Letter

Fate of multiparticle resonances: From Q-balls to ^{3}He droplets

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We study a system of N nonrelativistic particles which form a near-threshold resonance. Assuming no subset of these particles can form a bound state, the resonance can only decay through an "explosion" into N particles. We find that the decay width of the resonance scales as $E^{\Delta-5/2}$ in the limit when the energy E of the resonance goes to zero, where Δ is the ground-state energy of a system of N particles in a spherical harmonic trap with unit frequency. The formula remains valid when some pairs of final particles have zero-energy s-wave resonance, but the Efimov effect is not present. In the limit of large N, we show that the final particles follow a Maxwell-Boltzmann distribution if they are bosons and a semicirclelike law if they are fermions. We expect our general result to be applicable to various systems that exist in nature. In particular, we argue that metastable ^{3}He droplets exist with the lifetime varying over many orders of magnitude ranging from a fraction of a nanosecond to values greatly exceeding the age of the Universe.

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Introduction.—The existence of "Borromean" states—bound states of three particles, of which no pair is capable of forming a bound state—and their generalization to more than three particles (the "Brunnian" states) are of great interest in nuclear and atomic physics [1]. In particular, much effort has been dedicated to the search for universal properties of these systems. The limit of zero-range interaction, where the Efimov effect is at play, has received the most attention; it has been shown that three- and four-particle Efimov states have universal properties [2,3]. For more than four particles, our knowledge is much more limited (see Ref. [4] for a review and further references).

In this Letter we are concerned not with many-particle *bound* states, but with many-particle *resonances* [5]. We address here a sharp question concerning the width of multiparticle resonances in the near-threshold regime: what is the behavior of the width of a resonance when its energy crosses zero, i.e., when the resonance is just about to become a bound state (for example, when a parameter characterizing the interaction is varied)? Our result shows that the asymptotic behavior of the decay width is universal,

$$\Gamma(E) \sim E^{\Delta-5/2}, \tag{1}$$

where Δ is the ground-state energy of a system of N "surrogate" particles in a spherical harmonic potential with unit oscillator frequency for all particles, provided that $\Delta > \frac{7}{2}$. The "surrogate" particles have the same properties as the particles that make up the resonance (mass, spin, statistics). The interaction between the surrogate particles is turned off, unless a pair of the original particles have infinite s-wave scattering length, in which case the corresponding surrogate particles have zero-range, infinite scattering length (i.e., unitarity) interaction [6]. The significance of Δ is that it is the conformal dimension [7] of the lowest-dimensional operator that creates the resonance from the vacuum.

Our result can be applied to various physical contexts where multiparticle resonances appear. For bosons interacting through a potential of the Lennard-Jones type, in a certain range of the de Boer parameter, bound clusters exist but only when the number of particles exceeds a critical value [8,9]. Metastable droplets then should appear at particle numbers slightly smaller than the critical value. Another example is droplets of ^{3}He atoms. It is known that ^{3}He atoms form a bound droplet only when there is a sufficient number of them. The minimal number of atoms in a bound ^{3}He droplet, N_0, has been estimated to be between 20 and 40 [10–14]. Metastable ^{3}He droplets can then appear when the number of atoms N is slightly smaller than N_0, e.g., for $N = N_0 - 1$. We are not aware of any previous estimate of the lifetimes of such metastable nanodroplets of ^{3}He. Quantum droplets may exist in weakly coupled bosonic mixtures [15]. In relativistic quantum field theory, a scalar quantum field theory that supports Q-balls [16,17] also allows for metastable Q-balls [18].

Previously known results.—Before presenting arguments leading to Eq. (1), let us check that it is consistent with all previously known results. For a two-body resonance with angular momentum ℓ, the energy of the surrogate system in the spherical harmonic trap with unit frequency is $3 + \ell$, and Eq. (1) then reproduces the known result $\Gamma \sim E^{\ell+1/2}$ for $\ell \geqslant 1$. For three bosons with no resonant interaction, the ground-state energy of the surrogate system is $\frac{9}{2}$, giving rise to the behavior $\Gamma \sim E^2$ previously found in Ref. [19]. When two of the three particles have infinite scattering length, the resonance interaction reduces the ground-state energy of the surrogate system by 1. Now Eq. (1) yields $\Gamma/E \sim E^0$, but as we will see, a more careful analysis reveals that there is a logarithmic modification which makes Γ/E decrease logarithmically as $E \to 0$, as first found in Ref. [20].

New results.—We can now read out the behavior of Γ for some cases which have not been solved before. The most nontrivial predictions involve spin-$\frac{1}{2}$ fermions at unitarity. For a resonance formed from two spin-up and one spin-down fermions of the same mass, with infinite s-wave scattering length between two fermions of different spins (an approximation for neutrons), the ground state in a harmonic trap has energy $\Delta = 4.27272$ for $\ell = 1$ and $\Delta = 4.66622$ for $\ell = 0$ [21,22]. The width of a near-threshold resonance then behaves as

$$\Gamma(E) \sim \begin{cases} E^{1.773}, & \ell = 1, \\ E^{2.166}, & \ell = 0. \end{cases} \quad (2)$$

In the case of neutrons, this behavior should hold for the trineutron resonance if such a resonance exists with energy between $E_a = \hbar^2/m_n a^2 \approx 0.1$ MeV and $E_{r_0} = \hbar^2/m_n r_0^2 \approx 5$ MeV, where m_n is the neutron mass, a and r_0 are the scattering length and the effective range of the nn scattering, respectively. If the energy of the resonance is less than E_a, the behavior of Γ is dictated by the ground-state energy of three *free* particles in the harmonic potential, which is $\frac{11}{2}$ for $\ell = 1$ and $\frac{13}{2}$ for $\ell = 0$. We find $\Gamma \sim E^3$ and $\Gamma \sim E^4$ for these two cases. A near-threshold three-neutron resonance does not seem to exist in the real world [23,24], but in model calculations it appears when a sufficiently strong three-body attraction is added to the forces between neutrons [25]. These behaviors are similar to the "un-nuclear" behavior of nuclear reactions with the emission of a few neutrons [26].

For a four-neutron resonance (which appears if sufficiently strong four-body attraction is added [27]) with energy in the regime $E_a \ll E \ll E_{r_0}$, the behavior of the width is controlled by the energy of the ground state of four unitary fermions in a spherical harmonic trap, which was numerically determined to be $\Delta \approx 5.0$ [28–34], so $\Gamma \sim E^{2.5}$. At energies much lower than E_a the behavior becomes $E^{11/2}$.

Weakly coupled bosonic droplets.—To gain intuition on the problem, let us first consider metastable droplets of bosons with small negative scattering length and effective three-body repulsion [9,35,36]. The Hamiltonian of the model reads

$$H[\psi] = \int d\boldsymbol{x}\left(\frac{|\nabla\psi|^2}{2} - \frac{g}{4}|\psi|^4 + \frac{G}{6}|\psi|^6\right). \quad (3)$$

(Here we set $\hbar = m = 1$.) When $G/g^4 \gg 1$, the droplets contain a large number of bosons (which are the nonrelativistic version of Q-balls) and can be found by minimizing the functional $H[\psi]$ at a fixed number of particles. Solving the problem numerically, we find that H has a local minimum with positive energy for $N_1 < N < N_0$ where

$$N_1 \approx 189.4\frac{G^{1/2}}{g^2}, \quad N_0 \approx 240.4\frac{G^{1/2}}{g^2}. \quad (4)$$

The parametric dependence of N_0 on g and G has been previously predicted in Ref. [9]. One can visualize the metastable droplet as the local minimum of the function that gives the energy as a function of the size of the droplet (Fig. 1).

The decay of a metastable droplet is described by an instanton, i.e., a solution to the equation of motion in Euclidean time. The instanton can be found mostly analytically for N near N_1 or N_0. For $N = N_1$, there is a flat direction in the functional space of the droplet density profiles. For $N > N_1$, moving along this direction towards larger droplet size R one encounters a potential barrier, as shown in Fig. 1. For small $N - N_1$, the width ΔR of the barrier shrinks as $\Delta R \sim (N - N_1)^{1/2}$ and vanishes at $N = N_1$. In this regime one can calculate the tunneling amplitude using the WKB approximation for the effective action in the collective coordinate R with the potential $U(R) \sim N[\Delta R(R - R_{\rm eq})^2 - (R - R_{\rm eq})^3]$ which has a metastable minimum at $R = R_{\rm eq}$ and a point of exit from the "tunnel" at $R = R_{\rm eq} + \Delta R$ as shown in Fig. 1. The imaginary action for classically forbidden tunneling is given by $S_I \sim \int_{R_{\rm eq}}^{R_{\rm eq}+\Delta R} dR\sqrt{NU(R)} \sim N(\Delta R)^{5/2}$, resulting in the exponentially suppressed decay rate $\Gamma \sim \omega\sqrt{S_I}\exp(-2S_I)$, where $\omega \sim \sqrt{\Delta R}$ is the frequency of the harmonic motion near the local minimum of $U(R)$, or

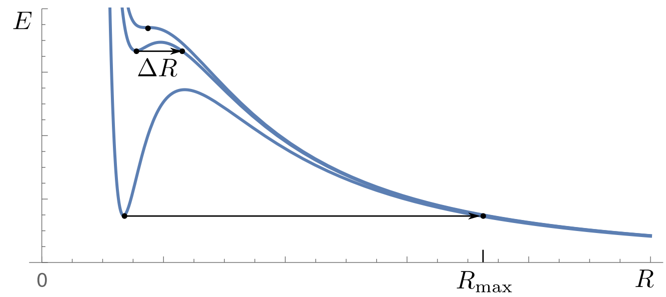

FIG. 1. The droplet's energy as a function of its size R for three values of N. The upper curve corresponds to $N = N_1$, the middle curve to $N = N_1(1 + \epsilon)$ where $0 < \epsilon \ll 1$, and the lowest curve to $N = N_0(1 - \epsilon)$.

$$\Gamma = \frac{c_2\sqrt{N_1}}{\sqrt{G}}\left(\frac{N - N_1}{N_1}\right)^{7/8}\exp\left[-c_1 N_1\left(\frac{N - N_1}{N_1}\right)^{5/4}\right], \quad (5)$$

where $c_1 \approx 1.58$ and $c_2 \approx 0.570$ [37].

At the other end of the window of metastability, near $N = N_0$, the system has to tunnel in Euclidean time to a droplet of a very large size before it can expand classically in real time. The energy of a cloud of N bosons with size R is $E \sim N/(mR^2)$, so coming out from under the barrier, the cloud of particles has size

$$R_{\max} \sim \sqrt{\frac{N}{mE}}, \quad (6)$$

which diverges as $E \to 0$. In contrast, the size of the system at the beginning of the tunneling process, $R_{\min}$, remains finite as $E \to 0$ (see Fig. 1). Most of the tunneling thus occurs in the regime where the interparticle interaction can be neglected. Since the potential energy behaves like $1/R^2$, the WKB exponent is proportional to $\ln(R_{\max}/R_{\min})$. To find the exact numerical coefficient, we need to solve the Euclidean equations of motion. Writing $\psi = fe^{i\theta}$, in Euclidean time $\tau = it$ and $\varphi = -i\theta$, the Euclidean action becomes

$$S_E = \int d\tau d\boldsymbol{x}\left(-f^2\partial_\tau\varphi - \frac{f^2}{2}(\nabla\varphi)^2 + \frac{(\nabla f)^2}{2}\right). \quad (7)$$

One can check that the following configuration is a solution to the Euclidean field equations with $E \to 0$: $f = (2\pi)^{-3/4}\sqrt{N}\exp(-\varphi)$ and $\varphi = r^2/(4\tau) + (3/4)\ln\tau$. This solution corresponds to a "Hubble expansion," $\nabla\varphi = \boldsymbol{r}/(2\tau)$. The solution applies in the intermediate regime when the size of the droplet $\tau^{1/2}$ is larger than the original size, but much smaller than the droplet size when it exits from under the barrier. Evaluating the Euclidean action of the solution, we find

$$S_E = \frac{3N}{2}\ln\frac{R_{\rm max}}{R_{\rm min}}. \tag{8}$$

The decay rate is $\Gamma \sim e^{-2S_E} \sim R_{\rm max}^{-3N}$. Since $R_{\rm max} \sim E^{-1/2}$, we find that $\Gamma \sim E^{3N/2}$. At large N, where the semiclassical instanton calculation applies, the resonance is narrow, i.e., $\Gamma \ll E$.

Field theory approach.—The above approach is not applicable when the number of particles in the droplet is small or when they are fermions. In these cases one can still find the behavior of the width of the resonance when its energy is small using a low-energy effective field theory. Let Ψ be the field describing the resonance, and ψ_a are the particles that constitute this resonance (which may belong to different species a). The effective field theory describing the system is

$$\mathcal{L} = \Psi^\dagger + \left(i\partial_t + \frac{\nabla^2}{2m_\Psi}\right)\Psi + L[\psi_a] + \mu_0\Psi^\dagger + \Psi + g(O^\dagger\Psi + \Psi^\dagger O), \tag{9}$$

where $L[\psi_a]$ is the Lagrangian of nonrelativistic conformal field theory (NRCFT) [7] of the ψ particles (the simplest version of a NRCFT is a free field theory), O is an (composite) operator with conformal dimension Δ in the theory described by $L[\psi_a]$, and g and μ_0 are some parameters. The simplest example of O is $O = \psi^N$ in the case where ψ is a boson field, where $\Delta = \frac{3}{2}N$. The coupling can be considered pointlike if the excitation energy of the droplet is larger than the typical energy of the final particles, which is the case when the resonance is near threshold. The field theory is assumed to have an ultraviolet cutoff at the momentum scale Λ (energy scale Λ^2).

The self-energy of Ψ obtained by integrating out ψ is

$$\Sigma(\omega, \boldsymbol{q}) = -ig^2\langle OO^\dagger\rangle_{\omega,\boldsymbol{q}}. \tag{10}$$

Galilean invariance implies that Σ is a function of $E = \omega - q^2/(2m_\Psi)$. The correlator of O, in general, contains ultraviolet divergences which are regularized by the cutoff Λ. These ultraviolet divergences contribute to the real (but not the imaginary) part of Σ.

We expand Σ in powers of E, keeping only the first two terms in the real part and the leading term in the imaginary part. When $\Delta > \frac{7}{2}$, the first two terms in the real part have power-law divergences, and the result reads

$$\Sigma = -g^2[a_0\Lambda^{2\Delta-5} + a_1\Lambda^{2\Delta-7}E + ib_0E^{\Delta-5/2}\theta(E)], \tag{11}$$

where a_0, a_1, and b_0 are some numbers. The existence of a low-energy resonance means that $\mu = \mu_0 + g^2a_0\Lambda^{2\Delta-5}$ is fine-tuned to an unnaturally small value; the a_1 term leads to a wave-function renormalization for Ψ: $Z^{-1} = 1 + g^2a_1\Lambda^{2\Delta-7}$. The propagator of Ψ is now

$$\langle\Psi\Psi^\dagger+\rangle \sim [Z^{-1}E + \mu + ig^2b_0E^{\Delta-5/2}\theta(E)]^{-1}. \tag{12}$$

When μ is small, the propagator's pole is located at $E_* = \mathrm{Re}\,E_* + i\,\mathrm{Im}\,E_*$, where $\mathrm{Re}\,E_* = -Z\mu$ and

$$\mathrm{Im}\,E_* = -g^2b_0Z(\mathrm{Re}\,E_*)^{\Delta-5/2}, \tag{13}$$

which goes to zero faster than $\mathrm{Re}\,E$ for $\Delta > \frac{7}{2}$. For example, for three bosons in an s-wave resonance, $\Delta = \frac{9}{2}$ and $\mathrm{Im}\,E \sim (\mathrm{Re}\,E)^2$, as found in Ref. [19] using a different method.

Consider now the case $\Delta = 7/2$. This case corresponds to two particles in s-wave resonance and a third particle of a different type that does not interact resonantly with any of the first two. The resonant pair is described by a "dimer" field with dimension 2 [7], and the third particle is described by a free field of dimension $\frac{3}{2}$, so the total dimension of O is $2 + \frac{3}{2} = \frac{7}{2}$. In this case,

$$\Sigma(\omega, \boldsymbol{q}) = -\left[a_0\Lambda^2 + a_1E\ln\frac{\Lambda^2}{|E|} + i\pi a_1E\,\theta(E)\right], \tag{14}$$

and redoing the analysis one sees that the ratio between the imaginary and real parts of the position of the pole decreases logarithmically with the energy. This was previously found in Ref. [20].

We now rederive Eq. (1) using a different method, which allows us to gain additional intuition for the behavior and also gives us additional information about the decay. In particular, for a resonance of $N \gg 1$ particles we will find the momentum distribution of the final particles.

Decay as tunneling through a centrifugal barrier.—The suppression of the decay rate as $E \to 0$ can be interpreted as the result of tunneling under a barrier. Instead of the position of N particles one can introduce the center-of-mass coordinate, one hyperradius, and $N-2$ hyperangles. Factoring out the center-of-mass motion, the Schrödinger equation with no interaction can then be written as

$$\frac{\partial^2\psi}{\partial R^2} + \frac{3N-4}{R}\frac{\partial\psi}{\partial R} + \frac{\Delta_\Omega\psi}{R^2} = 0, \tag{15}$$

where Δ_Ω is the Laplacian operator in hyperangles. For bosons the lowest eigenvalue of $-\Delta_\Omega$ is 0, which corresponds to the solution $\psi \sim R^{-3N+5}$. The decay rate can be obtained by evaluating the probability flux at $R = R_{\rm max}$: $R^{3N-4}\psi^*\partial_R\psi \sim R^{-(3N-5)}$. For $R_{\rm max} \sim E^{-1/2}$, this implies $\Gamma \sim E^{(3N-5)/2}$ [38].

For fermions or particles interacting via an s-wave resonance in general, the picture of the decay as tunneling under a $1/R^2$ barrier is still valid [37]. From the mapping between the dimension of the primary operator and the energy in a harmonic trap, the coefficient of the $1/R^2$ potential is determined to be $(\Delta-2)(\Delta-3)/2$. It has been computed in the context of three-body scattering in Refs. [39,40]. The same discussion as in the bosonic case then gives the decay rate scaling as $E^{\Delta-5/2}$ in agreement with the field theory approach. One can also make use of the SO(2,1) symmetry of NRCFT to arrive at the same conclusion.

Momentum distribution of final particles.—We now ask the following question: What is the momentum distribution of the final decay products of a metastable droplet of N particles, where $N \gg 1$?

For bosons, this distribution can be derived from the following argument: The amplitude of the decay of one resonance into N bosons $\Psi \to N\psi$ should be independent of the momenta of final particles when the latter are small. This implies that the distribution of final particles over momentum is the same as that in a microcanonical ensemble of N bosons where the energy is fixed to the energy of the resonance, E, and the momentum is fixed to 0. In the limit of a large number of particles, the ensemble is equivalent to the canonical ensemble; hence, the final particles should follow the Maxwell-Boltzmann distribution,

$$\frac{dN}{d\boldsymbol{p}} \sim \exp\left(-\frac{\boldsymbol{p}^2}{2mT_{\text{eff}}}\right), \quad (16)$$

with the effective temperature determined by the total energy, i.e., $T_{\text{eff}} = \frac{2}{3}E/N$. The same result follows from the Gaussian shape of the droplet's wave function with width $R_{\max} = \sqrt{3N/4E}$ at the end of the tunneling described by the Euclidean action in Eq. (7).

Now consider a resonance consisting of N fermions. For simplicity, let us consider spinless fermions. The vertex describing the decay of the resonance is now

$$\mathcal{L}_{\text{int}} = g\Psi^\dagger + \psi\,\partial_x\psi\,\partial_y\psi\,\partial_z\psi\,\partial_x^2\psi\,\partial_x\partial_y\psi\cdots. \quad (17)$$

The probability distribution function of the particles over their momenta is then

$$\rho(\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_n) \sim \begin{vmatrix} P_1(\boldsymbol{p}_1) & P_1(\boldsymbol{p}_2) & \dots & P_1(\boldsymbol{p}_N) \\ P_2(\boldsymbol{p}_1) & P_2(\boldsymbol{p}_2) & \dots & P_2(\boldsymbol{p}_N) \\ \cdots & \cdots & \cdots & \cdots \\ P_N(\boldsymbol{p}_1) & P_N(\boldsymbol{p}_2) & \dots & P_N(\boldsymbol{p}_N) \end{vmatrix}^2 \times \delta\left(\sum_a \boldsymbol{p}_a\right)\delta\left(\sum_a \frac{\boldsymbol{p}_a^2}{2m} - E\right), \quad (18)$$

where $P_1(\boldsymbol{p}) = 1$, $P_2(\boldsymbol{p}) = p_x$, $P_3(\boldsymbol{p}) = p_y, \dots$ are monomials of $\boldsymbol{p}$, each corresponding to a factor in the vertex (17). In the limit of large N we can replace the δ functions by the exponential factor $\exp[-\beta\sum_a \boldsymbol{p}_a^2/2m]$. It can be seen that the probability distribution function (18) is the square of the wave function of a ground state of N fermions in a harmonic potential with a suitable frequency. The distribution of final particles over momentum can be obtained through the Thomas-Fermi approximation of particles in a harmonic trap. The result is (the formula also works for spin-$\frac{1}{2}$ fermions)

$$\frac{dN}{d\boldsymbol{p}} \sim \left(p_{\max}^2 - \boldsymbol{p}^2\right)^{3/2}, \quad (19)$$

where $p_{\max}^2 = 16mE/(3N)$.

Lifetime of metastable ^{3}He *droplets*.—According to Monte Carlo calculations, a cluster of N ^{3}He atoms becomes bound at some $N = N_0$ between 20 and 40 [10–14]. Thus, there must be a range of N, $N_1 < N < N_0$, where the droplet has positive energy and is metastable. For N slightly smaller than N_0, the energy of the droplet is small, so its lifetime must be large. For example, for $N = 20$ the energy per atom in the droplet was estimated to be about 0.2 K [10–12], much smaller than the binding energy per particle of infinite ^{3}He liquid (2.4 K). For N just below N_0 the energy of the droplet is even smaller, typically less than 1 K for the whole droplet.

To estimate the lifetime of a metastable ^{3}He droplet, we note that the energy of noninteracting N spin-$\frac{1}{2}$ particles in a harmonic trap of unit frequency is

$$\Delta = 60 + \tfrac{9}{2}(N - 20) \quad (20)$$

for $20 \leqslant N \leqslant 40$. Taking N_0 to be the smallest number quoted in the literature, $N_0 = 29$, we consider the metastable droplet with $N_0 - 1 = 28$ atoms, $\Delta = 96$. This leads to a huge power in the dependence of the width on the energy: $\Gamma \sim (E/E_0)^{93.5}$.

E_0 can be estimated to be the kinetic energy of a free Fermi gas of N particles, confined by a harmonic potential with frequency chosen so that the rms size of that cloud of particles is equal to the rms size of the metastable droplet. In the Thomas-Fermi approximation, the kinetic energy of a cloud of N particles in a harmonic potential is related to its rms size $\langle r^2\rangle^{1/2}$ by

$$E_0 = \frac{3^{8/3}}{32}\frac{N^{5/3}}{m\langle r^2\rangle}. \quad (21)$$

For $N = 28$ and $\langle r^2\rangle^{1/2} \sim 8$ Å [10], we find $E_0 \sim 40$ K. For $E \sim 1$ K, the suppression factor $(E/E_0)^{93.5}$ becomes 10^{-150}. Even with the large uncertainty in the estimate, it is obvious a ^{3}He droplet containing one or a few particles less than the smallest stable droplet should live longer than the age of the Universe. These droplets, though having positive energy (relative to the free atoms), are essentially stable.

As the number of atoms in the droplet decreases, the lifetime becomes shorter and, at some value, must become comparable to $\hbar/\text{K} \sim 10^{-11}$ s. By varying the number of particles, the lifetime of the ^{3}He droplet can vary from a fraction of a nanosecond to values much larger than the age of the Universe. Unfortunately at this moment we have no method that can tell us reliably the lifetime of a ^{3}He droplet with a given N, nor can we say for which numbers of atoms the lifetime of the droplet may be in the experimentally interesting range.

Conclusion.—In this Letter we have shown that the near-threshold N-body resonances have certain universal properties. The lifetime of the resonance scales with energy with a universal exponent. The momentum distribution of the final decay products is also universal.

We have shown that, in nature, metastable ^{3}He droplets exist in a range of sizes. It would be useful to quantitatively determine that range and the lifetime of droplets with size therein. While the energetics of small ^{3}He droplets can be determined using various numerical methods, the study of the lifetime will likely require the development of new approaches. We hope that metastable droplets of ultracold atoms that decay into individual atoms can also be created and studied in the laboratory.

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