambda)B(y_1, \lambda) \left[p(\lambda|x_2, y_1) \right. \\
 & \left. - \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right] \\
 & + \alpha \int d\lambda A(x_1, \lambda)B(y_1, \lambda) \left[p(\lambda|x_2, y_1) \right. \\
 & \left. + \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right]. \tag{A25}
 \end{aligned}$$

We observe that Eq. (A25) is bounded by $I_\alpha^\beta \leq \max[t_1] + \max[t_2] + \max[t_3]$. We also note that as $p(\lambda|x_1)$ is a normalized probability distribution, $\int d\lambda p(\lambda|x_1) = 1$. This implies that the maximum value of the quantity $\beta \int d\lambda p(\lambda|x_1)A(x_1, \lambda)$ then takes the value of β when $A(x_1, \lambda)$ is set to 1. With these

observations, t_1 can be simplified as

$$\begin{aligned} t_1 &= \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) \\ &\quad + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) [p(\lambda|x_1, y_1) - p(\lambda|x_2, y_1)] \\ &\leq \beta + \alpha M_1. \end{aligned} \quad (\text{A26})$$

Similarly, t_2 can be simplified as

$$\begin{aligned} t_2 &= \alpha \int d\lambda A(x_1, \lambda) B(y_2, \lambda) [p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2)] \\ &\leq \alpha M_1, \end{aligned} \quad (\text{A27})$$

and for t_3 we have the expression

$$\begin{aligned} t_3 &= \int d\lambda A(x_2, \lambda) B(y_1, \lambda) \left[p(\lambda|x_2, y_1) - \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right] \\ &\quad + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) \left[p(\lambda|x_2, y_1) \right. \\ &\quad \left. + \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right]. \end{aligned} \quad (\text{A28})$$

We set the values of $A(x_1, \lambda)$, $B(y_1, \lambda)$, $A(x_2, \lambda)$, $B(y_2, \lambda)$ to 1 and obtain

$$\begin{aligned} t_3 &= \int d\lambda [p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2)] \\ &\quad + \alpha \int d\lambda [p(\lambda|x_2, y_1) + p(\lambda|x_2, y_2)]. \end{aligned} \quad (\text{A29})$$

The first term in Eq. (A29) is bounded by

$$\int d\lambda [p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2)] \leq M_2. \quad (\text{A30})$$

We note that as $p(\lambda|x_2, y_1)$ and $p(\lambda|x_2, y_2)$ are normalized probability distributions, $\int d\lambda p(\lambda|x_2, y_1) = 1$ and $\int d\lambda p(\lambda|x_2, y_2) = 1$. Equation (A29) is then expressed as

$$t_3 \leq M_2 + 2\alpha. \quad (\text{A31})$$

Now combining the values of t_1 , t_2 , and t_3 , we have the bound on I_α^β as

$$I_\alpha^\beta \leq t_1 + t_2 + t_3 \quad (\text{A32})$$

$$\leq \beta + \alpha M_1 + \alpha M_1 + M_2 + 2\alpha \quad (\text{A33})$$

$$\leq \beta + 2\alpha + 2\alpha M_1 + M_2. \quad (\text{A34})$$

We note that the bound on I_α^β must be the minimum of Eqs. (A23) and (A34) and is expressed as

$$I_\alpha^\beta \leq \beta + 2\alpha + \alpha[M_1 + \min\{M_1, M_2\}] + M_2. \quad (\text{A35})$$

We observe that for the case of $\beta = 0$ and $\alpha = 1$,

$$I_1^0 \leq 2 + M_1 + M_2 + \min\{M_1, M_2\}, \quad (\text{A36})$$

which is in agreement with that obtained in [49]. We note that the maximum value that I_α^β [Eq. (A1)] can take is $\beta + 2\alpha + 2$ (we get this bound by setting $\langle x_1 \rangle = 1$, $\langle x_1 y_1 \rangle = 1$, $\langle x_1 y_2 \rangle = 1$,

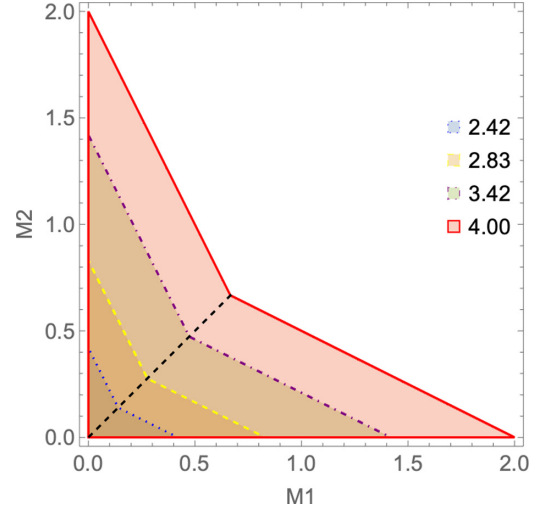


FIG. 7. In this figure, we plot the values of the measurement dependence parameters M_1 and M_2 for which the violation of Eq. (34) given by $I_\alpha^\beta = I'(\alpha = 1, \beta = 0, \mathbf{P}') \in \{4.00, 3.42, 2.83, 2.42\}$ cannot be described by a deterministic MDL model. The correlations violating Eq. (34) with (a) $I_\alpha^\beta = 4.00$ belong to the no-signaling boundary (shown enclosed by a red line), (b) $I_\alpha^\beta = 2.83$ belong to the quantum boundary (shown enclosed by a dashed yellow line), (c) $I_\alpha^\beta = 2.42$ belong to the quantum set (shown enclosed by a blue dotted line), and (d) $I_\alpha^\beta = 3.42$ belong to the no-signaling set (shown enclosed by a dot-dashed purple line). The black line in the figure denotes equal values of M_1 and M_2 in the regions (a), (b), (c), and (d).

$\langle x_2 y_1 \rangle = 1$, $\langle x_2 y_2 \rangle = -1$) and arrive at

$$I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\}. \quad (\text{A37})$$

APPENDIX B: BOUNDS ON THE MEASUREMENT DEPENDENCE TO CERTIFY NONLOCALITY

We observe in Proposition 1 that in the presence of relaxed measurement dependence, the behaviors $\{p(ab|xy)\}$ that violate Eq. (B1) are nonlocal,

$$I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\}. \quad (\text{B1})$$

Consider that there exists some behavior $\mathbf{P}' = \{p(ab|xy)\}$ that violates Eq. (34) with the amount of violation given by $I_\alpha^\beta = I'(\alpha, \beta, \mathbf{P}')$. We show in Fig. 7 the possible values of measurement dependence parameters M_1 and M_2 for which the amount of violation of Eq. (34) given by I' cannot be described by a deterministic measurement-dependent local model. We observe that Fig. 7 is in agreement with that obtained in [49].

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- which is independent of w . The probabilities are always non-negative, hence the above inequality can never be violated.
- [58] For $w = 0$, violation of Eqs. (9) and (8) requires
- $$l > \frac{p(+ - |01) + p(- + |10) + p(+ + |11)}{p(+ + |00) + 3[p(+ - |01) + p(- + |10) + p(+ + |11)]},$$
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