No contextual advantage in nonparadoxical scenarios of the two-state vector formalism

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The two-state vector formalism (TSVF) proposed by Aharonov, Bergmann, and Lebowitz [A. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964)] allows a counterfactual assignment of probabilities of outcomes of contemplated (but unperformed) measurements on quantum systems. The probabilities assigned by the Aharonov-Bergmann-Lebowitz (ABL) rule and the associated weak values have been used to provide insights into quantum situations and to unearth underlying quantum contextuality. We apply the principle of exclusivity on ABL probabilities which are assigned to mutually orthogonal projectors to define paradoxical and nonparadoxical scenarios. Given a pre- and a postselected pair of states, we consider the nonparadoxical sector with a view to explore the demonstration of quantum contextuality. For the Klyachko-Can-Binicioğlu-Shumovsky scenario, we numerically show that the ABL probabilities of the TSVF in the nonparadoxical sector do not offer any contextual advantage. Our approach can be easily generalized to other contextual scenarios as well. We thus argue that several previous proofs of the emergence of contextuality in pre- and postselected scenarios are only possible if the principle of exclusivity is violated and are therefore classified as paradoxical.

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I. INTRODUCTION

Standard quantum theory does not provide an adequate framework to make predictions about measurements in the past (retrodiction) of a quantum system, once it has been measured in a definite state. Aharonov, Bergmann, and Lebowitz, in their seminal work on time symmetry in successive quantum measurements, introduced a reformulation of standard quantum theory wherein one can meaningfully talk about statistical predictions of a measurement on a pre- and a postselected (PPS) ensemble at intermediate times [1]. The retrodiction formula derived by Aharonov *et al.* [the Aharonov-Bergmann-Lebowitz (ABL) rule] is the probability of a measurement outcome conditioned on the outcomes of a preceding and a succeeding measurement.

A generalized framework for PPS ensembles in terms of "weak values" was formulated as the two-state vector formalism (TSVF) [2–4] in order to experimentally validate the ABL formulation [5,6]. In the TSVF, the complete description of a quantum system is specified by two-state vectors, one evolving forward in time and the other one evolving backward. Here the arrow of time is described by the order of preceding and succeeding measurements. The TSVF finds intriguing applications in quantum foundations [7–13] and quantum information processing [14]. The ABL retrodiction, more specifically the TSVF, has resulted in various counterintuitive results commonly called PPS paradoxes [15–23]. In a recent work [24] counterfactual use of the ABL retrodiction was shown to run into direct contradiction with operational quantum mechanics, thus challenging the completeness of the TSVF. Therefore, further investigations about the appropriateness of the ABL retrodiction in connection with various nonclassical aspects of quantum theory are critical in order to pinpoint the exact role of the TSVF in the studies of quantum foundations.

Contrary to the Born rule, probabilities assigned by the ABL formula are determined by the specification of the measurement setting of an observable and on pre- and postselected states in the context of which the observable is being measured. This kind of context dependence of measurements has led to connections between PPS paradoxes and contextuality [25]: Since the probabilities assigned to measurement outcomes are explicitly context dependent, there is no motivation to consider a noncontextual hidden variable theory as a realistic extension of operational quantum theory. Nevertheless, this reasoning has been disputed based on the fact that Bell-type correlations can be simulated using postselection in local hidden variable theories [26]. Therefore, the mere presence of context-dependent elements in the ABL formula will not suffice to prove the Bell-Kochen-Specker (BKS) theorem [27,28] or the various statistical versions of contextuality [29]. More effort is required in order to establish a valid connection between contextuality and the ABL retrodiction formula.

Mermin [30] showed the existence of a strong connection between the ABL retrodiction formula and contextuality, by illustrating how measurements used in a proof of the BKS theorem can give birth to unsolvable PPS paradoxes, indicating the impossibility of noncontextual hidden variable theories. Leifer and Spekkens [31] later reasoned that for every

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PPS paradox with a scenario involving nonorthogonal pre- and post-selection states there exists an associated proof of the BKS theorem. Their proof is based on the fact that ABL probability assignments of certain sets of projectors in a variant of the three-box paradox violate algebraic constraints dictated by classical probability theory. An exhaustive discussion on the same in relation to weak values was presented by Tollaksen [32]. Another important contribution in this direction was recently made by establishing a direct connection between anomalous weak values and contextuality, where it was suggested that anomalous weak values can be considered as proofs of contextuality [33–35]. Thus far, the studies in this research direction have focused on logical proofs of contextuality invoking only the paradoxical nature of ABL probability assignments. In these logical proofs of contextuality, one arrives at a contradiction while making assignments of probabilities to various outcomes following the ABL rule. Such proofs generally involve the paradoxes generated by the application of the ABL rule. Therefore, it is natural to ask whether there is any contextual advantage in situations with nonparadoxical assignments of ABL probabilities.

In this paper, by analyzing the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) scenario [36] (which comprises a statistical proof of contextuality) within the framework of the ABL formula, we show that nonparadoxical ABL probability assignments do not give rise to any contextual advantage. Furthermore, in order to produce an advantage, one needs to renounce the exclusivity principle, which is central to any operational theory [37–39]. Our result raises serious questions about the connections between the ABL rule and contextuality, which have been advocated by previous authors: Is this connection merely an illusion created by postselection, similar to the detection efficiency loophole in Bell nonlocality tests? Since the paradoxical sector of ABL probabilities requires abandoning the notion of the principle of exclusivity, can ABL retrodiction be considered an appropriate description of quantum systems at all?

The paper is organized as follows. In Sec. II A we introduce the concept of PPS scenarios and briefly discuss the ABL formula and consequently the TSVF. In Sec. II C we present our main result that the ABL rule is unable to correctly predict the statistics of the KCBS scenario. In Sec. V we offer some concluding remarks.

II. ABL RULE, TSVF, AND PPS PARADOXES

In this section we describe the TSVF and ABL retrodiction rule. We define PPS scenarios and introduce the notion of the paradoxical and nonparadoxical nature of them. This classification depends on whether the probability assignments are properly conditioned by the exclusivity principle or not.

A. TSVF and ABL retrodiction

In this section we illustrate a general pre- and postselected scenario and elucidate how the ABL rule can be used to assign probabilities to intermediate measurements.

A pre- and postselection scenario deals with statistical assignment of probabilities to the outcomes of the measurement of an observable A at time t when the system is preselected

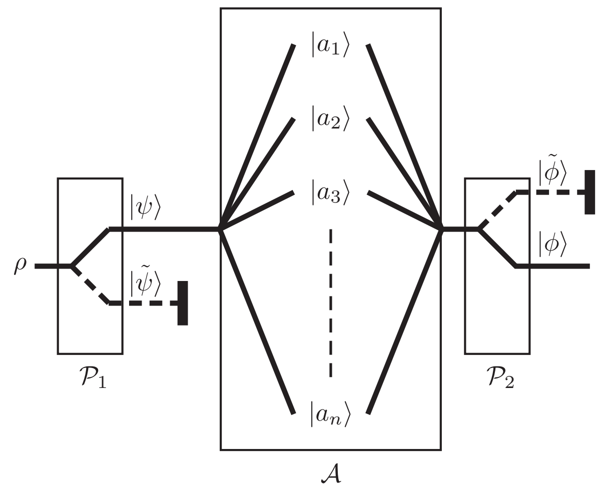

FIG. 1. A PPS scenario with a measurement of an observable $\mathcal{A}$ at an intermediate instance of time. The system is preselected in state $|\psi\rangle$ by performing a projective measurement $\mathcal{P}_1$ and filtering the outcome corresponding to the state $|\psi\rangle$ and postselected in state $|\phi\rangle$ by filtering outcome $|\tilde{\phi}\rangle$ of measurement $\mathcal{P}_2$. Here $\{|a\rangle_i\}$ is the set of all possible outcomes of an intermediate observable $\mathcal{A}$.

to be in the state $|\psi\rangle$ at some time $t_i < t$ and postselected in the state $|\phi\rangle$ at a later moment in time $t_f > t$. Preselection is achieved by performing a projective measurement $\mathcal{P}_1 \equiv \{|\psi\rangle\langle\psi|, \mathbb{1} - |\psi\rangle\langle\psi|\}$ at time t_i on the initial state of the system ρ (which can be chosen arbitrarily) and selecting only the outcomes corresponding to $|\psi\rangle\langle\psi|$. Similarly, for postselection one can perform a projective measurement of $\mathcal{P}_2 \equiv \{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ at time t_f where outcomes corresponding to $|\phi\rangle\langle\phi|$ are filtered (see Fig. 1). It is apparent that such probability assignments are conditioned on PPS states $|\psi\rangle$ and $|\phi\rangle$ and therefore are time symmetric.

Consider an observable $\mathcal{A}$ with outcomes $\{|a_i\rangle\langle a_i|\}$ which is measured after preselecting a state $|\psi\rangle$ and afterward postselecting a state $|\phi\rangle$. The probability of obtaining the outcome $|a_i\rangle\langle a_i|$ conditioned on pre- and postselections is given as

$$\zeta_i = \frac{|\langle\phi|a_i\rangle|^2|\langle a_i|\psi\rangle|^2}{\sum_j |\langle\phi|a_j\rangle|^2|\langle a_j|\psi\rangle|^2}, \tag{1}$$

which can be simplified as

$$\zeta_i = \frac{|\langle\phi|\Pi_i|\psi\rangle|^2}{\sum_j |\langle\phi|\Pi_j|\psi\rangle|^2}, \tag{2}$$

where $\Pi_i = |a_i\rangle\langle a_i|$. As one can see, ζ_i for a projector Π_i is dependent on PPS states $|\psi\rangle$ and $|\phi\rangle$. A different choice of these will yield different probability assignments. Furthermore, the measurement context of the projector Π_i also plays a major role. If the set of measurement settings in which Π_i appears is chosen differently, the term $\sum_j |\langle\phi|\Pi_j|\psi\rangle|^2$ will also change. All of the aforementioned choices form a context for the projector Π_i. This makes the ABL formula given in Eq. (2) inherently context dependent and led Albert and coworkers [2,25] to draw a parallel between ABL retrodiction and quantum contextuality.

B. Counterfactual vs noncounterfactual ABL retrodiction

There are two ways to interpret Eq. (1), namely, noncounterfactual and counterfactual. In the former, the measurement of an observable $\mathcal{A}$ is actually performed, while in the latter one only imagines such a measurement. In the noncounterfactual case the observable $\mathcal{A}$ is actually measured after performing a preselection and before postselection. In this case there is a total of three different sequential measurements being performed. This case is known as noncounterfactual assignment of probabilities [40–42]. In the second case, the observable $\mathcal{A}$ is not actually measured, but rather a probability distribution over its outcomes is assigned counterfactually depending on the PPS states [4]. This case is known as counterfactual assignment of probabilities. It has been argued by Aharonov *et al.* [1,4] that simultaneous counterfactual probability assignments to even noncommuting observables is possible.

It should be noted that Eq. (1) is in fact a classical Bayes rule expressing the conditional probability of getting outcome a_i given that measurement of an observable $\mathcal{A}$ was performed on a system that was pre- and postselected in states $|\psi\rangle$ and $|\phi\rangle$, respectively. The order of the events follows as $|\psi\rangle \rightarrow \mathcal{A} \rightarrow |\phi\rangle$, where the arrows indicate passage of time. In such a scenario, the observable $\mathcal{A}$ is actually performed. According to standard quantum mechanics, in order to evaluate the probability distribution over another observable $\mathcal{B}$ that does not commute with $\mathcal{A}$ requires using a separate ensemble of pre- and postselected states with a new order of events as $|\psi\rangle \rightarrow \mathcal{B} \rightarrow |\phi\rangle$. This scenario is reminiscent of standard quantum theory with probability distributions evaluated using the Born rule. We call such PPS scenarios noncounterfactual. This interpretation of Eq. (1) has been used as an argument by various authors to resolve the apparent PPS paradoxes [40–43].

However, proponents of the TSVF have used a counterfactual interpretation of Eq. (1) to construct a series of PPS paradoxes which are supported by experimental demonstrations using weak measurements [2,10,11,17,19,20,22,23,44]. Contrary to noncounterfactual PPS scenarios, there is no actual measurement between the pre- and postselection procedures. The order of events is just $|\psi\rangle \rightarrow |\phi\rangle$. It has been hypothesized in the literature that probability distributions can be assigned to outcomes of unperformed intermediate measurements. Such counterfactual probability assignment makes ABL retrodiction a suitable candidate for an ontological model [45].

The question whether quantum theory is a complete description of physical reality is still an open problem. In case it is incomplete, the description of the state is supplemented or substituted with certain ontological variables. Such a model should reproduce all statistical predictions of quantum mechanics. No-go theorems on nonlocality and quantum contextuality have ruled out certain models which appear to be classically reasonable [28,36,46]. The importance of such theorems lies in the characterization of the nonclassical features of quantum mechanics. Any new candidate for an ontological model must show nonlocal and contextual characteristics. Since the counterfactual ABL rule is a way to assign values to observables prior to actually performing a measurement, it becomes natural to ask whether such an assignment is contextual.

It has been demonstrated in the literature that counterfactual ABL retrodiction is contextual as required by the BKS theorem [30–32]; however, this could be achieved only by invoking PPS paradoxical scenarios. Here we investigate only nonparadoxical scenarios and demonstrate that these do not exhibit quantum contextuality.

For the remainder of this paper we consider only the counterfactual measurement setting of the observable $\mathcal{A}$. Moreover, we use the TSVF and counterfactual PPS scenarios inter-changeably. This rule [Eq. (1)] is eponymously known as the ABL rule and is the same for both the aforementioned cases. However, the interpretation for both the cases is entirely different and leads to some interesting results, especially when linked to counterfactual assignments of projectors in the KCBS scenario.

C. Paradoxical and nonparadoxical PPS scenarios

In this section we provide a classification of PPS scenarios into paradoxical and nonparadoxical. We then proceed to show how the KCBS scenario can be modified to fit within the paradigm of the TSVF and whether the latter can help in predicting the correct statistics of the former. We then provide a general algorithm to check for contextual advantages for other contextuality scenarios.

Definition 1 (counterfactual PPS scenario). A counterfactual PPS scenario is specified by $(\langle\phi||\psi\rangle, \mathcal{M})$, where $\langle\phi||\psi\rangle$ is a two-state describing the PPS ensemble and a projective valued measure (PVM) and $\mathcal{M} = \{\Pi_i\}$ $(i = 1, 2, \ldots, n)$ is the counterfactual measurement setting at an intermediate time. The corresponding $\{\zeta_i\}$ given by Eq. (2) are then the counterfactual probability assignments for the PVM $\mathcal{M}$.

For certain pre- and postselections counterfactual probability assignments can lead to paradoxical situations. The three-box paradox is a case in point. Consider a particle that is preselected in the state $(|A\rangle + |B\rangle + |C\rangle)/\sqrt{3}$ and postselected in the state $(|A\rangle + |B\rangle - |C\rangle)/\sqrt{3}$, where $|A\rangle$, $|B\rangle$, and $|C\rangle$ represent the states of the particle in boxes A, B, and C, respectively. Now consider two possible counterfactual measurement settings $\mathcal{A} = \{|A\rangle\langle A|, \mathbb{1} - |A\rangle\langle A|\}$ and $\mathcal{B} = \{|B\rangle\langle B|, \mathbb{1} - |B\rangle\langle B|\}$. It is easy to visualize $\mathcal{A}$ and $\mathcal{B}$ as the actions of opening the boxes A and B, respectively, at an intermediate time in order to check whether the particle was present there. A counterfactual probability assignment to both the projectors $|A\rangle\langle A|$ and $|B\rangle\langle B|$ can be made according to the ABL formula (2). However, it can be seen that such an assignment leads to a situation in which the particle is present in box A with unit probability and box B with unit probability [40].

The exclusivity principle states that the sum of probabilities of mutually exclusive events cannot be greater than 1. Therefore, such scenarios in which two mutually exclusive events are assigned unit probabilities are paradoxical. This motivates the following definition.

Definition 2 (logical PPS paradox). A logical PPS paradox consists of at least two counterfactual PPS scenarios $(\langle\phi||\psi\rangle, \mathcal{M}_1)$ and $(\langle\phi||\psi\rangle, \mathcal{M}_2)$, where $\mathcal{M}_i = \{\Pi_i, \mathbb{1} - \Pi_i\}$ for $i = 1, 2$ and $\mathrm{tr}(\Pi_1\Pi_2) = 0$ such that $\zeta_1 + \zeta_2 > 1$, where ζ_i is the counterfactual probability assigned to Π_i.

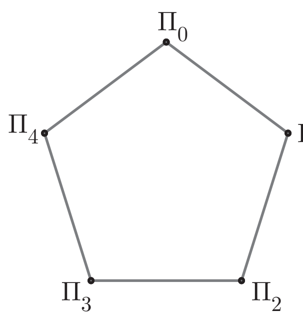

FIG. 2. The KCBS scenario with five projectors. Projectors connected by an edge are orthogonal.

The logical PPS paradox is defined for PPS scenarios which violate those shown in Ref. [31], that for every corresponding logical PPS paradox there exists a proof of the BKS theorem. However, the relation between nonparadoxical PPS scenarios and contextuality is still left unexplored. To that end we make the following definition to distinguish between the paradoxical and nonparadoxical sectors of PPS scenarios.

Definition 3 (paradoxical and nonparadoxical sectors). The set of all two-states that generate logical PPS paradoxes for given two counterfactual measurement settings $\mathcal{M}_1 = \{\Pi_1, \mathbb{1} - \Pi_1\}$ and $\mathcal{M}_2 = \{\Pi_2, \mathbb{1} - \Pi_2\}$ such that $\mathrm{tr}(\Pi_1\Pi_2) = 0$ is called the paradoxical sector corresponding to the pair $\{\mathcal{M}_1, \mathcal{M}_2\}$. We refer to the set of all two-states that are not elements of the above as the nonparadoxical sector corresponding to the pair $\{\mathcal{M}_1, \mathcal{M}_2\}$.

III. KCBS CONSTRUCTION AND NONCONTEXTUALITY

We now analyze whether the nonparadoxical sector of PPS scenarios can offer a proof of contextuality. We first focus on the minimal proof of state-dependent contextuality, namely, the KCBS scenario (Fig. 2), and construct an ontological description of the same via the TSVF.

Consider a scenario consisting of five tests e_i, $i \in \{0, 1, 2, 3, 4\}$. A test is an experiment which yields some statistics for a given preparation. These tests are assumed to be cyclically exclusive, i.e.,

$$P(e_i) + P(e_{i\oplus 1}) \leqslant 1, \tag{3}$$

where $i \oplus 1$ is taken modulo 5. This scenario is eponymously referred to as the KCBS scenario, in honor of the people who first studied it. The KCBS scenario can be represented on a graph whose vertices correspond to tests and two vertices are connected by an edge if they are exclusive. This scenario is capable of revealing quantum contextuality if the following inequality is violated:

$$\mathcal{K} := \sum_{i=0} P(e_i) \leqslant 2. \tag{4}$$

Here the underlying ontic probability distribution $P(e_i)$ is assumed to be noncontextual. This is the well-known KCBS inequality.

A valid construction of the KCBS scenario for the quantum case is as follows. Consider five different PVMs: $\mathcal{M}_i = \{\Pi_i, \mathbb{1} - \Pi_i\}$ ($i \in \{0, 1, 2, 3, 4\}$) and $\mathrm{tr}(\Pi_i\Pi_{i\oplus 1}) = 0$. Each projector corresponds to a test in the KCBS scenario and cyclic orthogonality ensures the required exclusivity conditions given in Eq. (3). The maximum quantum value of the KCBS inequality (4) for the aforementioned settings and a state $|\psi\rangle$ is

$$\max(\mathcal{K}) := \max\left(\sum_{i=0} P(\Pi_i = 1)\right) = \sqrt{5}, \tag{5}$$

which is greater than the noncontextual bound. This is an indication of contextual advantage of quantum probability distributions.

It should be noted that any valid construction of the KCBS scenario in any formalism must necessarily ensure the exclusivity conditions (3).

IV. ABL RULE AND KCBS INEQUALITY

The foremost requirement to check whether the nonparadoxical sector of the ABL formalism offers any contextual advantage is to set up the KCBS scenario with the proper exclusivity conditions given in Eq. (3) by assigning a probability distribution ζ_i to the projectors under a PPS scenario. We choose the PPS as $|\psi\rangle$ and $|\phi\rangle$, respectively, to assign a counterfactual probability distribution to the projector Π_i according to the ABL rule as

$$\zeta_i = \frac{|\langle\phi|\Pi_i|\psi\rangle|^2}{|\langle\phi|\Pi_i|\psi\rangle|^2 + |\langle\phi|(\mathbb{1} - \Pi_i)|\psi\rangle|^2}, \tag{6}$$

where the measurement setting is of the form $\{\Pi_i, \mathbb{1} - \Pi_i\}$.

By careful selection of a PPS, such counterfactual assignments can lead to a logical paradox in which Π_i and $\Pi_{i\oplus 1}$ are assigned probabilities leading to $\zeta_i + \zeta_{i\oplus 1} > 1$. This is a direct violation of the exclusivity conditions (3). Furthermore, in order to analyze the nonparadoxical sector of the KCBS scenario, it is required that $\zeta_i + \zeta_{i\oplus 1} \leqslant 1$ for all two-states $\langle\phi||\psi\rangle$.

We now propose the following setup to test the KCBS inequality using the ABL formalism. Without loss of generality we assume a preselected state $|\psi\rangle$ as

$$|\psi\rangle = (0, 0, 1)^T \tag{7}$$

and a postselected state as

$$|\phi\rangle = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)^T, \tag{8}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The projectors $\Pi_i = |v_i\rangle\langle v_i|$ are of the form

$$\begin{aligned}
|v_0\rangle &= (1, 0, \sqrt{\cos\pi/5})^T,\\
|v_1\rangle &= (\cos 4\pi/5, -\sin 4\pi/5, \sqrt{\cos\pi/5})^T,\\
|v_2\rangle &= (\cos 2\pi/5, \sin 2\pi/5, \sqrt{\cos\pi/5})^T,\\
|v_3\rangle &= (\cos 2\pi/5, -\sin 2\pi/5, \sqrt{\cos\pi/5})^T,\\
|v_4\rangle &= (\cos 4\pi/5, \sin 4\pi/5, \sqrt{\cos\pi/5})^T.
\end{aligned} \tag{9}$$

We then optimize $|\phi\rangle$ for the maximum value of $\mathcal{K}$ with the exclusivity conditions (3) imposed on ζ_i. We evaluate $\mathcal{K}$ using the rule (6) for the measurement $\{\Pi_i, \mathbb{1} - \Pi_i\}$ to assign $P(\Pi_i = 1) = \zeta_i$ accordingly.

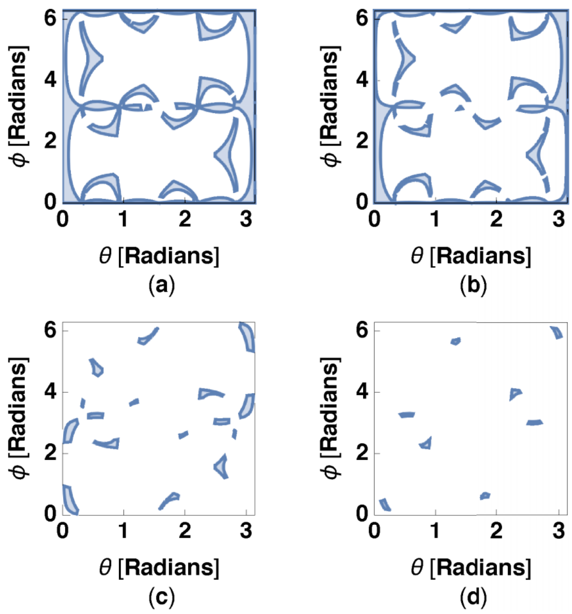

FIG. 3. Region plot corresponding to a set of postselected states (8) (shaded blue) for which Eq. (3) is satisfied for all i with (a) $\mathcal{K} > 1.4$, (b) $\mathcal{K} > 1.5$, (c) $\mathcal{K} > 1.6$, and (d) $\mathcal{K} > 1.7$. No set of states was found for $\mathcal{K} > 2.0$.

By imposing the exclusivity conditions we found that no postselection can lead to a violation of the KCBS inequality. In Fig. 3 we plot the intersection of the solutions that satisfy the inequalities (3) and (4) for various minimum values of $\mathcal{K}$ over the entire region of postselected states. Incidentally, we do not find any set of states for which the KCBS inequality is violated. This provides clear evidence of the fact that the ABL formalism does not provide a complete description under the nonparadoxical sector of the PPS scenario.

In order to achieve a violation it is necessary to violate at least a single exclusivity constraint. If all the constraints are satisfied, the resultant distribution, even from the ABL retrodiction formula, is necessarily noncontextual.

Therefore, any violation of the KCBS inequality observed via the ABL rule must necessarily arise from the paradoxical sector of PPS scenarios. As a consequence, the maximum violation can even go above the algebraic bound. This is because the exclusivity conditions are not properly satisfied.

It is a natural consequence of this work that a valid PPS KCBS scenario can be modeled using a noncontextual ontological model.

Our analysis can be extended to arbitrary contextuality scenarios too [47]. Following our analysis, it is required to first identify the proper exclusivity conditions according to Eq. (3). These conditions demarcate the set of nonparadoxical PPS scenarios from the paradoxical ones according to Definition 3. Within this set of counterfactual PPS scenarios one can then vary the preselected and postselected states for a given set of PVMs (which define the corresponding contextuality scenario) to make counterfactual probability assignments. A violation of the corresponding contextuality inequality would then indicate a contextual advantage of the TSVF.

The KCBS scenario requires a set of five PVMs and imposes five exclusivity constraints on the ABL rule. Any n-cycle scenario [47] would then consequently impose n such exclusivity constraints. This in turn reduces the nonparadoxical sector of PPS scenarios possible for this contextual inequality. While a solution for all n-cycle scenarios with n exclusivity constraints applied to TSVF is not possible, we conjecture with good confidence that no n-cycle contextual inequality can exhibit a violation under the TSVF paradigm.

V. CONCLUSION

In this work we have focused on unearthing quantum contextuality as identified by the violation of the KCBS inequality in PPS scenarios where the ABL rule provides a way to assign counterfactual probabilities to measurement outcomes. We provide a classification of PPS scenarios into paradoxical and nonparadoxical sectors. We then show that the nonparadoxical sector of the ABL rule to evaluate the probability distribution over the outcomes of an observable in a PPS scenario does not provide a contextual violation of the KCBS inequality. Since the ABL rule is applied in a counterfactual manner, the ABL rule acts as an ontic model of the KCBS inequality. By imposing proper exclusivity conditions on the ABL probabilities, we find that it is not able to reproduce the statistics that are observed in nature.

It should be noted that the KCBS scenario and the pentagram graph in general underlie many other contextual scenarios as well [48]. Apart from the KCBS scenario, our result also implies a noncontextual behavior of these scenarios under the paradigm of the TSVF.

At first glance it may appear that our method of classification of correlations into paradoxical and nonparadoxical sectors may also carry over to state-independent tests of quantum contextuality. In reality, the situation is not as straightforward. For instance, the state-independent exclusivity graph of the Yu-Oh scenario [49,50] can be used to define the exclusivity conditions that need to be imposed on the ABL probabilities. These conditions then define the paradoxical and nonparadoxical sectors. However, it should be noted that the resultant classification is no longer state independent as it implicitly depends on PPS states (as per Definitions 1–3 in our classification). Therefore, our analysis becomes state dependent irrespective of the contextuality scenario.

Our results show that the ABL rule is essentially noncontextual, contrary to recent studies [31,33,34]. Most of the recent studies deal with probability assignments which are not properly conditioned and lead to scenarios where the sum of probabilities of exclusive events can be greater than 1, leading to false signatures of contextuality. Any such signature arises from Definition 2 and therefore violates the principle of exclusivity which is at the heart of operational theories [37–39]. Therefore, in order to observe a violation, the principle of exclusivity needs to be abandoned.

Furthermore, it is well known that the ABL rule and weak values are deeply inter-connected, and the conclusions made on the former should also translate in some manner to the latter. While it has already been shown that anomalous weak values are a proof of contextuality [33–35], imposing exclusivity conditions on them following our work is not as trivial as they can take on imaginary values which can be arbitrarily large as opposed to real, positive, and normalized ABL probabilities. Since the exclusivity conditions cannot be directly imposed on weak values, it is not clear how our classification can be performed. Therefore, one needs to identify better conditions that can help identify paradoxical and nonparadoxical correlations arising from weak values.

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